Application of Symmetry Analysis and Conservation Laws to Fractional-Order Nonlinear Conduction-Diffusion Model

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Abstract: This study is aimed to perform Lie symmetry analysis of the nonlinear fractional-order conduction-diffusion Buckmaster model (BM), which involves the Riemann-Liouville (R-L) derivative of fractional-order 'β'. We are going through symmetry reduction to convert the fractional partial differential equation into a fractional ordinary differential equation. The fractional derivatives of the converted differential equations are evaluated with the help of Erdelyi-Kober (E-K) fractional operators. The power series solution and its convergence are analyzed by using Implicit function theorem. Conservation laws of the physical model are obtained for consistency of system by Noether's theorem.

Keywords: Convection-diffusion equation, Buckmaster model, Riemann-Liouville derivatives, Fractional differential equation, Lie Symmetry Analysis.

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1. Introduction

The study of fractional systems as a generalization of classical systems has attracted much attention of the scientists and mathematicians to know exact description of nonlinear phenomenon in fluid dynamics, mechanics, biological modelling, physics, engineering and areas of medical and tool science etc. Podlubny [1], Oldham [2], Debnath [3] and Kilbas et al. [4] has been described the importance and applications of local generalized derivatives or fractional order derivatives in real phenomenon. The nonlinear convection-diffusion equations have great contribution to model of the evolution of thermal waves in plasma (Rosenau and

Kamin [5]). The movement caused with in a fluid by propensity of hotter or less dense material to become colder, denser material to sink under impact of gravity, which in consequence shows in transfer of heat is called convection. The action of distributing matter by natural movement of particles is called diffusion. The classical nonlinear convection-diffusion equation is given by Edward [6].

$$v_t = [D(v)\, v_x]_x + C(v)\, v_x, \tag{1}$$

where $v(x,t)$ represents the density of particles and D (v) is diffusive term, C (v) is convective term; both D (v) and C (v) are non-zero terms. In present article, the nonlinear time fractional convection-diffusion equation is formed by replacing classical derivative by fractional derivative in equation (1).

$$\partial_t^{\beta} u = D(v)\, v_{xx} + D^{'}(v)\,(v_x)^2 + C(v)\, v_x. \tag{2}$$

We have considered special case of conduction-dispersion phenomenon, when D (v) = $4v^3$ and C (v) = $3v^2$ in equation (2), which is known as Buckmaster Model (BM) and it is extremely effective and relevant to explore the propagation of sound, electricity and electrodynamics in physical systems. As we know that buckling is the process of uncertainty that originates in thin materials due to pressure exceeds and makes the material bend out of plane. The BM equation (2) is also meant for dynamical modelling of thin sheet fluid flows to draw buckling, suggested by Buckmaster [7]. Mathematicians have been discussed the relevance of classical and fractional order systems in real dynamical problems with the various application of methodologies. Wazwaz [8-9] applied the variation iterative method, Tanh method and sine-cosine analysis to linear and nonlinear systems. Gardner equation have been solved Lin et al. [10] with imposition of tanh-coth method and Iyiola et al. [11-12] have described applications of Caputo fractional derivatives in different nonlinear time-fractional homogeneous and non-homogeneous models. Jafari et al. [16] explored the numerical scheme to study the system of fractional PDEs. Gandhi et al. [30-34] provided the explicit solution of fifth-order and fourth-order fractional systems by Lie symmetry analysis. He has been discussed about brain cancer tumor growth model and its analytic solution by the application of fractional reduced differential transform method on

Burgess equation. Recently, application of homotopy analysis has been imposed on linear and nonlinear fractional Newell-Whitehead-Segel Equation and ordinary differential equations to obtain the exact and approximate solutions.

It is well known that the Lie symmetry theory plays significant role in invariant analysis of differential equations. The basic observation of methodology is that infinitesimal transformation leaves the set of manifold considered differential equations invariant. This method is given by Sophus Lie, which involves lengthy symbolic process but systematically unifies and extends well known techniques to construct the explicit solutions to nonlinear problems. Olver [13] emphasized on wide range applications of Lie group symmetries analysis to partial differential equations (PDEs). Bakkyaraj & Sahadevan [14] illustrated on Lie group transformation to solve the fractional-order system. Moyo & Leach [15] presented the mathematical cancer model by symmetry analysis. The time-fractional Korteweg-de-Varies equations have been solved by Zhang [17]. Biswas et al. [18-19] organised multiple objectives like solitons, bifurcation analysis, conservation analysis, dual dispersion, and nonlinearity laws of Boussinesq equation. Bansal et al. [20] has designed optical perturbation, Lie group invariants to Fokas-Lenells equation. The symmetry reduction has been applied to clarify the soliton solution of time fractional KdV and K(m,n) equations by Wang et al. [21-22]. The Harry-Dym equation with Riemann-Liouville fractional derivative has been studied by Huang et al. [23]. Garrido et al. [24] suggested Lie point symmetry along with travelling wave solution to generalised Drinfeld-Sokolov system, Bokhari et al. [25] illustrated fundamentals of symmetries to time fractional tumour growth in brain. Liu et al. [26] and Singla et al. [27] declared that the Lie symmetry reduction is robust and authentic technique to solve higher order nonlinear systems. The extensive use of Erdelyi–Kober fractional operators are helpful in converting FPDEs into fractional ODEs has been stated by Sneddon [28]. Balsar et al. [29] attempted sum ability of the series solution of PDEs with constant coefficients. Shi et al. [36] and Razborova et al. [41] explained the additional conservation laws and exact solution to Boussinesq-Burgers system and Rosenau-KdV-RLW equation with nonlinearity by Lie symmetry. The study of diffusion and sub-diffusion wave equations with conservation laws has been concluded by Lukashuk et al. [42]. Anco et al. [43] focused on direct construction of conservation laws of linear and nonlinear PDEs. The concept of nonlinear self-adjointness to time-fractional Kompaneets equation has been obtained by Gazizov et al. [44]. In addition, recently, Gandhi et al. [45] focused on invariant analysis, exact series solution, convergence of solution by Implicit theorem and conservation laws by Noether's theorem on fractional-order Hirota-Satsuma Coupled KdV system. The comparative study for solving Laplace fractional equation has been produced by Dubey et al. [46]. Chatibi et al. [47] has done the discrete symmetry analysis of some global and local systems. The invariant solution of generalized fractional order (2+1)-Dimensional Date-Jimbo-Kashiwara-Miwa equation has been evaluated by Chauhan et al. [48]. Zhang et al. [49] imposed power diffusion and conservation analysis to Fokkar-Planck equation. Bruzon et al. [50] found similarity solution of the Cooper-Shepard-Sodano equation along with utilization of conservation analysis. Using Lie group theory, the exact solutions for certain time-fractional evolution equations and modified Khokhlov–Zabolotskaya–Kuznetsov equation are presented by Bira et al. [51] and Satapathy et al. [52] respectively.

Our research article is organized as: some basic definitions in section 2, Lie symmetry methodology algorithm for BM is explained in part 3 and infinitesimal generators have been deduced by using symmetry reduction in section 4, followed by reduction of FPDEs into FODEs with utilization of Erdelyi-Kober operators in section 5, the power series solutions of respective FODE of BM and their convergence have been studied in subsequent sections 6 and 7 respectively and finally conservation laws have been defined in section 8, which impart great information about physical BM system.

2. Preliminaries

Here, we explained the basic definitions of fractional derivatives and integrals. There are many ways to define fractional derivatives like Grunwald-Letnikov (GL), Riemann-Liouville (RL), Caputo definitions. Each has great advantages and sometimes disadvantages under different circumstance but we are interested in RL fractional derivative to explore symmetry reduction and the exact solution of FPDEs. Some of the required definitions are explained under:

2.1. Definition: The R-L fractional partial derivative of order '$\beta > 0$' for arbitrary function $v(x,t)$ with time variable 't' is given as

$$D_t^{\beta}(u(x,t)) = \begin{cases} \dfrac{1}{\Gamma(m-\beta)} \dfrac{\partial^m}{\partial t^m} \displaystyle\int_0^t (t-\xi)^{m-\beta-1} v(\xi,x) d\xi \text{ ;for } m-1<\beta<m,\ t>0,\ m\in N, \\ \dfrac{\partial^m u}{\partial t^m} \qquad ;\ for\ \beta = m. \end{cases} \tag{3}$$

2.2. Definition: The R-L integrals of fractional order '$\beta > 0$'and '$0 < t <$ T' are defined as

$$_0 J_t^{\beta} g(t,x) = \frac{1}{\Gamma(\beta)} \int_0^t (t-\xi)^{\beta-1} g(\xi,x) d\xi; \tag{4}$$

$$_t J_T^{\beta} g(t,x) = \frac{1}{\Gamma(\beta)} \int_t^T (t-\xi)^{\beta-1} g(\xi,x) d\xi, \tag{5}$$

some important results associated with above operators and applicable in this paper are:

$$D_t^{\beta}(t^{\alpha}) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha-\beta+1)} t^{\alpha-\beta}; \tag{6}$$

$$J_t^{\beta}(t^{\alpha}) = \frac{\Gamma(\alpha+1)}{\Gamma(\alpha+\beta+1)} t^{\alpha+\beta}; \tag{7}$$

$$J_t^{\beta}(D_t^{\beta}(\phi(t)) = \phi(t) - \sum_{k=0}^{\lambda-1} \frac{\phi^{(k)}(0)}{k!} t^k \ ;\lambda-1<\beta\le\lambda; \tag{8}$$

$$D_t^{\beta}(g(t)) = D_t^n J_t^{n-\beta}(g(t));\ n-1<\beta\le n. \tag{9}$$

3. Methodology:

We can impose distinct techniques for obtaining nearly exact or exact solutions of linear and nonlinear FPDEs but Lie symmetry reduction method is supposed to be best procedure to evaluate conservation laws and exact solutions of wide range of class of the symmetries of FPDEs, since symmetry reduction can also be schemed to solve problems by transforming them into fractional ODEs.

Let us assume a FPDE is in following manner

$$\partial_t^{\beta} v = F(x,t,v,v_x,v_{xx},...) \ ;\ \beta \in (0,1) \tag{10}$$

Lie group of infinitesimal transformations is invariant under one-parameter 'ε' and it satisfies

$$
\begin{aligned}
\bar{t} &= t + \varepsilon\,\tau(v;x,t) + O(\varepsilon^2);\\
\bar{x} &= x + \varepsilon\,\xi(v;x,t) + O(\varepsilon^2);\\
\bar{v} &= v + \varepsilon\,\eta(v;x,t) + O(\varepsilon^2);\\
\partial_t^{\beta}\bar{v} &= \partial_t^{\beta} v + \varepsilon\,\eta^{\beta,t}(v;x,t) + O(\varepsilon^2);\\
\partial_t^{1}\bar{v} &= \partial_t^{1} v + \varepsilon\,\eta^{x}(v;x,t) + O(\varepsilon^2);\\
\partial_t^{2}\bar{v} &= \partial_t^{2} v + \varepsilon\,\eta^{xx}(v;x,t) + O(\varepsilon^2),
\end{aligned}
\tag{11}
$$

where $\tau,\ \xi$ *and* η are required infinitesimals η^x , η^{xx} are extended infinitesimals and $\eta^{\beta,t}$ is extended infinitesimal of fractional parameter of order 'β' associated to Lie algebra of (10) is spanned by vector fields

$$
X = \eta\frac{\partial}{\partial v} + \xi\frac{\partial}{\partial x} + \tau\frac{\partial}{\partial t} \quad with\ \tau = \left.\frac{d\bar{t}}{d\varepsilon}\right|_{\varepsilon=0},\ \xi = \left.\frac{d\bar{x}}{d\varepsilon}\right|_{\varepsilon=0},\ \text{and}\ \eta = \left.\frac{d\bar{v}}{d\varepsilon}\right|_{\varepsilon=0}. \tag{12}
$$

Prolongation to (10) carried

$$
pr^{(\beta,2)}(\partial_t^{\beta} v - F)\,|_{\Delta=0} = 0, \tag{13}
$$

where prolongation operator is defined by

$$
pr^{(\beta,2)}(\Delta) = X + \eta^{\beta,t}\frac{\partial}{\partial(\partial_t^{\beta} v)} + \eta^{x}\frac{\partial}{\partial v_x} + \eta^{xx}\frac{\partial}{\partial v_{xx}}. \tag{14}
$$

The expressions for extended infinitesimals are given as

$$
\begin{aligned}
\eta^{x} &= D_x(\eta) - v_t D_x(\tau) - v_x D_x(\xi)\\
&= \eta_x - (\xi_x - \eta_v)v_x - v_t\tau_x - \xi_v v_x^2 - \tau_v v_t v_x,
\end{aligned}
\tag{15}
$$

$$
\begin{aligned}
\eta^{xx} &= D_x(\eta^{x}) - v_{xt} D_x(\tau) - v_{xx} D_x(\xi)\\
&= \eta_{xx} - (\xi_{xv} - 2\eta_{xv})v_x - \tau_{xx}v_t + (\eta_{vv} - 2\xi_{xv}){v_x}^2 - 2\tau_{xv}v_x v_t - \xi_{vv}{v_x}^3\\
&\quad - \tau_{vx}{v_x}^2 v_t - 2\tau_x v_{xt} + (\eta_v - 2\xi_x)v_{xx} - \tau_v v_{xx}v_t - 2\tau_v v_{xt}v_x - 3\xi_v v_x v_{xx}.
\end{aligned}
\tag{16}
$$

$$
D_t^{\beta}(\eta) = \partial_t^{\beta}\eta + \eta_v\,\partial_t^{\beta} v - v\,\partial_t^{\beta}\eta_v + \sum_{m=1}^{\infty}\binom{\beta}{n}\partial_t^{n}(\eta_v)\partial_t^{\beta-n}(v) + \mu, \tag{17}
$$

where,

$$
\mu = \sum_{\lambda=2}^{\infty}\sum_{n=2}^{\lambda}\sum_{k=2}^{n}\sum_{r=0}^{k-1}\binom{\beta}{\lambda}\binom{\lambda}{n}\binom{k}{r}\frac{t^{\lambda-\beta}(-v)^r}{\Gamma(k+1)\Gamma(\lambda+1-\beta)}\frac{\partial^n(v^{k-r})}{\partial t^n}\frac{\partial^{\lambda-n+k}\eta}{\partial t^{\lambda-n}\partial v^k}. \tag{18}
$$

As 'η' is linear function of 'v' then $\mu\to 0$

$$\eta^{\beta,t} = D_t^{\beta}(\eta) + \xi D_t^{\beta}(v_x) - D_t^{\beta}(\xi v_x) + D_t^{\beta}(v.D_t(\tau)) - D_t^{\beta+1}(\tau v) + \tau D_t^{\beta+1} v$$

$$= \partial_t^{\beta}(\eta) + (\eta_v - \alpha D_t \tau)\partial_t^{\beta}(v) - v\partial_t^{\beta}(\eta_v) + \sum_{\lambda=1}^{\infty}\left[\binom{\beta}{\lambda}\partial_t^{n}(\eta_v) - \binom{\beta}{\lambda+1} D_t^{\lambda}(\xi)\partial_t^{\beta-\lambda}(v)\right]\partial_t^{\beta-\lambda} \quad (19)$$

$$-\sum_{\lambda=1}^{\infty}\binom{\beta}{\lambda} D_t^{\lambda}(\xi)\partial_t^{\beta-\lambda}(v_x) + \mu.$$

Finally, we use equations (14-19) in prolonged equation (13), split the coefficients of v_x and v_{xx} and equate to zero then solve the obtained system linear or nonlinear fractional PDEs and ODEs.

4. Fractional-Order Convection-Diffusion Buckmaster Model

Applying Lie symmetry method on BM (2), using Lie symmetry analysis to obtain following set of PDEs

$$\sum_{\lambda=1}^{\infty}\left[\binom{\beta}{\lambda}\partial_t^{\lambda}(\eta_v) - \binom{\beta}{\lambda+1} D_t^{\lambda}(\xi)\,\partial_t^{\beta-\lambda}(v)\right] = 0; \quad (20)$$

$$\sum_{\lambda=1}^{\infty}\binom{\beta}{\lambda} D_t^{\lambda}(\xi) = 0; \quad (21)$$

$$\tau_x = 0,\ \tau_v = 0 : \eta_{vv} = 0,\ \eta_{vx} = 0\ ; \xi_v = 0,\ \xi_t = 0; \quad (22)$$

$$(24v)\eta + (12v^2)(\eta_v - 2\xi_x + \beta\tau_t) = 0; \quad (23)$$

$$(3v^2)(\xi_x - \beta\tau_t) - (6v)\eta - 24v^2\eta_x - 4v^3.(2\eta_{vx} - \xi_{xx}) = 0; \quad (24)$$

$$(4v^3)(2\xi_x - \beta\tau_t) - (12v^2)\eta = 0. \quad (25)$$

In order to solve set of equations (20-25), infinitesimals in explicit form with arbitrary constants '*p*' and '*q*' are given by

$$\xi = px + q, \tau = \frac{-p}{\beta}t, \text{and } \eta = pv\,. \quad (26)$$

Infinitesimal generators are described as

$$S_1 = \frac{\partial}{\partial x}\, and\, S_2 = x\frac{\partial}{\partial x} - \frac{t}{\beta}\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}. \quad (27)$$

The set {S_1, S_2} forms Lie Algebra of obtained infinitesimal generators with the Lie braces operator $[X,Y] = XY - YX$ as we have explained in all above cases. The characteristic equation for S_2 is

$$\frac{dx}{x}=\frac{\beta dt}{-t}=-\frac{dv}{v}. \tag{28}$$

Solving (28), we obtain similarity transformation

$$\zeta = x t^{\beta} \text{and } v = t^{-\beta} J(\zeta), \tag{29}$$

and related FODE with time fractional conduction-diffusion buckmaster equation is

$$\frac{\partial^{\beta} v}{\partial t^{\beta}} = t^{-2\beta}[4(J(\zeta))^{3}(J''(\zeta))+12(J(\zeta)^{2}).(J'(\zeta))^{2}+3(J(\zeta))^{2}.J'(\zeta)] \tag{30}$$

5. Applications of Erdelyi-Kober Operators:

Here, we illustrated the relevance of Erdelyi-Kober fractional differential and integral operators in solving FODEs. Before calculations of reduction of fractional operator $\frac{\partial^{\beta} v}{\partial t^{\beta}}$, let us define the Erdelyi-Kober operators as

$$\left(E_{\partial}^{\tau,\beta} J\right)(\zeta)=\prod_{k=0}^{n-1}\left(\tau+k-\frac{1}{\partial}\zeta\frac{d}{d\zeta}\right)(K_{\partial}^{\tau+\beta,n-\beta}J)(\zeta) ;$$

$$(K_{\partial}^{\tau,\beta}J)(\zeta)=\begin{cases}\frac{1}{\Gamma(\beta)}\int_{1}^{\infty}(w-1)^{\beta-1}w^{-(\tau+\beta)}g(\zeta\, w^{1/\partial})dw, \beta>0; \\ J(\zeta) \qquad , \ \beta=0;\end{cases} \tag{31}$$

with $z>0, \partial>0$ and $\beta>0$; and $n=\begin{cases}[\beta]+1, & \beta\notin N; \\ \beta\ , & \text{otherwise.}\end{cases}$

Theorem: Under the similarity transformations (29) for vector field X_2 the reduced FODE (30) is

$$\frac{\partial^{\beta} v}{\partial t^{\beta}} = 4(J(\zeta))^{3}(J''(z)+12(J(\zeta))^{2}.(J'(\zeta))^{2}+3(J(\zeta))^{2}.J'(\zeta).$$

Now we will solve left hand side $\frac{\partial^{\beta} u}{\partial t^{\beta}}$ with application of E-K operator.

Riemann Liouville derivative for similarity reduction is

$$D_t^{\beta} v = D_t^{\lambda}\left(\frac{1}{\Gamma(\lambda-\beta)}\int_{0}^{t}(t-s)^{\lambda-\beta-1}s^{-\beta}J(xs^{\beta})ds\right). \tag{32}$$

Substituting $s=\frac{t}{\gamma}$ in (32), it reduces to

$$D_t^{\beta} v = D_t^{\lambda}\left(\frac{1}{\Gamma(\lambda-\beta)}\int_1^{\infty}(t-\frac{t}{\gamma})^{\lambda-\beta-1}\left(\frac{t}{\gamma}\right)^{-\beta} J(x(t/\gamma)^{\beta})\frac{t}{\gamma^2}\,dv\right),$$
$$= D_t^{\lambda}\left(\frac{t^{\lambda-2\beta}}{\Gamma(\lambda-\beta)}\int_1^{\infty}(\gamma-1)^{\lambda-\beta-1}\gamma^{-(\lambda+1-2\beta)} J(\zeta\gamma^{-\beta})d\gamma\right). \tag{33}$$

Using equation (32), we obtain

$$D_t^{\beta} v = D_t^{\lambda}\left(t^{\lambda-2\beta}\left[\left(K_{-1/\beta}^{1-\beta,\lambda-\beta}J\right)(\zeta)\right]\right) \tag{34}$$

$$if\ \ \zeta = xt^{-\beta},\ J\in C'(0,\infty)$$
$$t\,D_t J(\zeta) = tx(-\beta)t^{-\beta-1}D_{\zeta}J(\zeta) = -\beta\,\zeta\,D_{\zeta}J(\zeta)$$

$$D_t^{\beta} v = D_t^{\lambda-1}\,D_t\left(t^{\lambda-2\beta}\left[\left(K_{-1/\beta}^{1-\beta,\lambda-\beta}J\right)(\zeta)\right]\right)$$
$$= D_t^{\lambda-1}\left(t^{\lambda-2\beta-1}\left(\lambda-2\beta+\beta\,\zeta\,D_{\zeta}\right)\left[\left(K_{-1/\beta}^{1-\beta,\lambda-\beta}J\right)(\zeta)\right]\right) \tag{35}$$

Reconsider similar arguments (λ-1) times, to get

$$D_t^{\beta} v = t^{-2\beta}\prod_{j=0}^{\lambda-1}\left(1+j-2\beta+\beta\,\zeta\,D_{\zeta}\right)\left(K_{-1/\beta}^{1-\beta,\lambda-\beta}J\right)(\zeta) = t^{-2\beta}\left(P_{-1/\beta}^{1-2\beta,\beta}J\right)(\zeta) \tag{36}$$

Finally, FODE becomes

$$\left(P_{-1/\beta}^{1-2\beta,\beta}J\right)(\zeta) = [4(J(\zeta))^3 J''(\zeta)+12(J(\zeta))^2.(J'(\zeta))^2+3(J(\zeta))^2.J'(\zeta)] \tag{37}$$

6. Power Series Solution of BM:

Now for further solution of FODEs, we want to explore the explicit power series solution [30, 31], which can be applied to solve FODE (37).

Set the power series $$J(\zeta)=\sum_{n=0}^{\infty}a_n\zeta^n, \tag{38}$$

substituting (38) in (37), it yields

$$\sum_{n=0}^{\infty}\frac{\Gamma(2-\beta-n\beta)}{\Gamma(3-2\beta-n\beta)}.a_n.\zeta^n = 4\sum_{n=0}^{\infty}\sum_{k=o}^{n}\sum_{i=0}^{k}\sum_{j=0}^{i}(n+2-k).(n+1-k)\,a_i\,a_{k-i}\,a_{i-j}a_{n+2-k}.\zeta^n$$
$$+12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(n+1-k)^2.a_i a_{k-i}a^2{}_{n+1-k}.\zeta^n + 3\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(n-k+1).a_i a_{k-i}a_{n+k+1}\,\zeta^n. \tag{39}$$

Put n = 0 in (25) and comparing coefficients of ζ^n, we get

$$a_2 = -\frac{1}{8a^3{}_0}\left(\frac{(1-2\beta)\Gamma(2-\beta)}{\Gamma(3-2\beta)}a_0 - 12a_0{}^2.a_1{}^2 - 3a_0{}^2a_1\right),\ a_0\ \ and\ \ a_1 \neq 0.$$

$$a_{n+2} = \frac{1}{(n+1)(n+2)(4a_n{}^3)}\left(\begin{array}{l}\frac{(1-2\beta+n\beta)\Gamma(2-(n+1)\beta)}{\Gamma(3-(n+2)\beta)}a_n - 12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(i+1-k)(i+1)a_i a_{k-i}a^2{}_{n+1-k} \\ -3\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}(i+1-k).a_i a_{k-i}a_{n+1-k}\end{array}\right)$$

As

$$J(\zeta) = a_0 + a_1(\zeta) + a_2(\zeta)^2 + \sum_{n=1}^{\infty}a_{n+2}(\zeta)^{n+2},$$

$$v(x,t) = a_0 + a_1xt^{-\alpha/3} + a_2x^2t^{-2\alpha/3} + \sum_{n=1}^{\infty}a_{n+2}x^{n+2}t^{-(n+2)\alpha/3}. \tag{40}$$

Hence, we found the exact power series solution (40).

Now we are expecting the convergence of solution of BM, so a_{n+2} in equation (39) taken as

$$|a_{n+2}| \leq \frac{1}{|d|}\left(\begin{array}{l}\left|\frac{(1-2\beta+n\beta)\Gamma(2-(n+2)\beta)}{\Gamma(3-(2+n)\beta)}\right||a_n| + 12\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}|a_i||a_{k-i}||a^2{}_{n+1-k}| \\ +3\sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}|a_i||a_{n+1-k}||a_{k-i}|\end{array}\right). \tag{41}$$

We can find $\left|\frac{(1-2\beta+n\beta)\Gamma(2-(n-1)\beta)}{\Gamma(2-2\beta+n\beta)}\right| < 1$, for large arbitrary value of *n*.

$$|a_{n+2}| \leq M\left(|a_n| + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}|a_i||a^2{}_{n+1-k}||a_{k-i}| + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}|a_i||a_{k-i}||a_{n+1-k}|\right); \tag{42}$$

where $M = \text{greatest}\left\{\frac{1}{|d|}, \frac{12}{|d|}, \frac{3}{|d|}\right\}$.

Introduce another majorant series

$$G(\zeta) = \sum_{n=0}^{\infty}c_n\zeta^n; c_i = |a_i|, i = 0,1,2,3...,\tag{43}$$

where

$$c_{n+3} = M\left(c_n + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}c_ic_{k-i}c^2{}_{n+1-k}. + \sum_{n=0}^{\infty}\sum_{k=0}^{n}\sum_{i=0}^{k}c_ic_{k-i}c_{n+1-k}\right). \tag{44}$$

It can observe that $|a_n| \leq c_n; n = 0,1,\ 2...,$

Further, the series function G (ζ) has non-negative convergence radius and it presents

$$G(\zeta)=c_0+c_1\zeta+c_2\zeta^2+M\sum_{n=1}^{\infty}\left(c_n+\sum_{k=0}^{n}\sum_{i=0}^{k}c_ic_{k-i}c_{n+1-k}{}^2+\sum_{k=0}^{n}\sum_{i=0}^{k}c_ic_{k-i}c_{n-k+1}\right)\zeta^{n+2}. \quad (45)$$

Now, implicit function system is defined with the variable ζ.

$$I(\zeta,G)=G-c_0-c_1\zeta-c_2\zeta^2-M\sum_{n=1}^{\infty}\left(c_n+\sum_{k=0}^{n}\sum_{i=0}^{k}c_ic_{k-i}c_{n+1-k}{}^2+\sum_{k=0}^{n}\sum_{i=0}^{k}c_ic_{k-i}c_{n-k+1}\right)\zeta^{n+2}. \quad (46)$$

As I (ζ, G) is regular in vicinity of (0, c_0) and I (*0*, c_0) = 0 with $\frac{\partial}{\partial G}I(0,c_0)\neq 0$, by implicit function theorem explained in Rudin [35]. We observed that G (ζ) is regular in the vicinity of the point (0, c_0) and have real positive radius and the series solution (40), converges in the vicinity of the (0, c_0).

7. Construction of Conservation Laws for BFM:

In physical and mathematical vision, conservation laws plays key role in the analysis of time fractional PDEs. To obtain the conservation laws of convection-diffusion BM, we are generalizing the Noether's theorem suggested by Ibragimov [37-38], which have been discussed in Bourdin et al. [39] and Malinowaska et al. [40]. The applications of conservation laws in FPDEs are almost alike to the application of these laws in classical order PDEs. These conservation laws can be extended from PDEs to FDEs.

Let us define a conserved vector for BM (2), where λ^t and λ^x are components of vector

$$\lambda=(\lambda^t,\lambda^x), \quad (47)$$

which satisfy the continuity or conservation equation given by

$$D_t(\lambda^t)+D_x(\lambda^x)\,|_{\Delta=0}=0\ . \quad (48)$$

A formal Lagrangian form with '*u'* as new independent variable described as

$$\ell=u\left[\partial_t^{\beta}v-4v^3v_{xx}-12v^2v_x{}^2-3v^2v_x\right], \quad (49)$$

where $\delta/\delta v$ is Euler-Lagrangian operator, is defined as

$$\frac{\delta}{\delta v}=\frac{\partial}{\partial v}+(D_t^{\beta})^*\frac{\partial}{\partial(D_t^{\beta}v)}+\sum_{k=1}^{\infty}(-1)^kD_{i_1}D_{i_2}...D_{i_k}\frac{\partial}{\partial v_{i_1},_{i_2},_{i_3}\cdots_{i_k}}, \quad (50)$$

where $(D_t^{\beta})^*$ is adjoint of R-L fractional differential operator (D_t^{β}).

Adjoint equation of (2), is given by

$$({D}_t^{\beta})^* = (-1)^n \, {}_tJ_T^{n-\beta}(D_t^n) = {}_t^cD_T^{\beta}\,; \;\; \Delta^* = \frac{\delta\ell}{\delta v} = 0, \tag{51}$$

where ${}_tJ_T^{n-\beta}$ is right-handed fractional integral of order *(n-β)* and ${}_t^cD_T^{\beta}$ is Caputo right handed derivative operator of fractional order β.

The idea of physical property of self-adjointness for establishing these laws have been discussed in [44] and this concept can also be applied and expanded to fractional PDEs. Here (2) will be self-adjoint if the adjoint equation (51) is well pleased for obtained solution of system (2).

For further discussion the basic Noether expression defined as

$$\overline{X} + D_t(\tau) + D_x(\xi) = W\frac{\delta}{\delta v} + D_t(N^t) + D_x(N^x), \tag{52}$$

where N^t and N^x are noether operators. As N^x in (2) doesn't have the non-integer or fractional derivatives with variable 'x' so, general expression is

$$N^x = \xi\ell + W\left(\frac{\partial}{\partial v_x} - D_x\frac{\partial}{\partial v_{xx}}\right) + D_x(W)\left(\frac{\partial}{\partial v_{xx}}\right), \tag{53}$$

and N^t involves fractional derivative so, this can be expressed by RL derivatives as

$$N^t = \ell\tau + \sum_{j=0}^{n-1}(-1)^j \, D_t^j \frac{\partial}{\partial(D_t^{\beta}v)} D_t^{\beta-1-j}(W) - (-1)^n I\left(W, \, D_t^n \frac{\partial}{\partial(D_t^{\beta}v)}\right), \tag{54}$$

$\overline{X}$ is prolongation of symmetry reduction with characteristic of the vector field $W = \eta - \tau\, v_t - \xi\, v_x$ in (52) and operator I in (54) is described as the following integral

$$I(g, f) = \frac{1}{\Gamma(n-\beta)}\int_0^t\int_t^T \frac{g(\tau, x) f(\mu, x)}{(\mu-\tau)} d\mu d\tau. \tag{55}$$

Apply Lagrangian operator 'ℓ' on both sides of (52) for any vector X of (2) and its solution, after that we observed

$$\overline{X}\,\ell + D_t(\tau)\ell + D_x(\xi)\ell\,|_{\Delta=0} = 0, \text{ also } \frac{\delta\ell}{\delta v} = 0. \tag{56}$$

Thus, we obtained the conservation law of (2)

$$D_t(N^t\ell) + D_x(N^x\ell) = 0. \tag{57}$$

The components λ^t and λ^x of conserved vector fields in (31) can be expressed by

$$\lambda^t = N^t \ell = \tau \ell + \sum_{j=0}^{m-1} (-1)^j D_t^j \frac{\partial \ell}{\partial (D_t^\beta v)} D_t^{\beta-1-j}(W) - (-1)^m I\left(W, D_t^m \frac{\partial \ell}{\partial (D_t^\beta v)}\right),$$
$$\lambda^x = N^x \ell = \xi \ell + W\left(\frac{\partial \ell}{\partial v_x} - D_x \frac{\partial \ell}{\partial v_{xx}}\right) + D_x(W)\left(\frac{\partial \ell}{\partial v_{xx}}\right). \tag{58}$$

The adjoint equation for (2) is found as

$$\begin{aligned}\Delta^* &= D_t^{\beta^*}(v) - 12uv^2 v_{xx} - 24uvv_x^2 - 6uvv_x + D_x(24uv^2 v_x) + D_x(3uv^2) - D_x^2(4uv^3)\\ &= D_t^{\beta^*}(v) + 3u_x v^2 - 4u_{xx} v^3\\ &= 0\end{aligned} \tag{59}$$

If adjoint equation (59) satisfied for all solutions of (2), is said to be nonlinear self adjoint. It shows

$$D_t^{\beta^*}(v) + 3u_x v^2 - 4u_{xx} v^3 = \lambda\left[\partial_t^\beta v - 4v^3 v_{xx} - 12v^2 {v_x}^2 - 3v^2 v_x\right] \tag{60}$$

Substituting $u = \psi(t,x) = \omega(t)\rho(x) \neq 0$ represents the nonlinear self adjointness of (2). By using above (60), we obtain

$$D_t^{\beta^*}(\omega(t)) = {}_t^c D_t^{\beta^*}(\omega(t)) = 0,$$
$$3\rho_x(x) v^2 - 4\rho_{xx}(x) v^3 = 0.$$

Which implies that $v = \psi(t,x) = C$, C is arbitrary constant.

So, Lagrangian operator for (2) is $\ell = C\left[\partial_t^\beta v - 4v^3 v_{xx} - 12v^2 {v_x}^2 - 3v^2 v_x\right]$

Now, we proceed with the calculation of conservation laws of BM by using (58).

Case 1. For $0 \le \beta < 1, S_1 = \dfrac{\partial}{\partial x}$, the Lie characteristic is $W_1 = -v_x$, so the components of conserved vectors are as follows

$$\begin{aligned}\lambda^x &= \xi \ell + W_1\left(-D_x \frac{\partial}{\partial v_{xx}} + \frac{\partial}{\partial v_x}\right)\ell + D_x(W_1)\left(\frac{\partial \ell}{\partial v_{xx}}\right)\\ &= (24c.v^2 v_x + 3c.v^2 - 4cD_x(v^3))v_x - D_x(v_x)(-4cv^3)\\ &= c.v^2(3v_x + 12{v_x}^2 + 4vv_{xx}),\end{aligned} \tag{61}$$

$$\begin{aligned}\lambda^t &= c.D_t^{\beta-1}(-v_x) + I(-v_x, 0)\\ &= -c.D_t^{\beta-1}(v_x).\end{aligned} \tag{62}$$

Case 2. For $S_2 = x\frac{\partial}{\partial x} - \frac{t}{\beta}\frac{\partial}{\partial t} + v\frac{\partial}{\partial v}$ the Lie characteristic is $W_2 = v - \frac{t}{\beta}v_t + xv_x$, so the components of conserved vectors are as follows

$$\begin{aligned}\lambda^x &= \xi\,\ell + W_2\left(-D_x\frac{\partial}{\partial v_{xx}} + \frac{\partial}{\partial v_x}\right)\ell + D_x(W_2)\left(\frac{\partial \ell}{\partial v_{xx}}\right)\\ &= (v + \frac{t}{\beta}v_t - xv_x)(-24cv^2v_x - 3cv^2 - D_x(-4cv^3)) - 4cv^3 D_x(u + \frac{t}{\beta}v_t - xv_x)\\ &= -3cv^2(4v_x + 1)(v + \frac{t}{\beta}v_t - xv_x) - 4cv^3(v_x - \frac{t}{\beta}v_{tx} - v_x - xv_{xx}),\end{aligned} \tag{63}$$

$$\begin{aligned}\lambda^t &= \sum_{k=0}^{n-1}(-1)^k D_t^k \frac{\partial \ell}{\partial(D_t^\beta v)} D_t^{\beta-1-k}(W_2) - (-1)^n I\left(W_2,\, D_t^n \frac{\partial \ell}{\partial(D_t^\beta v)}\right)\\ &= c\,D_t^{\beta-1}(v + \frac{t}{\beta}v_t - xv_x) + I(v + \frac{t}{\beta}v_t - xv_x, 0)\\ &= c\,D_t^{\beta-1}(v + \frac{t}{\beta}v_t - xv_x).\end{aligned} \tag{64}$$

8. Conclusions:

In this article, we have utilized the symmetry reduction to fractional-order convection-diffusion Buckmaster model. The Lie point infinitesimal generators and Lie algebra have been constructed and the Erdelyi-Kober operators are used to transform the fractional-PDE into fractional-ODE. Finally, the power series solutions of model obtained with its convergence by implicit function theorem. Also, Ibragimov's method and Noether's theorem have been used for construction of conservation laws of the model.

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