Generalized inequalities of the Mercer type for strongly convex functions

Muhammad Adil Khan[1]
adilswati@gmail.com

Fayaz Alam[1]
fayazalam896@gmail.com

Eze R. Nwaeze[2,*]
enwaeze@alasu.edu

Sajid Iqbal[3]
sajid_uos2000@yahoo.com

[1] Department of Mathematics, University of Peshawar, Peshawar, Pakistan

[2] Department of Mathematics and Computer Science, Alabama State University, Montgomery, AL 36101, USA

[3] Department of Mathematics, Riphah International University, Faisalabad Campus, Satyana Road, Faisalabad, Pakistan

Abstract

A generalization of the Mercer type inequality, for strongly convex functions with modulus $c > 0$, is hereby established. Let $\mathfrak{h} : [\delta, \zeta] \to \mathbb{R}$ be a strongly convex function on the interval $[\delta, \zeta] \subset \mathbb{R}$. Let $\mathbf{a} = (a_1, ..., a_s)$, $\mathbf{b} = (b_1, ..., b_s)$ and $\mathbf{p} = (p_1, ..., p_s)$, where $a_k, b_k \in [\delta, \zeta], p_k > 0$ for each $k = \overline{1, s}$. If $\mathbf{n} \in \mathbb{R}^s$, $\langle \mathbf{a} - \mathbf{b}, \mathbf{n} \rangle = 0$ and under some separability assumptions, then we prove that

$$\sum_{l=1}^{s} p_l \mathfrak{h}(b_l) \leq \sum_{l=1}^{s} p_l \mathfrak{h}(a_l) - c \sum_{l=1}^{s} p_l (a_l - b_l)^2.$$

Using the above result, we derive loads of inequalities for similarly separable vectors. We further applied our results to different types of tuples. Our results extend, complement and generalize known results in the literature.

Keywords: Strongly convex function; Jensen–Mercer type inequality; Majorization; Separable tuple; Monotone tuple

2010 Mathematics Subject Classification: 26D15; 26A51; 39B62

*Corresponding author

1 Introduction

Let $\mathfrak{h} : [\delta, \zeta] \to \mathbb{R}$, $\upsilon, \nu \in [\delta, \zeta]$ and $t \in [0, 1]$. We call $\mathfrak{h}$ convex on $[\delta, \zeta]$ if it satisfies the following inequality:

$$\mathfrak{h}(t\upsilon + (1-t)\nu) \leq t\mathfrak{h}(\upsilon) + (1-t)\mathfrak{h}(\nu).$$

In the early twentieth century, D. Jensen proved the following generalization of the above statement – which is now known as the Jensen's inequality: if $\mathfrak{h}$ is convex on $[\delta, \zeta]$, then

$$\mathfrak{h}\left(\sum_{h=1}^{t} q_h x_h\right) \leq \sum_{h=1}^{t} q_h \mathfrak{h}(x_h),$$

where $\sum_{h=1}^{t} q_h = 1$ with $q_h > 0$ and $x_h \in [\delta, \zeta]$ for each h.

In 2003, Mercer established the following variant of the Jensen's inequality:

Theorem 1 ([7]). *If $\mathfrak{h} : [\delta, \zeta] \to \mathbb{R}$ is convex with $x_h \in [\delta, \zeta]$ for $h = 1, \cdots, t$, then*

$$\mathfrak{h}\left(x_1 + x_t - \sum_{h=1}^{t} q_h x_h\right) \leq \mathfrak{h}(x_1) + \mathfrak{h}(x_t) - \sum_{h=1}^{t} q_h \mathfrak{h}(x_h),$$

where $\sum_{h=1}^{t} q_h = 1$ with $q_h \geq 0$.

Now, let $\mathbf{d} = (d_1, ...d_r)$ and $\mathbf{b} = (b_1, ..., b_r)$ be two r-tuples such that

$$d_1 \geq d_2 \geq \cdots \geq d_r \qquad \text{and} \qquad b_1 \geq b_2 \geq \cdots \geq b_r.$$

We say that the r-tuple $\mathbf{d}$ majorizes $\mathbf{b}$, and write $\mathbf{b} \prec \mathbf{d}$, if

$$\begin{cases} \displaystyle\sum_{l=1}^{k} d_l \geq \sum_{l=1}^{k} b_l \qquad \text{holds for } k = 1, 2, \cdots, r-1; \\ \text{and} \\ \displaystyle\sum_{l=1}^{r} d_l = \sum_{l=1}^{r} b_l. \end{cases}$$

By means of the theory of majorization, Niezgoda, among other things, proved the succeeding generalization of Theorem 1:

Theorem 2 ([12]). *Let $\mathfrak{h} : [\delta, \zeta] \to \mathbb{R}$ be a continuous convex function on interval $[\delta, \zeta] \subset \mathbb{R}$. Suppose $\mathbf{d} = (d_1, d_2, ..., d_s)$ with $d_l \in [\delta, \zeta]$, and $\mathbf{Z} = (z_{hl})$ is a real $t \times s$ matrix such that $z_{hl} \in [\delta, \zeta]$ for all h, l. If $\mathbf{d}$ majorizes each row of $\mathbf{X}$, that is;*

$$\mathbf{z}_{h.} = (z_{h1}, ..., z_{hs}) \prec (d_1, ..., d_s) = \mathbf{d} \quad \textit{for each} \quad h = 1, ..., t,$$

then we have the inequality

$$\mathfrak{h}\left(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} \sum_{h=1}^{t} q_h z_{hl}\right) \leq \sum_{l=1}^{s} \mathfrak{h}(d_l) - \sum_{l=1}^{s-1} \sum_{h=1}^{t} q_h \mathfrak{h}(z_{hl}),$$

where $\sum_{h=1}^{t} q_h = 1$ with $q_h \geq 0$ for each h.

The notion of convex function has been generalized in the following sense:

Definition 1 ([13]). A function $\mathfrak{h} : [\delta, \zeta] \to \mathbb{R}$ is strongly convex with modulus $c > 0$ if

$$\mathfrak{h}\left(t\upsilon + (1-t)\nu\right) \leq t\mathfrak{h}(\upsilon) + (1-t)\mathfrak{h}(\nu) - ct(1-t)(\upsilon - \nu)^2$$

for all υ, $\nu \in [\delta, \zeta]$ and $t \in [0, 1]$.

It is known that "a function $\mathfrak{g}$ is strongly convex with modulus c on an interval if and only if the function $h = \mathfrak{g} - c(\cdot)^2$ is convex on the same interval." So, to show that the function $\mathfrak{g}$ is strongly convex with modulus c, it suffices to show h' in nondecreasing in the interval under consideration. Strongly convex functions have been found to be applicable in the theories of optimization and approximation, and mathematical economics. The literature is filled with abundance of work around this class of functions. For example, see [5, 9]. In 2010, Merentes and Nikodem [8] proved the following version of the classical discrete Jensen inequality for the class of strongly convex function.

Theorem 3 ([8]). *If $\mathfrak{h} : [\delta, \zeta] \to \mathbb{R}$ is strongly convex with modulus c, then*

$$\mathfrak{h}\left(\sum_{h=1}^{t} q_h x_h\right) \leq \sum_{h=1}^{t} q_h \mathfrak{h}(x_h) - c\sum_{h=1}^{t} q_h \left(x_h - \sum_{h=1}^{t} q_h x_h\right)^2$$

for all $x_1, x_2, \cdots, x_t \in [\delta, \zeta]$ and all $q_1, \cdots, q_t > 0$ such that $\sum_{h=1}^{t} q_h = 1$.

In [15], Zaheer Ullah et al. established loads of majorization results for strongly convex functions. Worthy of mention, are the following two results:

Theorem 4 ([15]). *Let $\mathfrak{h} : [\delta, \zeta] \to \mathbb{R}$ be a strongly convex function with respect to modulus c. Suppose $\mathbf{d} = (d_1, ...d_s)$ and $\mathbf{b} = (b_1, ..., b_s)$ are s-tuples, $d_l, b_l \in [\delta, \zeta], l = 1, ..., s$ and the s-tuple $\mathbf{d}$ majorizes $\mathbf{b}$. Then the following inequality holds:*

$$\sum_{l=1}^{s} \mathfrak{h}(b_l) \leq \sum_{l=1}^{s} \mathfrak{h}(d_l) - c\sum_{l=1}^{s} (d_l - b_l)^2. \tag{1}$$

Theorem 5 ([15]). *Let $\mathfrak{h} : [\delta, \zeta] \to \mathbb{R}$ be a strongly convex function with respect to modulus c. Suppose $\mathbf{d} = (d_1, ...d_s)$, $\mathbf{b} = (b_1, ..., b_s)$ and $\mathbf{p} = (p_1, ..., p_s)$ are s-tuples, $d_l, b_l \in [\delta, \zeta], l = 1, ..., s$. Then the following inequality holds:*

$$\sum_{l=1}^{s} p_l \mathfrak{h}(d_l) \geq \sum_{l=1}^{s} p_l \mathfrak{h}(b_l) + \sum_{l=1}^{s} p_l \mathfrak{h}'(b_l)(d_l - b_l) + c\sum_{l=1}^{s} p_l (d_l - b_l)^2. \tag{2}$$

For some more results related to majorization we recommend [1–3].

Inspired by the work described above, our goal in this article is twofold. Namely,

1. Extend Theorem 2 to the family of strongly convex functions.
2. Establish loads of Mercer type inequalities for similarly separable vectors within the framework of strongly convex functions.

This work is arranged as follows: We prove our main results in Section 2. In Section 3, we applied our main results to selected vectors and discuss nonincreasing mean tuples, convex tuples and star-shaped tuples with regards to our results.

2 Main results

We now state and prove an inequality of the Mercer kind by using the majorization technique.

Theorem 6. *Let $\mathfrak{h} : [\delta, \zeta] \to \mathbb{R}$ be a continuous strongly convex function on interval $[\delta, \zeta] \subset \mathbb{R}$. Suppose $\mathbf{d} = (d_1, d_2, ..., d_s)$ with $d_l \in [\delta, \zeta]$, and $\mathbf{Z} = (z_{hl})$ is a real $t \times s$ matrix such that $z_{hl} \in [\delta, \zeta]$ for all h, l. If $\mathbf{d}$ majorizes each row of $\mathbf{Z}$, that is;*

$$\mathbf{z}_{h.} = (z_{h1}, ..., z_{hs}) \prec (d_1, ..., d_s) = \mathbf{d} \quad \textit{for each} \;\; h = 1, ..., t, \tag{3}$$

then we have the inequality

$$\begin{aligned} &\mathfrak{h}\Big(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1}\sum_{h=1}^{t} q_h z_{hl}\Big) \\ &\le \sum_{l=1}^{s} \mathfrak{h}(d_l) - \sum_{l=1}^{s-1}\sum_{h=1}^{t} q_h \mathfrak{h}(z_{hl}) - c\sum_{l=1}^{s}\sum_{h=1}^{t} q_h (d_l - z_{hl})^2 \\ &\qquad - c\sum_{h=1}^{t} q_h \Bigg[\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl} - \sum_{h=1}^{t} q_h\Big(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl}\Big)\Bigg]^2, \end{aligned} \tag{4}$$

where $\sum_{h=1}^{t} q_h = 1$ with $q_h \ge 0$ for each h.

Proof. We start by noticing that:

$$\begin{aligned} \mathfrak{h}\Big(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1}\sum_{h=1}^{t} q_h z_{hl}\Big) &= \mathfrak{h}\Big(\sum_{h=1}^{t} q_h \sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1}\sum_{h=1}^{t} q_h z_{hl}\Big) \\ &= \mathfrak{h}\Big(\sum_{h=1}^{t} q_h \Big(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl}\Big)\Big). \end{aligned}$$

Now, using Theorem 3 we obtained that:

$$\begin{aligned} &\mathfrak{h}\Big(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1}\sum_{h=1}^{t} q_h z_{hl}\Big) \\ &\le \sum_{h=1}^{t} q_h \mathfrak{h}\Big(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl}\Big) \\ &\qquad - c\sum_{h=1}^{t} q_h \Bigg(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl} - \sum_{h=1}^{t} q_h\Big(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl}\Big)\Bigg)^2. \end{aligned} \tag{5}$$

By (3) we have

$$\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl} = z_{hs} \quad \text{for each} \quad h = 1, ..., t. \tag{6}$$

Using (6) and (1) with $\mathbf{b} = \mathbf{z_{h.}}$ we have that for each $h = 1, ..., t$

$$\mathfrak{h}\Big(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl}\Big) = \mathfrak{h}(z_{hs}) \le \sum_{l=1}^{s} \mathfrak{h}(d_l) - \sum_{l=1}^{s-1} \mathfrak{h}(z_{hl}) - c\sum_{l=1}^{s}(d_l - z_{hl})^2. \tag{7}$$

Multiplying both sides of (7) by $\sum_{h=1}^{t} q_h$, we have

$$\begin{aligned}
&\sum_{h=1}^{t} q_h \mathfrak{h}\bigg(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl}\bigg) \\
&\leq \sum_{h=1}^{t} q_h \sum_{l=1}^{s} \mathfrak{h}(d_l) - \sum_{h=1}^{t} q_h \sum_{l=1}^{s-1} \mathfrak{h}(z_{hl}) - c \sum_{h=1}^{t} q_h \sum_{l=1}^{s} (d_l - z_{hl})^2.
\end{aligned}$$

This implies that:

$$\begin{aligned}
&\sum_{h=1}^{t} q_h \mathfrak{h}\bigg(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl}\bigg) \\
&\leq \sum_{l=1}^{s} \mathfrak{h}(d_l) - \sum_{l=1}^{s-1}\sum_{h=1}^{t} q_h \mathfrak{h}(z_{hl}) - c \sum_{l=1}^{s}\sum_{h=1}^{t} q_h (d_l - z_{hl})^2.
\end{aligned}$$

Subtracting both sides of the above inequality by

$$c \sum_{h=1}^{t} q_h \left(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl} - \sum_{h=1}^{t} q_h \bigg(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl} \bigg) \right)^2,$$

one gets:

$$\begin{aligned}
&\sum_{h=1}^{t} q_h \mathfrak{h}\bigg(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl}\bigg) - c \sum_{h=1}^{t} q_h \left(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl} - \sum_{h=1}^{t} q_h \bigg(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl} \bigg) \right)^2 \\
&\leq \sum_{l=1}^{s} \mathfrak{h}(d_l) - \sum_{l=1}^{s-1}\sum_{h=1}^{t} q_h \mathfrak{h}(z_{hl}) - c \sum_{l=1}^{s}\sum_{h=1}^{t} q_h (d_l - z_{hl})^2 \\
&\quad - c \sum_{h=1}^{t} q_h \left(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl} - \sum_{h=1}^{t} q_h \bigg(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl} \bigg) \right)^2,
\end{aligned} \tag{8}$$

which implies from (5) and (8) that

$$\begin{aligned}
&\mathfrak{h}\bigg(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1}\sum_{h=1}^{t} q_h z_{hl}\bigg) \\
&\leq \sum_{l=1}^{s} \mathfrak{h}(d_l) - \sum_{l=1}^{s-1}\sum_{h=1}^{t} q_h \mathfrak{h}(z_{hl}) - c \sum_{l=1}^{s}\sum_{h=1}^{t} q_h (d_l - z_{hl})^2 \\
&\quad - c \sum_{h=1}^{t} q_h \left(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl} - \sum_{h=1}^{t} q_h \bigg(\sum_{l=1}^{s} d_l - \sum_{l=1}^{s-1} z_{hl} \bigg) \right)^2.
\end{aligned}$$

This completes the proof. □

Remark 1. If $c \to 0^+$, then Theorem 6 becomes Theorem 2. By setting $s = 2, d_1 = a_1, d_2 = a_t$ with $d_1 \leq d_2$, and $z_{h1} = a_h$ and $z_{h2} = d_1 + d_2 - a_h$ for $h = 1, ..., t$, then the inequality in Theorem 6 amounts to:

$$\begin{aligned}
&\mathfrak{h}\Bigg(a_1 + a_t - \sum_{h=1}^{t} q_h a_h\Bigg) \\
&\leq \mathfrak{h}(a_1) + \mathfrak{h}(a_t) - \sum_{h=1}^{t} q_h \mathfrak{h}(a_h) \\
&\qquad - c\Bigg(2\sum_{h=1}^{t} q_h \big(a_1 - a_h\big)^2 + \sum_{h=1}^{t} q_h \Bigg(a_h - \sum_{h=1}^{t} q_h a_h\Bigg)^2\Bigg).
\end{aligned}$$

We recall that an $r \times r$ matrix $\mathbf{D} = (d_{lj})$ is said to be doubly stochastic, if $d_{lj} \geq 0$ and $\sum_{l=1}^{r} a_{lj} = \sum_{j=1}^{r} a_{lj} = 1$ for all $l, j = 1, ..., m$. For matrices of this kind, the following relation was established in [6, p.20]:

$$\mathbf{dD} \prec \mathbf{d} \quad \text{for each real } r-\text{tuple } \mathbf{d} = (d_1, ..., d_r). \tag{9}$$

By using Theorem 6 and (9), the following corollary can be easily deduced:

Corollary 1. *Let $\mathfrak{h} : [\delta, \zeta] \to \mathbb{R}$ be a continous strongly convex function on interval $[\delta, \zeta] \subset \mathbb{R}$. Suppose $\mathbf{d} = (d_1, ..., d_s) \in [\delta, \zeta]^s$ and $\mathbf{D}_1, ..., \mathbf{D}_t$ are $s \times s$ doubly stochastic matrices. If we set*

$$\mathbf{Z} = (z_{hl}) = \begin{pmatrix} \mathbf{dD}_1 \\ . \\ . \\ . \\ \mathbf{dD}_t \end{pmatrix},$$

then the inequality (4) *holds.*

Now, given that $\mathbf{d} = (d_1, ..., d_s)$ and $\mathbf{b} = (b_1, ..., b_s)$, we define the following inner product on $\mathbb{R}^s$ by

$$\langle \mathbf{d}, \mathbf{b} \rangle = \sum_{l=1}^{s} p_l d_l b_l, \tag{10}$$

where $\mathbf{p} = (p_1, ..., p_s)$ is a positive s-tuple. For $t = 1, ..., s$, we denote

$$P_t = \sum_{l=1}^{t} p_l, \quad \hat{P}_t = \sum_{l=1}^{t} l p_l, \quad \tilde{P}_t = \sum_{l=1}^{t} l^2 p_l.$$

Except where otherwise noted, $\mathfrak{E} = \{\mathfrak{e}_1, ..., \mathfrak{e}_t\}$ is a basis in $\mathbb{R}^s$ and $\mathfrak{D} = \{\mathfrak{d}_1, ..., \mathfrak{d}_t\}$ is the dual basis of $\mathfrak{E}$; that is, $\langle \mathfrak{e}_l, \mathfrak{d}_j \rangle = \delta_{lj}$ (kronecker delta), $l, j = 1, ..., s$. We now collate the following definitions that will be needed in the sequel.

Definition 2 ([10, 11])**.** A vector $\mathbf{n} \in \mathbb{R}^s$ is called $\mathfrak{E}$-positive if $\langle \mathfrak{e}_l, \mathbf{n} \rangle > 0$ for all $l = 1, ..., s$. Let $H = \{1, ..., s\}$ and suppose H_1 and H_2 are two indexing sets such that $H_1 \cup H_2 = H$. Given $\mu \in \mathbb{R}$ and $\mathbf{n} \in \mathbb{R}^s$, we say that a vector $\mathbf{a} \in \mathbb{R}^s$ is $\mu, \mathbf{n}$-separable on H_1 and H_2 w.r.t the basis $\mathfrak{E}$, if

$$\langle \mathfrak{e}_l, \mathbf{a} - \mu \mathbf{n} \rangle \geq 0 \quad \text{for} \quad l \in H_1, \quad \text{and} \quad \langle \mathfrak{e}_j, \mathbf{a} - \mu \mathbf{n} \rangle \leq 0 \quad \text{for} \quad j \in H_2. \tag{11}$$

Equivalently, we say that $\mathbf{a} \in \mathbb{R}^s$ is $\mu, \mathbf{n}$-separable on H_1 and H_2 w.r.t the basis $\mathfrak{E}$, if and only if for any $\mathfrak{E}$-positive vector $\mathbf{n} \in \mathbb{R}^s$, the following double inequality holds:

$$\max_{j \in H_2} \frac{\langle \mathfrak{e}_j, \mathbf{a} \rangle}{\langle \mathfrak{e}_j, \mathbf{n} \rangle} \leq \mu \leq \min_{l \in H_1} \frac{\langle \mathfrak{e}_l, \mathbf{a} \rangle}{\langle \mathfrak{e}_l, \mathbf{n} \rangle}. \tag{12}$$

A vector $\mathbf{a} \in \mathbb{R}^s$ is termed $\mathbf{n}$-separable on H_1 and H_2 w.r.t the basis $\mathfrak{E}$, if for some $\mu \in \mathbb{R}$, $\mathbf{a}$ is $\mu, \mathbf{n}$-separable on H_1 and H_2. A map $\psi : [\delta, \zeta] \to \mathbb{R}$ preserves $\mathbf{n}$-separability on H_1 and H_2 w.r.t the basis $\mathfrak{E}$, if $\psi(\mathbf{a}) = (\psi(a_1), ..., \psi(a_s))$ is $\mathbf{n}$-separable on H_1 and H_2 w.r.t the basis $\mathfrak{E}$, whenever $\mathbf{a} = (a_1, ..., a_s) \in [\delta, \zeta]^s$ is $\mathbf{n}$-separable on H_1 and H_2 w.r.t the basis $\mathfrak{E}$. In the situation where $\mathbf{n}$ is $\mathfrak{E}$-positive on $H_1 = \{j_0\}$ and $H_2 = H/\{j_0\}$, the $\mathbf{n}$-separability of $\mathbf{a}$ is implied by the condition

$$\frac{\langle \mathfrak{e}_j, \mathbf{a} \rangle}{\langle \mathfrak{e}_j, \mathbf{n} \rangle} \leq \frac{\langle \mathfrak{e}_{j_0}, \mathbf{a} \rangle}{\langle \mathfrak{e}_{j_0}, \mathbf{n} \rangle} \quad \text{for } j = 1, ..., s, \tag{13}$$

that is the function $H \ni j \to \frac{\langle \mathfrak{e}_j, \mathbf{a} \rangle}{\langle \mathfrak{e}_j, \mathbf{n} \rangle} \in \mathbb{R}$ takes its maximum at $j = j_0$.

The second main result of this paper shall be anchored on the following lemma:

Lemma 1 ([10]). *Let $\mathfrak{E} = \{\mathfrak{e}_1, ..., \mathfrak{e}_t\}$ be a basis in V and $\mathfrak{D} = \{\mathfrak{d}_1, ..., \mathfrak{d}_t\}$ be the dual basis of $\mathfrak{E}$. Suppose $\mathbf{e}, \mathbf{n}, \mathbf{m}$, and $\mathbf{z}$ are the vectors in V with $\langle \mathbf{e}, \mathbf{n} \rangle > 0$. Denote $\lambda = \langle \mathbf{m}, \mathbf{n} \rangle / \langle \mathbf{e}, \mathbf{n} \rangle$. If there exist index sets H_1 and H_2 with $H_1 \cup H_2 = H$, where $H = \{1, 2, ..., t\}$ such that*

(i). *$\mathbf{z}$ is $\mathbf{n}$-separable on H_1 and H_2 w.r.t. $\mathfrak{E}$;*

(ii). *$\mathbf{m}$ is $\lambda, \mathbf{e}$-separable on H_1 and H_2 w.r.t. $\mathfrak{D}$,*

then the inequality

$$\langle \mathbf{z}, \mathbf{e} \rangle \langle \mathbf{m}, \mathbf{n} \rangle \leq \langle \mathbf{z}, \mathbf{m} \rangle \langle \mathbf{e}, \mathbf{n} \rangle$$

holds.

Next, we prove some inequalities of the Mercer type for similarly separable vectors.

Theorem 7. *Let $\mathfrak{h} : [\delta, \zeta] \to \mathbb{R}$ be a strongly convex function on open interval $[\delta, \zeta] \subset \mathbb{R}$. Suppose $\mathbf{a} = (a_1, ..., a_t)$, $\mathbf{b} = (b_1, ..., b_t)$ and $\mathbf{p} = (p_1, ..., p_t)$, where $a_h, b_h \in [\delta, \zeta], p_h > 0$ for $h \in H = \{1, ..., t\}$. Let $\partial\mathfrak{h} : [\delta, \zeta] \to \mathbb{R}$ be the subdifferential of $\mathfrak{h}$, and suppose $\psi \in \partial\mathfrak{h}$. Define*

$$\psi(\mathbf{z}) = (\psi(z_1), ..., \psi(z_t)) \quad \textit{for} \quad \mathbf{z} = (z_1, ..., z_t) \in [\delta, \zeta]^t.$$

Let $\mathfrak{E}, \mathfrak{D}, \mathbf{e}, \mathbf{n}$ be as in Lemma 1 for $\mathbf{n} = \mathbb{R}^t$ with inner product given by (10). Denote $\lambda = \langle \mathbf{a} - \mathbf{b}, \mathbf{n} \rangle / \langle \mathbf{e}, \mathbf{n} \rangle$ with $\langle \mathbf{e}, \mathbf{n} \rangle > 0$. If there exist index sets H_1 and H_2 with $H_1 \cup H_2 = H$ such that

(i). *$\mathbf{b}$ is $\mathbf{n}$-separable on H_1 and H_2 w.r.t. $\mathfrak{E}$;*

(ii). *$\mathbf{a} - \mathbf{b}$ is λ, $\mathbf{e}$-separable on H_1 and H_2 w.r.t. $\mathfrak{D}$; and*

(iii). *ψ preserves-$\mathbf{n}$-separablity on H_1 and H_2 w.r.t. $\mathfrak{E}$.*

Then the following statements take hold, under the above condtions.

A. *If $\langle \mathbf{a} - \mathbf{b}, \mathbf{n} \rangle = 0$, then*

$$\sum_{l=1}^{t} p_l \mathfrak{h}(b_l) \leq \sum_{l=1}^{t} p_l \mathfrak{h}(a_l) - c \sum_{l=1}^{t} p_l (a_l - b_l)^2 \quad \textit{for each } l = 1, ..., t. \tag{14}$$

B. *If* $\langle \mathbf{a}-\mathbf{b}, \mathbf{n}\rangle \geq 0$ *and* $\langle \psi(\mathbf{b}), \mathbf{e}\rangle \geq 0$*, then* (14) *holds.*

Proof. Using (10) and a consequence of (2), we have that

$$\sum_{l=1}^{t} p_l(\mathfrak{h}(a_l)-\mathfrak{h}(b_l)) - c\sum_{l=1}^{t} p_l(a_l-b_l)^2 \geq \sum_{l=1}^{t} p_l(a_l-b_l)\psi(b_l) = \langle \mathbf{a}-\mathbf{b}, \psi(\mathbf{b})\rangle. \quad (15)$$

We deduce from combining (i) and (iii) that the vector $\psi(y)$ is $\mathbf{n}$-separable on H_1 and H_2 w.r.t. $\mathfrak{E}$. Using Lemma 1, we are going to get

$$\langle \mathbf{a}-\mathbf{b}, \psi(\mathbf{b})\rangle \geq \frac{1}{\langle \mathbf{e}, \mathbf{n}\rangle}\langle \mathbf{a}-\mathbf{b}, \mathbf{n}\rangle\langle \psi(\mathbf{b}), \mathbf{e}\rangle.$$

Since $\langle \mathbf{e}, \mathbf{n}\rangle > 0$. So, if $\langle \mathbf{a}-\mathbf{b}, \mathbf{n}\rangle = 0$ then $\langle \mathbf{a}-\mathbf{b}, \psi(\mathbf{b})\rangle \geq 0$, which implies from (15) that

$$\sum_{l=1}^{t} p_l(\mathfrak{h}(a_l)-\mathfrak{h}(b_l)) - c\sum_{l=1}^{t} p_l(a_l-b_l)^2 \geq 0.$$

This implies that

$$\sum_{l=1}^{t} p_l\mathfrak{h}(b_l) \leq \sum_{l=1}^{t} p_l\mathfrak{h}(a_l) - c\sum_{l=1}^{t} p_l(a_l-b_l)^2.$$

Hence, the desired inequality is established. □

Remark 2. If we let $c \to 0^+$ in Theorem 7, then we recover [11, Theorem 2.2].

Theorem 8. *Let* $\mathfrak{h} : [\delta, \zeta] \to \mathbb{R}$ *be a strongly convex function on open interval* $[\delta, \zeta] \subset \mathbb{R}$*. Let* $\partial\mathfrak{h} : [\delta, \zeta] \to \mathbb{R}$ *be the subdifferential of* $\mathfrak{h}$*, and suppose* $\psi \in \partial\mathfrak{h}$*. Suppose* $\mathbf{d} = (d_1, ...d_s) \in [\delta, \zeta]^s$*, and* $\mathbf{Z} = (z_{hl})$ *is a real* $t \times s$ *matrix such that* $z_{hl} \in [\delta, \zeta]$ *for all* h, l*. Let* $\mathbf{m}, \mathbf{n} \in \mathbb{R}^s$ *with* $\langle \mathbf{m}, \mathbf{n}\rangle > 0$*. For each* $h = 1, 2, ..., t$*, if there exist index sets* H_1 *and* H_2 *with* $H_1 \cup H_2 = H$ *such that*

(i). $\mathbf{z_h}.$ *is* $\mathbf{n}$*-separable on* H_1 *and* H_2 *w.r.t.* $\mathfrak{E}$*;*

(ii). $\mathbf{d}-\mathbf{z}_h.$ *is* 0*,* $\mathbf{m}$*-separable on* H_1 *and* H_2 *w.r.t.* $\mathfrak{D}$*;*

(iii). $\langle \mathbf{d}-\mathbf{z}_h., \mathbf{n}\rangle = 0$*;*

(iv). ψ *preserves-*$\mathbf{n}$*-separablity on* H_1 *and* H_2 *w.r.t.* $\mathfrak{E}$*.*

Then we have the inequality that follows

$$\begin{aligned} & p_s\mathfrak{h}\left(\sum_{l=1}^{s} \varepsilon p_l n_l d_l - \sum_{l=1}^{s-1}\sum_{h=1}^{t} \varepsilon q_h p_l n_l z_{hl}\right) \\ & \leq \sum_{l=1}^{s} p_l\mathfrak{h}(d_l) - \sum_{l=1}^{s-1}\sum_{h=1}^{t} q_h p_l\mathfrak{h}(z_{hl}) - c\sum_{l=1}^{s}\sum_{h=1}^{t} q_h p_l(d_l - z_{hl})^2 \\ & \quad - cp_s\sum_{h=1}^{t} q_h\left[\varepsilon\left(\sum_{l=1}^{s} p_l n_l d_l - \sum_{l=1}^{s-1} p_l n_l z_{hl}\right) - \sum_{h=1}^{t} q_h\varepsilon\left(\sum_{l=1}^{s} p_l n_l d_l - \sum_{l=1}^{s-1} p_l n_l z_{hl}\right)\right]^2, \end{aligned} \quad (16)$$

where $\varepsilon = \frac{1}{p_s n_s}$ *with* $n_s \neq 0$ *and* $\sum_{h=1}^{t} q_h = 1$ *with* $q_h \geq 0$*.*

Proof. For $h = 1, ..., t$, denote $\lambda_h = \langle \mathbf{d} - \mathbf{z}_{h\cdot}, \mathbf{n}\rangle / \langle \mathbf{m}, \mathbf{n}\rangle$. By using (iii) we get $\lambda_h = 0$. Under the conditions of the theorem, with $\lambda_h = 0$, we deduce from Theorem 7 the following inequality:

$$\sum_{l=1}^{s} p_l \mathfrak{h}(z_{hl}) \leq \sum_{l=1}^{s} p_l \mathfrak{h}(d_l) - c \sum_{l=1}^{s} p_l (d_l - z_{hl})^2 \quad \text{for each } h = 1, ..., t. \tag{17}$$

First, we observe that:

$$\begin{aligned}\mathfrak{h}\left(\sum_{l=1}^{s} \varepsilon p_l n_l d_l - \sum_{l=1}^{s-1}\sum_{h=1}^{t} \varepsilon q_h p_l n_l z_{hl}\right) &= \mathfrak{h}\left(\sum_{h=1}^{t} q_h \sum_{l=1}^{s} \varepsilon p_l n_l d_l - \sum_{l=1}^{s-1}\sum_{h=1}^{t} \varepsilon q_h p_l n_l z_{hl}\right) \\ &= \mathfrak{h}\left(\sum_{h=1}^{t} q_h \varepsilon\left(\sum_{l=1}^{s} p_l n_l d_l - \sum_{l=1}^{s-1} p_l n_l z_{hl}\right)\right).\end{aligned}$$

Utilizing Theorem 3, we get:

$$\begin{aligned}&\mathfrak{h}\left(\sum_{l=1}^{s} \varepsilon p_l n_l d_l - \sum_{l=1}^{s-1}\sum_{h=1}^{t} \varepsilon q_h p_l n_l z_{hl}\right) \\ &\quad \leq \sum_{h=1}^{t} q_h \mathfrak{h}\left(\varepsilon\left(\sum_{l=1}^{s} p_l n_l d_l - \sum_{l=1}^{s-1} p_l n_l z_{hl}\right)\right) \\ &\quad\quad - c\sum_{h=1}^{t} q_h \left(\varepsilon\left(\sum_{l=1}^{s} p_l n_l d_l - \sum_{l=1}^{s-1} p_l n_l z_{hl}\right) - \sum_{h=1}^{t} q_h \varepsilon\left(\sum_{l=1}^{s} p_l n_l d_l - \sum_{l=1}^{s-1} p_l n_l z_{hl}\right)\right)^2,\end{aligned}$$

which is equivalent to

$$\begin{aligned}&p_s\mathfrak{h}\left(\sum_{l=1}^{s} \varepsilon p_l n_l d_l - \sum_{l=1}^{s-1}\sum_{h=1}^{t} \varepsilon q_h p_l n_l z_{hl}\right) \\ &\quad \leq p_s\sum_{h=1}^{t} q_h \mathfrak{h}\left(\varepsilon\left(\sum_{l=1}^{s} p_l n_l d_l - \sum_{l=1}^{s-1} p_l n_l z_{hl}\right)\right) \\ &\quad\quad - cp_s\sum_{h=1}^{t} q_h \left(\varepsilon\left(\sum_{l=1}^{s} p_l n_l d_l - \sum_{l=1}^{s-1} p_l n_l z_{hl}\right) - \sum_{h=1}^{t} q_h \varepsilon\left(\sum_{l=1}^{s} p_l n_l d_l - \sum_{l=1}^{s-1} p_l n_l z_{hl}\right)\right)^2.\end{aligned} \tag{18}$$

Given that $\langle \mathbf{d} - \mathbf{z}_{h\cdot}, \mathbf{n}\rangle = 0$ for each $h = 1, ..., t$, we get from (10) the following identities:

$$\begin{aligned}\varepsilon\left(\sum_{l=1}^{s} p_l n_l d_l - \sum_{l=1}^{s-1} p_l n_l z_{hl}\right) &= z_{hs} \\ p_s\mathfrak{h}\left(\varepsilon\left(\sum_{l=1}^{s} p_l n_l d_l - \sum_{l=1}^{s-1} p_l n_l z_{hl}\right)\right) &= p_s\mathfrak{h}(z_{hs}).\end{aligned}$$

From (17), we have

$$p_s\mathfrak{h}(z_{hs}) \leq \sum_{l=1}^{s} p_l \mathfrak{h}(d_l) - \sum_{l=1}^{s-1} p_l \mathfrak{h}(z_{hl}) - c\sum_{l=1}^{s} p_l (d_l - z_{hl})^2 \quad \text{for each } h = 1, ..., t.$$

So,

$$p_s\mathfrak{h}\left(\varepsilon\left(\sum_{l=1}^{s}p_ln_ld_l-\sum_{l=1}^{s-1}p_ln_lz_{hl}\right)\right)\leq\sum_{l=1}^{s}p_l\mathfrak{h}(d_l)-\sum_{l=1}^{s-1}p_l\mathfrak{h}(z_{hl})-c\sum_{l=1}^{s}p_l(d_l-z_{hl})^2.$$

Multiplying $\sum_{h=1}^{t}q_h$ to both sides of the above inequality, one obtains:

$$\begin{aligned}
&p_s\sum_{h=1}^{t}q_h\mathfrak{h}\left(\varepsilon\left(\sum_{l=1}^{s}p_ln_ld_l-\sum_{l=1}^{s-1}p_ln_lz_{hl}\right)\right)\\
&\leq\sum_{h=1}^{t}q_h\sum_{l=1}^{s}p_l\mathfrak{h}(d_l)-\sum_{l=1}^{s-1}\sum_{h=1}^{t}q_hp_l\mathfrak{h}(z_{hl})-c\sum_{l=1}^{s}\sum_{h=1}^{t}q_hp_l(d_l-z_{hl})^2\\
&=\sum_{l=1}^{s}p_l\mathfrak{h}(d_l)-\sum_{l=1}^{s-1}\sum_{h=1}^{t}q_hp_l\mathfrak{h}(z_{hl})-c\sum_{l=1}^{s}\sum_{h=1}^{t}q_hp_l(d_l-z_{hl})^2.
\end{aligned}$$

This implies:

$$\begin{aligned}
&p_s\sum_{h=1}^{t}q_h\mathfrak{h}\left(\varepsilon\left(\sum_{l=1}^{s}p_ln_ld_l-\sum_{l=1}^{s-1}p_ln_lz_{hl}\right)\right)\\
&\quad-cp_s\sum_{h=1}^{t}q_h\left(\varepsilon\left(\sum_{l=1}^{s}p_ln_ld_l-\sum_{l=1}^{s-1}p_ln_lz_{hl}\right)-\sum_{h=1}^{t}q_h\varepsilon\left(\sum_{l=1}^{s}p_ln_ld_l-\sum_{l=1}^{s-1}p_ln_lz_{hl}\right)\right)^2\\
&\leq\sum_{l=1}^{s}p_l\mathfrak{h}(d_l)-\sum_{l=1}^{s-1}\sum_{h=1}^{t}q_hp_l\mathfrak{h}(z_{hl})-c\sum_{l=1}^{s}\sum_{h=1}^{t}q_hp_l(d_l-z_{hl})^2\\
&\quad-cp_s\sum_{h=1}^{t}q_h\left(\varepsilon\left(\sum_{l=1}^{s}p_ln_ld_l-\sum_{l=1}^{s-1}p_ln_lz_{hl}\right)-\sum_{h=1}^{t}q_h\varepsilon\left(\sum_{l=1}^{s}p_ln_ld_l-\sum_{l=1}^{s-1}p_ln_lz_{hl}\right)\right)^2.
\end{aligned}\tag{19}$$

It therefore follows from (18) and (19) that

$$\begin{aligned}
&p_s\mathfrak{h}\left(\sum_{l=1}^{s}\varepsilon p_ln_ld_l-\sum_{l=1}^{s-1}\sum_{h=1}^{t}\varepsilon q_hp_ln_lz_{hl}\right)\\
&\leq\sum_{l=1}^{s}p_l\mathfrak{h}(d_l)-\sum_{l=1}^{s-1}\sum_{h=1}^{t}q_hp_l\mathfrak{h}(z_{hl})-c\sum_{l=1}^{s}\sum_{h=1}^{t}q_hp_l(d_l-z_{hl})^2\\
&\quad-cp_s\sum_{h=1}^{t}q_h\left(\varepsilon\left(\sum_{l=1}^{s}p_ln_ld_l-\sum_{l=1}^{s-1}p_ln_lz_{hl}\right)-\sum_{h=1}^{t}q_h\varepsilon\left(\sum_{l=1}^{s}p_ln_ld_l-\sum_{l=1}^{s-1}p_ln_lz_{hl}\right)\right)^2.
\end{aligned}$$

This makes the proof complete. □

Remark 3. If we let $c\to 0^+$ in Theorem 8, then we recapture [12, Theorem 3.1].

Corollary 2. *Let all the conditions of Theorem 8 hold and assume there exist $j_0\in H=\{1,....,s\}$ such that $\mathbf{n}=\mathbf{b}_{j_0}$. Suppose $H_1=\{j_0\}$ and $H_2=H\setminus\{j_0\}$, and substitute the conditions* (i), (ii) *in Theorem 8 by*

(i). *$\mathbf{z}_{h.}$ is $\mathbf{n}$-separable on H_1 and H_2 w.r.t $\mathfrak{E}$, e.g., $\mathbf{n}$ is $\mathfrak{E}$-positive and*

$$\frac{\langle\mathfrak{e}_j,\mathbf{z}_{h.}\rangle}{\langle\mathfrak{e}_j,\mathbf{n}\rangle}\leq\frac{\langle\mathfrak{e}_{j_0},\mathbf{z}_{h.}\rangle}{\langle\mathfrak{e}_{j_0},\mathbf{n}\rangle}\quad\textit{for}\quad j=1,....,s;$$

(ii). $\mathbf{d}-\mathbf{z}_{h.}$ *is* $0,\mathbf{m}$*-separable on* H_1 *and* H_2 *w.r.t* $\mathfrak{D}$, *that is,*

$$\langle \mathbf{b}_j, \mathbf{d}-\mathbf{z}_{h.}\rangle \leq \langle \mathbf{b}_{j_0}, \mathbf{d}-\mathbf{z}_{h.}\rangle \quad \textit{for} \quad j=1,....,s. \tag{20}$$

Then the inequality (16) *holds.*

Proof. By (13) it can be easily seen that the conditions (i) and (ii) of Theorem 8 reduce to (i) and (ii) of Corollary 2 respectively. Clearly, from (iii) we have $\langle \mathbf{b}_{j_0}, \mathbf{d}-\mathbf{z}_{h.}\rangle = \langle \mathbf{n}, \mathbf{d}-\mathbf{z}_{h.}\rangle = 0$. Hence (20) gives

$$\langle \mathbf{b}_j, \mathbf{d}-\mathbf{z}_{h.}\rangle \leq 0 = \langle \mathbf{b}_{j_0}, \mathbf{d}-\mathbf{z}_{h.}\rangle \quad \text{for} \quad j=1,....,s,$$

which means that $\mathbf{d}-\mathbf{z}_{h.}$ is $0,\mathbf{m}$-separable on H_1 and H_2 w.r.t. $\mathfrak{D}$. The statement is now derived from Theorem 8. □

3 Applications

Here, we apply Theorem 8 and Corollary 2 to different vectors $\mathbf{m}$ and $\mathbf{n}$. For this, the following pair of dual basis are considered: $\mathfrak{E}=\{\mathfrak{e}_1,....,\mathfrak{e}_t\}$ and $\mathfrak{D}=\{\mathfrak{d}_1,....,\mathfrak{d}_t\}$. Take,

$$\mathfrak{e}_\tau = \mathfrak{d}_\tau = \frac{1}{\sqrt{p_\tau}}(\underbrace{0,...,0,}_{\tau-1\ times}\ 1,0,...,0) \quad \text{for} \quad \tau=1,...,s \tag{21}$$

and

$$\begin{cases} \mathfrak{e}_\tau = \left(\underbrace{0,...,0,}_{\tau-1\ times}\ \frac{1}{p_\tau}, -\frac{1}{P_{\tau+1}}, 0,...,0\right) & \text{for} \quad \tau=1,...,s-1, \\ \mathfrak{e}_r = \left(0,...,0,\frac{1}{p_s}\right), & \end{cases} \tag{22}$$

$$\mathfrak{d}_\tau = \left(\underbrace{1,...,1,}_{\tau\ times}0,...,0\right) \quad \text{for} \quad \tau=1,...,s. \tag{23}$$

The pair given by (21) gives an orthonormal basis in $\mathbb{R}^s$ with respect to the inner product defined in (10). The latter corresponds to weak majorization ordering, whenever $p_1=...=p_s=1$

Corollary 3. *Let all the conditions of Theorem* 8 *hold. Let* $\mathbf{m}=\mathbf{n}=(1,\cdots,1)$ *and suppose that* $\mathfrak{E}=\mathfrak{D}$ *are the basis in* $\mathbb{R}^s$ *given by* (21). *For each* $h=1,....,t$, *if there exist index sets* H_1 *and* H_2 *with* $H_1\cup H_2=H$ *such that*

(i). $\mathbf{z}_{h.}$ *is* $\mathbf{n}$*-separable on* H_1 *and* H_2 *w.r.t* $\mathfrak{E}$, *that is,*

$$z_{hj} \leq z_{hl} \quad \textit{for} \quad l\in H_1 \quad \textit{and} \quad j\in H_2; \tag{24}$$

(ii). $\mathbf{d}-\mathbf{z}_{h.}$ *is* $0,\mathbf{m}$*-separable on* H_1 *and* H_2 *w.r.t* $\mathfrak{D}=\mathfrak{E}$, *that is,*

$$d_j - z_{hj} \leq 0 \leq d_l - z_{hl} \quad \textit{for} \quad l\in H_1 \quad \textit{and} \quad j\in H_2; \tag{25}$$

(iii). $\sum_{i=1}^{s}(d_i-z_{hi})p_i = 0$.

Then the following inequality holds

$$\begin{aligned}
&\mathfrak{h}\left(\sum_{l=1}^{s}\hat{p}_l d_l-\sum_{l=1}^{s-1}\sum_{h=1}^{t}\hat{p}_l q_h z_{hl}\right)\\
&\leq \sum_{l=1}^{s}\hat{p}_l\mathfrak{h}(d_l)-\sum_{l=1}^{s-1}\sum_{h=1}^{t}\hat{p}_l q_h\mathfrak{h}(z_{hl})-c\sum_{l=1}^{s}\sum_{h=1}^{t}\hat{p}_l q_h(d_l-z_{hl})^2 \qquad (26)\\
&\quad -c\sum_{h=1}^{t}q_h\left[\left(\sum_{l=1}^{s}\hat{p}_l d_l-\sum_{l=1}^{s-1}\hat{p}_l z_{hl}\right)-\sum_{h=1}^{t}q_h\left(\sum_{l=1}^{s}\hat{p}_l d_l-\sum_{l=1}^{s-1}\hat{p}_l z_{hl}\right)\right]^2,
\end{aligned}$$

where $\hat{p}_l=\frac{p_l}{p_s}$ *and* $\sum_{h=1}^{t}q_h=1$ *with* $q_h\geq 0$.

Proof. Using the double inequality in (12) and the vector given in (21), one can deduce that a vector $\mathbf{a}=(a_1,...,a_t)$ is $\mathbf{n}$-separable on H_1 and H_2 w.r.t $\mathfrak{C}$ if and only if

$$a_j\leq a_l \quad \text{for} \quad l\in H_1 \quad \text{and} \quad j\in H_2. \qquad (27)$$

Therefore, (24) and (25) imply the conditions (i) and (ii) of Theorem 8. Since ψ is nondecreasing, it thus follows from (38) that

$$\psi(a_j)\leq\psi(a_l) \quad \text{for} \quad l\in H_1 \quad \text{and} \quad j\in H_2.$$

Hence, $\psi(\mathbf{a})$ is $\mathbf{n}$-separable on H_1 and H_2 w.r.t $\mathfrak{C}$. So the condition (iv) of Theorem 8 is satisfied. Moreover, $\langle\mathbf{d}-\mathbf{z}_{h.},\mathbf{n}\rangle=\sum_{i=1}^{s}(d_i-z_{hi})p_i=0$, which implies (iii) of Theorem 8. We now obtain the desired inequality (26) by using (16). $\square$

Remark 4. We note that if both $\mathbf{z}_{h.}$ and $\mathbf{d}-\mathbf{z}_{h.}$ are nondecreasing, i.e.,

$$z_{h1}\leq ...\leq z_{hs} \quad \textit{and} \quad d_1-z_{h1}\leq ...\leq d_s-z_{hs},$$

then the conditions (24) and (25) hold for index sets

$$H_1=\{i+1,...,s\} \quad \text{and} \quad H_2=\{1,2,...,i\} \text{ for some } i.$$

If $\hat{p}_l=1$ (i.e $p_1=...=p_s$), then (26) becomes

$$\begin{aligned}
&\mathfrak{h}\left(\sum_{l=1}^{s}d_l-\sum_{l=1}^{s-1}\sum_{h=1}^{t}q_h z_{hl}\right)\\
&\leq \sum_{l=1}^{s}\mathfrak{h}(d_l)-\sum_{l=1}^{s-1}\sum_{h=1}^{t}q_h\mathfrak{h}(z_{hl})-c\sum_{l=1}^{s}\sum_{h=1}^{t}q_h(d_l-z_{hl})^2 \qquad (28)\\
&\quad -c\sum_{h=1}^{t}q_h\left[\left(\sum_{l=1}^{s}d_l-\sum_{l=1}^{s-1}z_{hl}\right)-\sum_{h=1}^{t}q_h\left(\sum_{l=1}^{s}d_l-\sum_{l=1}^{s-1}z_{hl}\right)\right]^2.
\end{aligned}$$

Remark 5. If $s=2$, $z_{h1}=a_h$ and $z_{h2}=d_1+d_2-a_h$ for $h=1,...,t$, then (28) reduces to

$$\begin{aligned}
&\mathfrak{h}\left(d_1+d_2-\sum_{h=1}^{t}q_h a_h\right)\\
&\leq \mathfrak{h}(d_1)+\mathfrak{h}(d_2)-\sum_{h=1}^{t}q_h\mathfrak{h}(a_h)-2c\sum_{h=1}^{t}q_h\big(d_1-a_h\big)^2 \qquad (29)\\
&\quad -c\sum_{h=1}^{t}q_h\left[a_h-\sum_{h=1}^{t}q_h a_h\right]^2.
\end{aligned}$$

Corollary 4. *Let all the conditions of Theorem* 8 *hold. Suppose* $\mathbf{m} = \mathbf{n} = (1, ..., 1)$ *and* $\mathfrak{E}$ *and* $\mathfrak{D}$ *are the basis in* $\mathbb{R}^s$ *defined by* (22) *and* (23)*, respectively. For each* $h = 1, ..., t$ *if there exist index sets* H_1 *and* H_2 *with* $H_1 \cup H_2 = H$ *such that*

(i). $\mathbf{z}_{h.}$ *is* $\mathbf{n}$*-separable on* H_1 *and* H_2 *w.r.t* $\mathfrak{E}$*, that is, there exist* $\mu \in \mathbb{R}$ *satisfying,*

$$z_{hj} - z_{h,j+h} \le 0 \le z_{hl} - z_{h,l+1} \quad \textit{for} \quad l \in H_1 \quad \textit{and} \quad j \in H_2 \tag{30}$$

with the convention $x_{h,r+1} = \mu$*,*

(ii). $\mathbf{d} - \mathbf{z}_{h.}$ *is* $0, \mathbf{m}$*-separable on* H_1 *and* H_2 *w.r.t* $\mathfrak{D} = \mathfrak{E}$*; that is,*

$$\sum_{i=1}^{j} (d_i - z_{hi}) p_i \le 0 \le \sum_{i=1}^{l} (d_i - z_{hi}) p_i \quad \textit{for} \quad l \in H_1 \quad \textit{and} \quad j \in H_2; \tag{31}$$

(iii). $\sum_{i=1}^{s} (d_i - z_{hi}) p_i = 0.$

Then the inequalities (26)*,* (28) *and* (29) *hold.*

Proof. Employing (22) and (23) together with (11), one can easily show that the vector $\mathbf{a} = (a_1, ..., a_t)$ is $\mathbf{n}$-separable on H_1 and H_2 w.r.t $\mathfrak{E}$ if and only if there exists $\mu \in \mathbb{R}$ such that

$$a_j - a_{j+1} \le 0 \le a_l - a_{l+1} \quad \text{for} \quad l \in H_1 \quad \text{and} \quad j \in H_2 \tag{32}$$

with the convention $a_{s+1} = \mu$. Also by (12) we deduce that the vector $\mathbf{a} = (a_1, ..., a_t)$ is $0, \mathbf{m}$-separable on H_1 and H_2 w.r.t $\mathfrak{D}$ if and only if

$$\sum_{k=1}^{l} a_h p_h \le 0 \le \sum_{k=1}^{j} a_h p_h \quad \text{for} \quad l \in H_1 \quad \text{and} \quad j \in H_2.$$

Hence (30) and (31) imply the conditions (i) and (ii) of Theorem 8. Furthermore, since ψ is nondecreasing, one gets from (43) the following relation:

$$\psi(a_j) - \psi(a_{j+1}) \le 0 \le \psi(a_l) - \psi(a_{l+1}) \quad \text{for} \quad l \in H_1 \quad \text{and} \quad j \in H_2.$$

Hence, ψ preserves $\mathbf{n}$-separability on on H_1 and H_2 w.r.t $\mathfrak{E}$, and thus the condition (iv) of Theorem 8 is satisfied. Moreover, $\langle \mathbf{d} - \mathbf{z}_{h.}, \mathbf{n} \rangle = \sum_{i=1}^{s} (d_i - z_{hi}) p_i = 0$, which implies (iii) of Theorem 8. By using the inequality (16) of Theorem 8, one obtains (26). Also (28) and (29) follow from (26). □

Remark 6. It is pertinent to note that under assumption (iii) of Corollary 4, conditions (30) and (31) are satisfied for

$$H_1 = \{s\} \quad \textit{and} \quad H_2 = \{1, 2, ..., s-1\}$$

provided $\mathbf{z}_{h.}$ is nondecreasing, i.e $z_{h1} \le z_{h2} \le ... \le z_{hs}$, and $\mathbf{d} - \mathbf{z}_{h.}$ is nondecreasing in P-mean [14, p. 318]. That is,

$$\frac{1}{P_j} \sum_{i=1}^{j} (d_i - z_{hi}) p_i \le \frac{1}{P_{j+1}} \sum_{i=1}^{j+1} (d_i - z_{hi}) p_i \quad \text{for} \quad j = 1, 2, ..., s-1.$$

An s-tuple $\mathbf{a} = (a_1, ..., a_s) \in \mathbb{R}^s$ is called star-shaped [14, p. 318], if

$$\frac{a_j}{j} \le \frac{a_{j+1}}{j+1} \quad \text{for } j = 1, 2, ..., s-1. \tag{33}$$

A function $\psi : [\delta, \zeta] \to \mathbb{R}, h \in [\delta, \zeta]$, where $[\delta, \zeta] \subset \mathbb{R}_+$, is called star-shaped, if the function $h \to \frac{\psi(h)}{h}$ is nondecreasing.

Since every strongly convex function is convex, therefore the following lemma also holds for strongly convex function.

Lemma 2 ([11]). *Let $\psi : [\delta, \zeta] \to \mathbb{R}$ be a convex and differentiable positive nondecreasing function on a positive open interval $[\delta, \zeta] \subset \mathbb{R}_+$. If ψ is star-shaped, then it preserves star-shapeness of s-tuples in the sense that* (33) *implies*

$$\frac{\psi(a_j)}{j} \leq \frac{\psi(a_{j+1})}{j+1} \quad \textit{for } j = 1, 2, ..., s-1.$$

Corollary 5. *Let all the conditions of Theorem 8 hold. Suppose $\mathbf{m} = \mathbf{n} = (1, 2, ..., s)$ and $\mathfrak{E} = \mathfrak{D}$ are the basis in $\mathbb{R}^s$ given by* (21). *For each $h = 1, ..., t$ if there exist index sets H_1 and H_2 with $H_1 \cup H_2 = H$ such that*

(i). *$\mathbf{z}_{h.}$ is $\mathbf{n}$-separable on H_1 and H_2 w.r.t $\mathfrak{E}$, that is,*

$$\frac{z_{hj}}{j} \leq \frac{z_{hl}}{l} \quad \textit{for} \;\; l \in H_1 \quad \textit{and} \quad j \in H_2; \tag{34}$$

(ii). *$\mathbf{d} - \mathbf{z}_{h.}$ is $0, \mathbf{m}$-separable on H_1 and H_2 w.r.t $\mathfrak{D} = \mathfrak{E}$, that is,*

$$\frac{d_j - z_{hj}}{j} \leq 0 \leq \frac{d_l - z_{hl}}{l} \quad \textit{for} \;\; l \in H_1 \quad \textit{and} \quad j \in H_2; \tag{35}$$

(iii). $\displaystyle\sum_{i=1}^{s} (d_i - z_{hi}) i p_i = 0$;

(iv). *ψ preserves $\mathbf{n}$-separability on H_1 and H_2 w.r.t $\mathfrak{E}$, that is,* (34) *implies*

$$\frac{\psi(z_{hj})}{j} \leq \frac{\psi(z_{hl})}{l} \quad \textit{for} \;\; l \in H_1 \quad \textit{and} \quad j \in H_2. \tag{36}$$

Then the following inequality holds

$$\begin{aligned}
&\mathfrak{h}\bigg(\sum_{l=1}^{s} \hat{p}_l \hat{n}_l d_l - \sum_{l=1}^{s-1}\sum_{h=1}^{t} \hat{p}_l \hat{n}_l q_h z_{hl}\bigg) \\
&\leq \sum_{l=1}^{s} \hat{p}_l \mathfrak{h}(d_l) - \sum_{l=1}^{s-1}\sum_{h=1}^{t} \hat{p}_l q_h \mathfrak{h}(z_{hl}) - c\sum_{l=1}^{s}\sum_{h=1}^{t} \hat{p}_l q_h (d_l - z_{hl})^2 \\
&\quad - c\sum_{h=1}^{t} q_h \Bigg(\bigg(\sum_{l=1}^{s} \hat{p}_l \hat{n}_l d_l - \sum_{l=1}^{s-1} \hat{p}_l \hat{n}_l z_{hl}\bigg) - \sum_{h=1}^{t} q_h \bigg(\sum_{l=1}^{s} \hat{p}_l \hat{n}_l d_l - \sum_{l=1}^{s-1} \hat{p}_l \hat{n}_l z_{hl}\bigg)\Bigg)^2,
\end{aligned} \tag{37}$$

where $\hat{p}_l = \frac{p_l}{p_s}$, $\hat{n}_l = \frac{l}{s}$ and $\sum_{h=1}^{t} q_h = 1$ with $q_h \geq 0$.

Proof. Clearly from (12) and (21), it can be seen that a vector $\mathbf{a} = (a_1, ..., a_t)$ is $\mathbf{n}$-separable on H_1 and H_2 w.r.t $\mathfrak{E}$ if and only if

$$\frac{a_j}{j} \leq \frac{a_l}{l} \quad \text{for} \;\; l \in H_1 \;\; \text{and} \;\; j \in H_2. \tag{38}$$

Also, (34) and (35) imply the conditions (i) and (ii) of Theorem 8. Also, the conditions (iii) and (iv) of Theorem 8 can be easily derived from the assumptions (iii) and (iv) of Corollary 5 respectively. We therefore obtain (37) by using (16). □

Remark 7. If $\mathbf{z}_{h.}$ and $\mathbf{d}-\mathbf{z}_{h.}$ are star-shaped tuples. And if the map ψ preserves star-shaped tuples, then the conditions (34), (35) and (36) are satisfied for the index sets

$$H_1=\{i+1,...,s\} \text{ and } H_2=\{1,2,...,i\} \text{ for some } i.$$

For instance, if $\hat{p_l}=1$ (i.e $p_1=...=p_s$), then (37) becomes

$$\begin{aligned}
&\mathfrak{h}\left(\sum_{l=1}^{s}\frac{l}{s}d_l-\sum_{l=1}^{s-1}\sum_{h=1}^{t}\frac{l}{s}q_h z_{hl}\right)\\
&\leq\sum_{l=1}^{s}\mathfrak{h}(d_l)-\sum_{l=1}^{s-1}\sum_{h=1}^{t}q_h\mathfrak{h}(z_{hl})-c\sum_{l=1}^{s}\sum_{h=1}^{t}q_h(d_l-z_{hl})^2 \qquad (39)\\
&\quad-c\sum_{h=1}^{t}q_h\left(\left(\sum_{l=1}^{s}\frac{l}{s}d_l-\sum_{l=1}^{s-1}\frac{l}{s}z_{hl}\right)-\sum_{h=1}^{t}q_h\left(\sum_{l=1}^{s}\frac{l}{s}d_l-\sum_{l=1}^{s-1}\frac{l}{s}z_{hl}\right)\right)^2.
\end{aligned}$$

If $s=2$, $z_{h1}=a_h$ and $z_{h2}=d_1+d_2-a_h$ for $h=1,...,t$, then (39) becomes

$$\begin{aligned}
&\mathfrak{h}\left(\frac{1}{2}d_1+d_2-\frac{1}{2}\sum_{h=1}^{t}q_h a_h\right)\\
&\leq\mathfrak{h}(d_1)+\mathfrak{h}(d_2)-\sum_{h=1}^{t}q_h\mathfrak{h}(a_h) \qquad (40)\\
&\quad-c\left(2\sum_{h=1}^{t}q_h\left(d_1-a_h\right)^2+\sum_{h=1}^{t}q_h\left(\frac{1}{2}a_h-\frac{1}{2}\sum_{h=1}^{t}q_h a_h\right)^2\right).
\end{aligned}$$

Corollary 6. *Let all the conditions of Theorem 8 hold. Suppose* $\mathbf{m}=\mathbf{n}=(1,2,...,s)$ *and* $\mathfrak{E}$ *and* $\mathfrak{D}$ *are the basis in* $\mathbb{R}^s$ *given by* (22) *and* (23)*. For each* $h=1,....,t$ *if there exist index sets* H_1 *and* H_2 *with* $H_1\cup H_2=H$ *such that*

(i). $\mathbf{z}_{h.}$ *is* $\mathbf{n}$*-separable on* H_1 *and* H_2 *w.r.t* $\mathfrak{E}$*, that is, there exist* $\mu\in\mathbb{R}$ *satisfying*

$$z_{h,j+1}-z_{h,j}\geq\mu\geq z_{h,l+1}-z_{hl} \quad \text{for } l\in H_1 \quad \text{and} \quad j\in H_2 \qquad (41)$$

with the convention $z_{h,s+1}=\mu(s+1)$*;*

(ii). $\mathbf{d}-\mathbf{z}_{h.}$ *is* $0,\mathbf{m}$*-separable on* H_1 *and* H_2 *w.r.t* $\mathfrak{D}$*, that is,*

$$\sum_{i=1}^{j}(d_i-z_{hi})p_i\leq 0\leq\sum_{i=1}^{l}(d_i-z_{hi})p_i \quad \text{for } l\in H_1 \quad \text{and} \quad j\in H_2; \qquad (42)$$

(iii). $\sum_{i=1}^{s}(d_i-z_{hi})ip_i=0$*;*

(iv). ψ *preserves* $\mathbf{n}$*-separability on* H_1 *and* H_2 *w.r.t* $\mathfrak{E}$*, that is,* (41) *implies that there exists* $\upsilon\in\mathbb{R}$ *satisfying*

$$\psi(z_{h,j+1})-\psi(z_{hj})\geq\upsilon\geq\psi(z_{h,l+1})-\psi(z_{hl}) \quad \text{for } l\in H_1 \quad \text{and} \quad j\in H_2$$

with the convention $\psi(z_{h,s+1})=\upsilon(m+1)$*.*

Then the inequalities (37)*,* (39) *and* (40) *hold.*

Proof. Using (11) and the vectors given in (22) and (23), it can be seen that a vector $\mathbf{a} = (a_1, ..., a_t)$ is $\mathbf{n}$-separable on H_1 and H_2 w.r.t $\mathfrak{C}$ if and only if there exists $\mu \in \mathbb{R}$ such that

$$a_j - a_{j+1} \leq \mu \leq a_l - a_{l+1} \quad \text{for} \quad l \in H_1 \quad \text{and} \quad j \in H_2 \tag{43}$$

with the convention $a_{s+1} = \mu$. In the other hand, using (12) we deduce that a vector $\mathbf{a} = (a_1, ..., a_t)$ is $\mathbf{n}$-separable on H_1 and H_2 w.r.t $\mathfrak{D}$ if and only if

$$\sum_{i=1}^{j} a_i p_i \leq 0 \leq \sum_{i=1}^{l} a_i p_i \quad \text{for} \quad l \in H_1 \quad \text{and} \quad j \in H_2.$$

Hence, (41) and (42) imply the conditions (i) and (ii) of Theorem 8. Also the conditions (iii) and (iv) of Theorem 8 can be easily derived from the assumptions (iii) and (iv) of Corollary 6 respectively. Therefore, using the inequality (16) of Theorem 8 we get (37). Also (39) and (40) are simply derived from (37). □

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

Authors' Contribution

All authors contributed equally to writing of this paper. All authors read and approved the final manuscript.

References

[1] S. Zaheer Ullah, M. Adil Khan, Z. A. Khan and Y.-M. Chu, Integral majorization type inequalities for the functions in the sense of strong convexity, *J. Funct. Spaces*, **2019** (2019), Article ID 9487823, 11 pages.

[2] S.S. Dragomir, Some majorization type discrete inequalities for convex functions, Math. Ineq. Appl., **7**(2) (2004) 207-216.

[3] N. Latif, J. Pečarić and I. Perić, On discrete Favard's and Berwald's inequalities, Commun. Math. Anal., **12**(2) (2012), 34-57.

[4] A.W. Marshall and I. Olkin, Inequalities, Theory of Majorization and its Applications, Academic Press, New York, (1979).

[5] M. Klaričić Bakula, Jensen–Steffensen inequality for strongly convex functions, J. Inequal. Appl., **2018:306** (2018).

[6] A.W. Marshall, I. Olkin, Inequalities: Theory of Majorization and its Applications, Academic Press, New York, 1979.

[7] A.McD. Mercer, A variant of Jensen's inequality, J. Inequal. Pure & Appl. Math., **4** (4) (2003) Art. 73.

[8] N. Merentes, K. Nikodem, Remarks on strongly convex functions, Aequ. Math., **80** (1–2) (2010) 193–199.

[9] K. Nikodem, On strongly convex functions and related classes of functions. In: Handbook of Functional Equations. Springer Optim. Appl., **95** (2014) 365–405. Springer, New York.

[10] M. Niezgoda, Bifractional inequalities and convex cones, Discrete Math., **306** (2006) 231–243.

[11] M. Niezgoda, Remarks on convex functions and separable sequences, Discrete Math., **308** (2008) 1765–1773.

[12] M. Niezgoda, A generalization of Mercer's result on convex functions, Nonlinear Anal., **71** (7) (2009) 2771–2779.

[13] B. T. Polyak, Existence theorems and convergence of minimizing sequences in extremum problems with restrictions, Sov. Math. Dokl., **7** (1966) 72–75.

[14] Gh. Toader, On Chebyshev's inequality for sequences, Discrete Math., **161** (1996) 317–322.

[15] S. Zaheer Ullah, M. Adil Khan, Y. Chu, Majorization theorems for strongly convex functions, J. Inequal. Appl., **2019** (2019), Article ID 58, 13 pages.