Ground state solutions for asymptotically linear Schrödinger equations on locally finite graphs

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Abstract

We are considered with the following nonlinear Schrödinger equation

$$-\Delta u + (\lambda a(x) + 1)u = f(u), x \in V,$$

on a locally finite graph $G = (V, E)$, where V denotes the vertex set, E denotes the edge set, $\lambda > 1$ is a parameter, $f(s)$ is asymptotically linear with respect to s at infinity, and the potential $a : V \to [0, +\infty)$ has a nonempty well Ω. By using variational methods we prove that the above problem has a ground state solution u_λ for any $\lambda > 1$. Moreover, we show that as $\lambda \to \infty$, the ground state solution u_λ converges to a ground state solution of a Dirichlet problem defined on the potential well Ω.

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1. Introduction

In this paper, we consider the following asymptotically linear Schrödinger equation

$$-\Delta u + (\lambda a(x) + 1)u = f(u), x \in V, \tag{1.1}$$

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on a connected locally finite graph $G = (V, E)$, where V denotes the vertex set, E denotes the edge set, and $\lambda > 1$ is a parameter. We call G a locally finite graph if for any $x \in V$, there are only finite $y \in V$ such that $xy \in E$. A graph is connected if any two vertices x and y can be connected via finite edges. For any edge $xy \in E$, we assume that the weight ω_{xy} satisfies $\omega_{xy} = \omega_{yx} > 0$. The measure $\mu : V \to \mathbb{R}^+$ is a finite positive function. For any function $u : V \to \mathbb{R}$, the graph Laplacian of u is defined as

$$\Delta u(x) := \frac{1}{\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x)), \tag{1.2}$$

where $y \sim x$ stands for any vertex y connected with x by an edge $xy \in E$.

In the past decade, the study of partial differential equations on graphs, which has a wide range of applications in image processing, data analysis and neural networks, has been receiving great attention, see [1, 2, 3] for more practical backgrounds. Recently, there are some mathematical works on geometric inequalities on graphs [4, 5], and the heat equation on graphs [6, 7, 8, 9, 10]. In the case of elliptic differential equations on graphs, Grigor'yan et al. [11, 12, 13] obtained the existence results by using variational methods. Specially in [13], Grigor'yan, Lin and Yang proved the existence of strictly positive solutions for (1.1) when the nonlinearity f satisfies the so-called Ambrosetti-Rabinowitz ((AR) for short) condition:

(AR): there exists a constant $\theta > 2$ such that for all $x \in V$ and $s > 0$,

$$0 < \theta F(s) \leq s f(s).$$

By using the Nehari manifold method, Zhang and Zhao [14] proved the existence and asymptotical behavior of ground state solutions for (1.1) when the nonlinearity f satisfies $f = |u|^{p-1}u$. The aim of this paper is to investigate the asymptotically linear Schrödinger equation on locally finite graphs and extend the results of [14].

Before introducing the assumptions on the nonlinearity f and the potential function a in (1.1), we give some preliminary settings. For any two functions $u, v : V \to \mathbb{R}$, the gradient form of u and v on the graph is defined by

$$\Gamma(u, v)(x) := \frac{1}{2\mu(x)} \sum_{y \sim x} \omega_{xy}(u(y) - u(x))(v(y) - v(x)). \tag{1.3}$$

Setting $\Gamma(u) = \Gamma(u,u)$, we denote the length of its gradient by

$$|\nabla u|(x) := \sqrt{\Gamma(u)(x)} = \left(\frac{1}{2\mu(x)} \sum_{y\sim x} \omega_{xy}(u(y)-u(x))^2\right)^{1/2}. \tag{1.4}$$

The integral of a function $u : V \to \mathbb{R}$ is defined by

$$\int_V u d\mu = \sum_{x\in V} \mu(x)u(x). \tag{1.5}$$

Let $C_c(V)$ be the set of all functions with compact support, and $W^{1,2}(V)$ be the completion of $C_c(V)$ under the norm

$$\|u\|_{W^{1,2}(V)} = \left(\int_V \left(|\nabla u|^2 + u^2\right) d\mu\right)^{1/2}. \tag{1.6}$$

We define a space of functions

$$H_\lambda = \left\{u \in W^{1,2}(V) : \int_V \lambda a(x)u^2 d\mu < +\infty\right\}, \tag{1.7}$$

with a norm

$$\|u\|_{H_\lambda} = \left(\int_V \left(|\nabla u|^2 + (\lambda a(x)+1)u^2\right) d\mu\right)^{1/2}, \tag{1.8}$$

and the inner product

$$\langle u,v\rangle_{H_\lambda} = \int_V \left(\Gamma(u,v) + (\lambda a(x)+1)uv\right) d\mu,\ \forall u,v \in H_\lambda.$$

Then, H_λ is a Hilbert space. For any $x, y \in V$, the distance $d(x,y)$ is defined by the minimal number of edges which connect x and y. Given a subset $\Omega \subset V$, we call Ω a bounded domain in V if the distance $d(x,y)$ is uniformly bounded from above for any $x, y \in \Omega$. The boundary of Ω is defined as

$$\partial\Omega := \{x \in V\backslash\Omega : \exists\, y \in \Omega \text{ such that } xy \in E\}. \tag{1.9}$$

Now, we give the following conditions on f and a in problem (1.1):

(F_1) $f : V \times \mathbb{R} \to \mathbb{R}$, $f(s)$ is continuous in s, $f(s) \equiv 0$ for all $s \leq 0$ and $f(s)s^{-1} \to 0$ as $s \to 0^+$.

(F_2) Let

$$\mu^* = \inf\left\{\int_V \left[|\nabla u|^2 + (\lambda a(x)+1)u^2\right] d\mu : u \in H_\lambda,\ \int_V u^2 d\mu = 1\right\}. \quad (1.10)$$

There exists $l \in (\mu^*, +\infty)$ such that $f(s)s^{-1} \to l$ as $s \to +\infty$.

(F_3) Let $F(s) = \int_0^s f(t)dt$ and $\Phi(s) = \frac{1}{2}f(s)s - F(s)$. $F(s), \Phi(s) \geq 0$ for all $s \in \mathbb{R}$, and $\Phi(s) \to +\infty$ as $s \to +\infty$.

(F_4) $f(s)s^{-1}$ is strictly increasing in $s > 0$.

(A_1) $a : V \to [0, +\infty)$, and the potential well $\Omega = \{x \in V : a(x) = 0\}$ is a nonempty, connected and bounded domain in V.

(A_2) There exists a vertex $x_0 \in V$ such that $a(x) \to +\infty$ as $d(x, x_0) \to +\infty$.

The functional related to (1.1) is defined by

$$I_\lambda(u) = \frac{1}{2}\int_V \left(|\nabla u|^2 + (\lambda a(x)+1)u^2\right) d\mu \ - \int_V F(u)d\mu,\ u \in H_\lambda. \quad (1.11)$$

From Proposition 2.1, we deduce that $I_\lambda \in C^1(H_\lambda, \mathbb{R})$ and

$$\langle I_\lambda'(u), v\rangle = \int_V \left(\Gamma(u,v) + (\lambda a(x)+1)uv\right) d\mu - \int_V f(u)v d\mu,\ \forall\, v \in H_\lambda. \quad (1.12)$$

A function $u \in H_\lambda$ is said to be a nonzero solution of problem (1.1) if $\mu\{x \in V : u(x) \neq 0\} > 0$ and $\langle I_\lambda'(u), v\rangle = 0$ for any $v \in H_\lambda$. We call a nonzero solution u_0 of (1.1) as a ground state solution if $I_\lambda(u_0) \leq I_\lambda(u)$ for any nonzero solution u of (1.1).

Throughout this paper, we always assume that there exists a constant $\mu_{min} > 0$ such that $\mu(x) \geq \mu_{min}$ for all $x \in V$. The main results of this paper are as follows:

Theorem 1.1. *Let $G = (V, E)$ be a locally finite and connected graph. Assume that (A_1), (A_2), (F_1), (F_2) and (F_3) hold. Then for any $\lambda > 1$, there exists a nonzero solution u of (1.1).*

Theorem 1.2. *Under the same conditions in Theorem 1.1, then for any $\lambda > 1$, there exists a ground state solution u_λ of (1.1).*

To investigate the behavior of u_λ as $\lambda \to \infty$, we introduce the following Dirichlet problem

$$\begin{cases} -\Delta u + u = f(u), x \in \Omega, \\ u = 0, x \in \partial\Omega, \end{cases} \quad (1.13)$$

where Ω is the potential well given by (A_2). It is suitable to study (1.13) in the Hilbert space $W_0^{1,2}(\Omega)$, which is the completion of $C_c(\Omega)$ under the norm

$$\|u\|_{W_0^{1,2}(\Omega)} = \left(\int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu + \int_\Omega u^2 d\mu\right)^{1/2}.$$

The functional related to (1.13) is defined by

$$I_\Omega(u) = \frac{1}{2}\left(\int_{\Omega\cup\partial\Omega} |\nabla u|^2 d\mu + \int_\Omega u^2 d\mu\right) - \int_\Omega F(u) d\mu,\ u \in W_0^{1,2}(\Omega). \tag{1.14}$$

We remark that unlike in Euclidean Spaces, we do not have $|\nabla u|(x) = 0$ for $x \in \partial\Omega$ by the definitions of (1.4) and (1.9). Moreover, we find that (1.13) is some kind of limit problem for (1.1) as $\lambda \to +\infty$. Precisely, we have the following theorem.

Theorem 1.3. *Let $G = (V, E)$ be a locally finite and connected graph. Assume that (A_1), (A_2), (F_1)-(F_4) and $\lambda > 1$ hold. Then, for any sequence $\lambda_k \to \infty$, up to a subsequence, the corresponding ground state solutions u_{λ_k} of (1.1) strongly converge in $W^{1,2}(V)$ to a ground state solution u_0 of (1.13).*

2. Existence of a ground state solution

In this section, we firstly prove that (1.1) has a nonzero solution. For that, we introduce an embedding result from [14] Lemma 2.6.

Proposition 2.1. *Assume that $\lambda > 1$ and $a(x)$ satisfies (A_1) and(A_2). Then H_λ is continuously embedded into $L^q(V)$ for any $q \in [2, +\infty)$ and there exists a constant $C > 0$ independent of λ such that for any $u \in H_\lambda$, $\|u\|_{q,V} \le C\|u\|_{H_\lambda}$. Moreover, for any bounded sequence $\{u_k\} \subset H_\lambda$, there exists $u \in H_\lambda$ such that, up to a subsequence,*

$$\begin{cases} u_k \rightharpoonup u & \text{in } H_\lambda, \\ u_k(x) \to u(x) & \forall\ x \in V, \\ u_k \to u & \text{in } L^q(V). \end{cases}$$

Then we show that the functional I_λ defined by (1.11) has a mountain pass geometry.

Lemma 2.1. *Let the conditions (A_1),(A_2), (F_1) and (F_2) hold. Then for $\lambda > 1$, there exist two constants $\rho, \eta > 0$ such that*

$$\inf\{I_\lambda(u) : u \in H_\lambda, \|u\|_{H_\lambda} = \rho\} \ge \eta.$$

Proof. For any $\varepsilon > 0$, it follows from (F_1), (F_2) that there exists $C_\varepsilon > 0$ such that

$$|f(s)| \leq \varepsilon|s| + C_\varepsilon|s|^2, \ \forall s \in \mathbb{R},$$

Then by Proposition 2.1, we have for any $u \in H_\lambda$,

$$\begin{aligned}\int_V F(u)d\mu &\leq \frac{\varepsilon}{2}\int_V |u|^2 d\mu + \frac{C_\varepsilon}{3}\int_V |u|^3 d\mu \\ &\leq \frac{\varepsilon C_1}{2}\|u\|_{H_\lambda}^2 + \widetilde{C}_\varepsilon\|u\|_{H_\lambda}^3.\end{aligned} \tag{2.1}$$

This yields

$$I_\lambda(u) \geq \frac{1-\varepsilon C_1}{2}\|u\|_{H_\lambda}^2 - \widetilde{C}_\varepsilon\|u\|_{H_\lambda}^3. \tag{2.2}$$

Choosing $\varepsilon \in (0, \frac{1}{C_1})$ and $\rho > 0$ small enough, we see that there is $\eta > 0$ such that this lemma holds. □

Lemma 2.2. *Let the conditions* (A_1),(A_2), (F_1) *and* (F_2) *hold. Then for* $\lambda > 1$, *there exists* $e \in H_\lambda$ *with* $\|e\|_{H_\lambda} > \rho$ *such that* $I_\lambda(e) < 0$, *where* ρ *is given by Lemma 2.1.*

Proof. By the definition of μ^* in (F_2), there exists $\phi \in H_\lambda$ such that $\int_V \phi^2 d\mu = 1$ and $\mu^* \leq \|\phi\|_{H_\lambda}^2 < l$. Then, by (F_1), (F_2) and Fatou's lemma we get

$$\begin{aligned}\lim_{t\to+\infty}\frac{I_\lambda(t\phi)}{t^2} &= \frac{1}{2}\|\phi\|_{H_\lambda}^2 - \lim_{t\to+\infty}\int_V \frac{F(t\phi)}{t^2\phi^2}\phi^2 d\mu \\ &\leq \frac{1}{2}\|\phi\|_{H_\lambda}^2 - \frac{1}{2}l\int_V \phi^2 d\mu \\ &\leq \frac{1}{2}(\mu^* - l)\int_V \phi^2 d\mu < 0,\end{aligned}$$

and the lemma is proved by taking $e = t_0\phi$ with $t_0 > 0$ large enough. □

Based on Lemmas 2.1 and 2.2, by Mountain Pass Theorem there is a sequence $\{u_n\} \subset H_\lambda$ such that

$$I_\lambda(u_n) \xrightarrow{n} c \geq \eta \quad \text{and} \quad (1 + \|u_n\|)\|I_\lambda'(u_n)\|_{H_\lambda^*} \xrightarrow{n} 0, \tag{2.3}$$

where H_λ^* denotes the dual space of H_λ.

In the following lemma, we show that $\{u_n\}$ is bounded in H_λ.

Lemma 2.3. *Let the conditions* (A_1),(A_2) *and* (F_1)-(F_3) *hold. Then for* $\lambda > 1$, $\{u_n\}$ *is bounded in* H_λ.

Proof. Suppose by contradiction that $\|u_n\|_{H_\lambda} \xrightarrow{n} +\infty$. Letting $w_n = \dfrac{u_n}{\|u_n\|_{H_\lambda}}$, then $\|w_n\|_{H_\lambda} = 1$ and there exists $w \in H_\lambda$ such that, up to a subsequence, $w_n \stackrel{n}{\rightharpoonup} w$ weakly in H_λ. From (2.3) we get

$$o(1) = \frac{\langle I_\lambda'(u_n), u_n\rangle}{\|u_n\|_{H_\lambda}^2} = 1 - \int_V \frac{f\left(u_n(x)\right)}{u_n(x)}|w_n(x)|^2 d\mu. \tag{2.4}$$

Here, and in what follows, $o(1)$ denotes a quantity which tends to zero as $n \to \infty$.

Set

$$L := \sup\left\{\frac{f(s)}{s} : s \neq 0\right\}. \tag{2.5}$$

By $(F_1)(F_2)$ we have $0 < L < +\infty$. Since $w_n \stackrel{n}{\rightharpoonup} w$ weakly in H_λ, by Proposition 2.1, we get $w_n \xrightarrow{n} w$ strongly in $L^2(V)$. Therefore,

$$\int_V \frac{f\left(u_n(x)\right)}{u_n(x)}|w_n(x)|^2 d\mu \leq \int_V L|w_n(x)|^2 d\mu = o(1) + L\int_V |w(x)|^2 d\mu. \tag{2.6}$$

From (2.4), we deduce that $\int_V |w(x)|^2 d\mu > 0$.

Let $A = \{x \in V : w(x) \neq 0\}$. Then $\mu(A) > 0$. Noting that $u_n(x) \xrightarrow{n} +\infty$ for $x \in A$, from (2.3) and (F_3) we get a contradiction that

$$\begin{aligned} c + o(1) &= I_\lambda(u_n) - \frac{1}{2}\langle I_\lambda'(u_n), u_n\rangle \\ &= \int_V \left(\frac{1}{2}f(u_n)u_n - F(u_n)\right) d\mu \\ &= \int_V \Phi(u_n) d\mu \geq \int_A \Phi(u_n) d\mu \xrightarrow{n} +\infty. \end{aligned} \tag{2.7}$$

So, $\{u_n\}$ is bounded in H_λ. $\square$

Proof of Theorem 1.1 By Lemma 2.3, $\{u_n\}$ is bounded in H_λ. Then there exists a subsequence, still denoted by $\{u_n\}$, such that $u_n \stackrel{n}{\rightharpoonup} u$ weakly in H_λ. By Proposition 2.1, we get $u_n \xrightarrow{n} u$ strongly in $L^q(V)$ for $q \in [2, +\infty)$. Then from (2.3) we deduce that

$$\left|\langle u_n, \varphi\rangle_{H_\lambda} - \int_V f(u_n)\varphi d\mu\right| = o(1),\ \forall\ \varphi \in H_\lambda, \tag{2.8}$$

which implies that $\langle I_\lambda'(u), \varphi\rangle = 0$ for any $\varphi \in H_\lambda$.

By (2.5) and Hölder inequality, we have

$$\begin{aligned}\left|\int_V f(u_n)(u_n-u)d\mu\right| &= \left|\int_V \frac{f(u_n)}{u_n}(u_n)(u_n-u)d\mu\right| \\ &\leq L\int_V (u_n)(u_n-u)d\mu \\ &\leq L\left(\int_V (u_n-u)^2 d\mu\right)^{1/2}\left(\int_V u_n^2 d\mu\right)^{1/2} = o(1).\end{aligned} \tag{2.9}$$

Taking $\varphi = u_n - u$ in (2.8), we have

$$\langle u_n, u_n-u\rangle_{H_\lambda} = \int_V f(u_n)(u_n-u)d\mu + o(1) = o(1). \tag{2.10}$$

On the other hand, we have $\langle u, u_n-u\rangle_{H_\lambda} = o(1)$ by $u_n \rightharpoonup u$ weakly in H_λ. This and (2.10) lead to $u_n \to u$ strongly in H_λ. Then $I_\lambda(u) = c > 0$, which implies that $\mu\{x \in V : u(x) \neq 0\} > 0$. Thus, u is a nonzero solution of (1.1). □

Proof of Theorem 1.2 In order to get the ground state of (1.1), we consider the following minimization problem

$$c_0 = \inf\left\{I_\lambda(u) : u \in \mathcal{M}_\lambda\right\}, \tag{2.11}$$

where

$$\mathcal{M}_\lambda = \left\{u \in H_\lambda : \mu\{x \in V : u(x) \neq 0\} > 0, \text{ and } \langle I_\lambda'(u), u\rangle = 0\right\}.$$

Theorem 1.1 implies $\mathcal{M}_\lambda \neq \emptyset$. For any $u \in \mathcal{M}_\lambda$, by (F_3) we have

$$I_\lambda(u) = I_\lambda(u) - \frac{1}{2}\langle I_\lambda'(u), u\rangle = \int_V \Phi(u)d\mu \geq 0,$$

which implies that $c_0 \geq 0$.

From $(F_1)(F_2)$, for any $\varepsilon > 0$, there exists $C(\varepsilon) > 0$ such that

$$\|u\|_{H_\lambda}^2 = \int_V f(u)ud\mu \leq \varepsilon\|u\|_{H_\lambda}^2 + C(\varepsilon)\|u\|_{H_\lambda}^3.$$

Thus, there exists $\delta > 0$ such that $\|u\|_{H_\lambda} \geq \delta > 0$ for any $u \in \mathcal{M}_\lambda$.

Choosing a minimization sequence $\{u_k\} \subset \mathcal{M}_\lambda$ such that $I_\lambda(u_k) \xrightarrow{k} c_0$, by the proof of Theorem 1.1 there exists $u_\lambda \in H_\lambda$ such that $u_k \xrightarrow{k} u_\lambda$ strongly in H_λ. Then we deduce that $I_\lambda(u_\lambda) = c_0$, $\langle I'_\lambda(u_\lambda), \varphi\rangle = 0$ for any $\varphi \in H_\lambda$, and $\|u_\lambda\|_{H_\lambda} \geq \delta > 0$. Therefore, u_λ is a ground state of (1.1). $\square$

3. Asymptotic behavior of ground state solutions

In this section, we show the asymptotic behavior of ground state solutions u_λ as $\lambda \to +\infty$. Firstly, we give the following embedding result from [21, Lemma 2.7].

Proposition 3.1. *Assume that Ω is a bounded domain in V. Then $W_0^{1,2}(\Omega)$ is continuously embedded into $L^q(\Omega)$ for any $q \in [1, +\infty)$. Moreover, for any bounded sequence $\{u_k\} \subset W_0^{1,2}(\Omega)$, there exists $u \in W_0^{1,2}(\Omega)$ such that, up to a subsequence,*

$$\begin{cases} u_k \rightharpoonup u & \text{in } W_0^{1,2}(\Omega), \\ u_k(x) \to u(x) & \forall\ x \in \Omega, \\ u_k \to u & \text{in } L^q(\Omega). \end{cases}$$

Next, we prove that (1.13) is some kind of limit problem for (1.1) as $\lambda \to +\infty$.

Lemma 3.1. *Set*

$$m_\lambda = \inf\{I_\lambda(u),\ u \in \mathcal{M}_\lambda\}, \text{ and } m_\Omega = \inf\{I_\Omega(u),\ u \in \mathcal{M}_\Omega\},$$

where

$$\mathcal{M}_\Omega = \left\{u \in W_0^{1,2}(\Omega) : \mu\{x \in \Omega : u(x) \neq 0\} > 0, \text{ and } \langle I'_\Omega(u), u\rangle = 0\right\}.$$

Then under the conditions of Theorem 1.2, we have $m_\lambda \to m_\Omega$ as $\lambda \to +\infty$.

Proof. Since $W_0^{1,2}(\Omega) \subset H_\lambda$, we have $m_\lambda \leq m_\Omega$ for any $\lambda > 1$. Take a sequence $\lambda_k \to +\infty$ such that

$$\lim_{k\to\infty} m_{\lambda_k} = M \leq m_\Omega, \tag{3.1}$$

and let $u_{\lambda_k} \in \mathcal{M}_{\lambda_k}$ be such that $I_\lambda(u_{\lambda_k}) = m_{\lambda_k}$. By the proof of Theorem 1.2, we get $M \geq 0$. Similar to the proof of Lemma 2.3, we deduce that $\{u_{\lambda_k}\}$ is bounded in $W^{1,2}(V)$. Then, up to a subsequence, there exists $u_0 \in W^{1,2}(V)$ such that $u_{\lambda_k} \rightharpoonup u_0$ weakly in $W^{1,2}(V)$. Moreover, by $u_{\lambda_k} \in \mathcal{M}_{\lambda_k}$ and Proposition 2.1, we deduce that there exists a constant $\delta_1 > 0$, which is independent of λ_k such that

$$\int_V u_{\lambda_k}^2 d\mu \geq \delta_1. \tag{3.2}$$

Thus, from $u_{\lambda_k} \to u_0$ strongly in $L^2(V)$, we get $\int_V u_0^2 d\mu \geq \delta_1 > 0$.

We claim that $u_0|_{\Omega^c} = 0$. Otherwise, there exists a vertex $x_0 \notin \Omega$ such that $u_0(x_0) \neq 0$. Since $u_{\lambda_k} \in \mathcal{M}_{\lambda_k}$, we get

$$\begin{aligned} I_{\lambda_k}(u_{\lambda_k}) &= I_{\lambda_k}(u_{\lambda_k}) + \frac{1}{2}\left\langle I'_{\lambda_k}(u_{\lambda_k}), u_{\lambda_k}\right\rangle \\ &= \|u_{\lambda_k}\|_{H_\lambda}^2 + \int_V \left(\frac{1}{2} f(u_{\lambda_k})u_{\lambda_k} - F(u_{\lambda_k})\right) d\mu \\ &= \|u_{\lambda_k}\|_{H_\lambda}^2 + \int_V \Phi(u_{\lambda_k}) d\mu \geq \|u_{\lambda_k}\|_{H_\lambda}^2 \\ &\geq \int_V \lambda_k a(x) u_{\lambda_k}^2 d\mu \geq \lambda_k a(x_0)\mu(x_0) u_{\lambda_k}^2(x_0). \end{aligned} \tag{3.3}$$

Since $a(x_0) > 0$ for $x_0 \notin \Omega$, $\mu(x_0) \geq \mu_{\min} > 0$, $u_{\lambda_k}(x_0) \to u_0(x_0) \neq 0$ as $\lambda_k \to +\infty$, from (3.3) we get

$$m_{\lambda_k} = \lim_{k\to\infty} I_{\lambda_k}(u_{\lambda_k}) = +\infty,$$

which is a contradiction with (3.1).

Since $u_{\lambda_k} \rightharpoonup u_0$ in $W^{1,2}(V)$ and $u_{\lambda_k} \to u_0$ in $L^q(V)$ for any $q \in [2, +\infty)$, by Fatou's lemma and Lebesgue dominated convergence theorem we get

$$\begin{aligned} &\int_{\Omega\cup\partial\Omega} \left(|\nabla u_0|^2 + u_0^2\right) d\mu \leq \int_V \left(|\nabla u_0|^2 + u_0^2\right) d\mu \\ &\leq \liminf_{k\to\infty} \int_V \left(|\nabla u_{\lambda_k}|^2 + u_{\lambda_k}^2\right) d\mu \\ &\leq \liminf_{k\to\infty} \int_V \left(|\nabla u_{\lambda_k}|^2 + (\lambda_k a(x) + 1) u_{\lambda_k}^2\right) d\mu \\ &= \liminf_{k\to\infty} \int_V f(u_{\lambda_k}) u_{\lambda_k} d\mu = \int_V f(u_0) u_0 d\mu. \end{aligned}$$

Noting that $u_0|_{\Omega^c} = 0$, we get

$$\int_{\Omega\cup\partial\Omega} \left(|\nabla u_0|^2 + u_0^2\right) d\mu \le \int_\Omega f(u_0)u_0 d\mu.$$

Then there exists $\alpha \in (0,1]$ such that

$$\int_{\Omega\cup\partial\Omega} \left(|\alpha\nabla u_0|^2 + |\alpha u_0|^2\right) d\mu = \int_\Omega f(\alpha u_0)\alpha u_0 d\mu.$$

So, we have $\alpha u_0 \in \mathcal{M}_\Omega$, and

$$\begin{aligned}
m_\Omega &\le I_\Omega(\alpha u_0) = \frac{1}{2}\int_{\Omega\cup\partial\Omega} \left(|\nabla \alpha u_0|^2 + |\alpha u_0|^2\right) d\mu - \int_\Omega F(\alpha u_0) d\mu \\
&= \int_V \frac{1}{2}\left(f(\alpha u_0)\alpha u_0\right) d\mu - \int_V F(\alpha u_0) d\mu \\
&\le \int_V \frac{1}{2} f(u_0)u_0 d\mu - \int_V F(u_0) d\mu \\
&= \lim_{k\to\infty} [\int_V \frac{1}{2} f(u_{\lambda_k})u_{\lambda_k} d\mu - \int_V F(u_{\lambda_k}) d\mu] \\
&= \lim_{k\to\infty} [\int_V \frac{1}{2}\left(|\nabla u_{\lambda_k}|^2 + u_{\lambda_k}^2\right) d\mu - \int_V F(u_{\lambda_k}) d\mu \\
&= \lim_{k\to\infty} I_{\lambda_k}(u_{\lambda_k}) = \lim_{k\to\infty} m_{\lambda_k} = M
\end{aligned}$$

By (3.1), we get $M = m_\Omega$. Thus, $\lim_{\lambda\to+\infty} m_\lambda = m_\Omega$. $\square$

Proof of Theorem 1.3 By the proof of Lemma 3.1, we see that $\{u_{\lambda_k}\}$ is bounded in $W^{1,2}(V)$, and we may assume that $u_{\lambda_k} \rightharpoonup u_0$ in $W^{1,2}(V)$. Then we have $\int_V u_0^2 d\mu > 0$ and $u_0|_{\Omega^c} = 0$.

Now, we claim that $\lambda_k \int_V a(x)u_{\lambda_k}^2 d\mu \to 0$ and $\int_V |\nabla u_{\lambda_k}|^2 d\mu \to \int_V |\nabla u_0|^2 d\mu$, as $k \to +\infty$. Suppose by contradiction that $\lim\limits_{k\to\infty} \lambda_k \int_V a(x)u_{\lambda_k}^2 d\mu = \delta > 0$. We have

$$\begin{aligned}
&\int_{\Omega\cup\partial\Omega} \left(|\nabla u_0|^2 + |u_0|^2\right) d\mu < \int_V \left(|\nabla u_0|^2 + |u_0|^2\right) d\mu + \delta \\
&\le \liminf_{k\to\infty} \int_V \left[|\nabla u_{\lambda_k}|^2 + (\lambda_k a(x) + 1)u_{\lambda_k}^2\right] d\mu \\
&= \liminf_{k\to\infty} \int_V f(u_{\lambda_k})u_{\lambda_k} d\mu = \int_\Omega f(u_0)u_0 d\mu.
\end{aligned}$$

Then there exists $\alpha \in (0,1)$ such that $\alpha u_0 \in \mathcal{M}_\Omega$. Similarly, if $\liminf\limits_{k\to\infty} \int_V |\nabla u_{\lambda_k}|^2 d\mu > \int_V |\nabla u_0|^2 d\mu$, we also have $\int_{\Omega\cup\partial\Omega} \left(|\nabla u_0|^2 + |u_0|^2\right) d\mu < \int_\Omega f(u_0)u_0 d\mu$. Then in both cases, we can find $\alpha \in (0,1)$ such that $\alpha u_0 \in \mathcal{M}_\Omega$. Therefore, by (F_4) we have

$$
\begin{aligned}
m_\Omega \leq I_\Omega(\alpha u_0) &= \frac{1}{2}\int_{\Omega\cup\partial\Omega} \left(|\nabla \alpha u_0|^2 + |\alpha u_0|^2\right) d\mu - \int_\Omega F(\alpha u_0) d\mu \\
&= \frac{1}{2}\int_{\Omega\cup\partial\Omega} f(\alpha u_0)\alpha u_0 d\mu - \int_\Omega F(\alpha u_0) d\mu \\
&= \int_\Omega \left[\frac{1}{2} f(\alpha u_0)\alpha u_0 - F(\alpha u_0)\right] d\mu = \int_\Omega \Phi(\alpha u_0) d\mu \\
&< \int_\Omega \Phi(u_0) d\mu \leq \int_V \Phi(u_0) d\mu \\
&\leq \liminf_{k\to\infty} \int_V \left[\frac{1}{2} f(u_{\lambda_k})u_{\lambda_k} - F(u_{\lambda_k})\right] d\mu \\
&= \liminf_{k\to\infty} \left[\int_V \frac{1}{2}\left(|\nabla u_{\lambda_k}|^2 + (\lambda_k a(x)+1)u_{\lambda_k}^2\right) d\mu\right] - \int_V F(u_{\lambda_k}) d\mu \\
&= \liminf_{k\to\infty} I_{\lambda_k}(u_{\lambda_k}) = \lim_{k\to\infty} m_{\lambda_k} = m_\Omega,
\end{aligned}
$$

which leads to a contradiction. Thus, we get $\|u_{\lambda_k}\|_{W^{1,2}(V)} \to \|u_0\|_{W^{1,2}(V)}$ as $\lambda_k \to +\infty$. Moreover, similar to the proof of Theorem 1.2, by Proposition 3.1 and Lemma 3.1, we can deduce that u_0 is a ground state solution of (1.13). $\square$

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