xxx
() : –

Variation of parameters and initial time difference Lipschitz stability of impulsive differential equations

Saliha DEMİRBÜKEN[1,*], **Coşkun YAKAR**[1,2]
[1]Department of Mathematics, Faculty of Science, Gebze Technical University, Kocaeli, Turkey
ORCID iD: https://orcid.org/0000-0002-1394-8621
[2]Department of Mathematics, Faculty of Science, Gebze Technical University, Kocaeli, Turkey
ORCID iD: https://orcid.org/0000-0002-7759-7939

Abstract: In this paper, we investigate the Lipschitz stability of a perturbed impulsive differential system concerning the unperturbed system. We employ the variation of parameters or the constant of variation for impulsive differential systems with an initial time difference.

Key words: Lipschitz stability. variation of parameters. initial time difference. impulsive differential equations

1. Introduction

In real-time applications, many evolutionary processes are abruptly modified under the influence of short-lasting disruptions, whose period is insignificant compared to the process's duration. It is natural to assume that these perturbations in time and position take effect immediately in the form of impulses. Observing such "leaps and bounds" in developing dynamical situations is significant in various areas, including control theory, population dynamics, pharmacokinetics, mechanics, epidemiology, economics, ecology, and more [7,8,12]. Therefore, impulsive differential equations (IDEs) are practical mathematical models for behaviours and processes, such as the operation of a damper subjected to impact effects, variation of valve shutter speed in the transition from an open to a closed state, variations of the pendulum system under external repulsive effects, percussion models of a clock mechanism, vibratory percussion systems, electronic schemes, intermittent oscillators subject to impulsive influences, control of satellite orbits using radial acceleration, optimization problems in population dynamics of impulsive tamper and predator-prey species, and population death due to impulsive influences [3].

Moreover, stability is one of the most significant features to examine in differential equations. Many studies on this issue with initial time differences (ITD) employ various approaches such as Lyapunov functions, in terms of two measures, comparison principles, monotone iterative techniques, and quasilinearization methods, etc. [17–23].

In addition to these, the literature discusses two solutions starting at different times instead of analyzing only the change or perturbation of the dependent variable that keeps the initial time constant for initial value problems. There is a detailed relation between the unperturbed system and its perturbed system. However, this relation may be unfamiliar. Shifting the perturbed system to the left by μ times makes it uncertain whether there is a solution for the perturbed system in the time interval $t \in [t_0, \tau_0]$, where $\tau_0 > t_0$, and $\tau_0 - t_0 = \mu$. To address this difficulty, this study shifts the unperturbed system to the right by μ units. Consequently, it

*Correspondence: salihademirbuken@gmail.com

is known that the unperturbed system has a solution for $t > \tau_0$. This approach proves to be more useful and applicable than previous study [16].

In this paper, we have some basic definitions and theorems. Then, in section 2, we present main lemmas and theorems. In section 3, we discuss the variation of parameters formula for IDEs relative to ITD. Subsequently, the Lipschitz stability of one of the obtained results is examined theoretically and numerically in section 4.

2. Basic definition and theorems

Let the n-dimensional Euclidean space is indicated by $\mathbb{R}^n$ with suitable norm $||.||$ and $\mathbb{R}^+ = [0, +\infty)$. Consider the subsequent unperturbed IDEs [2,7,16,17]

$$\begin{aligned} z' &= F(t,z),\, t \neq t_j, \\ z(t_0^+) &= z_0, \\ z(t_j^+) &= z(t_j) + I_{t_j}(z(t_j)), \end{aligned} \tag{2.1}$$

$$\begin{aligned} z' &= F(t,z),\, t \neq t_j, \\ z(\tau_0^+) &= w_0, \\ z(t_j^+) &= z(t_j) + I_{t_j}(z(t_j)), \quad t_j \geq \tau_0, \end{aligned} \tag{2.2}$$

with corresponding the perturbed system of (2.2)

$$\begin{aligned} w' &= h(t,w),\, t \neq t_j, \\ w(\tau_0^+) &= w_0, \\ w(t_j^+) &= w(t_j) + I_{t_j}(w(t_j)), \quad t_j \geq \tau_0, \end{aligned} \tag{2.3}$$

where $h(t,w) = F(t-\mu, w) + g(t,w)$ and

(1) $0 \leq t_0 < t_1 < t_2 < \dots < t_j < \dots$ and $\lim_{j\to\infty} t_j = \infty$ $j = 1, 2, \dots$;

(2) $\tau_0 > t_0$, $\mu = \tau_0 - t_0$;

(3) $\bar{t_j} = t_j - \mu \geq 0$;

(4) $X_1 = \{t_j\}$, $X_1^* = \{\bar{t_j}\}$, $X = X_1 \cup X_1^*$;

(5) $t \in \mathbb{R}^+, z \in \Lambda \subset \mathbb{R}^n$, Λ-open;

(6) $F, h : \mathbb{R}^+ \times \Lambda \to \mathbb{R}^n$;

(7) $I_{t_j} : \Lambda \to \mathbb{R}^n$;

(8) $F(t,0) = 0$, $I_{t_j}(0) = 0$, for all t_j.

Hereby, we investigate nonlinear variation of parameter for IDEs with ITD. Before the necessary lemmas, let give the main theorem for variation of parameters.

Theorem 2.1 (Variation of parameters) *Assume that* $F \in C[I \times \mathbb{R}^n, \mathbb{R}^n]$ *and possess continuous partial derivatives* $\frac{\partial F}{\partial z}$ *on* $I \times \mathbb{R}^n$. *Let the solution* $z(t) = z(t, \tau_0, z_0)$ *of IVP of* $z' = F(t, z), z(\tau_0) = z_0$ *exist for* $t \geq \tau_0$ *and let*

$$G(t, \tau_0, z_0) = \frac{\partial F}{\partial z}(t, z(t, \tau_0, z_0)) \tag{2.4}$$

Then (i)

$$\Psi(t, \tau_0, z_0) = \frac{\partial z(t, \tau_0, z_0)}{\partial z_0} \tag{2.5}$$

exists and is a solution of the IVP for the linear system

$$u' = G(t, \tau_0, z_0)u, \tag{2.6}$$

$$u(\tau_0) = E_n, \tag{2.7}$$

where E_n *is* $n \times n$ *identity matrix.*
(ii) $\frac{\partial z(t,\tau_0,z_0)}{\partial \tau_0}$ *exists and is the solution of (2.6) and satisfies the relation*

$$\frac{\partial z(t, \tau_0, z_0)}{\partial \tau_0} = -\Psi(t, \tau_0, z_0)F(\tau_0, z_0), t \geq \tau_0 \tag{2.8}$$

and this implies that

$$\frac{\partial z(t, \tau_0, z_0)}{\partial \tau_0} + \Psi(t, \tau_0, z_0)F(\tau_0, z_0) \equiv 0, t \geq \tau_0. \tag{2.9}$$

The Equation (2.6) is called variational equation.

For more information, see the reference [2]. Now, we can give essential lemmas to reach the principal consequences.

Lemma 2.2 *Let the given conditions be satisfied:*

(A_1) the function is a element of $C[\mathbb{R}^+ \times \Lambda, \mathbb{R}^n]$ *in* $(t_{j-1}, t_j] \times \Lambda$, $j = 1, 2, \dots$ *and* $F(t, z)$ *has a finite limit as* $(t, z) \to (t_j, z_0(t_j))$, $t > t_j$, *for each* j *and* $z_0 \in \mathbb{R}^n$;

(A_2) the function F *satisfies being locally Lipschitzian (if for each* $z_0 \in \mathbb{R}^+ \times \Lambda$, *constants* $L > 0$ *exist and* $\lambda_0 > 0$ *such that* $||z - z_0|| < \lambda_0$ *implies that* $||F(z) - F(z_0)|| \leq L||z - z_0||$*);*

(A_3) for $j = 1, 2, \dots$ *the mapping* $\phi_j : \Lambda \to \Lambda, z \to u, u = \phi_j(z) \equiv z + I_{t_j}(z)$ *is a homeomorphism;*

(A_4) $\psi(t)$ *is a solution of the system (2.1) in* $[\alpha, \beta]$, $(\alpha, \beta \neq t_j, j = 1, 2, \dots)$.

Then, $\exists\ \epsilon > 0$ *and a set*

$$V = \{(t, z) \in \mathbb{R}^+ \times \Lambda, \alpha \leq t \leq \beta, |z - \psi(t^+)| < \epsilon\}, \tag{2.10}$$

such that

(i) There exists the system (2.1) has a unique solution $z(t,t_0,z_0)$ *for every* $(t_0,z_0) \in V$*, that is described on* $[\alpha,\beta]$*;*

(ii) the function $z(t,t_0,z_0)$ *is continous for*

$$t \in [\alpha,\beta], \quad (t_0,z_0) \in V, \quad t,t_0 \notin X_1; \tag{2.11}$$

(iii) t,t_0 *depend on the range of existence of solution* $z(t,t_0,z_0)$ *of system (2.1), for* $j = 1,2,\ldots, z_0 \in \Lambda$*,* $t \notin X_1$*,*

$$\lim_{\substack{\eta\to t_0\\ \zeta\to z_0}} z(t,\eta,\zeta) = z(t,t_0,z_0). \tag{2.12}$$

Lemma 2.3 *Let the given conditions be satisfied:*

(A_5) *the function* $F : \mathbb{R}^+ \times \Lambda \to \mathbb{R}^n$ *is continuous in* $(t_{j-1},t_j] \times \Lambda$*,* $j = 1,2,\ldots$ *and* $F_z(t,z)$ *is continuous in* $(t_{j-1},t_j) \times \Lambda, j = 1,2,\ldots$ *;*

(A_6) F *and* F_z *have finite limits of as* $(t,z) \to (t_j, z_0(t_j))$*,* $t > t_j$*, for every* $z_0 \in \Lambda, j = 1,2,\ldots$ *;*

(A_7) *The mapping* $\phi_j : \Lambda \to \Lambda, z \to u, u = \phi_j(z) \equiv z + I_{t_j}(z)$ *is a diffeomorphism for* $j = 1,2,\ldots$ *and for* $z \in \Lambda$

$$det\left(I + \frac{\partial I_{t_j}}{\partial z}(z)\right) \neq 0, \quad j = 1,2,\ldots. \tag{2.13}$$

Then,

(i) there exists $\lambda > 0$ *such that the solution* $z(t,t_0,z_0)$ *of system (2.1) has continuous derivatives* $\frac{\partial z}{\partial t}$*,* $\frac{\partial z}{\partial t_0}$*,* $\frac{\partial z}{\partial z_0}$*, in the domain*

$$V : \alpha < t < \beta, \quad \alpha < t_0 < \beta, \quad t,t_0 \neq t_j, j = 1,2,\ldots \quad |z_0 - \psi(t_0^+)| < \lambda; \tag{2.14}$$

(ii) the derivative $\Psi(t,t_0,z_0) = (\frac{\partial z}{\partial z_0})(t,t_0,z_0)$ *is a solution of the initial value problem*

$$\begin{aligned} v' &= F_z(t,\psi(t))v, \ t \neq t_j, \\ \nabla v &= \frac{\partial I_{t_j}}{\partial z}(\psi(t_j))v, \ t = t_j, \\ v(t_0^+) &= I, \end{aligned} \tag{2.15}$$

where $\psi(t)$ *is the solution of system (2.1) in* $[\alpha,\beta]$ $\alpha,\beta \neq t_j$*,* $j = 1,2,\ldots$*;*

(iii) the derivative $\frac{\partial z}{\partial t_0}$ *provides the following statement*

$$\frac{\partial z}{\partial t_0}(t,t_0,z_0) + \Psi(t,t_0,z_0)F(t_0,z_0) \equiv 0. \tag{2.16}$$

Please see in details [7,16].

Definition 2.4 *The solution* $w(t,\tau_0,w_0)$ *of the system (2.3) is said to be initial time difference Lipschitz stable (ITDLS) with respect to the solution* $z(t-\mu,t_0,z_0)$ *for* $t\geq\tau_0$ *, where* $z(t,t_0,z_0)$ *is any solution of the system (2.1), if and only if there exists a* $\kappa=\kappa(\tau_0)>0$ *such that*

$$||w(t,\tau_0,w_0)-z(t-\mu,t_0,z_0)||\leq\kappa(||w_0-z_0||+(\tau_0-t_0)). \tag{2.17}$$

Please see in details [16,17].

According to the study of Kulev and Bainov (1990) [6] under some assumptions zero solution of the perturbed impulsive equation is (uniformly) Lipschitz stable. Let give this theorem. Firstly, let define the necessary conditions, say (A).

(A1) $0\leq t_0<t_1<t_2<\cdots<t_j<\ldots$ and $\lim_{j\to\infty}t_j=\infty$ $j=1,2,\ldots$.

(A2) The functions F and $F_z(t,z)$ are continuous in $(t_{j-1},t_j]\times\Lambda$, $j=1,2,\ldots$ and $(t_{j-1},t_j)\times\Lambda$, $j=1,2,\ldots$, respectively and $F(t,0)\equiv0$.

(A3) For any $z_0\in\Lambda, j=1,2,\ldots$, the functions F and F_z have finite limits as $(t,w)\to(t_j,z)$, $t>t_j$.

(A4) The functions $I_{t_j}:\Lambda\to\Lambda$, $j=1,2,\ldots$ are continuously differentiable in Λ and $I_{t_j}(0)=0$, $j=1,2,\ldots$.

(A5) The solution $z(t;t_0,z_0)$ of system (2.1) satisfying the initial condition $z(t_0+0;t_0,z_0)=z_0$ is defined in the interval $\alpha<t<\beta$.

Theorem 2.5 *Let the following conditions be fulfilled:*

(1) Conditions (A) hold.

(2) The zero of system (2.1) is uniformly Lipschitz stable.

(3) $|\Psi(t,x,u)f(x,u)|\leq\alpha(x)|u|$ *for* $t\geq x>t_0\geq0, u\in\mathbb{R}^n$*, where* Ψ *is the fundamental solution of (2.5) and the function* $f:J\times\mathbb{R}^n\to\mathbb{R}^n$ *satisfies condition* $(A2)$ *and* $(A3)$

(4) $|Q_j(w)|\leq b_j|w|$ *for* $w\in\mathbb{R}^n, j=1,2,\ldots,$ *where* $Q_j:\mathbb{R}^n\to\mathbb{R}$ *satisfy condition* $(A4)$ *and* $b_j\geq0, j=1,2,\ldots,$ *are constants.*

(5) $||\Psi(t,t_j,w+I_{t_j}(w)+xQ_j(w))||\leq a_j$ *for* $t>t_0\geq0, w\in\mathbb{R}^n, 0\leq x\leq1,$ *and* $j=1,2,\ldots,$ *where* $a_j\geq0, j=1,2,\ldots,$ *are constants.*

(6) $\int_\sigma^\infty\alpha(t)dt<\infty$ *and* $\prod_{\sigma<t_j<\infty}(1+a_jb_j)<\infty$ *for* $\sigma>t_0\geq0$.

Then, the zero solution of the perturbed impulsive system (2.3) is uniformly Lipschitz stable.

For the detail of the proof, see the study of Kulev and Bainov (1990) [6].

3. Main results of nonlinear variation of parameters

In this section, the relation between the unperturbed IDEs system and its perturbed system will be shown by using nonlinear variation of parameters formula. Accordingly, it is crucial to examine the stability features.

Theorem 3.1 *Let the system (2.1) provide the properties of Lemma (2.3) and let $z(t,t_0,z_0)$ be a solution of system (2.1). Then, for any solution $w(t)=w(t,\tau_0,t_0)$ of the system (2.3). The sequent statement is acceptable.*

$$\begin{aligned} w(t,\tau_0,w_0) &= z(t-\mu,t_0,z_0) + \int_0^1 \Theta(t,\tau_0,\gamma(x))(w_0-z_0)dx \\ &+ \int_{\tau_0}^t \tilde{h}(x,\tilde{z}(x),\mu)dx \qquad , \\ &+ \sum_{\tau_0<\bar{t_j}<t} \int_0^1 \Theta(t,\bar{t_j},\tilde{z}(\bar{t_j}) + xI_{\bar{t_j}-\mu}(\tilde{z}(\bar{t_j})))dx.I_{\bar{t_j}-\mu}(\tilde{z}(\bar{t_j})) \end{aligned} \tag{3.1}$$

where $\Theta(t,\tau_0,\gamma(x)) = \frac{\partial w}{\partial \gamma}(t,\tau_0,\gamma(x))$, $\tilde{h}(x,\tilde{z}(x),\mu) = \frac{\partial w}{\partial x}(t,x,\tilde{z}(x)) + \frac{\partial w}{\partial \tilde{z}}(t,x,\tilde{z}(x))F(x,\tilde{z}(x))$ *and* $\tilde{z}(x) = z(x-\mu,\tau_0,z_0)$, $\tau_0<x<t$.

Proof Set $m(x)=w(t,x,\tilde{z}(x))$, where $\tilde{z}(x)=z(x-\mu,\tau_0,z_0)$, $\tau_0<x<t$. Then for $x\notin X$, we get

$$\begin{aligned} m'(x) &= \frac{\partial w}{\partial x}(t,x,\tilde{z}(x)) + \frac{\partial w}{\partial \tilde{z}}(t,x,\tilde{z}(x))F(x,\tilde{z}(x)) \\ &\equiv \tilde{h}(x,\tilde{z}(x),\mu). \end{aligned} \tag{3.2}$$

When $x\in X$, there are two cases.

Case 1. Consider

$$\begin{aligned} \Delta m(x)|_{x=\bar{t_j}} &= w(t,\bar{t_j}^+,\tilde{z}(\bar{t_j}^+)) - w(t,\bar{t_j}^-,\tilde{z}(\bar{t_j}^-)) \\ &= w(t,\bar{t_j},\tilde{z}(\bar{t_j}) + I_{\bar{t_j}-\mu}(\tilde{z}(\bar{t_j}))) - w(t,\bar{t_j},\tilde{z}(\bar{t_j})) \\ &= \int_0^1 \Theta(t,\bar{t_j},\tilde{z}(\bar{t_j}) + xI_{\bar{t_j}-\mu}(\tilde{z}(\bar{t_j})))dxI_{\bar{t_j}-\mu}(\tilde{w}(\bar{t_j})). \end{aligned} \tag{3.3}$$

Case 2. Consider

$$\Delta m(x)|_{x=t_j\in X_1-X_1^*} = w(t,{t_j}^+,\tilde{z}({t_j}^+)) - w(t,{t_j}^-,\tilde{z}({t_j}^-)) = 0. \tag{3.4}$$

Integrating (3.2) from τ_0 to t and using (3.3) and (3.4), the following equality is obtained.

$$\begin{aligned} w(t,\tau_0,z_0) &= z(t-\mu,t_0,z_0) + \int_{\tau_0}^t \tilde{h}(x,\tilde{z}(x),\mu)dx \\ &+ \sum_{\tau_0<\bar{t_j}<t} \int_0^1 \Theta(t,\bar{t_j},\tilde{z}(\bar{t_j}) + xI_{\bar{t_j}-\mu}(\tilde{z}(\bar{t_j})))dxI_{\bar{t_j}-\mu}(\tilde{z}(\bar{t_j})). \end{aligned} \tag{3.5}$$

Now, let $p(x) = w(t, \tau_0, \gamma(x))$, where $\gamma(x) = w_0 x + (1-x)z_0, 0 \leq x \leq 1$. Then,

$$\frac{dp(x)}{dx} = \frac{\partial w}{\partial \gamma}(t, \tau_0, \gamma(x))(w_0 - z_0). \tag{3.6}$$

After integrating (3.6) from 0 to 1, we get

$$w(t, \tau_0, w_0) - w(t, \tau_0, z_0) = \int_0^1 \frac{\partial w}{\partial \gamma}(t, \tau_0, \gamma(x))(w_0 - z_0)dx. \tag{3.7}$$

Combining (3.5) and (3.7)

$$\begin{aligned} w(t, \tau_0, w_0) &= z(t-\mu, t_0, z_0) - \int_{\tau_0}^t \tilde{h}(x, \tilde{z}(x), \mu)dx \\ &+ \sum_{\tau_0 < \bar{t_j} < t} \int_0^1 \Theta(t, \bar{t_j}, \tilde{z}(\bar{t_j}) + xI_{\bar{t_j}-\mu}(\tilde{z}(\bar{t_j})))dx.I_{\bar{t_j}-\mu}(\tilde{z}(\bar{t_j})) \\ &+ \int_0^1 \Theta(t, \tau_0, \gamma(x))(w_0 - z_0)dx, \end{aligned} \tag{3.8}$$

where $\Theta(t, \tau_0, \gamma(x)) = \frac{\partial w}{\partial \gamma}(t, \tau_0, \gamma(x))$, $\tilde{h}(x, \tilde{z}(x), \mu) = \frac{\partial w}{\partial x}(t, x, \tilde{z}(x)) + \frac{\partial w}{\partial \tilde{z}}(t, x, \tilde{z}(x))F(x, \tilde{z}(x))$ and $\tilde{z}(x) = z(x-\mu, \tau_0, z_0)$, $t_0 < x < t$. □

Corollary 3.2 *Assume that the conditions of the previous theorem satisfy except that $F(t-\mu, z)$ being replaced with $h(t, z) + g(t, z)$, then the given statement is prevalent:*

$$\begin{aligned} w(t, \tau_0, w_0) &= z(t-\mu, t_0, z_0) + \int_0^1 \Theta(t, \tau_0, \gamma(x))(w_0 - z_0)dx \\ &- \sum_{\tau_0 < \bar{t_j} < t} \int_0^1 \Theta(t, \bar{t_j}, \tilde{z}(\bar{t_j}) + xI_{\bar{t_j}-\mu}(\tilde{z}(\bar{t_j})))dx.I_{\bar{t_j}-\mu}(\tilde{z}(\bar{t_j})) \\ &+ \int_{\tau_0}^t \Theta(t, x, \tilde{z}(x))g(x, \tilde{z})dx. \end{aligned} \tag{3.9}$$

Theorem 3.3 *Assume that the conditions of the previous theorem hold; then the following formula is valid:*

$$
\begin{aligned}
w(t,\tau_0,w_0)-z(t-\mu,t_0,z_0) = z(t,\tau_0,z_0-w_0)+\int_{\tau_0}^{t}\tilde{G}(x,\nu(x),\mu)dx \\
-\sum_{\substack{\tau_0<\bar{t_j}<t\\ \bar{t_j}\in X_1^*-X_1}}\int_0^1 \Psi\big(t,\bar{t_j},w(\bar{t_j})-z(\bar{t_j}-\mu) \\
-xI_{\bar{t_j}-\mu}(z(\bar{t_j}-\mu))\big)dxI_{\bar{t_j}-\mu}(z(\bar{t_j}-\mu)) \\
+\sum_{\substack{\tau_0<t_j<t\\ t_j\in X_1-X_1^*}}\int_0^1 \Psi\big(t,t_j,w(t_j)-z(t_j-\mu) \\
+xI_{t_j}(w(t_j))\big)dxI_{t_j}(w(t_j)) \\
+\sum_{\substack{\tau_0<t_j<t\\ \bar{t_j}\in X_1^*\cap X_1}}\int_0^1 \Psi\big(t,t_j,w(t_j)-z(t_j-\mu) \\
+xI_{t_j}(w(t_j))-xI_{t_j-\mu}(z(t_j)-\mu)\big)dx \\
.\big(I_{t_j}(w(t_j))-I_{t_j-\mu}(z(t_j)-\mu)\big),
\end{aligned}
\tag{3.10}
$$

where $\Psi(t,\tau_0,u_0)=(\frac{\partial z}{\partial z_0})(t,\tau_0,u_0)$, $\tilde{G}(x,\nu(x),\mu)=\frac{\partial z}{\partial x}(t,x,\nu(x))+\frac{\partial z}{\partial \nu}(t,x,\nu(x))G(x,\nu(x),\mu)$ *and* $G(x,\nu(x),\mu)=h(x,w(x))-F(x-\mu,w(x)-\nu(x))$.

Proof Set $m(x)=z(t,x,\nu(x))$, where $\nu(x)=w(x,\tau_0,w_0)-z(x-\mu,t_0,z_0)$, $t_0<x<t$. Then for $x\notin X$,

$$
m'(x)=\frac{\partial z}{\partial x}(t,x,\nu(x))+\frac{\partial z}{\partial \nu}(t,x,\nu(x))G(x,\nu(x),\mu)\equiv\tilde{G}(x,\nu(x),\mu), \tag{3.11}
$$

where $G(x,\nu(x),\mu)=h(x,w(x))-F(x-\mu,w(x)-\nu(x))$. If $x\in X$, there are three cases.

Case 1. Consider

$$
\begin{aligned}
\Delta m(x)|_{x=\bar{t_j}\in X_1^*-X_1} &= z(t,\bar{t_j}^+,\nu(\bar{t_j}^+))-z(t,\bar{t_j}^-,\nu(\bar{t_j}^-)) \\
&= z(t,\bar{t_j},w(\bar{t_j}^+)-z(\bar{t_j}-\mu^+))-z(t,\bar{t_j},w(\bar{t_j}^-)-z(\bar{t_j}^- -\mu)) \\
&= z(t,\bar{t_j},w(\bar{t_j})-z(\bar{t_j}-\mu)-I_{\bar{t_j}-\mu}(z(\bar{t_j}-\mu)))-z(t,\bar{t_j},w(\bar{t_j}+\mu)-z(\bar{t_j})) \\
&= \int_0^1 \Psi\big(t,\bar{t_j},w(\bar{t_j})-z(\bar{t_j}-\mu)-xI_{\bar{t_j}-\mu}(z(\bar{t_j}-\mu))\big)dxI_{\bar{t_j}-\mu}(z(\bar{t_j}-\mu)).
\end{aligned}
\tag{3.12}
$$

Case 2. Consider

$$
\begin{aligned}
\Delta m(x)|_{x=t_j\in X_1-X_1^*} &= z(t,t_j^+,\nu(t_j^+))-z(t,t_j^-,\nu(t_j^-)) \\
&= z(t,t_j,w(t_j^+)-z(t_j^+-\mu))-z(t,t_j,w(t_j^-)-z(t_j^- -\mu)) \\
&= z(t,t_j,w(t_j)+I_{t_j}(w(t_j))-z(t_j-\mu))-z(t,t_j,w(t_j)-z(t_j-\mu)) \\
&= -\int_0^1 \Psi\big(t,t_j,w(t_j)-z(t_j-\mu)+xI_{t_j}(w(t_j))\big)dxI_{t_j}(w(t_j)).
\end{aligned}
\tag{3.13}
$$

Case 3. Consider

$$
\begin{aligned}
\Delta m(x)|_{x=t_j\in X_1\cap X_1^*} &= z(t,{t_j}^+,\nu({t_j}^+)) - z(t,{t_j}^-,\nu({t_j}^-))\\
&= z(t,t_j,w({t_j}^+) - z({t_j}^+-\mu)) - z(t,t_j,w({t_j}^-) - z({t_j}^- - \mu))\\
&= z(t,t_j,w(t_j) + I_{t_j}(w(t_j)) - z(t_j-\mu) - I_{t_j-\mu}(z(t_j-\mu)))\\
&\quad - z(t,t_j,w(t_j) - z(t_j-\mu))\\
&= \int_0^1 \Psi\big(t,t_j,w(t_j) - z(t_j-\mu) + xI_{t_j}(w(t_j)) - xI_{t_j-\mu}(z(t_j-\mu))\big)dx\\
&\quad .\big(I_{t_j}(w(t_j)) - I_{t_j-\mu}(z(t_j-\mu))\big).
\end{aligned}
\tag{3.14}
$$

Integrating (3.11) from τ_0 to t and using Equation (3.12) - (3.14),

$$
\begin{aligned}
w(t,\tau_0,w_0) - z(t-\mu,t_0,z_0) = z(t,\tau_0,z_0-w_0) &+ \int_{\tau_0}^t \tilde{G}(x,\nu(x),\mu)dx\\
&- \sum_{\substack{\tau_0<\bar{t_j}<t\\ \bar{t_j}\in X_1^*-X_1}} \int_0^1 \Psi\big(t,\bar{t_j},w(\bar{t_j}) - z(\bar{t_j}-\mu)\\
&\qquad - xI_{\bar{t_j}-\mu}(z(\bar{t_j}-\mu))\big)dxI_{\bar{t_j}-\mu}(z(\bar{t_j}-\mu))\\
&+ \sum_{\substack{\tau_0<t_j<t\\ t_j\in X_1-X_1^*}} \int_0^1 \Psi\big(t,t_j,w(t_j) - z(t_j-\mu)\\
&\qquad + xI_{t_j}(w(t_j))\big)dxI_{t_j}(w(t_j))\\
&+ \sum_{\substack{\tau_0<t_j<t\\ \bar{t_j}\in X_1^*\cap X_1}} \int_0^1 \Psi\big(t,t_j,w(t_j) - z(t_j-\mu)\\
&\qquad + xI_{t_j}(w(t_j)) - xI_{t_j-\mu}(z(t_j)-\mu)\big)dx\\
&\qquad .\big(I_{t_j}(w(t_j)) - I_{t_j-\mu}(z(t_j)-\mu)\big),
\end{aligned}
\tag{3.15}
$$

where $\Psi(t,\tau_0,u_0) = (\frac{\partial z}{\partial z_0})(t,\tau_0,u_0)$, $\tilde{G}(x,\nu(x),\mu) = \frac{\partial z}{\partial x}(t,x,\nu(x)) + \frac{\partial z}{\partial \nu}(t,x,\nu(x))G(x,\nu(x),\mu)$ and $G(x,\nu(x),\mu) = h(x,w(x))) - F(x-\mu,w(x)-\nu(x))$.

The desired result is obtained [7,8,16]. □

4. An application of the Lipschitz stability of perturbed system with respect to the unperturbed system

In this section, we have investigated that the Lipschitz stability of perturbed system (2.3) with respect to the unperturbed system (2.2) by using the Corollary 3.2. Additionally, we demonstrate this application with a simple numerical example. Moreover, considering Theorem 2.5, condition (B_2) is acceptable in the following theorem.

Theorem 4.1 *Let the following conditions be fulfilled.*

(B_1) the hypotheses of Corollary 3.2 are provided;

(B_2) the zero solution of system (2.3) is Lipschitz stable;

(B_3) $||\Theta(t,x,\tilde{z}(x))g(x,\tilde{z}(x))|| \leq \alpha(x)||\tilde{z}(x)||$ *for* $\tau_0 < x \leq t$*;*

(B_4) $||\Theta(t,t_0,\tau(x)|| \leq \kappa_1(||w_0-z_0||+\mu)/||w_0-z_0||$ *and* κ_1 *is a constant;*

(B_5) $||I_{\bar{t_j}-\mu}(\tilde{z}(\bar{t_j}))|| \leq b_j||\tilde{z}(\bar{t_j})||$ *and* $b_j \geq 0$ *are constants;*

(B_6) $||\Theta(t,\bar{t_j},\tilde{z}(\bar{t_j})+xI_{\bar{t_j}-\mu}(\tilde{z}(\bar{t_j})))|| \leq a_j$ *and* $a_j \geq 0$ *are constants;*

(B_7) $\int_{\tau_0}^{\infty}\alpha(x)dx < \infty, \alpha(x) \in C[\mathbb{R}^+,\mathbb{R}^+]$ *and* $\prod_{t_0<\bar{t_j}<t}(1+a_jb_j) < \infty$.

Then, the solution $w(t,\tau_0,w_0)$ *of the system (2.3) for* $t \geq \tau_0$ *is ITDLS with respect to the solution* $z(t-\mu,t_0,z_0)$ *for* $t \geq \tau_0$ *.*

Proof Using Corollary (3.2), the following statement is attained.

$$\begin{aligned} w(t,\tau_0,w_0) = z(t-\mu,t_0,z_0) &+ \int_0^1 \Theta(t,t_0,\gamma(x))(w_0-z_0)dx \\ &- \sum_{t_0<\bar{t_j}<t}\int_0^1 \Theta(t,\bar{t_j},\tilde{z}(\bar{t_j})+xI_{\bar{t_j}-\mu}(\tilde{z}(\bar{t_j})))dx.I_{\bar{t_j}-\mu}(\tilde{z}(\bar{t_j})) \\ &+ \int_{\tau_0}^t \Theta(t,x,\tilde{z}(x))g(x,\tilde{z})dx. \end{aligned} \tag{4.1}$$

For both sides, applying the norm and employing the triangle inequality gives the following inequality.

$$\begin{aligned} ||w(t,\tau_0,w_0)-\tilde{z}(t)|| \leq &\int_0^1 ||\Theta(t,t_0,\gamma(x))||||(w_0-z_0)||dx \\ &+ \sum_{\tau_0<\bar{t_j}<t}\int_0^1 ||\Theta(t,\bar{t_j},\tilde{z}(\bar{t_j})+xI_{\bar{t_j}-\mu}(\tilde{z}(\bar{t_j})))||dx.||I_{\bar{t_j}-\mu}(\tilde{z}(\bar{t_j}))|| \\ &+ \int_{\tau_0}^t ||\Theta(t,x,\tilde{z}(x))g(x,\tilde{z})||dx. \end{aligned} \tag{4.2}$$

Conditions (B_2)-(B_5) implies that

$$||w(t,\tau_0,w_0)-\tilde{z}(t)|| \leq \kappa_1(||w_0-z_0||+\mu) + \int_{\tau_0}^t \alpha(x)||\tilde{z}(x)||dx + \sum_{\tau_0<\bar{t_j}<t} a_jb_j||\tilde{z}(\bar{t_j})||. \tag{4.3}$$

Setting $\kappa^*(t) = ||w(t,\tau_0,w_0)-\tilde{z}(t)||$, then it gives that

$$\begin{aligned} \kappa^*(t) \leq \kappa_1(||w_0-z_0||+\mu) &- \int_{\tau_0}^t \alpha(x)\kappa^*(x)dx + \int_{\tau_0}^t \alpha(x)||w(x,\tau_0,w_0)||dx \\ &- \sum_{\tau_0<\bar{t_j}<t} a_jb_j\kappa^*(\bar{t_j}) + \sum_{\tau_0<\bar{t_j}<t} a_jb_j||w(\bar{t_j})||. \end{aligned} \tag{4.4}$$

Since $||w(t,\tau_0,w_0)|| \leq \kappa_2||w_0||$ as long as $||w_0|| < \epsilon$, then $||w(t,\tau_0,w_0)|| \leq \kappa_2\epsilon$ for $t \geq \tau_0$,

$$\begin{aligned}\kappa^*(t) \leq \kappa_1(||w_0 - z_0|| + \mu) + \int_{\tau_0}^{t} \alpha(x)\kappa^*(x)dx + \kappa_2\epsilon \int_{\tau_0}^{t} \alpha(x)dx \\ + \kappa_2\epsilon \sum_{t_0<\bar{t_j}<t} a_j b_j \kappa^*(\bar{t_j}) + \kappa_2\epsilon \sum_{t_0<\bar{t_j}<t} a_j b_j.\end{aligned} \tag{4.5}$$

Applying the Gronwall's Lemma to (4.5), [6,13–15],

$$\begin{aligned}\kappa^*(t) \leq \left(\kappa_1(||w_0 - z_0|| + \mu) + \kappa_2\epsilon \int_{t_0}^{t} \alpha(x)dx + \kappa_2\epsilon \sum_{t_0<\bar{t_j}<t} a_j b_j\right) \\ . \prod_{t_0<\bar{t_j}<t} (1 + \kappa_2\epsilon a_j b_j) exp\left\{ \int_{t_0}^{t} \alpha(x)dx \right\}.\end{aligned} \tag{4.6}$$

Setting $\kappa_3 = \{\kappa_1 + (\kappa_2\epsilon \int_{t_0}^{t} \alpha(x)dx / ||w_0 - z_0|| + \mu) + (\kappa_2\epsilon \sum_{t_0<\bar{t_j}<t} a_j b_j / ||w_0 - z_0|| + \mu)\} \prod_{t_0<\bar{t_j}<t} (1 + \kappa_2\epsilon a_j b_j) exp\{\int_{t_0}^{t} \alpha(x)dx\}$,

$$\kappa^*(t) \leq \kappa_3(||w_0 - z_0|| + \mu). \tag{4.7}$$

Finally, by the condition (B_7), the solution $w(t,\tau_0,w_0)$ of the system (2.3) for $t \geq \tau_0$ is ITDLS with respect to the solution $z(t-\mu,t_0,z_0)$ for $t \geq \tau_0$.
The desired result is obtained. □

Followingly, we have an illustrative application of the main theorem as follows how to apply some of the results. The figures were generated using MATLAB R2022a.

Example 4.2 *Let's consider the following systems.*

$$\begin{aligned} z' &= -z,\, t \neq t_j, \\ z(0) &= 1, \\ z(t_j^+) &= z(t_j) + \frac{z(t_j)}{2}, \end{aligned} \tag{4.8}$$

and

$$\begin{aligned} z' &= -z,\, t \neq t_j, \\ z(1) &= e, \\ z(t_j^+) &= z(t_j) + \frac{z(t_j)}{2}, \quad t_j \geq \tau_0, \end{aligned} \tag{4.9}$$

corresponding its perturbed system of (4.9)

$$\begin{aligned} w' &= -w + e^{-t},\, t \neq t_j, \\ w(e) &= e, \\ w(t_j^+) &= w(t_j) + \frac{w(t_j)}{2}, \quad t_j \geq \tau_0, \end{aligned} \tag{4.10}$$

The solution of the systems for each interval can be obtained as follows. For the system (4.8), we get that

$$z(t) = \begin{cases} z_0(t) = e^{-t} & , t \in [0,2] \\ z_1(t) = \frac{3}{2}e^{-t} & , t \in (1,2] \\ z_2(t) = \left(\frac{3}{2}\right)^2 e^{-t} & , t \in (2,3] \\ \vdots & \\ z_n(t) = \left(\frac{3}{2}\right)^n e^{-t} & , t \in (n, n+1]. \end{cases}$$

Similarly, the solutions of the unperturbed system (4.9) and its perturbed system (4.10) can be found in the following.

$$z(t) = \begin{cases} z_1(t) = e^{2-t} & , t \in [1,2] \\ z_2(t) = \frac{3}{2}e^{2-t} & , t \in (2,3] \\ z_3(t) = \left(\frac{3}{2}\right)^2 e^{2-t} & , t \in (3,4] \\ \vdots & \\ z_n(t) = \left(\frac{3}{2}\right)^{n-1} e^{2-t} & , t \in (n, n+1] \end{cases}$$

and

$$w(t) = \begin{cases} w_1(t) = (t + e^2 - 1)e^{-t} & , t \in [1,2] \\ w_2(t) = (t + \frac{3}{2}e^2 + 1)e^{-t} & , t \in (2,3] \\ w_3(t) = (t + \left(\frac{3}{2}\right)^2 e^2 + 1)e^{-t} & , t \in (3,4] \\ \vdots & \\ w_n(t) = (t + \left(\frac{3}{2}\right)^n e^2 + 2n - 3)e^{-t} & , t \in (n, n+1]. \end{cases}$$

Now, our aim is to show that they satisfy Lipschitz condition with respect to in time and in position. That is, $||w(t, \tau_0, w_0) - \tilde{z}(t)|| < \kappa(||w_0 - z_0|| + \mu)$ *for* $t \geq \tau_0$. *Regarding each interval, the following inequality can be obtained and easily proved by mathematical induction.*

$$||w_n(t, n, w_n(n)) - z_n(t-1, n-1, z_{n-1}(n-1))|| \leq e^{-n}(||(3n-3)e + \left(\frac{3}{2}\right)^{n-1}|| + 1), \quad n \geq 1. \tag{4.11}$$

Then, by choosing the Lipschitz constant κ *such that* $\kappa = e^{-1} = max_{n \geq 1}\{e^{-n}\}$, *we have that*

$$||w_n(t, n, w_n(n)) - z_n(t-1, n-1, z_{n-1}(n-1))|| \leq e^{-1}(||(3n-3)e + \left(\frac{3}{2}\right)^{n-1}|| + 1), \quad n \geq 1.$$

where $||w_0 - z_0|| = ||(3n-3)e + \left(\frac{3}{2}\right)^{n-1}||$ *and* $\mu = 1$.
Hence, it is seen easily that the unperturbed system (4.8) has the initial time difference Lipschitz stability with respect to the perturbed system (4.10).
Moreover, the behaviour of the given systems is represented in $t \in [0,6]$ *in the following Figure 1.*

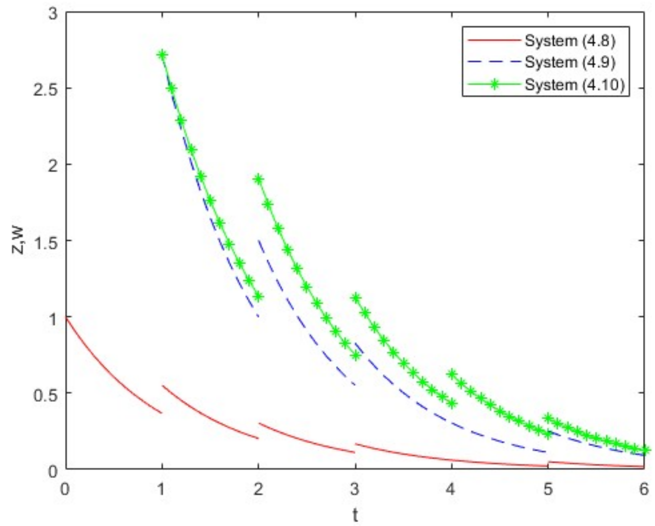

Figure 1. The graph of the perturbed systems (4.8), (4.9) and the unperturbed system (4.10).

Additionally, in the Equation (4.11), it can be observed that

$$\textit{for } n = 50, \quad ||w_{50}(t, 50, w_{50}(50)) - z_{50}(t-1, 49, z_{49}(49))|| \leq 1,4 \times 10^{-13},$$
$$\textit{for } n = 100, \quad ||w_{100}(t, 100, w_{100}(100)) - z_{100}(t-1, 99, z_{99}(99))|| \leq 1,7 \times 10^{-26}.$$

Thus, it can be inferred that behaviour of perturbed system (4.8) with respect to unperturbed system (4.10) in terms of initial time difference goes to 0 *as time* t *approaches to* ∞*. In other words, this implies that these systems are an asymptotically stability. It is also presented in Figure 2 and Figure 3.*

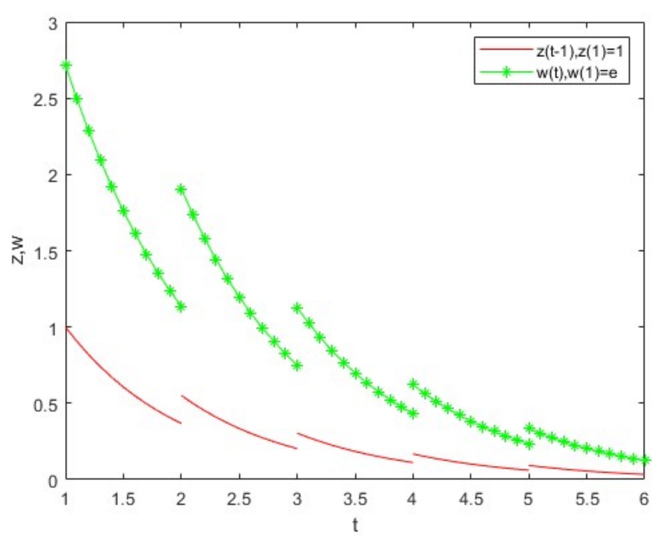

Figure 2. The graph of the perturbed system (4.8) with shifted $\mu = 1$ and the unperturbed system (4.10).

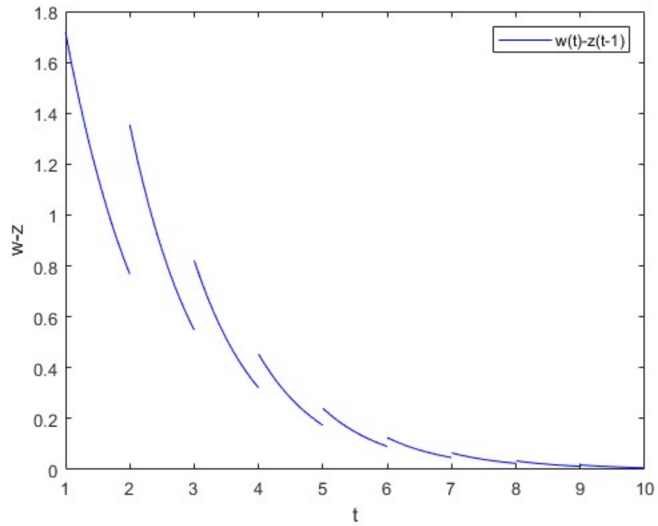

Figure 3. The graph of the difference between perturbed system (4.8) with shifted $\mu = 1$ and the unperturbed system (4.10).

5. Conclusion

IDEs can be described as differential equations containing impulse effects, that is, they seem as spontaneous description of investigated modification of some real world problems and applications. While examining the manner of these systems, the stability is of great importance. In this paper, variation of parameter formulas, which is a beneficial tool, for impulsive differential equations with initial time difference have been improved. In addition to these, we have attained the one of the most important result and shown that the initial time difference Lipschitz stability of unperturbed system with respect to the perturbed system.

In addition to these, recent studies by F.Karakoç, A.Unal and H. Bereketoğlu (2018), A. Elbori, R.M.N. Al-wahishi and O. Mohammed (2021), M.L. Büyükkahraman (2022), A. Moumen, A.C. Benaissa M. Ferhat, M. Bouye and K. Zennir (2023) and R. Liu, J. Wang and D. O'Regan (2023) provide comprehensive insights into IDEs across various approaches [1,4,5,9,11].

Furthermore, as a future work, we will plan to investigate the stability, boundedness, controllability and observability of IDEs with ITD and application of these for two measures.

References

[1] Büyükkahraman, Mehtap Lafçı. "Existence of periodic solutions to a certain impulsive differential equation with piecewise constant arguments." Eurasian Mathematical Journal 13.4 (2022): 54-60.

[2] Deo, Sadashiv G., et al. Textbook of ordinary differential equations. McGraw-Hill Education, 1997.

[3] Dishlieva, Katya. "Impulsive differential equations and applications." Journal of Applied and Computational Mathematics 1.6 (2012): 1-6.

[4] Elbori, Abdalftah, et al. "Generalized systems of impulsive differential equations". 2021.

[5] Karakoç, Fatma, Arzu Ünal, and Hüseyin Bereketoğlu. "Oscillation of a nonlinear impulsive differential equation system with piecewise constant argument." Advances in Difference Equations 2018.1 (2018): 1-11.

[6] Kulev, G. K., and D. D. Bainov. "Lipschitz stability of impulsive systems of differential equations." International journal of theoretical physics 30.5 (1991): 737-756.

[7] Lakshmikantham, Vangipuram, and Sadashiv G. Deo. Method of Variation of Parameters for Dynamic Systems. Routledge, 2019.

[8] Lakshmikantham, Vangipuram, and Pavel S. Simeonov. Theory of impulsive differential equations. Vol. 6. World scientific, 1989.

[9] Liu, Rui, Jinrong Wang, and Donal O'Regan. "Existence of solutions to nonlinear impulsive fuzzy differential equations." Filomat 37.4 (2023): 1223-1240.

[10] Mahajan, Pallvi, Sanjay Kumar Srivastava, and Rakesh Dogra. "Practical stability and boundedness criteria of impulsive differential system with initial time difference." (2021).

[11] Moumen, Abdelkader, et al. "Existence Results for Systems of Nonlinear Second-Order and Impulsive Differential Equations with Periodic Boundary." Mathematics 11.24 (2023): 4907.

[12] Samoilenko, Anatolii Mikhailovich, and N. A. Perestyuk. Impulsive differential equations. World scientific, 1995.

[13] Shao, Jing, and Fanwei Meng. "Nonlinear impulsive differential and integral inequalities with integral jump conditions." Advances in Difference Equations 2016.1 (2016): 1-13.

[14] Simeonov, P.S., and D.D. (1987). Differentiability of solutions of systems with impulse effect with respect to initial data and parameter, Bulletin of the Institute of Mathematics, Academia Sinica, 15(2), 251-269.

[15] Thiramanus, Phollakrit, and Jessada Tariboon. "Impulsive differential and impulsive integral inequalities with integral jump conditions." Journal of Inequalities and Applications 2012.1 (2012): 1-11.

[16] Wang, Peiguang, and Xiaowei Liu. "Nonlinear variation of parameters formula for impulsive differential equations with initial time difference and application." Abstract and Applied Analysis. Vol. 2014. Hindawi, 2014.

[17] Yakar, Coşkun. "Variation-of-parameters formulae and Lipschitz stability criteria for nonlinear matrix differential equations with initial time difference." Differential and Difference (2006): 1201.

[18] Yakar, Coşkun, and Michael D. Shaw. "Practical stability in terms of two measures with initial time difference." Nonlinear Analysis: Theory, Methods and Applications 71.12 (2009): e781-e785.

[19] Yakar, Coşkun, and Mohammed Sami. "Stability, boundedness and Lagrange stability with several Lyapunov functions in terms of two measures with initial time difference". Dynamics of Continuous, Discrete and Impulsive Systems. Series B: Applications and Algorithms (2022) 29: 411-421.

[20] Yakar, Coşkun, and Mohammed Sami. "Stability and asymptotic stability in terms of two measures with initial time difference". Dynamics of Continuous, Discrete and Impulsive Systems. Series A: Mathematical Analysis (2009) 29: 225-232.

[21] Yakar, Coșkun, and Hazm Talab. "Stability of perturbed set differential equations involving causal operators in regard to their unperturbed ones considering difference in initial conditions." Advances in Mathematical Physics 2021 (2021): 1-12.

[22] Yakar, Coşkun, and Hazm Talab. "Practical Stability in Terms of Two Measures with Initial Time Difference for Set Differential Equations involving Causal Operators." Konuralp Journal of Mathematics 10.1 (2022): 149-158.

[23] Yakar, Coşkun, and Betül Özbay Elibüyük. "Practical stability analysis of perturbed fuzzy control system related to unperturbed fuzzy control system." Turkish Journal of Mathematics 45.6 (2021): 2441-2454.