MAXWELL EQUATIONS IN DISTRIBUTIONAL SENSE AND APPLICATIONS

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ABSTRACT. This paper discusses the Maxwell system of electrodynamics in the context of distributions. It is used to establish boundary conditions for fields at the interface when the charge and current densities are concentrated measures on the interface. The paper derives the generalized Snell's law from this analysis.

CONTENTS

1. INTRODUCTION

Maxwell's equations are well understood in the classical sense. However, when dealing with discontinuous fields across a surface, it is often convenient to analyze them in the context of generalized functions (or distributions), see for example [Ide11] and [Gut17].

The purpose of this paper is to explore Maxwell's equations from a distributional perspective and derive relationships between the electric and magnetic fields on either side of the boundary or interface (i.e. boundary conditions) when the current and charge densities are measures concentrated on the interface.

From this analysis, the second part of the paper focuses on metasurfaces or metalenses, which are extremely thin interfaces containing nano materials that can steer light in

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unusual ways. The use of metasurfaces for controlling beam shaping applications is a thriving field of research with numerous applications. At its core is the generalized Snell law of refraction and reflection, explaining how beams propagate across metasurfaces. This law was first introduced in the influential work [YGK+11] and [AKG+12] for planar geometries. The formulation of the law involves a function defined in a small neighborhood of the metasurface or metalens, referred to as the phase discontinuity, and it is further discussed in [YC14]. A rigorous mathematical derivation of the law for non-planar geometries was initially obtained using wave fronts in [GPS17, Sect. 3] and later on using the Fermat principle of minimum action in [GS21, Sect. 2]. These two papers also demonstrate the existence of phase discontinuities for various geometric configurations.

A main objective of this paper is to derive the generalized Snell law using Maxwell's equations by representing the transmitted electric field as a nonlinear wave. Specifically, we model the transmitted field as a perturbation of a plane wave involving the phase discontinuity and taking the form as given in equation (4.1). The generalized Snell law is then deduced from the boundary conditions for the Maxwell system, as obtained in Theorem 3.1, and is explicitly stated by equation (4.4). Ensuring that the electric field satisfies the Maxwell system translates into conditions on the phase discontinuity and the current density.

To place the results in perspective, it is worth mentioning that metasurfaces refracting or reflecting beams according to energy patterns, are closely related to solving Monge-Ampère type partial differential equations, as discussed in [GP18]. Furthermore, the analysis of chromatic aberration in metalenses is conducted in [GS21], and additional insights can be found in [YGK+11]. For applications related to the construction of tunable metasurfaces using graphene, we refer to [BGN+18]. Recent results and applications in the field of metasurfaces can be also found in [RF20], [JCX+23], and [YSC+23].

The paper is organized as follows. In Section 2, we present results related to distributions and outline assumptions regarding the fields. These results are then applied in Section 3 to establish the proof of Theorem 3.1. The deduction of the generalized Snell law is the central focus of Section 4, where we leverage Theorem 3.1. Lastly, Section 4.1 addresses the verification of the transmitted electric field representation given by (4.1). Specifically, it demonstrates that the phase discontinuity φ must satisfy (4.6), and the current density must adhere to (4.9).

2. Preliminaries

In this section, we begin introducing the notions needed to analyze the Maxwell system in the sense of distributions. Let $\Omega \subset \mathbb{R}^3$ be an open and bounded domain. A generalized function or distribution in Ω is a complex-valued continuous linear functional defined in the class of test functions $\mathcal{D}(\Omega) = C_0^\infty(\Omega)$. As usual, $\mathcal{D}'(\Omega)$ denotes the class of distributions in Ω [Sch66]. If $g \in \mathcal{D}'(\Omega)$, then $\langle g, \varphi\rangle$ denotes the value of the distribution g on the test function $\varphi \in \mathcal{D}(\Omega)$.

We say that $\mathbf{G} = (G_1, G_2, G_3)$ is a vector valued distribution in Ω if each component $G_i \in \mathcal{D}'(\Omega)$, $1 \le i \le 3$. The divergence of $\mathbf{G}$ is the scalar distribution defined by

$$\langle \nabla \cdot \mathbf{G}, \varphi\rangle = -\sum_{i=1}^{3} \langle G_i, \partial_{x_i}\varphi\rangle, \tag{2.1}$$

and the curl of $\mathbf{G}$ is the vector valued distribution in Ω defined by

$$\langle \nabla \times \mathbf{G}, \varphi\rangle = \left(\langle G_2, \varphi_{x_3}\rangle - \langle G_3, \varphi_{x_2}\rangle\right)\mathbf{i} - \left(\langle G_1, \varphi_{x_3}\rangle - \langle G_3, \varphi_{x_1}\rangle\right)\mathbf{j} + \left(\langle G_1, \varphi_{x_2}\rangle - \langle G_2, \varphi_{x_1}\rangle\right)\mathbf{k}. \tag{2.2}$$

Then it follows that

$$\nabla \cdot (\nabla \times \mathbf{G}) = 0, \tag{2.3}$$

in the sense of distributions. When the distribution $\mathbf{G} = (G_1, G_2, G_3)$ is locally integrable in Ω we obtain from (2.1), and (2.2) that

$$\langle \nabla \cdot \mathbf{G}, \varphi\rangle = -\int_\Omega \mathbf{G} \cdot \nabla\varphi \, dx, \qquad \langle \nabla \times \mathbf{G}, \varphi\rangle = \int_\Omega \mathbf{G} \times \nabla\varphi \, dx.$$

We consider the following setup. Ω is a smooth open and bounded domain in $\mathbb{R}^3$ and Γ is a smooth surface that splits Ω into two open parts Ω_+ and Ω_- as follows: for every $x_0 \in \Gamma$ there exists a ball $B(x_0, r) \subset \Omega$ and $\psi \in C^1(B(x_0, r))$ such that

$$\Omega_- \cap B(x_0, r) = \{(x_1, x_2, x_3) \in B(x_0, r) : x_3 < \psi(x_1, x_2)\}$$
$$\Omega_+ \cap B(x_0, r) = \{(x_1, x_2, x_3) \in B(x_0, r) : x_3 > \psi(x_1, x_2)\}.$$

We are given fields $\mathbf{G}_-$ in Ω_- and $\mathbf{G}_+$ in Ω_+ satisfying the following properties

(F1) $\mathbf{G}_- \in C^1(\Omega_-), \mathbf{G}_+ \in C^1(\Omega_+)$.

(F2) The first order derivatives of $\mathbf{G}_\pm$ are in $L^1(\Omega_\pm)$, respectively.

(F3) For every $x \in \Gamma$, $\lim_{y\to x, y\in\Omega_-} \mathbf{G}_-(y)$ and $\lim_{y\to x, y\in\Omega_+} \mathbf{G}_+(y)$ exist and are finite.

As a consequence, each $\mathbf{G}_-$ and $\mathbf{G}_+$ can be extended continuously to Γ by setting

$$\mathbf{G}_-(x) = \lim_{y\to x, y\in\Omega_-} \mathbf{G}_-(y); \qquad \mathbf{G}_+(x) = \lim_{y\to x, y\in\Omega_+} \mathbf{G}_+(y),$$

for each $x \in \Gamma$. For such fields $\mathbf{G}_-$ and $\mathbf{G}_+$, the linear functional $\mathbf{G}$ given by

$$\langle \mathbf{G}, \varphi\rangle = \int_{\Omega_-} \mathbf{G}_-(x)\,\varphi(x)\,dx + \int_{\Omega_+} \mathbf{G}_+(x)\,\varphi(x)\,dx \tag{2.4}$$

is a well defined distribution for $\varphi \in \mathcal{D}(\Omega)$. The jump of the fields in Γ is defined by

$$[[\mathbf{G}(x)]] = \mathbf{G}_+(x) - \mathbf{G}_-(x), \quad \text{for } x \in \Gamma. \tag{2.5}$$

We then have the following expressions for the curl and divergence of $\mathbf{G}$.

Proposition 2.1. *If the field* $\mathbf{G}$ *satisfies (F1)–(F3), then for each* $\varphi \in \mathcal{D}(\Omega)$ *we have*

$$\langle \nabla \cdot \mathbf{G}, \varphi \rangle = \int_\Gamma \varphi(x)\, [[\mathbf{G}(x)]] \cdot \mathbf{n}\, d\sigma + \int_{\Omega_-} \varphi\, \nabla \cdot \mathbf{G}_-\, dx + \int_{\Omega_+} \varphi \nabla \cdot \mathbf{G}_+\, dx \tag{2.6}$$

$$\langle \nabla \times \mathbf{G}, \varphi \rangle = -\int_\Gamma \varphi(x)\, [[\mathbf{G}(x)]] \times \mathbf{n}\, d\sigma + \int_{\Omega_-} \varphi \nabla \times \mathbf{G}_-\, dx + \int_{\Omega_+} \varphi \nabla \times \mathbf{G}_+\, dx \tag{2.7}$$

with $\mathbf{n}$ *the unit normal to* Γ *pointing toward* Ω_+*.*

Proof. Given $\varepsilon > 0$, define

$$\Omega_-^\varepsilon = \{x \in \Omega_- : \operatorname{dist}(x, \Gamma) > \varepsilon\} \qquad\qquad \Omega_+^\varepsilon = \{x \in \Omega_+ : \operatorname{dist}(x, \Gamma) > \varepsilon\}.$$

For $\varphi \in \mathcal{D}(\Omega)$, we have from (2.1) and the definition of $\mathbf{G}$ in (2.4) that

$$\begin{aligned}\langle \nabla \cdot \mathbf{G}, \varphi \rangle &= -\int_{\Omega_-} \mathbf{G}_- \cdot \nabla\varphi\, dx - \int_{\Omega_+} \mathbf{G}_+ \cdot \nabla\varphi\, dx \\ &= -\int_{\Omega_-^\varepsilon} \mathbf{G}_- \cdot \nabla\varphi\, dx - \int_{\Omega_- \setminus \Omega_-^\varepsilon} \mathbf{G}_- \cdot \nabla\varphi\, dx - \int_{\Omega_+^\varepsilon} \mathbf{G}_+ \cdot \nabla\varphi\, dx - \int_{\Omega_+ \setminus \Omega_+^\varepsilon} \mathbf{G}_+ \cdot \nabla\varphi\, dx \\ &= -(I + II + III + IV).\end{aligned}$$

$II, IV \to 0$ as $\varepsilon \to 0^+$ because the extensions $\mathbf{G}_\pm$ are locally bounded in $\Omega_\pm \cup \Gamma$ and φ is compactly supported in Ω. Since $\mathbf{G}_-$ is C^1 in Ω_-^ε, using the divergence theorem we obtain

$$I = \int_{\partial\Omega_-^\varepsilon} \varphi \mathbf{G}_- \cdot \mathbf{n}_-^\varepsilon\, d\sigma - \int_{\Omega_-^\varepsilon} \varphi \nabla \cdot \mathbf{G}_-\, dx$$

with $\mathbf{n}_-^\varepsilon$ the outward unit normal to $\partial\Omega_-^\varepsilon$. Since φ has compact support in Ω, $\nabla \cdot \mathbf{G}_- \in L^1(\Omega_-)$, and $\mathbf{G}_-$ is locally bounded in $\Omega_- \cup \Gamma$, it follows that

$$I \to \int_\Gamma \varphi \mathbf{G}_- \cdot \mathbf{n}\, d\sigma - \int_{\Omega_-} \varphi \nabla \cdot \mathbf{G}_-\, dx,$$

with $\mathbf{n}$ is the unit normal to Γ toward Ω_+. Similarly we get

$$III \to \int_\Gamma \varphi \mathbf{G}_+ \cdot (-\mathbf{n})\, d\sigma - \int_{\Omega_+} \varphi \nabla \cdot \mathbf{G}_+\, dx,$$

with $-\mathbf{n}$ the unit normal to Γ toward Ω_-. Hence (2.6) follows.

We next prove (2.7). Let $\varphi \in \mathcal{D}(\Omega)$, we have from 2.2 and the definition of $\mathbf{G}$ in (2.4) that

$$\begin{aligned}\langle \nabla \times \mathbf{G}, \varphi \rangle &= \int_{\Omega_-} \mathbf{G}_- \times \nabla\varphi\, dx + \int_{\Omega_+} \mathbf{G}_+ \times \nabla\varphi\, dx \\ &= \int_{\Omega_-^\varepsilon} \mathbf{G}_- \times \nabla\varphi\, dx + \int_{\Omega_- \setminus \Omega_-^\varepsilon} \mathbf{G}_- \times \nabla\varphi\, dx + \int_{\Omega_+^\varepsilon} \mathbf{G}_+ \times \nabla\varphi\, dx + \int_{\Omega_+ \setminus \Omega_+^\varepsilon} \mathbf{G}_+ \times \nabla\varphi\, dx \\ &= I + II + III + IV\end{aligned}$$

$II, IV \to 0$ as $\varepsilon \to 0^+$ because the extensions $\mathbf{G}_\pm$ are locally bounded in $\Omega_\pm \cup \Gamma$ and φ is compactly supported in Ω. We write $I = (I_1, I_2, I_3)$, $\mathbf{G}_- = (G_1^-, G_2^-, G_3^-)$, and $\mathbf{n}_-^\varepsilon = (n_1^\varepsilon, n_2^\varepsilon, n_3^\varepsilon)$ the the outward unit normal to $\partial\Omega_-^\varepsilon$. Using the divergence theorem

$$\begin{aligned} I_1 &= \int_{\Omega_-^\varepsilon} G_2^- \varphi_{x_3} - G_3^- \varphi_{x_2}\, dx \\ &= \left(\int_{\partial\Omega_-^\varepsilon} \varphi G_2^- n_3^\varepsilon \, d\sigma - \int_{\Omega_-^\varepsilon} (G_2^-)_{x_3} \varphi \, dx \right) - \left(\int_{\partial\Omega_-^\varepsilon} \varphi G_3^- n_2^\varepsilon \, d\sigma - \int_{\Omega_-^\varepsilon} (G_3^-)_{x_2} \varphi \, dx \right) \\ &= \int_{\partial\Omega_-^\varepsilon} \varphi (G_2^- n_3^\varepsilon - G_3^- n_2^\varepsilon)\, d\sigma + \int_{\Omega_-^\varepsilon} \varphi ((G_3^-)_{x_2} - (G_2^-)_{x_3})\, dx. \end{aligned}$$

Similarly

$$I_2 = \int_{\Omega_-^\varepsilon} -G_1^- \varphi_{x_3} + G_3^- \varphi_{x_1}\, dx = \int_{\partial\Omega_-^\varepsilon} \varphi (G_3^- n_1^\varepsilon - G_1^- n_3^\varepsilon)\, d\sigma + \int_{\Omega_-^\varepsilon} \varphi ((G_1^-)_{x_3} - (G_3^-)_{x_1})\, dx.$$

and

$$I_3 = \int_{\Omega_-^\varepsilon} G_1^- \varphi_{x_2} - G_2^- \varphi_{x_1}\, dx = \int_{\partial\Omega_-^\varepsilon} \varphi (G_1^- n_2^\varepsilon - G_2^- n_1^\varepsilon)\, d\sigma + \int_{\Omega_-^\varepsilon} \varphi ((G_2^-)_{x_1} - (G_1^-)_{x_2})\, dx.$$

Combining the above calculations, we deduce that

$$I = \int_{\partial\Omega_-^\varepsilon} \varphi \mathbf{G}_- \times \mathbf{n}_-^\varepsilon \, d\sigma + \int_{\Omega_-^\varepsilon} \varphi \nabla \times \mathbf{G}_- \, dx.$$

Since φ is compactly supported in Ω, $\mathbf{G}_-$ is locally bounded in $\Omega_- \cup \Gamma_-$, and $\nabla \times \mathbf{G}_- \in L^1(\Omega_-)$ then

$$I \to \int_\Gamma \varphi \mathbf{G}_- \times \mathbf{n} \, d\sigma + \int_{\Omega_-} \varphi \nabla \times \mathbf{G}_- \, dx.$$

Similarly we get

$$III = \int_{\partial\Omega_+^\varepsilon} \varphi \mathbf{G}_+ \times \mathbf{n}_+^\varepsilon \, d\sigma + \int_{\Omega_+^\varepsilon} \varphi \nabla \times \mathbf{G}_+ \, dx \to \int_\Gamma \varphi \mathbf{G}_+ \times (-\mathbf{n}) \, d\sigma + \int_{\Omega_+} \varphi \nabla \times \mathbf{G}_+ \, dx.$$

Hence (2.7) follows. □

2.1. **Distributions depending on a parameter.** Since the fields satisfying Maxwell equations depend on time, in this section we consider distributions in Ω, possibly vector-valued, that depend on a parameter $t \in \mathbb{R}$, that is, for each $t \in \mathbb{R}$, $g(\cdot, t) \in \mathcal{D}'(\Omega)$, see [GS64, Appendix 2, p. 147].

Definition 2.2. *Let $g(\cdot, t)$ be a distribution in $\Omega \subseteq \mathbb{R}^n$ depending on the parameter $t \in \mathbb{R}$. We say that the derivative of $g(\cdot, t)$ with respect to the parameter t exists if for each test function $\varphi \in \mathcal{D}(\Omega)$, the function $\langle g(\cdot, t), \varphi \rangle$ is differentiable in t, and there exists a distribution $h(\cdot, t)$ depending on the parameter t such that*

$$\langle h(\cdot, t), \varphi \rangle = \frac{d}{dt} \langle g(\cdot, t), \varphi \rangle.$$

We write $h(x, t) = \dfrac{\partial g}{\partial t}(x, t)$.

Proposition 2.3. *Given a distribution $g(\cdot, t)$ in Ω, $t \in \mathbb{R}$, if $\dfrac{\partial g}{\partial t}(\cdot, t)$ exists for each $t \in \mathbb{R}$, then for every multi-index α, the derivative with respect to t of the distribution $D^\alpha g$ exists and we have*

$$\frac{\partial (D^\alpha g)}{\partial t} = D^\alpha \left(\frac{\partial g}{\partial t} \right).$$

Proof. If $h(\cdot, t) = \dfrac{\partial g}{\partial t}$ and $\varphi \in \mathcal{D}(\Omega)$, then $\langle g(\cdot, t), \varphi \rangle$ is differentiable in t. Since

$$\langle D^\alpha g(\cdot, t), \varphi \rangle = (-1)^{|\alpha|} \langle g(\cdot, t), D^\alpha \varphi \rangle,$$

and $D^\alpha \varphi \in \mathcal{D}(\Omega)$, then $\langle D^\alpha g(\cdot, t), \varphi \rangle$ is differentiable in t and

$$\frac{d}{dt} \langle D^\alpha g(\cdot, t), \varphi \rangle = (-1)^{|\alpha|} \frac{d}{dt} \langle g(\cdot, t), D^\alpha \varphi \rangle = (-1)^{|\alpha|} \langle h(\cdot, t), D^\alpha \varphi \rangle = \langle D^\alpha h, \varphi \rangle.$$

□

Recalling the set up at the beginning of this section, Ω is a smooth open and bounded domain in $\mathbb{R}^3$, and Γ is a smooth surface that splits Ω into two open parts Ω_+ and Ω_-. For $t \in \mathbb{R}$, we are given a function $g(\cdot, t)$ satisfying

(H1) $g(\cdot, t) \in L^1_{loc}(\Omega)$ for every t,

(H2) for each fixed $x \in \Omega_\pm$ the function $g(x, \cdot)$ is differentiable with respect to t and there exists a function $\psi \in L^1(\Omega_+ \cup \Omega_-)$ such that $\left| \dfrac{\partial g}{\partial t}(x, t) \right| \leq \psi(x)$ for a.e. $x \in \Omega_\pm$ and for each t.

For every t, the linear functional $g(\cdot, t)$ given by

$$\langle g(\cdot, t), \varphi \rangle = \int_\Omega g(x, t) \varphi(x) \, dx,$$

is then a well defined distribution by Item (H1).

Proposition 2.4. *Under the assumptions (H1) and (H2), the distribution $g(\cdot, t)$ has a derivative with respect to the parameter t, and*

$$\left\langle \frac{\partial g}{\partial t}(\cdot, t), \varphi \right\rangle = \int_{\Omega_-} \frac{\partial g}{\partial t}(x, t) \varphi(x) \, dx + \int_{\Omega_+} \frac{\partial g}{\partial t}(x, t) \varphi(x) \, dx.$$

Proof. We write for $\varphi \in \mathcal{D}(\Omega)$ and $t \in (a, b)$

$$\langle g(\cdot, t), \varphi \rangle = \int_\Omega g(x, t) \, \varphi(x) \, dx = \int_{\Omega_-} g(x, t) \, \varphi(x) \, dx + \int_{\Omega_+} g(x, t) \, \varphi(x) \, dx,$$

since from (H1), the integral $\int_\Gamma g(x, t) \, \varphi(x) \, dx = 0$. Using condition (H2) and the Lebesgue dominated convergence theorem, we can justify differentiation under the integral sign and obtain that $\langle g(\cdot, t), \varphi \rangle$ is differentiable in t, and that

$$\frac{d}{dt} \langle g(\cdot, t), \varphi \rangle = \int_{\Omega_-} \frac{\partial g}{\partial t}(x, t) \, \varphi(x) \, dx + \int_{\Omega_+} \frac{\partial g}{\partial t}(x, t) \, \varphi(x) \, dx.$$

From (H2), the linear functional $h(\cdot, t)$ given by

$$\langle h(\cdot, t), \varphi \rangle = \int_{\Omega_-} \frac{\partial g}{\partial t}(x, t)\, \varphi(x)\, dx + \int_{\Omega_+} \frac{\partial g}{\partial t}(x, t)\, \varphi(x)\, dx,$$

is a well defined distribution and hence we obtain $h(\cdot, t) = \dfrac{\partial g}{\partial t}(\cdot, t)$. □

3. Maxwell equations in distributional sense and general boundary conditions

Given Ω open and bounded domain in $\mathbb{R}^3$ and Γ a smooth surface separating Ω into two open parts Ω_+ and Ω_- as in Section 2, we are interested in the Maxwell system [BW06, Sections 1.1 and 1.2] which written in Gaussian (or cgs) units have the form

$$\begin{cases} \nabla \times \mathbf{H} = \dfrac{4\pi}{c}\mathbf{J} + \dfrac{1}{c}\dfrac{\partial \mathbf{D}}{\partial t} \\ \nabla \cdot \mathbf{D} = 4\pi\rho \\ \nabla \times \mathbf{E} = -\dfrac{1}{c}\dfrac{\partial \mathbf{B}}{\partial t} \\ \nabla \cdot \mathbf{B} = 0, \end{cases} \tag{3.1}$$

where the curl and divergence are understood in the sense of distributions as in Section 2, and the fields $\mathbf{H}, \mathbf{J}, \mathbf{D}, \mathbf{E}, \mathbf{B}$ are vector valued distributions in Ω depending on the parameter $t \in \mathbb{R}$ in the sense of Section 2.1, with $\mathbf{J}$ given, and ρ is scalar distribution in Ω, also given, depending also on the parameter $t \in \mathbb{R}$.

The purpose of this section is to show under general assumptions on the current density field $\mathbf{J}$ and the charge density ρ that the solutions to the Maxwell system satisfy boundary conditions at the interface Γ and are classical solutions away from Γ. This is the contents of the following theorem.

Theorem 3.1. *Suppose the fields $\mathbf{B}, \mathbf{D}, \mathbf{E}$ and $\mathbf{H}$ satisfy the Maxwell system* (3.1) *in the sense of distributions where $\mathbf{J}$ and ρ satisfy*

- *(a) $\mathbf{J}(x, t) = \mathbf{J}_0(x, t) + \nu_t$ with $\mathbf{J}_0(x, t)$ a locally integrable $\mathbb{C}^3$-valued function for $x \in \Omega$ for each t; and ν_t are $\mathbb{C}^3$-valued measures in Ω depending on the parameter t and all concentrated in Γ[1];*
- *(b) $\rho(x, t) = \rho_0(x, t) + \mu_t$ with $\rho_0(x, t)$ locally integrable in Ω for each t, and μ_t are measures in Ω depending on the parameter t that are all concentrated on the surface Γ.*

If $\mathbf{B}$ and $\mathbf{D}$ satisfy (F1), (F2), (F3), (H1), and (H2); and $\mathbf{E}$ and $\mathbf{H}$ satisfy (F1), (F2), and (F3), then we have the following

$$d\mu_t(x) = \frac{1}{4\pi}[[\mathbf{D}(x, t)]] \cdot \mathbf{n}(x)\, d\sigma(x), \tag{3.2}$$

$$d\nu_t(x) = -\frac{c}{4\pi}[[\mathbf{H}(x, t)]] \times \mathbf{n}(x)\, d\sigma(x), \tag{3.3}$$

[1] That is, the support of ν_t is contained in Γ.

where $d\sigma$ denotes the surface measure on Γ, and $\mathbf{n}$ is the unit normal toward Ω_+. We also have

$$[[\mathbf{E}(x,t)]] \times \mathbf{n}(x) = \mathbf{0} \quad \textit{for a.e. } x \in \Gamma \textit{ (with respect to surface measure) for all } t, \tag{3.4}$$

and

$$[[\mathbf{B}(x,t)]] \cdot \mathbf{n}(x) = 0 \quad \textit{for a.e. } x \in \Gamma \textit{ (with respect to surface measure) for all } t. \tag{3.5}$$

In addition, the fields $\mathbf{B}, \mathbf{D}, \mathbf{E}$ and $\mathbf{H}$ satisfy the Maxwell system (3.1) *in the classical sense for $x \in \Omega_\pm$ for all t.*

Proof. It will follow by analyzing each equation in (3.1) one at a time.

Analysis of the equation $\nabla \cdot \mathbf{D} = 4\pi\rho$

From (b), $\rho(\cdot,t)$ is a distribution depending on t given by

$$\langle \rho(\cdot,t), \varphi \rangle = \int_\Omega \rho_0(x,t)\,\varphi(x)\,dx + \int_\Omega \varphi(x)\,d\mu_t(x) = \int_\Omega \rho_0(x,t)\,\varphi(x)dx + \int_\Gamma \varphi(x)\,d\mu_t(x),$$

for each $\varphi \in \mathcal{D}(\Omega)$. The left hand side $\nabla \cdot \mathbf{D}$ is a distribution that acting on a test function φ is given by (2.6). We have

$$\langle \nabla \cdot \mathbf{D}(\cdot,t), \varphi \rangle = 4\pi \langle \rho(\cdot,t), \varphi \rangle$$

for each t. If $\operatorname{supp}(\varphi) \subset \Omega_-$ or $\operatorname{supp}(\varphi) \subset \Omega_+$, then from (2.6)

$$\int_{\Omega_-} \varphi(x)\,\nabla \cdot \mathbf{D}_-(x,t)\,dx + \int_{\Omega_+} \varphi(x)\,\nabla \cdot \mathbf{D}_+(x,t)\,dx = 4\pi \int_\Omega \rho_0(x,t)\,\varphi(x)\,dx,$$

which implies

$$\nabla \cdot \mathbf{D}_\pm(x,t) = 4\pi\rho_0(x,t) \quad \text{for a.e. } x \in \Omega_\pm \text{ and for each } t. \tag{3.6}$$

That is, the equation $\nabla \cdot \mathbf{D} = 4\pi\rho$ is satisfied pointwise a.e. in $\Omega_\pm$. If $\operatorname{supp}(\varphi) \cap \Gamma \neq \emptyset$, we then get again from (2.6) and from (3.6) that

$$\int_\Gamma \varphi(x)\,[[\mathbf{D}(x,t)]] \cdot \mathbf{n}(x)\,d\sigma(x) = 4\pi \int_\Gamma \varphi(x)\,d\mu_t(x),$$

that is, the measure μ_t has density $\dfrac{1}{4\pi}[[\mathbf{D}(x,t)]] \cdot \mathbf{n}(x)$, and (3.2) follows. Notice that this part only uses that $\mathbf{D}$ satisfies (F1)–(F3).

Analysis of the equation $\nabla \cdot \mathbf{B} = 0$

We proceed as in the last equation and in this way we obtain

$$\nabla \cdot \mathbf{B}_\pm(x,t) = 0 \quad \text{for a.e. } x \in \Omega_\pm \text{ and for each } t; \tag{3.7}$$

and (3.5).

Analysis of the equation $\nabla \times \mathbf{H} = \dfrac{4\pi}{c}\mathbf{J} + \dfrac{1}{c}\dfrac{\partial \mathbf{D}}{\partial t}$

The equation reads

$$\langle \nabla \times \mathbf{H}, \varphi \rangle = \frac{4\pi}{c}\,\langle \mathbf{J}, \varphi \rangle + \frac{1}{c}\left\langle \frac{\partial \mathbf{D}}{\partial t}, \varphi \right\rangle,$$

for each $\varphi \in \mathcal{D}(\Omega)$. From Proposition 2.4

$$\left\langle \frac{\partial \mathbf{D}}{\partial t}, \varphi \right\rangle = \int_{\Omega_-} \frac{\partial \mathbf{D}(x,t)}{\partial t} \varphi(x)\, dx + \int_{\Omega_+} \frac{\partial \mathbf{D}(x,t)}{\partial t} \varphi(x)\, dx.$$

If $\operatorname{supp}(\varphi) \subset \Omega_-$ or $\operatorname{supp}(\varphi) \subset \Omega_+$, then from (2.7)

$$\begin{aligned} &\int_{\Omega_-} \varphi(x) \nabla \times \mathbf{H}_-(x,t)\, dx + \int_{\Omega_+} \varphi(x) \nabla \times \mathbf{H}_+(x,t)\, dx \\ &\quad = \frac{4\pi}{c} \int_{\Omega} \mathbf{J}_0(x,t)\, \varphi(x)\, dx + \frac{1}{c} \int_{\Omega_-} \frac{\partial \mathbf{D}}{\partial t}(x,t) \varphi(x)\, dx + \frac{1}{c} \int_{\Omega_+} \frac{\partial \mathbf{D}}{\partial t}(x,t) \varphi(x)\, dx, \end{aligned}$$

which implies

$$\nabla \times \mathbf{H}_\pm(x,t) = \frac{4\pi}{c} \mathbf{J}_0(x,t) + \frac{1}{c} \frac{\partial \mathbf{D}}{\partial t}(x,t) \quad \text{for a.e. } x \in \Omega_\pm \text{ and all } t. \tag{3.8}$$

If $\operatorname{supp}(\varphi) \cap \Gamma \neq \emptyset$, we then get again from (2.7) and from (3.8) that

$$-\int_{\Gamma} \varphi(x)\, [[\mathbf{H}(x,t)]] \times \mathbf{n}(x)\, d\sigma(x) = \frac{4\pi}{c} \int_{\Gamma} \varphi(x)\, d\nu_t(x),$$

that is, the measure ν_t has density $-\frac{c}{4\pi}[[\mathbf{H}(x,t)]] \times \mathbf{n}(x)$,that is, (3.3) follows.

Analysis of the equation $\nabla \times \mathbf{E} = -\frac{1}{c}\frac{\partial \mathbf{B}}{\partial t}$

Using the analysis in the last equation we get similarly that

$$\nabla \times \mathbf{E}_\pm(x,t) = -\frac{1}{c} \frac{\partial \mathbf{B}}{\partial t}(x,t) \quad \text{for a.e. } x \in \Omega_\pm \text{ and all } t, \tag{3.9}$$

and also (3.4).

□

3.1. **Compatibility condition.** Let us assume that (3.1) holds with vector valued distributions $\mathbf{E}, \mathbf{H}, \mathbf{D}, \mathbf{B}, \mathbf{J}$ depending on the parameter t, with $\mathbf{B}, \mathbf{D}, \mathbf{J}$ also differentiable with respect to this parameter. Hence from the first and second Maxwell equations in (3.1), (2.3), and Proposition 2.3 we have in the distributional sense the continuity equation

$$0 = \nabla \cdot (\nabla \times \mathbf{H}) = \nabla \cdot \mathbf{J} + \nabla \cdot \frac{\partial \mathbf{D}}{\partial t} = \nabla \cdot \mathbf{J} + \frac{\partial \rho}{\partial t}. \tag{3.10}$$

When $\mathbf{J}$ and ρ satisfy the assumptions in Theorem 3.1, equation (3.10) leads to the following compatibility condition between the current $\mathbf{J}$ and the density ρ:

$$\nabla \cdot \mathbf{J}_0 + \nabla \cdot \nu_t + \frac{\partial \rho_0}{\partial t} + \frac{\partial \mu_t}{\partial t} = 0,$$

in the sense of distributions.

4. Generalized Snell's law deduced from Maxwell equations

Letting as usual [BW06, Section 1.1.2] the material or constitutive equations

$$\mathbf{D} = \epsilon\,\mathbf{E}, \qquad \mathbf{B} = \mu\,\mathbf{H},$$

we obtain from (3.1)

$$\nabla \cdot \mathbf{E} = \frac{4\,\pi\,\rho}{\epsilon}, \tag{M.1}$$

$$\nabla \cdot \mathbf{H} = 0, \tag{M.2}$$

$$\nabla \times \mathbf{E} = -\frac{\mu}{c}\frac{\partial \mathbf{H}}{\partial t}, \tag{M.3}$$

$$\nabla \times \mathbf{H} = \frac{4\pi}{c}\,\mathbf{J} + \frac{\epsilon}{c}\frac{\partial \mathbf{E}}{\partial t}, \tag{M.4}$$

where $\rho(x,t)$ is the charge density, $\mathbf{J}(x,t)$ is the current density vector, $\mathbf{E}$ is the electric field, $\mathbf{H}$ is the magnetic field, and ϵ, μ are constants, the permittivity and permeability of the media (isotropic), respectively. If ϵ_0, μ_0 are the permittivity and permeability of vacuum, then $\epsilon_r = \epsilon/\epsilon_0$ and , $\mu_r = \mu/\mu_0$ denote the relative permittivity and relative permeability, respectively, and $n = \sqrt{\epsilon_r \mu_r}$ is the refractive index of the media. Since the speed of light in vacuum is $c = 1/\sqrt{\epsilon_0 \mu_0}$, and the speed of light in the media is $v = 1/\sqrt{\epsilon\mu}$, then $v = c/n$.

We have two media I and II that are separated by a surface Γ with equation $x_3 = u(x_1, x_2)$, and let Ω^+, Ω^- denote the regions above Γ and below Γ respectively. The constants ϵ, μ the Maxwell system (M.1)–(M.4) may be different in media I and II. Suppose the incoming incident electric field in media I is a plane wave with the form

$$\mathbf{E}_i(x,t) = A_i\, e^{i\,\omega\left(\frac{k_i \cdot x}{v_1} - t\right)}$$

where k_i is the incident unit vector, v_1 is the velocity of propagation in medium I, A_i is a three dimensional constant vector, the amplitude, and ω is a constant (the angular frequency). This wave is defined for $x_3 < u(x_1, x_2)$, i.e., the field is incident to the surface Γ from below defined in Ω^-. This wave strikes the surface Γ at a point $(p_1, p_2, u(p_1, p_2)) = P$, and it is then transmitted into medium II as a nonlinear wave that we assume has the form

$$\mathbf{E}_r(x,t) = A_r\, e^{i\,\omega\left(\frac{k_r \cdot x}{v_2} - t\right)}\, e^{i\,\phi(x)} \tag{4.1}$$

where now k_r is the refracted unit vector, v_2 is the velocity of propagation in medium II, A_r is the amplitude, a constant vector, the wave is defined for $x_3 > u(x_1, x_2)$, i.e., on Ω^+, and $\phi(x)$ is a function called the phase discontinuity defined in a neighborhood of the surface Γ.

We shall now use the result from the previous section, Theorem 3.1, to obtain the generalized Snell's law. In fact, from the boundary condition (3.4) the vector

$$[[\mathbf{E}(X,t)]] = \lim_{x\to X, x\in\Omega^+} \mathbf{E}_r(x,t) - \lim_{x\to X, x\in\Omega^-} \mathbf{E}_i(x,t) \tag{4.2}$$

is parallel to the normal ν at $X = (x_1, x_2, u(x_1, x_2)) \in \Gamma$, for all x_1, x_2 and $t > 0$; $x = (x_1, x_2, x_3)$, and therefore the tangential component of the vector (4.2) is zero. Since $A_i = A_i^{\tan} + A_i^{\perp}$, and $A_r = A_r^{\tan} + A_r^{\perp}$, then from the form of the waves $\mathbf{E}_i$ and $\mathbf{E}_r$ it follows that

$$A_r^{\tan} e^{i\omega\left(\frac{k_r\cdot(x_1,x_2,u(x_1,x_2))}{v_2}-t\right)} e^{i\,\phi(x_1,x_2,u(x_1,x_2))} = \lim_{x\to X, x\in\Omega^+} A_r^{\tan} e^{i\,\omega\left(\frac{k_r\cdot x}{v_2}-t\right)} e^{i\,\phi(x)} \tag{4.3}$$

$$= \lim_{x\to X, x\in\Omega^-} A_i^{\tan} e^{i\omega\left(\frac{k_i\cdot x}{v_1}-t\right)} = A_i^{\tan} e^{i\omega\left(\frac{k_i\cdot(x_1,x_2,u(x_1,x_2))}{v_1}-t\right)}$$

which at $x_1 = p_1$, $x_2 = p_2$ means

$$A_r^{\tan} e^{i\omega\left(\frac{k_r\cdot(p_1,p_2,u(p_1,p_2))}{v_2}-t\right)} e^{i\,\phi(p_1,p_2,u(p_1,p_2))} = A_i^{\tan} e^{i\omega\left(\frac{k_i\cdot(p_1,p_2,u(p_1,p_2))}{v_1}-t\right)}.$$

This implies that the tangential component $A_r^{\tan}$ satisfies

$$A_r^{\tan} = A_i^{\tan} e^{i\omega\left(\frac{k_i\cdot(p_1,p_2,u(p_1,p_2))}{v_1}-\frac{k_r\cdot(p_1,p_2,u(p_1,p_2))}{v_2}\right)} e^{-i\,\phi(p_1,p_2,u(p_1,p_2))}$$

which substituted in (4.3) yields

$$e^{i\left(\omega\left(\frac{k_r\cdot(x_1-p_1,x_2-p_2,u(x_1,x_2)-u(p_1,p_2))}{v_2}\right)+\phi(x_1,x_2,u(x_1,x_2))-\phi(p_1,p_2,u(p_1,p_2))\right)}$$
$$= e^{i\omega\left(\frac{k_i\cdot(x_1-p_1,x_2-p_2,u(x_1,x_2)-u(p_1,p_2))}{v_1}\right)}$$

for all x_1, x_2. Letting

$$f(x_1,x_2) = \omega\,\frac{k_i\cdot(x_1-p_1,x_2-p_2,u(x_1,x_2)-u(p_1,p_2))}{v_1},$$
$$g(x_1,x_2) = \omega\,\frac{k_r\cdot(x_1-p_1,x_2-p_2,u(x_1,x_2)-u(p_1,p_2))}{v_2} + \phi(x_1,x_2,u(x_1,x_2)) - \phi(p_1,p_2,u(p_1,p_2)),$$

we get $e^{i(g(x_1,x_2)-f(x_1,x_2))} = 1$ which yields $g(x_1,x_2) - f(x_1,x_2) = 2m\pi$ for some m integer. Since $f(p_1,p_2) - g(p_1,p_2) = 0$, we obtain $f(x_1,x_2) - g(x_1,x_2) = 0$ for all x_1, x_2. Letting $x_2 = p_2$, dividing the last identity by $x_1 - p_1 \neq 0$, and letting $x_1 \to p_1$ (similarly, taking $x_1 = p_1$ and dividing by $x_2 - p_2 \neq 0$, and letting $x_2 \to p_2$) we obtain

$$\omega\frac{k_r}{v_2}\cdot(1,0,u_{x_1}(p_1,p_2)) + \nabla\phi(p_1,p_2,u(p_1,p_2))\cdot(1,0,u_{x_1}(p_1,p_2)) = \omega\frac{k_i}{v_1}\cdot(1,0,u_{x_1}(p_1,p_2)),$$
$$\omega\frac{k_r}{v_2}\cdot(0,1,u_{x_2}(p_1,p_2)) + \nabla\phi(p_1,p_2,u(p_1,p_2))\cdot(0,1,u_{x_2}(p_1,p_2)) = \omega\frac{k_i}{v_1}\cdot(0,1,u_{x_2}(p_1,p_2)).$$

If we set $n_i = c/v_i$, then this means

$$\Big(n_1\,k_i - n_2\,k_r - (c/\omega)\,\nabla\phi(p_1,p_2,u(p_1,p_2))\Big)\cdot(1,0,u_{x_1}(p_1,p_2)) = 0$$

and

$$\Big(n_1\,k_i - n_2\,k_r - (c/\omega)\,\nabla\phi(p_1,p_2,u(p_1,p_2))\Big)\cdot(0,1,u_{x_2}(p_1,p_2)) = 0.$$

Recall that $c/\omega = \lambda_0/2\pi$ with λ_0 the wavelength of a wave with frequency ω when traveling in vacuum. For a discussion about these constants see [BW06, Section 1.3.3] and [GS21, Sect. 2]. The last two equations imply that the vector $n_1\,k_i - n_2\,k_r - \nabla(\lambda_0\phi/2\pi)$ has the direction of the normal $\nu(P)$ to Γ, that is,

$$n_1\,k_i - n_2\,k_r = \lambda\,\nu(P) + \nabla(\lambda_0\phi/2\pi)\,(P) \tag{4.4}$$

for some $\lambda \in \mathbb{R}$, obtaining in this way the generalized Snell law, see [GS21, Equation (2.4)].

Since there is also a reflected back wave in medium I, which is also assumed as a nonlinear wave having the form

$$\mathbf{E}_b(x,t) = A_b\,e^{i\,\omega\left(\frac{k_b\cdot x}{v_1}-t\right)+i\phi(x)},$$

proceeding in the same way we obtain the generalized reflection law

$$n_1\,k_i - n_1\,k_b = \lambda'\,\nu(P) + \nabla(\lambda_0\phi/2\pi)\,(P)\,. \tag{4.5}$$

4.1. **Conditions for admissibility of the field** (4.1) **to the Maxwell system.** In this section, we prove that the transmitted field $\mathbf{E}(x,t)$ given in (4.1) satisfies the Maxwell system (M.1)–(M.4) for each $\phi \in C^2$ satisfying $A\cdot(\omega\,k + \nabla\phi) = 0$ in the region Ω^+, by calculating the corresponding magnetic field $\mathbf{H}$, and under the assumption that the current density $\mathbf{J}$ satisfies (4.9) and $\rho = 0$. To simplify the notation we have set $k = k_r/v_2$ and $A = A_r$.

We set $\Phi(x,t) = \omega\,(k\cdot x - t) + \phi(x)$. Since $\mathbf{E}(x,t) = Ae^{i\,\Phi(x,t)}$ must satisfy (M.1), and $\rho = 0$, we get $A\cdot\nabla\Phi = 0$ and so

$$A\cdot(\omega\,k + \nabla\phi) = 0; \tag{4.6}$$

this means that the directional derivative of ϕ in the direction of the vector A must be constant.

We next construct $\mathbf{H}$ that verifies (M.3). We have

$$\nabla\times\mathbf{E} = \nabla\times Ae^{i\,\Phi(x,t)} = \nabla(e^{i\,\Phi(x,t)})\times A = -(A\times(\omega\,k + \nabla\phi))\,i\,e^{i\,\Phi(x,t)}. \tag{4.7}$$

Hence by integration in t

$$\mathbf{H}(x,t) = -\frac{c}{\mu\,\omega} A \times (\omega\, k + \nabla\phi) e^{i\Phi(x,t)} + C(x) = -\frac{c}{\mu\,\omega}\mathbf{E} \times (\omega\, k + \nabla\phi) + C(x), \tag{4.8}$$

with C a C^1 vector field in the space variable.

To verify (M.2) write

$$\begin{aligned}\nabla\cdot\mathbf{H} &= -\frac{c}{\mu\,\omega}\nabla\cdot(\mathbf{E}\times(\omega\, k+\nabla\phi)) + \nabla\cdot C(x)\\ &= -\frac{c}{\mu\,\omega}\Big((\omega\, k+\nabla\phi)\cdot(\nabla\times\mathbf{E}) - \mathbf{E}\cdot\Big(\nabla\times(\omega\, k+\nabla\phi)\Big)\Big) + \nabla\cdot C(x),\end{aligned}$$

from the formula $\nabla\cdot(a\times b) = b\cdot(\nabla\times a) - a\cdot(\nabla\times b)$. From (4.7) the first summand in the last equation is zero. The second summand is also zero (curl of a gradient) and so **H** and **E** satisfy (M.2) when C is chosen with

$$\nabla\cdot C(x) = 0.$$

We next verify that (M.4) holds choosing **J** appropriately. From the definitions and the formula $\nabla\times(a\times b) = a(\nabla\cdot b) + (b\cdot\nabla)a - b(\nabla\cdot a) - (a\cdot\nabla)b$, with the notation for fields $a = (a_1, a_2, a_3)$, $b = (b_1, b_2, b_3)$

$$(a\cdot\nabla)b = \left(a_1\frac{\partial}{\partial x_1} + a_2\frac{\partial}{\partial x_2} + a_3\frac{\partial}{\partial x_3}\right)(b_1, b_2, b_3),$$

it follows that

$$\begin{aligned}\nabla\times\mathbf{H} &= -\frac{c}{\mu\,\omega}\nabla\times(\mathbf{E}\times(\omega\, k+\nabla\phi)) + \nabla\times C(x)\\ &= -\frac{c}{\mu\,\omega}\Big(\mathbf{E}\,\nabla\cdot(\omega\, k+\nabla\phi) + ((\omega\, k+\nabla\phi)\cdot\nabla)\mathbf{E} - (\omega\, k+\nabla\phi)\nabla\cdot\mathbf{E} - (\mathbf{E}\cdot\nabla)(\omega\, k+\nabla\phi)\Big) + \nabla\times C\\ &= -\frac{c}{\mu\,\omega}(I + II + III + IV) + \nabla\times C.\end{aligned}$$

From (M.1), $III = 0$. Also

$$\begin{aligned}IV = (\mathbf{E}\cdot\nabla)(\omega\, k+\nabla\phi) &= E_1(\phi_{x_1x_1}, \phi_{x_1x_2}, \phi_{x_1x_3}) + E_2(\phi_{x_2x_1}, \phi_{x_2x_2}, \phi_{x_2x_3}) + E_3(\phi_{x_3x_1}, \phi_{x_3x_2}, \phi_{x_3x_3})\\ &= (\mathbf{E}\cdot\nabla\phi_{x_1}, \mathbf{E}\cdot\nabla\phi_{x_2}, \mathbf{E}\cdot\nabla\phi_{x_3}).\end{aligned}$$

From (4.6)

$$\mathbf{E}\cdot\nabla\phi_{x_j} = (A\cdot\nabla\phi_{x_j})e^{i\,\Phi(x,t)} = (A\cdot\nabla\phi)_{x_j}e^{i\,\Phi(x,t)} = 0, \quad 1\le j\le 3,$$

hence $IV = 0$.

Next

$$II = ((\omega\, k+\nabla\phi)\cdot\nabla)\mathbf{E} = (\omega\, k_1 + \phi_{x_1})\mathbf{E}_{x_1} + (\omega\, k_2 + \phi_{x_2})\mathbf{E}_{x_2} + (\omega\, k_3 + \phi_{x_3})\mathbf{E}_{x_3}.$$

Since $\mathbf{E}_{x_j} = Ai(\omega\, k_j + \phi_{x_j})e^{i\,\Phi(x,t)}$, $1\le j\le 3$, it follows that

$$II = A\, i\,|\omega\, k+\nabla\phi|^2 e^{i\,\Phi(x,t)} = i\,|\omega\, k+\nabla\phi|^2\mathbf{E}.$$

Finally, $I = \mathbf{E}\Delta\phi$; Δ denotes the Laplacian. Hence (M.4) holds if

$$-\frac{c}{\mu\,\omega}(\mathbf{E}\Delta\phi + i\,|\omega\,k + \nabla\phi|^2\mathbf{E}) + \nabla\times C = \frac{4\pi}{c}\mathbf{J} - i\,\omega\frac{\epsilon}{c}\mathbf{E}, \tag{4.9}$$

which determines the current density $\mathbf{J}$, i.e., $\mathbf{J}$ must be chosen as a multiple (depending on ϕ) of $\mathbf{E}$ plus a field with curl zero.

Remark 4.1. We remark that if at the outset $\mathbf{J} = \sigma\,\mathbf{E}$, with σ a complex constant, $\sigma = \alpha + i\,\beta$, then $\alpha = 0$, $\beta \le \omega\epsilon/4\pi$ and the phase ϕ must be linear; see [BW06, Section 14.3, eq. (6)] related to optics in metals. In fact, if in (4.9) C is constant, then

$$\Delta\phi + i\,|\omega\,k + \nabla\phi|^2 = -\frac{4\pi(\alpha + i\,\beta)\mu\omega}{c^2} + i\,\omega^2\,\frac{\epsilon\,\mu}{c^2}.$$

Since ϕ is real valued we get

$$\Delta\phi + \frac{4\pi\mu\omega}{c^2}\alpha = 0, \quad |\omega\,k + \nabla\phi|^2 = \omega^2\,\frac{\epsilon\,\mu}{c^2} - \frac{4\pi}{c^2}\mu\omega\beta.$$

This implies $\beta \le \omega\epsilon/4\pi$. If we let $\psi(x) = \omega\,k\cdot x + \phi(x)$, we then get

$$\Delta\psi + \frac{4\pi\mu\omega}{c^2}\,\alpha = 0, \quad |\nabla\psi|^2 = \frac{\mu\omega}{c^2}\,(\omega\epsilon - 4\pi\beta)\,,$$

for $x_3 > 0$. Now

$$\begin{aligned}\Delta\left(|\nabla\psi|^2\right) &= \sum_{i=1}^{3}\partial_{x_ix_i}\left(\sum_{j=1}^{3}\left(\psi_{x_j}\right)^2\right) = \sum_{i=1}^{3}\partial_{x_i}\left(\sum_{j=1}^{3}2\psi_{x_j}\,\psi_{x_jx_i}\right)\\ &= 2\sum_{i,j=1}^{3}\left(\psi_{x_ix_j}\right)^2 + 2\sum_{j=1}^{3}\psi_{x_j}\sum_{i=1}^{3}\psi_{x_jx_ix_i} = 2\sum_{i,j=1}^{3}\left(\psi_{x_ix_j}\right)^2\end{aligned}$$

since $\Delta(\psi_{x_j}) = 0$ for $j = 1,2,3$. Since the gradient of ψ is constant we obtain that $\psi_{x_ix_j} = 0$ for all i, j and from the Taylor formula it follows that ψ is linear, i.e., $\psi(x) = \gamma + k'\cdot x$ with γ constant and $|k'|^2 = \dfrac{\mu\omega}{c^2}\,(\omega\epsilon - 4\pi\beta)$. Therefore

$$\phi(x) = \gamma + (k' - \omega\,k)\cdot x$$

and so $\Delta\phi = 0$ implying $\alpha = 0$.

Remark 4.2. We apply the calculation in Section 4.1 to find the solutions to the Maxwell system corresponding to the widely used phase ϕ on a plane focusing collimated monochromatic rays into one point $P_0 = (0,0,p_0)$. Let Γ be the horizontal plane $\{x_3 = a\}$ with $0 < a < p_0$ surrounded by vacuum (i.e. $v_1 = v_2 = c$, $\mu = \mu_0$, $\epsilon = \epsilon_0$), and vertical rays emanating from the plane $x_3 = 0$ with wavelength λ_0 strike Γ at (x_1, x_2, a). From the calculation in [GS21, Theorem 5.1], and [AGK+12, Eq.(1)], if the phase ϕ has the form

$$\phi(x_1,x_2,x_3) = \frac{2\pi}{\lambda_0}|(x_1,x_2,a) - P_0| + g(x_3), \tag{4.10}$$

with $g'(a) = 0$, then the metalens (Γ, ϕ) refracts all vertical rays into the point P_0. We assume that g is a constant function as in [AGK+12, Eq.(1)].

Modeling the electric and magnetic fields as in Section 4.1 corresponding to the phase ϕ we shall calculate the form of these fields and the current density $\mathbf{J}$ satisfying the Maxwell system (M.1)–(M.4). In fact, with the notation in Section 4.1 and since $v_2 = c$

$$k = \frac{k_r}{c} = -\frac{(x_1, x_2, a - p_0)}{c\,|(x_1, x_2, a - p_0)|} \quad \text{and} \quad \nabla\phi(x_1, x_2, x_3) = \frac{2\pi}{\lambda_0}\frac{(x_1, x_2, 0)}{|(x_1, x_2, a - p_0)|} = \frac{\omega}{c}\frac{(x_1, x_2, 0)}{|(x_1, x_2, a - p_0)|}.$$

Letting $\Phi(x, t) = \omega(k \cdot x - t) + \phi(x) = \omega\left(-\frac{(x_1, x_2, a - p_0)}{c\,|(x_1, x_2, a - p_0)|} \cdot (x_1, x_2, x_3) - t\right) + \phi(x)$, the transmitted electric field has the form $\mathbf{E}(x, t) = Ae^{i\Phi(x,t)}$ with $A = (A_1, A_2, A_3)$ satisfying (4.6), that is

$$0 = A \cdot \left(-\frac{\omega\,(x_1, x_2, a - p_0)}{c\,|(x_1, x_2, a - p_0)|} + \frac{\omega}{c}\frac{(x_1, x_2, 0)}{|(x_1, x_2, a - p_0)|}\right) = -\frac{\omega}{c}A \cdot \frac{(0, 0, a - p_0)}{|(x_1, x_2, a - p_0)|},$$

and hence $A_3 = 0$, concluding that $\mathbf{E}$ is horizontal. We next find the magnetic field $\mathbf{H}$. Noticing that

$$A \times (\omega k + \nabla\phi) = (A_1, A_2, 0) \times \left(-\frac{\omega}{c}\frac{(0, 0, a - p_0)}{|(x_1, x_2, a - p_0)|}\right) = -\frac{\omega}{c}\frac{a - p_0}{|(x_1, x_2, a - p_0)|}(A_2, -A_1, 0),$$

then from (4.8) we get

$$\mathbf{H}(x, t) = \frac{a - p_0}{\mu_0\,|(x_1, x_2, a - p_0)|}(A_2, -A_1, 0)e^{i\Phi(x,t)} + C(x),$$

with C a divergence free vector field.

Assuming for simplicity that $C(x)$ is constant, we calculate $\mathbf{J}$ from Formula (4.9). We have

$$|\omega k + \nabla\phi|^2 = \frac{\omega^2}{c^2}\frac{(a - p_0)^2}{|(x_1, x_2, a - p_0)|^2}.$$

$$\begin{aligned}
\Delta\phi &= \frac{\omega}{c}\nabla \cdot \left(\frac{(x_1, x_2, 0)}{|(x_1, x_2, a - p_0)|}\right) \\
&= \frac{\omega}{c|(x_1, x_2, a - p_0)|^2}\left(|(x_1, x_2, a - p_0)| - \frac{x_1^2}{|(x_1, x_2, a - p_0)|} + |(x_1, x_2, a - p_0)| - \frac{x_1^2}{|(x_1, x_2, a - p_0)|}\right) \\
&= \frac{\omega}{c|(x_1, x_2, a - p_0)|^3}\left(|(x_1, x_2, a - p_0)|^2 + (a - p_0)^2\right).
\end{aligned}$$

Hence (4.9) reads

$$-\frac{1}{\mu_0}\left(\frac{1}{|(x_1, x_2, a - p_0)|^3}\left(|(x_1, x_2, a - p_0)|^2 + (a - p_0)^2\right) + i\frac{\omega}{c}\frac{(a - p_0)^2}{|(x_1, x_2, a - p_0)|^2}\right)\mathbf{E} = \frac{4\pi}{c}\mathbf{J} - i\frac{\omega\,\epsilon_0}{c}\mathbf{E}.$$

Multiplying the last identity by μ_0 and since $\epsilon_0\mu_0 = 1/c^2$ we get

$$-\left(\frac{1}{|(x_1, x_2, a - p_0)|^3}\left(|(x_1, x_2, a - p_0)|^2 + (a - p_0)^2\right) + i\frac{\omega}{c}\frac{(a - p_0)^2}{|(x_1, x_2, a - p_0)|^2}\right)\mathbf{E} = \frac{4\pi\,\mu_0}{c}\mathbf{J} - i\frac{\omega}{c^3}\mathbf{E},$$

obtaining

$$\mathbf{J} = \frac{c}{4\pi\,\mu_0}\left(-\frac{1}{|(x_1,x_2,a-p_0)|^3}\left(|(x_1,x_2,a-p_0)|^2+(a-p_0)^2\right)+i\left(\frac{\omega}{c^3}-\frac{\omega}{c}\frac{(a-p_0)^2}{|(x_1,x_2,a-p_0)|^2}\right)\right)\mathbf{E}.$$

5. Conclusion

An analysis of the Maxwell system of electrodynamics in the sense of distributions is carried out and boundary conditions for the electromagnetic field are deduced when the current and charge densities are measures concentrated on the interface. As a consequence, and realizing the electric field as a non linear perturbation of a plane wave involving the phase discontinuity function, the generalized Snell law is obtained. Given that the electromagnetic field must satisfy the Maxwell system, this implies admissibility conditions for the phase that determines the current density. To illustrate, it is also shown that if the current density is a complex constant multiple of the electric field then the phase discontinuity must be linear. In addition, if one considers the phase focusing monochromatic rays into a fix point, then the current density must be a variable complex multiple of the electric field.

Conflict of interest statement

The authors declare that they do not have any conflict of interest related to the work reported in this paper, and no data is associated with this work.

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