# FRACTIONAL SIMPSON'S TYPE INEQUALITIES FOR TWICE DIFFERENTIABLE CONVEX FUNCTIONS WITH APPLICATIONS

JAMSHED NASIR, SHAHID QAISAR, SAAD IHSAN BUTT, AND MUHAMMAD AAMIR ALI

Abstract. In this paper, we prove a new identity involving the second derivative of the function and Riemann-Liouville fractional integrals. The newly established identity is then used to establish some new Simpson's type inequalities for twice differentiable convex functions. Finally, we give applications of special functions using the newly proved inequalities.

1

## 1. Introduction and Preliminaries

A fantastic dependence has been found among inequalities and theory of convex functions. This relationship is the essential mental security behind the tremendous information using convex functions. The Simpson type imbalances have been analyzed comprehensively throughout ongoing many years. Recall that the secant line joining the images of any two points in the domain of a convex function dominates the entire graph between the points. Formally, A function $h : J \subseteq R \to R$, is termed as convex on $J$, if the inequality

$$h\left(\eta\vartheta_1 + (1-\eta)\vartheta_2\right) \le \eta h\left(\vartheta_1\right) + (1-\eta) h\left(\vartheta_2\right), \quad , \tag{1.1}$$

holds for all $\vartheta_1, \vartheta_2 \in J, \vartheta_1 < \vartheta_2$, and $\eta \in [0,1]$. We say that $h$ is concave, if the inequality (1.1) holds in the reverse direction.

One of the marvelous result in the theory of inequalities, is the Simpson's inequality. The geometrical significance attracts the researcher to obtain the average integrals for generalized classes of convex functions. So far, many generalizations has become the part of literature. We refer to [1, 4, 5, 7, 11, 20] and the references therin.

In [4], Dragomir et al. established the Simpson's inequality.

**Theorem 1.1.** *Let $h : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be a differentiable mapping whose derivative is continuous on $(\vartheta_1, \vartheta_2)$ and $h' \in L[\vartheta_1, \vartheta_2]$. Then we have the following inequality:*

$$\left|\left[\frac{1}{6}h\left(\vartheta_1\right) + \frac{2}{3}h\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + \frac{1}{6}h\left(\vartheta_2\right)\right] - \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} h(\theta)\, d\theta\right| \le \frac{(\vartheta_2-\vartheta_1)}{3}\left\|h\right\|_1, \tag{1.2}$$

*where* $\|h'\|_1 = \int_{\vartheta_1}^{\vartheta_2} |h'(\vartheta)|\, d\vartheta$.

The bound of inequality (1.3) for L-Lipschitzian mapping was given in [4] by $\frac{5}{36}\left(\vartheta_2 - \vartheta_1\right)$. In [15], Kırmaci et al. established the some inequalities of the Hermite–Hadamard type inequality as follows

**Theorem 1.2.** *Let $h : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ be a differentiable mapping whose derivative is continuous on $(\vartheta_1, \vartheta_2)$ and $h' \in L[\vartheta_1, \vartheta_2]$. Then we have the following inequality:*

$$\left|h\left(\frac{\vartheta_1+\vartheta_2}{2}\right) - \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} h(x)\, dx\right| \le \frac{\vartheta_2-\vartheta_1}{4}\left[\frac{|h'(\vartheta_1)| + |h'(\vartheta_2)|}{2}\right], \tag{1.3}$$

The following Lemma is proved by Sarikaya et al. see [19]:

**Lemma 1.3.** *Suppose $h : [\vartheta_1, \vartheta_2] \to \mathbb{R}$ is twice differentiable mapping on $I^0$ such that $h'' \in L_1[\vartheta_1, \vartheta_2]$ where $\vartheta_1, \vartheta_2 \in I$ with $\vartheta_1 < \vartheta_2$. Then following equality holds:*

$$\left[\frac{1}{6}h\left(\vartheta_1\right) + \frac{2}{3}h\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + \frac{1}{6}h\left(\vartheta_2\right)\right] - \frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} h(\theta)\, d\theta$$

---

*Key words and phrases.* Simpson's inequality, Hölder– inequality, convexity, numerical integration.

[1]2010 Mathematics Subject Classification: Primary: 26A51; Secondary: 26A33, 26D07, 26D10, 26D15.

$$= \quad (\vartheta_2-\vartheta_1)^2\int_0^1 \lambda(\eta)\, h''(\eta\vartheta_2+(1-\eta)\vartheta_1)d\eta, \tag{1.4}$$

where

$$\lambda(\eta)=\begin{cases} \frac{\eta}{2}\left(\frac{1}{3}-\eta\right), & \eta\in\left[0,\frac{1}{2}\right) \\ (1-\eta)\left(\frac{\eta}{2}-\frac{1}{3}\right), & \eta\in\left[\frac{1}{2},1\right] \end{cases}$$

Using Lemma, Sarikaya et al. [19] established the following results holds for twice differentiable mapping

**Theorem 1.4.** *Suppose $h:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ is a differentiable mapping on $I^0$ such that $h''\in L_1\,[\vartheta_1,\vartheta_2]$ where $\vartheta_1,\vartheta_2\in I$ with $\vartheta_1<\vartheta_2$, then following equality holds:*

$$\left|\left[\frac{1}{6}h(\vartheta_1)+\frac{2}{3}h\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{1}{6}h(\vartheta_2)\right]-\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} h(\theta)\,d\theta\right|$$
$$=\frac{(\vartheta_2-\vartheta_1)^2}{162}\int_0^1 \lambda(\eta)\,h''(\eta\vartheta_2+(1-\eta)\vartheta_1)d\eta, \tag{1.5}$$

**Theorem 1.5.** *Let $h$ be defined as in Theorem 1 and if $|h''|$ is a convex on $[\vartheta_1,\vartheta_2]$, with $q\ge 1$, then we have:*

$$\left|\left[\frac{1}{6}h(\vartheta_1)+\frac{2}{3}h\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+\frac{1}{6}h(\vartheta_2)\right]-\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2} h(\theta)\,d\theta\right|\le(\vartheta_2-\vartheta_1)^2\left(\frac{1}{162}\right)^{1-1/q}$$
$$\left\{\left(\frac{59}{3^5.2^7}|h''(\vartheta_2)|^q+\frac{133}{3^5.2^7}|h''(\vartheta_1)|^q\right)^{1/q}+\left(\frac{133}{3^5.2^7}|h''(\vartheta_2)|^q+\frac{59}{3^5.2^7}|h''(\vartheta_1)|^q\right)^{1/q}\right\} \tag{1.6}$$

*where $\frac{1}{p}+\frac{1}{q}=1$.*

*Now we recall the important definition of Riemann−Liouville fractional integral*

**Definition 1.6.** Suppose $h\in L[\vartheta_1,\vartheta_2]$. The left−sided and right−sided Riemann−Liouville fractional integrals for left−sided and right−sided of order $\chi>0$ defined by

$$J^{\chi}_{\vartheta_1}h(\tau)=\frac{1}{\Gamma(\chi)}\int_{\vartheta_1}^{\tau}(\tau-\eta)^{\chi-1}h(\eta)d\eta, \qquad \vartheta_1<\tau$$

and

$$J^{\chi}_{\vartheta_2}h(\tau)=\frac{1}{\Gamma(\chi)}\int_{\tau}^{\vartheta_2}(\eta-\tau)^{\chi-1}h(\eta)d\eta, \qquad \tau<\vartheta_2$$

Gamma function is defined as $\Gamma(\chi)=\int_0^{\infty}e^{-u}u^{\chi-1}du$. Note that $J^0_{\vartheta_1}h(\tau)=J^0_{\vartheta_2}h(\tau)=h(\tau)$.

If $\chi=1$, the above integral becomes the classical integral. Numerous researchers have been demonstrated Simpson's type inequalities and obtained various outcomes. For more details(see, [10]- [8]). For the execution of differentiation and integration of real or complex number orders, fractional calculus, demonstrated as an accommodating device which exhibits its centrality. The topic had pulled in a great deal of thought from numerous authors who center around investigation of analysis during the latest couple of many years. For late results related to current study, one can see( [16]- [18]). Among a lot of the fractional integral which are grown up, the Riemann–Liouville fractional integral has been generally thought of, because of uses in various fields of sciences.

The aim of this paper is to establish some new inequalities of type Simpson's for twice differentiable for convex and concave functions via Riemann-Liouville fractional integral. Our results generalize the results obtained by Sarikaya [19] and provide new estimates on these types of inequalities for fractional integrals.

## 2. Inequalities for Simpson type

In this section, First we prove an important new Lemma for Riemann-Livouille fractional integrals, which plays a key role to prove our main results is as follows:

**Lemma 2.1.** *Let $h:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ be a twice differentiable mapping such that $h''$ is integrable on $(\vartheta_1,\vartheta_2)$ with $\vartheta_1<\vartheta_2$. If $|h''|$ is convex on $[\vartheta_1,\vartheta_2]$, then the identity holds for $0<\chi\le 1$*

$$\frac{\Gamma(\chi+1)}{2(\vartheta_2-\vartheta_1)^{\chi}}\left[J^{\chi}_{\vartheta_1}h(\vartheta_2)+J^{\chi}_{\vartheta_2}h(\vartheta_1)\right]$$
$$-\frac{1}{2}\left[\left(\frac{(2^{\chi}-1)(\chi+1)}{2^{\chi}(\chi+2)}\right)(h(\vartheta_1)+h(\vartheta_2))+\left(\frac{2^{\chi}+\chi+1}{2^{\chi-1}(\chi+2)}\right)h\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right]$$

$$= (\vartheta_2 - \vartheta_1)^2 \left[ D_1 + D_2 + D_3 + D_4 \right], \tag{2.1}$$

*where*

$$D_1 = \int_0^{1/2} \left( \frac{2^\chi (\chi + 2) \eta^{\chi+1} - \eta(\chi+1)}{(\chi + 1) 2^\chi (\chi + 2)} \right) h''(\eta\vartheta_1 + (1-\eta)\vartheta_2) d\eta,$$
$$D_2 = \int_{1/2}^1 \left( \frac{\eta^{\chi+1}}{\chi + 1} - \frac{\eta}{\chi + 1} - \frac{\eta}{2^\chi} + \frac{1}{\chi + 2} + \frac{1}{2^\chi} - \frac{1}{\chi + 1} \right) h''(\eta\vartheta_2 + (1-\eta)\vartheta_1) d\eta,$$
$$D_3 = \int_0^{1/2} \left( \frac{2^\chi (\chi + 2) \eta^{\chi+1} - \eta(\chi+1)}{(\chi + 1) 2^\chi (\chi + 2)} \right) h''(\eta\vartheta_2 + (1-\eta)\vartheta_1) d\eta,$$
$$D_4 = \int_{1/2}^1 \left( \frac{\eta^{\chi+1}}{\chi + 1} - \frac{\eta}{\chi + 1} - \frac{\eta}{2^\chi} + \frac{1}{\chi + 2} + \frac{1}{2^\chi} - \frac{1}{\chi + 1} \right) h''(\eta\vartheta_1 + (1-\eta)\vartheta_2) d\eta.$$

*Proof.* Using integration by parts, we obtain

$$\begin{aligned}
D_1 &= \int_0^{1/2} \left( \frac{\eta^{\chi+1}}{\chi + 1} - \frac{\eta}{2^\chi (\chi + 2)} \right) h''(\eta\vartheta_1 + (1-\eta)\vartheta_2) d\eta \\
&= \left( \frac{\eta^{\chi+1}}{\chi+1} - \frac{\eta}{2^\chi (\chi + 2)} \right) \frac{h'(\eta\vartheta_1 + (1-\eta)\vartheta_2)}{\vartheta_1 - \vartheta_2} \Bigg|_0^{1/2} \\
&- \frac{1}{\vartheta_1 - \vartheta_2} \int_0^1 \left( \eta^\chi - \frac{1}{2^\chi (\chi + 2)} \right) h'(\eta\vartheta_1 + (1-\eta)\vartheta_2) d\eta \\
&= -\frac{1}{\vartheta_2 - \vartheta_1} \left[ \frac{1}{2^{\chi+1} (\chi + 1)} - \frac{1}{2^{\chi+1} (\chi + 2)} \right] h'\left( \frac{\vartheta_1 + \vartheta_2}{2} \right) + \\
&\frac{1}{\vartheta_2 - \vartheta_1} \\
&\times \left[ \left( \eta^\chi - \frac{1}{2^\chi (\chi + 2)} \right) \frac{h(\eta\vartheta_2 + (1-\eta)\vartheta_1)}{\vartheta_1 - \vartheta_2} \Bigg|_0^{1/2} - \frac{1}{\vartheta_1 - \vartheta_2} \int_0^{1/2} \chi\eta^{\chi-1} h(\eta\vartheta_1 + (1-\eta)\vartheta_2) d\eta \right] \\
&= -\frac{1}{\vartheta_2 - \vartheta_1} . \frac{1}{2^{\chi+1}} \left( \frac{1}{(\chi + 1)(\chi + 2)} \right) h'\left( \frac{\vartheta_1 + \vartheta_2}{2} \right) \\
&- \frac{1}{(\vartheta_2 - \vartheta_1)^2} \left( \frac{1}{2^\chi} - \frac{1}{2^\chi (\chi + 2)} \right) h\left( \frac{\vartheta_1 + \vartheta_2}{2} \right) \\
&+ \frac{1}{(\vartheta_2 - \vartheta_1)^2} \left( \frac{-1}{2^\chi (\chi + 2)} \right) h(\vartheta_2) + \frac{\chi}{(\vartheta_2 - \vartheta_1)^2} \int_0^{1/2} \eta^{\chi-1} h(\eta\vartheta_1 + (1-\eta)\vartheta_2) d\eta \\
&= -\frac{1}{\vartheta_2 - \vartheta_1} . \frac{1}{2^{\chi+1} (\chi + 1)(\chi + 2)} h'\left( \frac{\vartheta_1 + \vartheta_2}{2} \right) \\
&- \frac{1}{(\vartheta_2 - \vartheta_1)^2} \left( \frac{(\chi + 1)}{2^\chi (\chi + 2)} \right) h\left( \frac{\vartheta_1 + \vartheta_2}{2} \right) \\
&- \frac{1}{2^\chi (\chi + 2)(\vartheta_2 - \vartheta_1)^2} h(\vartheta_2) + \frac{\chi}{(\vartheta_2 - \vartheta_1)^2} \int_0^{1/2} \eta^{\chi-1} h(\eta\vartheta_1 + (1-\eta)\vartheta_2) d\eta \\
&= -\frac{1}{\vartheta_2 - \vartheta_1} . \frac{1}{2^{\chi+1} (\chi + 1)(\chi + 2)} h'\left( \frac{\vartheta_1 + \vartheta_2}{2} \right) \\
&- \frac{1}{(\vartheta_2 - \vartheta_1)^2} \left( \frac{(\chi + 1)}{2^\chi (\chi + 2)} \right) h\left( \frac{\vartheta_1 + \vartheta_2}{2} \right) \\
&- \frac{1}{2^\chi (\chi + 2)(\vartheta_2 - \vartheta_1)^2} h(\vartheta_2) + \frac{\chi}{(\vartheta_2 - \vartheta_1)^2} \int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2} \frac{(\theta_2 - \vartheta)^{\chi-1}}{(\vartheta_2 - \vartheta_1)^\chi} h(\theta) d\theta.
\end{aligned}$$

$$\begin{aligned}
D_2 &= \frac{1}{\vartheta_2 - \vartheta_1} . \frac{1}{2^{\chi+1} (\chi + 1)(\chi + 2)} h'\left( \frac{\vartheta_1 + \vartheta_2}{2} \right) \\
&- \frac{1}{(\vartheta_2 - \vartheta_1)^2} \left( \frac{(\chi + 1)}{2^\chi (\chi + 2)} \right) h\left( \frac{\vartheta_1 + \vartheta_2}{2} \right) \\
&- \frac{1}{2^\chi (\chi + 2)(\vartheta_2 - \vartheta_1)^2} h(\vartheta_1) + \frac{\chi}{(\vartheta_2 - \vartheta_1)^2} \int_{\frac{\vartheta_1+\vartheta_2}{2}}^{\vartheta_2} \frac{(\theta - \vartheta_1)^{\chi-1}}{(\vartheta_2 - \vartheta_1)^\chi} h(\theta) d\theta.
\end{aligned}$$

$$
\begin{aligned}
D_3 &= \int_{1/2}^{1}\left(\frac{\eta^{\chi+1}}{\chi+1}-\frac{\eta}{\chi+1}-\frac{\eta}{2^{\chi}}+\frac{1}{\chi+2}+\frac{1}{2^{\chi}}-\frac{1}{\chi+1}\right)h''(\eta\vartheta_2+(1-\eta)\vartheta_1)d\eta \\
&= \left(\frac{\eta^{\chi+1}}{\chi+1}-\frac{\eta}{\chi+1}-\frac{\eta}{2^{\chi}}+\frac{1}{\chi+2}+\frac{1}{2^{\chi}}-\frac{1}{\chi+1}\right)\frac{h'(\eta\vartheta_2+(1-\eta)\vartheta_1)}{\vartheta_2-\vartheta_1}\Bigg|_{1/2}^{1} \\
&- \frac{1}{\vartheta_1-\vartheta_2}\int_{1/2}^{1}\left(\eta^{\chi}-\frac{1}{\chi+1}-\frac{1}{2^{\chi}}\right)h'(\eta\vartheta_2+(1-\eta)\vartheta_1)d\eta \\
&= -\frac{1}{\vartheta_2-\vartheta_1}\left[\frac{1}{2(\chi+2)}-\frac{\chi}{2^{\chi+1}(\chi+1)}+\frac{1}{2^{\chi}}-\frac{1}{\chi+1}\right]h'\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&- \frac{1}{\vartheta_2-\vartheta_1}\int_{1/2}^{1}\left(\eta^{\chi}-\frac{1}{\chi+1}-\frac{1}{2^{\chi}}\right)h'(\eta\vartheta_2+(1-\eta)\vartheta_1)d\eta \\
&= -\frac{1}{\vartheta_2-\vartheta_1}\left[\frac{1}{2(\chi+2)}-\frac{\chi}{2^{\chi+1}(\chi+1)}+\frac{1}{2^{\chi}}-\frac{1}{\chi+1}\right]h'\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&- \frac{1}{\vartheta_2-\vartheta_1}\left[\left(\eta^{\chi}-\frac{1}{\chi+1}-\frac{1}{2^{\chi}}\right)\frac{h'(\eta\vartheta_2+(1-\eta)\vartheta_1)}{\vartheta_2-\vartheta_1}\Bigg|_{1/2}^{1}+\frac{\chi}{\vartheta_2-\vartheta_1}\int_{1/2}^{1}\eta^{\chi-1}h(\eta\vartheta_2+(1-\eta)\vartheta_1)d\eta\right] \\
&= -\frac{1}{\vartheta_2-\vartheta_1}\left[\frac{1}{2(\chi+2)}-\frac{\chi}{2^{\chi+1}(\chi+1)}+\frac{1}{2^{\chi}}-\frac{1}{\chi+1}\right]h'\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&- \frac{1}{(\vartheta_2-\vartheta_1)^2}\left(1-\frac{1}{(\chi+2)}-\frac{1}{2^{\chi}}\right)h(\vartheta_2) - \frac{1}{(\vartheta_2-\vartheta_1)^2(\chi+2)}h\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&+ \frac{\chi}{\vartheta_2-\vartheta_1}\int_{1/2}^{1}\eta^{\chi-1}h(\eta\vartheta_2+(1-\eta)\vartheta_1)d\eta \\
&= -\frac{1}{\vartheta_2-\vartheta_1}\left[\frac{1}{2(\chi+2)}-\frac{\chi}{2^{\chi+1}(\chi+1)}+\frac{1}{2^{\chi}}-\frac{1}{\chi+1}\right]h'\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&- \frac{1}{(\vartheta_2-\vartheta_1)^2}\left(1-\frac{1}{(\chi+2)}-\frac{1}{2^{\chi}}\right)h(\vartheta_2) - \frac{1}{(\vartheta_2-\vartheta_1)^2(\chi+2)}h\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&+ \frac{\chi}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\frac{(\theta-\vartheta_1)^{\chi-1}}{(\vartheta_2-\vartheta_1)^{\chi}}h(\theta)d\theta.
\end{aligned}
$$

$$
\begin{aligned}
D_4 &= \frac{1}{\vartheta_2-\vartheta_1}\left[\frac{1}{2(\chi+2)}-\frac{1}{\chi+1}-\frac{\chi}{2^{\chi+1}(\chi+1)}+\frac{1}{2^{\chi}}\right]h'\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&- \frac{1}{(\vartheta_2-\vartheta_1)^2}\left(1-\frac{1}{(\chi+2)}-\frac{1}{2^{\chi}}\right)h(\vartheta_1) - \frac{1}{(\vartheta_2-\vartheta_1)^2(\chi+2)}h\left(\frac{\vartheta_1+\vartheta_2}{2}\right) \\
&+ \frac{\chi}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\frac{\vartheta_1+\vartheta_2}{2}}\frac{(\theta_2-\vartheta)^{\chi-1}}{(\vartheta_2-\vartheta_1)^{\chi}}h(\theta)d\theta.
\end{aligned}
$$

Which completes the proof. □

**Theorem 2.2.** *Let $h$ be defined as in Lemma 2.1 and if $|h''|$ is a convex on $[\vartheta_1, \vartheta_2]$, then we have following Simpson's type inequality:*

$$
\begin{aligned}
&\Bigg|\frac{\Gamma(\chi+1)}{2(\vartheta_2-\vartheta_1)^{\chi}}\left[J_{\vartheta_1}^{\chi}h(\vartheta_2)+J_{\vartheta_2}^{\chi}h(\vartheta_1)\right] \\
&-\frac{1}{2}\left[\left(\frac{(2^{\chi}-1)(\chi+1)}{2^{\chi}(\chi+2)}\right)(h(\vartheta_1)+h(\vartheta_2))+\left(\frac{2^{\chi}+\chi+1}{2^{\chi-1}(\chi+2)}\right)h\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\Bigg| \\
&\leq (\vartheta_2-\vartheta_1)^2\left\{(G_1+G_3)\ |h''(\vartheta_1)|+((G_2+G_4)\,|h''(\vartheta_2)|\right\}.
\end{aligned}
\tag{2.2}
$$

*Proof.* By using the properties of modulus on Lemma 2.1, we have

$$
\begin{aligned}
&\Bigg|\frac{\Gamma(\chi+1)}{2(\vartheta_2-\vartheta_1)^{\chi}}\left[J_{\vartheta_1}^{\chi}h(\vartheta_2)+J_{\vartheta_2}^{\chi}h(\vartheta_1)\right] \\
&-\frac{1}{2}\left[\left(\frac{(2^{\chi}-1)(\chi+1)}{2^{\chi}(\chi+2)}\right)(h(\vartheta_1)+h(\vartheta_2))+\left(\frac{2^{\chi}+\chi+1}{2^{\chi-1}(\chi+2)}\right)h\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\Bigg|
\end{aligned}
$$

$$\leq (\vartheta_2-\vartheta_1)^2\left[|D_1|+|D_2|+|D_3|+|D_4|\right].$$

By the convexity of $|h''|$, $\chi\in(0,1]$ and $\forall\ \eta\in[0,1]$, we have

$$\begin{aligned}
&\left|\frac{\Gamma(\chi+1)}{2(\vartheta_2-\vartheta_1)^{\chi}}\left[J^{\chi}_{\vartheta_1}h(\vartheta_2)+J^{\chi}_{\vartheta_2}h(\vartheta_1)\right]\right.\\
&\left.-\frac{1}{2}\left[\left(\frac{(2^{\chi}-1)(\chi+1)}{2^{\chi}(\chi+2)}\right)(h(\vartheta_1)+h(\vartheta_2))+\left(\frac{2^{\chi}+\chi+1)}{2^{\chi-1}(\chi+2)}\right)h\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\right|\\
&\leq(\vartheta_2-\vartheta_1)^2\int_0^{1/2}\left|\frac{2^{\chi}(\chi+2)\eta^{\chi+1}-\eta(\chi+1)}{(\chi+1)2^{\chi}(\chi+2)}\right|\left|h''(\eta\vartheta_2+(1-\eta)\vartheta_1)\right|d\eta\\
&+(\vartheta_2-\vartheta_1)^2\int_0^{1/2}\left|\frac{2^{\chi}(\chi+2)\eta^{\chi+1}-\eta(\chi+1)}{(\chi+1)2^{\chi}(\chi+2)}\right|\left|h''(\eta\vartheta_1+(1-\eta)\vartheta_2)\right|d\eta\\
&+(\vartheta_2-\vartheta_1)^2\int_{1/2}^1\left|\frac{\eta^{\chi+1}}{\chi+1}-\frac{\eta}{\chi+1}-\frac{\eta}{2^{\chi}}+\frac{1}{\chi+2}+\frac{1}{2^{\chi}}-\frac{1}{\chi+1}\right|\left|h''(\eta\vartheta_2+(1-\eta)\vartheta_1)\right|d\eta\\
&+(\vartheta_2-\vartheta_1)^2\int_{1/2}^1\left|\frac{\eta^{\chi+1}}{\chi+1}-\frac{\eta}{\chi+1}-\frac{\eta}{2^{\chi}}+\frac{1}{\chi+2}+\frac{1}{2^{\chi}}-\frac{1}{\chi+1}\right|\left|h''(\eta\vartheta_1+(1-\eta)\vartheta_2)\right|d\eta\\
&=(\vartheta_2-\vartheta_1)^2\int_0^{1/2}\left|\frac{2^{\chi}(\chi+2)\eta^{\chi+1}-\eta(\chi+1)}{(\chi+1)2^{\chi}(\chi+2)}\right|\left\{\eta\left|h''(\vartheta_2)\right|+(1-\eta)\left|h''(\vartheta_1)\right|\right\}d\eta\\
&+(\vartheta_2-\vartheta_1)^2\int_0^{1/2}\left|\frac{2^{\chi}(\chi+2)\eta^{\chi+1}-\eta(\chi+1)}{(\chi+1)2^{\chi}(\chi+2)}\right|\left\{\eta\left|h''(\vartheta_1)\right|+(1-\eta)\left|h''(\vartheta_2)\right|\right\}d\eta\\
&+(\vartheta_2-\vartheta_1)^2\int_{1/2}^1\left|\frac{\eta^{\chi+1}}{\chi+1}-\frac{\eta}{\chi+1}-\frac{\eta}{2^{\chi}}+\frac{1}{\chi+2}+\frac{1}{2^{\chi}}-\frac{1}{\chi+1}\right|\left\{\eta\left|h''(\vartheta_2)\right|+(1-\eta)\left|h''(\vartheta_1)\right|\right\}d\eta\\
&+(\vartheta_2-\vartheta_1)^2\int_{1/2}^1\left|\frac{\eta^{\chi+1}}{\chi+1}-\frac{\eta}{\chi+1}-\frac{\eta}{2^{\chi}}+\frac{1}{\chi+2}+\frac{1}{2^{\chi}}-\frac{1}{\chi+1}\right|\left\{\eta\left|h''(\vartheta_1)\right|+(1-\eta)\left|h''(\vartheta_2)\right|\right\}d\eta\\
&\leq(\vartheta_2-\vartheta_1)^2\left\{(G_1+G_3)\ |h''(\vartheta_1)|+((G_2+G_4)|h''(\vartheta_2)|\right\}.
\end{aligned}$$

Simple calculations yields

$$\begin{aligned}
G_1&=\int_0^{1/2}\left|\frac{2^{\chi}(\chi+2)\eta^{\chi+1}-\eta(\chi+1)}{(\chi+1)2^{\chi}(\chi+2)}\right|\eta d\eta\\
&=\frac{81\left[(\chi+2)^4(2^{-\chi-3}-3^{-\chi-3})-1\right]-19(\chi+1)(\chi+2)^3(\chi+3)2^{-\chi-3}+27(2^{-\chi})(\chi+1)(\chi+3)}{81(\chi+1)(\chi+3)(\chi+2)^4}\\
G_2&=\int_0^{1/2}\left|\frac{2^{\chi}(\chi+2)\eta^{\chi+1}-\eta(\chi+1)}{(\chi+1)2^{\chi}(\chi+2)}\right|(1-\eta)d\eta\\
&=\frac{2^{-\chi-3}.3^{-\chi-4}\left(3^{\chi}(\chi(2\chi+89)+330)-3\ 2^{\chi+4}(2\chi+7)\right)}{(\chi+1)(\chi+2)(\chi+3)}\\
G_3&=\int_{1/2}^1\left|\frac{\eta^{\chi+1}}{\chi+1}-\frac{\eta}{\chi+1}-\frac{\eta}{2^{\chi}}+\frac{1}{\chi+2}+\frac{1}{2^{\chi}}-\frac{1}{\chi+1}\right|\eta d\eta\\
&=\frac{1}{24}\left(\frac{(\chi+4)(\chi+2)^3+3\ 2^{\chi+2}(\chi+1)(\chi+2)^2-4^{\chi+1}(2\chi+5)(\chi+2)-8^{\chi+1}(\chi+1)}{(\chi+1)(\chi+2)^4}\right)\\
&+\frac{1}{24}\left(24\left(\frac{(2^{\chi+2})^{\chi}-2^{-\chi-2}(\chi+2)^{\chi+3}}{(\chi+3)(\chi+2)^3(\chi+1)}\right)-2^{1-\chi}\right)\\
&+\frac{1}{6}\left(\frac{(2\chi+5)(\chi+2)^3+3\ 2^{\chi}(\chi+1)(\chi+2)^2-4^{\chi}(2\chi+5)(\chi+2)-2^{3\chi+1}(\chi+1)}{(\chi+1)(\chi+2)^4}\right)\\
&+\frac{1}{6}\left(\frac{(\chi+2)^3(\chi+2)^{\chi}-(2^{\chi+3})^{\chi}}{(\chi+2)^{\chi+3}(\chi+1)(\chi+3)}-2^{\chi}\right)\\
G_4&=\int_{1/2}^1\left|\frac{\eta^{\chi+1}}{\chi+1}-\frac{\eta}{\chi+1}-\frac{\eta}{2^{\chi}}+\frac{1}{\chi+2}+\frac{1}{2^{\chi}}-\frac{1}{\chi+1}\right|(1-\eta)d\eta\\
&=-\frac{2^{\chi}(2^{\chi}(-\chi^2-3\chi+2^{\chi}-2)\cdot 3(\chi+2)^{\chi+2}\cdot(\chi+3)+32^{2\alpha-1}(2\chi^2)+(2^{\chi}+6)\chi+4)(\chi+2)(\chi+3)}{3(\chi+2)^{\chi+4}(\chi+3)(\chi+1)}\\
&+\frac{(2^{\chi+4})^{\chi}\cdot 3(\chi+3)(\chi+2)^{\chi}-(2^{\chi})^{2\chi+2}3(\chi+2)(\chi+3)-3^{\chi}(\chi+1)(\chi+2^{\chi}+2)(\chi+3)}{3(\chi+2)^{\chi+4}(\chi+3)(\chi+1)}
\end{aligned}$$

$$-2^{\chi}\frac{3.2^{\chi}(-\chi^2-3\chi+2^{\chi}-2)(\chi+3)(\chi+2)^{\chi+1}\ +\ 3\cdot 2^{2\xi-1}(2\chi^2+(2^{\chi}+6)\chi+4)(\chi+3)(\chi+2)^{\chi}}{3(\chi\ +\ 2)^{\chi+3}(\chi\ +\ 3)(\chi\ +\ 1)}$$
$$+\frac{3(2^{\chi+4})^{\chi}-3\cdot 2^{\chi}(2^{\chi})^{\chi+2}(\chi+3)\ -\ 8^{\chi}(\chi+1)(\chi+2^{\chi}+2)(\chi+3)(\chi+2)^{\chi-1}}{3(\chi\ +\ 2)^{\chi+3}(\chi\ +\ 3)(\chi\ +\ 1)}$$
$$-\frac{2^{-\chi-3}\left(7\chi^3-2\left(2^{\chi}-21\right)\chi^2\ +\ \left(80-17\ 2^{\chi}\right)\chi-33\ 2^{\chi}+54\right)}{3\left(\chi^3\ +\ 6\chi^2\ +\ 11\chi\ +\ 6\right)}$$

$$\begin{aligned}
G_5 &= \int_0^{1/2}\left|\frac{2^{\chi}\left(\chi\ +\ 2\right)\eta^{\chi+1}\ -\ \eta\left(\chi\ +\ 1\right)}{\left(\chi\ +\ 1\right)2^{\chi}\left(\chi\ +\ 2\right)}\right|d\eta\\
&= \frac{2^{-\chi-1}}{(\chi+2)^3}-\frac{(\chi+2)^{-\chi-3}}{\chi+1}-\frac{2^{-\chi-3}3^{-\chi-2}\left(5\ 3^{\chi}\chi-13\ 3^{\chi}+2^{\chi+3}\right)}{(\chi+1)(\chi+2)}\\
G_6 &= \int_{1/2}^{1}\left|\frac{\eta^{\chi+1}}{\chi\ +\ 1}-\frac{\eta}{\chi\ +\ 1}-\frac{\eta}{2^{\chi}}+\frac{1}{\chi\ +\ 2}+\frac{1}{2^{\chi}}-\frac{1}{\chi\ +\ 1}\right|d\eta\\
&= \frac{1}{8}\left(-\frac{2^{\chi+3}}{\chi^2\ +\ 3\chi\ +\ 2}-\frac{4\chi}{(\chi\ +\ 2)^2}-3\ 2^{-\chi}+8\frac{(2^{\chi+2})^{\chi}-2^{-\chi-2}(\chi\ +\ 2)^{2+\chi}}{(\chi\ +\ 2)^{\chi+4}(\chi\ +\ 1)}\right)\\
&\quad+\frac{1}{8}\left(+\frac{4}{\chi\ +\ 1}+\frac{9}{\chi\ +\ 2}+\frac{2^{\chi+2}}{(\chi\ +\ 2)^2}-\frac{8}{(\chi\ +\ 2)^2}-\frac{4^{\chi+1}}{(\chi\ +\ 2)^3}\right)\\
&\quad+\frac{1}{2}\left(\frac{2\left(\chi\ -\ 2^{\chi}\ +\ 2\right)}{\chi^2\ +\ 3\chi\ +\ 2}-2^{-\chi}-2\frac{(\chi\ +\ 2)^{\chi}\ -\ (2^{\chi+2})^{\chi}}{(\chi\ +\ 2)^{\chi+3}(\chi\ +\ 1)}+\frac{1}{\chi\ +\ 2}+\frac{2^{\chi}}{(\chi\ +\ 2)^2}-\frac{4^{\chi}}{(\chi\ +\ 2)^3}\right)
\end{aligned}$$

which completes the proof. □

**Corollary 2.3.** *By choosing $h\left(\vartheta_1\right)=h\left(\frac{\vartheta_1+\vartheta_2}{2}\right)=h\left(\vartheta_2\right)$ in Theorem 2.2, inequality (2.2), becomes*

$$\begin{aligned}
&\left|h\left(\frac{\vartheta_1+\vartheta_2}{2}\right)-\frac{\Gamma(\chi\ +\ 1)}{2(\vartheta_2\ -\ \vartheta_1)^{\chi}}\left[J^{\chi}_{\vartheta_1}h(\vartheta_2)+J^{\chi}_{\vartheta_2}h(\vartheta_1)\right]\right|\\
\leq\ &(\vartheta_2\ -\ \vartheta_1)^2\left\{(G_1+G_3)\ \left|h''(\vartheta_1)\right|+(G_2+G_4)\left|h''(\vartheta_2)\right|\right\}. \qquad (2.3)
\end{aligned}$$

*Remark* 2.4. By letting $\chi=1$ in above corollary , we get midpoint inequality (1.3), which looks better than the inequality presented by Kırmaci.

*Remark* 2.5. By letting $\chi=1$ in Theorem 2.2, we can get the inequality of [19, Theorem 2.2]:

$$\begin{aligned}
&\left|\frac{1}{6}\left\{h\left(\vartheta_1\right)\ +4h\left(\frac{\vartheta_1+\vartheta_2}{2}\right)+h\left(\vartheta_2\right)\right\}-\frac{1}{\vartheta_2-\vartheta_1}\int_{\vartheta_1}^{\vartheta_2}h\left(\theta\right)d\theta\right|\\
\leq\ &\frac{\left(\vartheta_2-\vartheta_1\right)^2}{162}\left\{\left|h''\left(\vartheta_1\right)\right|+\left|h''\left(\vartheta_2\right)\right|\right\}.
\end{aligned}$$

The corresponding version for powers of the absolute value of the derivative incorporates as.

**Theorem 2.6.** *Let $h$ be defined as in Lemma 2.1 and if $|h''|^q$ is a convex on $[\vartheta_1,\vartheta_2]$, with $q\geq 1$, then we have the following inequality*:

$$\begin{aligned}
&\left|\frac{\Gamma(\chi\ +\ 1)}{2(\vartheta_2\ -\ \vartheta_1)^{\chi}}\left[J^{\chi}_{\vartheta_1}h(\vartheta_2)+J^{\chi}_{\vartheta_2}h(\vartheta_1)\right]\right.\\
&\left.-\frac{1}{2}\left[\left(\frac{(2^{\chi}\ -\ 1)\left(\chi\ +\ 1\right)}{2^{\chi}(\chi\ +\ 2)}\right)\left(h\left(\vartheta_1\right)+h\left(\vartheta_2\right)\right)+\left(\frac{2^{\chi}\ +\ \chi\ +\ 1}{2^{\chi-1}(\chi\ +\ 2)}\right)h\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\right|\\
&\leq(\vartheta_2-\vartheta_1)^2\ \left[G_1\left\{\left(G_3\left|h''\left(\vartheta_2\right)\right|^q+\left(G_1-G_3\right)\left|h''\left(\vartheta_1\right)\right|^q\right)^{1/q}+\left(G_3\left|h''\left(\vartheta_1\right)\right|^q+\left(G_1-G_3\right)\left|h''\left(\vartheta_2\right)\right|^q\right)^{1/q}\right\}\right.\\
&\left.+G_2\left\{\left(G_4\left|h''\left(\vartheta_2\right)\right|^q+\left(G_2-G_4\right)\left|h''\left(\vartheta_1\right)\right|^q\right)^{1/q}+\left(G_4\left|h''\left(\vartheta_1\right)\right|^q+\left(G_2-G_4\right)\left|h''\left(\vartheta_2\right)\right|^q\right)^{1/q}\right\}\right]. \qquad (2.4)
\end{aligned}$$

*Proof.* By the use of power-mean integral inequality for $q\geq 1$, we obtain

$$\begin{aligned}
&\left|\frac{\Gamma(\chi\ +\ 1)}{2(\vartheta_2\ -\ \vartheta_1)^{\chi}}\left[J^{\chi}_{\vartheta_1}h(\vartheta_2)\ +\ J^{\chi}_{\vartheta_2}h(\vartheta_1)\right]\right.\\
&\left.-\frac{1}{2}\left[\left(\frac{(2^{\chi}\ -\ 1)\left(\chi\ +\ 1\right)}{2^{\chi}(\chi\ +\ 2)}\right)\left(h\left(\vartheta_1\right)+h\left(\vartheta_2\right)\right)\ +\ \left(\frac{2^{\chi}\ +\ \chi\ +\ 1}{2^{\chi-1}(\chi\ +\ 2)}\right)h\left(\frac{\vartheta_1\ +\ \vartheta_2}{2}\right)\right]\right|\\
&\leq(\vartheta_2\ -\ \vartheta_1)^2\int_0^{1/2}\left|\frac{2^{\chi}\left(\chi\ +\ 2\right)\eta^{\chi+1}\ -\ \eta\left(\chi\ +\ 1\right)}{\left(\chi\ +\ 1\right)2^{\chi}\left(\chi\ +\ 2\right)}\right|\left|h''(\eta\vartheta_2\ +\ (1-\eta)\vartheta_1)\right|d\eta
\end{aligned}$$

$$+(\vartheta_2-\vartheta_1)^2\int_{1/2}^{1}\left|\frac{\eta^{\chi+1}}{\chi+1}-\frac{\eta}{\chi+1}-\frac{\eta}{2^\chi}+\frac{1}{\chi+2}+\frac{1}{2^\chi}-\frac{1}{\chi+1}\right|h''(\eta\vartheta_2+(1-\eta)\vartheta_1)d\eta$$

$$\leq(\vartheta_2-\vartheta_1)^2\left(\int_0^{1/2}\left|\frac{2^\chi(\chi+2)\eta^{\chi+1}-\eta(\chi+1)}{(\chi+1)2^\chi(\chi+2)}\right|d\eta\right)^{1-1/q}$$

$$\times\left(\int_0^{1/2}|h''(\eta\vartheta_2+(1-\eta)\vartheta_1)|^q d\eta\right)^{1/q}$$

$$+(\vartheta_2-\vartheta_1)^2\left(\int_{1/2}^{1}\left|\frac{\eta^{\chi+1}}{\chi+1}-\frac{\eta}{\chi+1}-\frac{\eta}{2^\chi}+\frac{1}{\chi+2}+\frac{1}{2^\chi}-\frac{1}{\chi+1}\right|d\eta\right)^{1-1/q}\left(\int_{1/2}^{1}|h''(\eta\vartheta_2+(1-\eta)\vartheta_1)|^q d\eta\right)^{1/q}$$

One can notice that

$$\begin{aligned}&\int_0^{1/2}\left|\frac{2^\chi(\chi+2)\eta^{\chi+1}-\eta(\chi+1)}{(\chi+1)2^\chi(\chi+2)}\right||h''(\eta\vartheta_2+(1-\eta)\vartheta_1)|^q d\eta\\ \leq\;&\int_0^{1/\alpha+2}\left(\frac{2^\chi(\chi+2)\eta^{\chi+1}-\eta(\chi+1)}{(\chi+1)2^\chi(\chi+2)}\right)\left\{\eta\left|h''(\vartheta_1)\right|^q+(1-\eta)\left|h''(\vartheta_2)\right|^q\right\}d\eta\\ &+\int_{1/\alpha+2}^{1/2}\left(\frac{2^\chi(\chi+2)\eta^{\chi+1}-\eta(\chi+1)}{(\chi+1)2^\chi(\chi+2)}\right)\left\{\eta\left|h''(\vartheta_2)\right|^q+(1-\eta)\left|h''(\vartheta_1)\right|^q\right\}d\eta \end{aligned}\tag{2.5}$$

and

$$\begin{aligned}&\int_{1/2}^{1}\left|\frac{\eta^{\chi+1}}{\chi+1}-\frac{\eta}{\chi+1}-\frac{\eta}{2^\chi}+\frac{1}{\chi+2}+\frac{1}{2^\chi}-\frac{1}{\chi+1}\right||h''(\eta\vartheta_2+(1-\eta)\vartheta_1)|^q d\eta\\ \leq\;&\int_{1/2}^{2^\alpha/\alpha+2}\left(\frac{\eta^{\chi+1}}{\chi+1}-\frac{\eta}{\chi+1}-\frac{\eta}{2^\chi}+\frac{1}{\chi+2}+\frac{1}{2^\chi}-\frac{1}{\chi+1}\right)\left\{\eta\left|h''(\vartheta_1)\right|^q+(1-\eta)\left|h''(\vartheta_2)\right|^q\right\}d\eta\\ &+\int_{2^\alpha/\alpha+2}^{1}\left(\frac{\eta^{\chi+1}}{\chi+1}-\frac{\eta}{\chi+1}-\frac{\eta}{2^\chi}+\frac{1}{\chi+2}+\frac{1}{2^\chi}-\frac{1}{\chi+1}\right)\left\{\eta\left|h''(\vartheta_2)\right|^q+(1-\eta)\left|h''(\vartheta_1)\right|^q\right\}d\eta\end{aligned}$$

which completes the proof. □

In the following theorem, we obtain estimate of Simpson's inequality for concave functions.

**Theorem 2.7.** *Let $h:[\vartheta_1,\vartheta_2]\to\mathbb{R}$ be a differentiable function on $(\vartheta_1,\vartheta_2)$ such that $h''\in L^1[a,b]$. If $|h''|^q$ is concave on $[\vartheta_1,\vartheta_2]$, for some fixed $p>1$ with $q=\frac{p}{p-1}$, then we have*

$$\begin{aligned}&\Bigg|\frac{\Gamma(\chi+1)}{2(\vartheta_2-\vartheta_1)^\chi}\left[J^\chi_{\vartheta_1}h(\vartheta_2)+J^\chi_{\vartheta_2}h(\vartheta_1)\right]\\ &-\frac{1}{2}\left[\left(\frac{(2^\chi-1)(\chi+1)}{2^\chi(\chi+2)}\right)(h(\vartheta_1)+h(\vartheta_2))+\left(\frac{2^\chi+\chi+1}{2^{\chi-1}(\chi+2)}\right)h\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right]\Bigg|\\ &\leq(\vartheta_2-\vartheta_1)^2\times\\ &\left[G_1\left\{\left|h''\left(\frac{G_3\vartheta_2+(G_1-G_3)\vartheta_1}{G_1}\right)\right|+\left|h''\left(\frac{G_3\vartheta_1+(G_1-G_3)\vartheta_2}{G_1}\right)\right|\right\}\right.\\ &\left.+G_2\left|h''\left(\frac{G_4\vartheta_2+(G_2-G_4)\vartheta_1}{G_2}\right)\right|+\left|h''\left(\frac{G_4\vartheta_1+(G_2-G_4)\vartheta_2}{G_2}\right)\right|\right].\end{aligned}\tag{2.6}$$

*Proof.* Utilizing concavity of $|h''|^q$ and the power-mean inequality, we obtain

$$\begin{aligned}|h''(\eta\vartheta_1+(1-\eta)\vartheta_2)|^q &> \eta|h''(\vartheta_1)|^q+(1-\eta)|h''(\vartheta_2)|^q\\ &\geq \left(\eta|h''(\vartheta_1)|+(1-\eta)|h''(\vartheta_2)|\right)^q\end{aligned}$$

$$h''(\eta\vartheta_1+(1-\eta)\vartheta_2)|\geq\eta|h''(\vartheta_1)|+(1-\eta)|h''(\vartheta_2)|,$$

Thus, $|h''|$ is also concave. Jensen integral inequality follows that

$$\left|\frac{1}{2}\left[\left(\frac{(2^\chi-1)(\chi+1)}{2^\chi(\chi+2)}\right)(h(\vartheta_1)+h(\vartheta_2))+\left(\frac{2^\chi+\chi+1}{2^{\chi-1}(\chi+2)}\right)h\left(\frac{\vartheta_1+\vartheta_2}{2}\right)\right]-\frac{\Gamma(\chi+1)}{2(\vartheta_2-\vartheta_1)^\chi}\left[J^\chi_{\vartheta_1}h(\vartheta_2)+J^\chi_{\vartheta_2}h(\vartheta_1)\right]\right|$$

$$\leq (\vartheta_2 - \vartheta_1)^2 \left( \int_0^{1/2} \left| \frac{2^\chi (\chi + 2) \eta^{\chi+1} - \eta (\chi + 1)}{(\chi + 1) 2^\chi (\chi + 2)} \right| d\eta \right) \left| h'' \left( \frac{\int_0^{1/2} \left| \frac{2^\chi (\chi + 2)\eta^{\chi+1} - \eta(\chi + 1)}{(\chi + 1)2^\chi(\chi + 2)} \right| (\eta\vartheta_1 + (1-\eta)\vartheta_2) d\eta}{\int_0^{1/2} \left| \frac{2^\chi(\chi + 2)\eta^{\chi+1} - \eta(\chi + 1)}{(\chi + 1)2^\chi(\chi + 2)} \right| d\eta} \right) \right|^q$$

$$+ (\vartheta_2 - \vartheta_1)^2 \left( \int_{1/2}^1 \left| \frac{\eta^{\chi+1}}{\chi + 1} - \frac{\eta}{\chi + 1} - \frac{\eta}{2^\chi} + \frac{1}{\chi + 2} + \frac{1}{2^\chi} - \frac{1}{\chi + 1} \right| d\eta \right)$$

$$\times \left| h'' \left( \frac{\int_{1/2}^1 \left| \frac{\eta^{\chi+1}}{\chi + 1} - \frac{\eta}{\chi + 1} - \frac{\eta}{2^\chi} + \frac{1}{\chi + 2} + \frac{1}{2^\chi} - \frac{1}{\chi + 1} \right| (\eta\vartheta_2 + (1-\eta)\vartheta_1) d\eta}{\int_{1/2}^1 \left| \frac{\eta^{\chi+1}}{\chi + 1} - \frac{\eta}{\chi + 1} - \frac{\eta}{2^\chi} + \frac{1}{\chi + 2} + \frac{1}{2^\chi} - \frac{1}{\chi + 1} \right| d\eta} \right) \right|^q$$

$$+ (\vartheta_2 - \vartheta_1)^2 \left( \int_0^{1/2} \left| \frac{2^\chi (\chi + 2) \eta^{\chi+1} - \eta (\chi + 1)}{(\chi + 1) 2^\chi (\chi + 2)} \right| d\eta \right) \left| h'' \left( \frac{\int_0^{1/2} \left| \frac{2^\chi (\chi + 2)\eta^{\chi+1} - \eta(\chi + 1)}{(\chi + 1)2^\chi(\chi + 2)} \right| (\eta\vartheta_2 + (1-\eta)\vartheta_1) d\eta}{\int_0^{1/2} \left| \frac{2^\chi(\chi + 2)\eta^{\chi+1} - \eta(\chi + 1)}{(\chi + 1)2^\chi(\chi + 2)} \right| d\eta} \right) \right|^q$$

$$+ (\vartheta_2 - \vartheta_1)^2 \left( \int_{1/2}^1 \left| \frac{\eta^{\chi+1}}{\chi + 1} - \frac{\eta}{\chi + 1} - \frac{\eta}{2^\chi} + \frac{1}{\chi + 2} + \frac{1}{2^\chi} - \frac{1}{\chi + 1} \right| d\eta \right)$$

$$\times \left| h'' \left( \frac{\int_{1/2}^1 \left| \frac{\eta^{\chi+1}}{\chi + 1} - \frac{\eta}{\chi + 1} - \frac{\eta}{2^\chi} + \frac{1}{\chi + 2} + \frac{1}{2^\chi} - \frac{1}{\chi + 1} \right| (\eta\vartheta_1 + (1-\eta)\vartheta_2) d\eta}{\int_{1/2}^1 \left| \frac{\eta^{\chi+1}}{\chi + 1} - \frac{\eta}{\chi + 1} - \frac{\eta}{2^\chi} + \frac{1}{\chi + 2} + \frac{1}{2^\chi} - \frac{1}{\chi + 1} \right| d\eta} \right) \right|^q$$

$$= (\vartheta_2 - \vartheta_1)^2 G_5 \left| h'' \left( \frac{G_1\vartheta_1 + G_2\vartheta_2}{G_5} \right) \right| + (\vartheta_2 - \vartheta_1)^2 G_6 \left| h'' \left( \frac{G_3\vartheta_2 + G_4\vartheta_1}{G_6} \right) \right|$$

$$+ (\vartheta_2 - \vartheta_1)^2 G_5 \left| h'' \left( \frac{G_1\vartheta_2 + G_2\vartheta_1}{G_5} \right) \right| + (\vartheta_2 - \vartheta_1)^2 G_6 \left| h'' \left( \frac{G_3\vartheta_1 + G_4\vartheta_2}{G_6} \right) \right|,$$

which completes the proof. □

**Corollary 2.8.** *By putting $\chi = 1$ in Theorem 2.7, then inequality* (10) *becomes as:*

$$\left| \left[ \frac{1}{6} h(\vartheta_1) + \frac{2}{3} h\left( \frac{\vartheta_1 + \vartheta_2}{2} \right) + \frac{1}{6} h(\vartheta_2) \right] - \frac{1}{\vartheta_2 - \vartheta_1} \int_{\vartheta_1}^{\vartheta_2} h(\theta)\, d\theta \right| \tag{2.7}$$

$$\leq \frac{(\vartheta_2 - \vartheta_1)^2}{162} \left[ \left| h'' \left( \frac{59\vartheta_1 + 133\vartheta_2}{192} \right) \right| + \left| h'' \left( \frac{133\vartheta_1 + 59\vartheta_2}{192} \right) \right| \right]. \tag{2.8}$$

*Remark* 2.9. Inequality (2.7) is a generalization of obtained inequality as in [19, Theorem 8]

## 3. Application to Beta Function

In this section, let $\chi > 0$, $\chi \geq 3$, $\vartheta_1 = 0$, $\vartheta_2 = 1$, $\Gamma(\chi)$ be the gamma function, and $h(\theta) = \theta^{\chi-1}$ $(\theta \in [0,1])$. Then $|h''|$ is convex on $[0,1]$.

*Remark* 3.1. First of all we define the beta function

$$\eta(\gamma, \delta) = \int_0^1 \theta^{\gamma-1} (1-\theta)^{\delta-1} d\theta \qquad (\gamma, \delta > 0).$$

From section 3, we have

$$\frac{\Gamma(\chi+1)}{2(\vartheta_2 - \vartheta_1)^\chi} J_{\vartheta_1}^\chi h(\vartheta_2) = \chi \int_0^1 \theta^{\gamma-1} (1-\theta)^{\delta-1} d\theta = \frac{\chi}{2} \eta(\chi, \chi)$$

and

$$\frac{\Gamma(\chi+1)}{2(\vartheta_2 - \vartheta_1)^\chi} J_{\vartheta_2}^\chi h(\vartheta_1) = \chi \int_0^1 \theta^{\chi+\chi-2} d\theta = \frac{\chi}{2(\chi + \chi - 1)}$$

*Proposition* 3.2. *In Theorem 2.2, the following inequality holds*

$$\left| \left[ \frac{\chi}{2} \eta(\chi, \chi) + \frac{\chi}{2(\chi + \chi - 1)} \right] - \left( \frac{(2^\chi - 1)(\chi + 1)}{2^{\chi+1}(\chi + 2)} + \frac{2^\chi + \chi + 1)}{2^{\chi-1} 2^\chi (\chi + 2)} \right) \right|$$

$$\leq (\chi - 1) \left[ ((G_1 + G_3)) + ((G_2 + G_4)) \right]$$

3.1. **q-digamma function.** Suppose $0 < \mathbf{q} < 1$, the $\mathbf{q}-$digamma function $\varphi_{\mathbf{q}}$, is the $\mathbf{q}-$analogue of the digamma function $\varphi$ $\big($see [13, 21] $\big)$given as:

$$\begin{aligned}\varphi_{\mathbf{q}} &= -\ln(1-\mathbf{q}) + \ \ln \mathbf{q} \sum_{k=0}^{\infty} \frac{\mathbf{q}^{k+\chi}}{1-\mathbf{q}^{k+\chi}} \\ &= -\ln(1-\mathbf{q}) + \ \ln \mathbf{q} \sum_{k=0}^{\infty} \frac{\mathbf{q}^{k\chi}}{1-\mathbf{q}^{k\chi}}.\end{aligned}$$

For $\mathbf{q} > 1$ and $\chi > 0$, $\mathbf{q}-$digamma function $\varphi_{\mathbf{q}}$ can be given as:

$$\begin{aligned}\varphi_{\mathbf{q}} &= -\ln(\mathbf{q}-1) + \ \ln \mathbf{q} \left[\chi - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{\mathbf{q}^{-(k+\chi)}}{1-\mathbf{q}^{-(k+\chi)}}\right] \\ &= -\ln(\mathbf{q}-1) + \ \ln \mathbf{q} \left[\chi - \frac{1}{2} - \sum_{k=0}^{\infty} \frac{\mathbf{q}^{-k\chi}}{1-\mathbf{q}^{-k\chi}}\right].\end{aligned}$$

**Proposition 3.3.** *Suppose* $\vartheta_1, \vartheta_2, \mathbf{q}$ *are the real numbers such that* $0 < \vartheta_1 < \vartheta_2,\ 0 < \mathbf{q} < 1$. *Then the following inequality holds:*

$$\begin{aligned}&\left|\frac{1}{6}\left\{\varphi_{\mathbf{q}}'(\vartheta_1) \ + 4\varphi_{\mathbf{q}}'\left(\frac{\vartheta_1+\vartheta_2}{2}\right) + \varphi_{\mathbf{q}}'(\vartheta_2)\right\} - \frac{\varphi_{\mathbf{q}}(\vartheta_2) - \varphi_{\mathbf{q}}(\vartheta_1)}{\vartheta_2 - \vartheta_1}\right| \\ &\leq \frac{(\vartheta_2-\vartheta_1)^2}{162}\left\{\left|\varphi_{\mathbf{q}}^{(3)}(\vartheta_1)\right| + \left|\varphi_{\mathbf{q}}^{(3)}(\vartheta_2)\right|\right\}. \qquad (3.1)\end{aligned}$$

*Proof.* By employing the definition of $\mathbf{q}$-digamma function $\varphi_{\mathbf{q}}(\chi)$, it is easy to notice that $\mathbf{q}$-trigamma function $\chi \to \varphi_{\mathbf{q}}^{'}(\chi)$ is completely monotonic on $(0, \infty)$. This ensures that the function $\varphi_{\mathbf{q}}^{(3)}$ is again completely monotonic on $(0, \infty)$ for each $\mathbf{q} \in (0, 1)$ and consequently is convex $\big($see [22], p.167 $\big)$. Now by applying Remark 2.5, we extract that the inequality (3.1) is valid for $\mathbf{q} \in (0, 1)$. □

Presently another application of inequality (3.1), we can give the accompanying inequalities for the $\mathbf{q}-$triagamma and $\mathbf{q}-$polygamma functions and the simple of Harmonic numbers $\mathtt{H}_{n\mathbf{q}}$ characterized by

$$\mathtt{H}_{n\mathbf{q}} = \sum \frac{\mathbf{q}^k}{1-\mathbf{q}^k}, \quad n \in \mathrm{N}.$$

So, from inequality (3.1) and using the equation

$$\varphi_{\mathbf{q}}(n+1) = \varphi_{\mathbf{q}}(1) - Log(\mathbf{q}) H_{n\mathbf{q}},$$

Analogously, we obtain the required result.

**Corollary 3.4.** *Suppose* $n \in N$, $0 < \mathbf{q} < 1$. *Then the following inequality holds:*

$$\left|\frac{1}{6}\left\{\varphi_{\mathbf{q}}'(1) \ + 4\varphi_{\mathbf{q}}'\left(\frac{n}{2}+1\right) + \varphi_{\mathbf{q}}'(n+1)\right\} + \frac{Log(\mathbf{q})H_{n\mathbf{q}}}{n}\right| \leq \frac{n^2}{162}\left\{\left|\varphi_{\mathbf{q}}^{(3)}(1)\right| + \left|\varphi_{\mathbf{q}}^{(3)}(n+1)\right|\right\}. \qquad (3.2)$$

**Proposition 3.5.** *Suppose* $n$ *is an integer and* $q > 1$. *Then the following inequality holds:*

$$\left|\frac{1}{6}\left\{\varphi'(1) \ + 4\varphi'\left(\frac{n}{2}+1\right) + \varphi'(n+1)\right\} - \frac{H_n}{n}\right| \leq \frac{n^2}{162}\left\{\left|\varphi^{(3)}(1)\right| + \left|\varphi^{(3)}(n+1)\right|\right\},$$

*Proof.* From inequality (3.2), when $\mathbf{q} \to 1$, and using the relation

$$\begin{aligned}lim_{\mathbf{q}\to 1} \log(\mathbf{q}) H_{n\mathbf{q}} &= \ \lim_{\mathbf{q}\to 1}[\left(\frac{\log(\mathbf{q})}{\mathbf{q}-1}\right).(\mathbf{q}-1) H_{n\mathbf{q}}] \\ &= -\lim_{\mathbf{q}\to 1} \sum_{k=1}^{n} \frac{1-\mathbf{q}}{1-\mathbf{q}^k} = -H_n.\end{aligned}$$

We obtain the required result. □

## References

[1] Alomari, M., Darus, M., Dragomir, S. S, *New inequalities of Simpson's type for s-convex functions with applications*, RGMIA Res. Rep. Coll, **12**(4) (2009), 13-20.
[2] Baleanu D, Mohammed P. O, Shengda Z, *Inequalities of trapezoidal type involving generalized fractional integrals*, Alex. Eng. J. **59** (2020), 2975–2985.
[3] Chen J., Huang X., *Some new inequalities of Simpson type for convex functions via fractional integrals*,Filomat, **31**(201) (2017), 4989–4997.
[4] Dragomir, S.S., Agarwal, R.P.,Cerone.P., *On Simpson'sinequality and applications*, J. Inequal. Appl., **5** (2000), 533–579.
[5] Dragomir, S.S. and Agarwal, R.P., *Two inequalities for differentiable mappings and applications to special means of real numbers and trapezoidal formula*, Appl. Math. Lett., **11**(5) (1998), 91–95.
[6] Dragomir, S.S. and Pearce, C.E.M., *Selected topics on Hermite–Hadamard inequalities and applications*, Victoria University, Australia, (2000).
[7] Du, Tingsong., Li, Yujiao., and Yang, Zhiqiao, *A generalization of Simpson's inequality via differentiable mapping using extended $(s,m)$-convex functions*, Appl. Math. Comput., **293** (1) (2017), 358-369.
[8] Gorenflo R., Mainardi F.: *Fractional calculus: integral and differential equations of fractional order,* Springer. Verlag. Wien. New York **47** (1997), 223-276.
[9] Godunova, E.K., and Levin, V.I., *Inequalities for functions of a broad class that contains convex, monotone and some other forms of functions. Numerical mathematics and mathematical physics (Russion)*, Moskov. Gos. Ped. Inst., Moscow., **166** (1985), 138-142.
[10] Hussain S., Qaisar S., *More results on Simpson's type inequality through convexity for twice differentiable continuous mappings*, SpringerPlus, **5**(77) (2016).
[11] Hsu, K-C and Hwang, S-R and Tseng, K-L, *Some extended Simpson-type inequalities and applications*, Bull. iran. math. soc., **43** (2017), 670-691.
[12] Isçan, I., *New refinements for integral and sum forms of Hölder inequality*, J. Inequal. Appl., **2019**(304) (2019).
[13] Jain S., Mehrez K., Baleanu D. and Agarwal P., *Certain Hermite–Hadamard inequalities for logarithmically convex functions with applications*, Mathematics, 7(2) (2019), 163.
[14] Kashuri A, Mohammed P. O, Abdeljawad T, Hamasalh F, ChuY, *New Simpson Type Integral Inequalities for s -Convex Functions and Their Applications*, Math. Probl. Eng 2020 (2020).
[15] Kırmaci U. S.,: *Inequalities for differentiable mappings and applications to special means of real numbers and to midpoint formula,* Appl. Math.Comput. **147** (2004) 137–146.
[16] Kunt, M., İşcan, İ., Alp, N., M.Z. Sarikaya, M. Z., *$(p,q)$-Hermite-Hadamard inequalities and $(p,q)$-estimates for midpoint type inequalities via convex and quasi convex functions,* RACSAM, **112** (2018), 969-992.
[17] Mehrez K, Agarwal P, *New Hermite–Hadamard type integral inequalities for convex functions and their applications*, J. Comput. Appl. Math., **350** (2019), 274–285.
[18] Rashid S., Isçan, I., Baleanu D.,: *Generation of new fractional inequalities via n polynomials s-type convexixity with applications,* Adv. Differ. Equ. (2020) 1–20.
[19] Sarikaya, M.Z., Set, E., Özdemir, M.E, *On new inequalities of Simpson's type for functions whose second derivatives absolute values are convex*, J. Appl. Math. Stat and Inform, **9**(1) (2013), 37-45.
[20] Set E, Akdemir A. O, Özdemir M.E., *Simpson type integral inequalities for convex functions via Riemann–Liouville integrals*, Filomat., **31**(14) (2017), 4415–4420.
[21] Watson G. N., *A treatise on the theory of Bessel functions*, Cambridge university press, (1995).
[22] Widder D. V., *The Laplace Transform*, Prinsceton University Press: Princeton, NJ, USA, 1941.

Jamshed Nasir, Department of Mathematics, COMSATS University Islamabad Lahore, Campus Pakistan
*E-mail address*: jnasir143@gmail.com

Shahid Qaisar, Department of Mathematics, COMSATS University Islamabad Lahore, Campus Pakistan
*E-mail address*: shahidqaisar90@cuisahiwal.edu.pk

Saad Ihsan Butt, Department of Mathematics, COMSATS University Islamabad Lahore, Campus Pakistan
*E-mail address*: saadihsanbutt@gmail.com

Muhammad Aamir Ali, Jiangsu Key Laboratory for NSLSCS, School of Mathematical Sciences, Nanjing Normal University, Nanjing, 210023, China.
*E-mail address*: mahr.muhammad.aamir@gmail.com