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RESEARCH ARTICLE

The Generalized Alpha Power Exponentiated Inverse Exponential distribution and it's application to real data

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ABSTRACT

This paper proposes a new distribution named "The Generalized Alpha Power Exponentiated Inverse Exponential (GAPEIEx for short) distribution" with four parameters, from which one (1) scale and three (3) shape parameters and the statistical properties such as Survival function, Hazard function, Quantile function, rth Moment, Rényi Entropy, and Order Statistics of the new distribution are derived. The method of maximum likelihood estimation (MLE) is used to estimate the parameters of the distribution. The performance of the estimators is assessed through simulation, which shows that the maximum likelihood method works well in estimating the parameters.
The GAPEIEx distribution was applied to simulated and real data in order to access the flexibility and adaptability of the distribution, and it happens to perform better than its submodels.

KEYWORDS:
Generalized Alpha Power, Exponentiated Inverse Exponential, Inverse Exponential, Maximum Likelihood estimation

1 | INTRODUCTION

Finding an appropriate statistical model to handle practical problems is one of the main challenges in statistics. Probability distributions are used to model real-life phenomena that are characterized by uncertainty and are dangerous to human life. Since real-life occurrences are intricate and challenging to model using conventional distributions, the majority of probability distributions have been created. As a result, probability distributions are statistically modified, and in recent years' statisticians have given these distributions more attention. The reason behind the modification of distributions is that the traditional ones cannot handle more than one of the data characteristics, such as heavy tails (left or right), skewness, kurtosis, monotonic, and non-monotonic failure rates. Data with these features requires distributions that are more adaptable than conventional ones. This is due to a single shape parameter in the cumulative distribution function (CDF) and probability density function (PDF) of the traditional distributions and families of distributions. Examples of such distributions are the Exponential distribution, the Lindley distribution, the Poisson distribution, the Rayleigh distribution, the Gull Alpha Power Family, and the Alpha Power Family.

According to[1], incorporating additional parameters into established probability distributions increased both their applicability to actual occurrences and their precision in defining the distribution's tail shape.[2] as the inventor of the exponentiated

[0]**Abbreviations:** GAPEIEx, Generalized Alpha Power Exponentiated Inverse Exponential; GF, Generalized Family; GAPF, Generalized Alpha Power Family; APF, Alpha Power Family; EIEx, Exponentiated Inverse Exponential; Sf, Survival function; Hf, Hazard function; APIEx, Alpha Power Inverse Exponential; APEIEx, Alpha Power Exponentiated Inverse Exponential; GAPIEx, Generalized Alpha Power Inverse Exponential; $Q(\eta)$, Quantile function; CDF, Cumulative Distri bution Function; PDF, Probability Density Function; AB, Average Bias; RMSE, Root Mean Square Error; MLE, Maximum Likelihood Estimator

family technique.[3],[4] and[5] as the pioneers of model modification.

Many researchers have employed traditional distributions and families by extending the distributions and families with additional parameter(s) so that one can model real-life and simulated data. This is possible due to the pioneers of adding parameter(s) in a distribution(s), exponentiated families, and model modification.

The Inverse Weibull Inverse Exponential distribution was proposed by[6], the Alpha Power Exponentiated Inverse Rayleigh distribution by[7], the modified Rayleigh distribution for modeling COVID-19 mortality rates by[8], the Kumaraswamy-Gull Alpha Power Rayleigh distribution with it's properties and application to HIV/AIDS data by[9], the Gull-Alpha Power Weibull distribution with applications to real and simulated data by[10], the Generalized Exponential distribution by[11],[12], and[13], the two-parameter Inverse Exponential distribution with a decreasing failure rate by[14], a new family of generalized distributions based on Alpha Power Transformation with application to cancer data by[15], on the Exponentiated Generalized Exponentiated Exponential distribution with Properties and Application by[16], the Exponentiated Generalized Class of distributions by[17] and many more.

An extremely innovative and approachable generalized alpha power exponentiated inverse exponential distribution is presented in this paper. We describe the new distribution's development process and its submodels in Section 2; The statistical features of the proposed distribution are shown in Section 3; the MLEs of the parameters are derived in Sections 4 and 5 through simulation; the proposed distribution is applied to real data in Section 6; and the conclusions are given in Section 7.

2 | DEVELOPMENT OF THE GAPEIEX DISTRIBUTION

2.1 | Generalized Family (GF) of distribution

If X is a random variable, the CDF and PDF of the generalized family of distribution is given respectively as;

$$F_{GF}(x) = \Big(M(x)\Big)^{b} \quad ;\text{if} \quad x, b > 0 \tag{1}$$

$$\text{the corresponding PDF is;} \quad f_{GF}(x) = \frac{d}{dx}\Big(F_{GF}(x)\Big)$$

$$f_{GF}(x) = bm(x)\Big(M(x)\Big)^{b-1} \quad ;\text{if} \quad x, b > 0 \tag{2}$$

where $b > 0$ is a shape parameter

2.2 | Alpha Power Family (APF) of distribution

Since X is a continuous random variable and α is a shape parameter with CDF, $M(x)$ and PDF, $m(x)$, the Alpha Power transform is define by[7] as,

$$M(x) = \begin{cases} \frac{\alpha^{G(x)}-1}{\alpha-1} & ;\text{if} \quad x, \alpha > 0, \alpha \neq 1 \\ 0 & ;\text{otherwise} \end{cases} \tag{3}$$

$$\text{From equation (3) the corresponding PDF is;} \quad m(x) = \frac{d}{dx}\Big(M(x)\Big)$$

$$m(x) = \begin{cases} \left(\frac{\log(\alpha)}{\alpha-1}\right)\alpha^{G(x)}g(x) & ;\text{if} \quad x, \alpha > 0, \alpha \neq 1 \\ 0 & ;\text{otherwise} \end{cases} \tag{4}$$

2.3 | Generalized Alpha Power Family (GAPF) of distribution

In order to find the generalized alpha power family of distribution, substitute equation (3) into (1) and equations (3) & (4) into equation (2) respectively;

$$\text{CDF;}\quad F_{GAPF}(x) = \begin{cases} (\alpha-1)^{-b}\left(\alpha^{G(x)}-1\right)^{b} & ;\text{if}\ \ x,\alpha,b>0,\alpha\neq 1 \\ 0 & ;\text{otherwise} \end{cases} \tag{5}$$

$$\text{PDF;}\quad f_{GAPF}(x) = \begin{cases} b\log(\alpha)(\alpha-1)^{-b}g(x)\alpha^{G(x)}\left(\alpha^{G(x)}-1\right)^{b-1} & ;\text{if}\ \ x,\alpha,b>0,\alpha\neq 1 \\ 0 & ;\text{otherwise} \end{cases} \tag{6}$$

2.4 | Exponentiated Inverse Exponential (EIEx) distribution

The exponentiated inverse exponential distribution is define as,

$$G(x) = \begin{cases} e^{-\frac{ak}{x}} & ;\text{if}\quad x,a,k>0 \\ 0 & ;\text{otherwise} \end{cases} \tag{7}$$

From equation (7) the corresponding PDF is;

$$\begin{aligned} g(x) &= \frac{d}{dx}\bigg(G_{IEx}(x)\bigg)^{a} \\ &= a g_{IEx}(x)\bigg(G_{IEx}(x)\bigg)^{a-1} \end{aligned}$$

$$g(x) = \begin{cases} \frac{ak}{x^2}\left(e^{-\frac{ak}{x}}\right) & ;\text{if}\quad x,a,k>0 \\ 0 & ;\text{otherwise} \end{cases} \tag{8}$$

2.5 | The GAPEIEx distribution

The Exponentiated Inverse Exponential distribution is used as the baseline distribution in this special example of the Generalized Alpha Power Family of distributions, which has three shape parameters and one scale parameter.

Hence, in order to find the CDF and PDF of the GAPEIEx distribution, substitute equation (7) into equation (5) and equations (7) and (8) into equation (6);

$$\text{CDF:}\quad F_{GAPEIEx}(x) = \begin{cases} (\alpha-1)^{-b}\left(\alpha^{e^{-\frac{ak}{x}}}-1\right)^{b} & ;\text{if}\quad x,\alpha,a,b,k>0,\alpha\neq 1 \\ 0 & ;\text{otherwise} \end{cases} \tag{9}$$

The corresponding PDF of equation (9) is given as,

$$f_{GAPEIEx}(x) = \begin{cases} \phi x^{-2}e^{-\frac{ak}{x}}\alpha^{e^{-\frac{ak}{x}}}\left(\alpha^{e^{-\frac{ak}{x}}}-1\right)^{b-1} & ;\text{if}\quad x,\alpha,a,b,k>0,\alpha\neq 1 \\ 0 & ;\text{otherwise} \end{cases} \tag{10}$$

where $\phi = abk\log(\alpha)(\alpha-1)^{-b}$

The following formulae may be use to describe the survival function (Sf) and hazard function (Hf) of the GAPEIEx distribution respectively;

$$\begin{aligned} Sf &= 1 - F_{GAPEIEx}(x) \\ &= 1 - \left(\frac{\alpha^{e^{-\frac{ak}{x}}}-1}{\alpha-1}\right)^{b} \quad ;\alpha,a,b,k>0,\alpha\neq 1 \end{aligned} \tag{11}$$

and

$$Hf = \frac{f_{GAPEIEx}(x)}{Sf}$$
$$= \frac{abk\log(\alpha)x^{-2}e^{-\frac{ak}{x}}\alpha^{e^{-\frac{ak}{x}}}\left(\alpha^{e^{-\frac{ak}{x}}}-1\right)^{b-1}}{(\alpha-1)^b-\left(\alpha^{e^{-\frac{ak}{x}}}-1\right)^b} \quad ;\alpha, a, b, k > 0 \tag{12}$$

2.6 | Submodels of GAPEIEx distribution

There are several well-known submodels of the GAPEIEx distribution. We set a parameter or parameters in the CDF and PDF of the GAPEIEx distribution to one in order to find these submodels.

1. At $a = b = 1$, the CDF and PDF of the GAPEIEx distribution in equation (9) and equation (10) will reduces to ***Alpha Power Inverse Exponential (APIEx) distribution***.

$$F_{APIEx}(x) = (\alpha-1)^{-1}\left(\alpha^{e^{-\frac{k}{x}}}-1\right)$$

$$f_{APIEx}(x) = k\log(\alpha)(\alpha-1)^{-1}x^{-2}e^{-\frac{k}{x}}\alpha^{e^{-\frac{k}{x}}} \quad ;\text{if } \ x, \alpha, k > 0, \alpha \neq 1$$

2. At $b = 1$, the CDF and PDF of the GAPEIEx distribution in equation (9) and equation (10) will reduces to ***Alpha Power Exponentiated Inverse Exponential (APEIEx) distribution***.

$$F_{APEIEx}(x) = (\alpha-1)^{-1}\left(\alpha^{e^{-\frac{ak}{x}}}-1\right)$$

$$f_{APEIEx}(x) = ak\log(\alpha)(\alpha-1)^{-1}x^{-2}e^{-\frac{ak}{x}}\alpha^{e^{-\frac{ak}{x}}} \quad ;\text{if } \ x, \alpha, a, k > 0, \alpha \neq 1$$

3. At $a = 1$, the CDF and PDF of the GAPEIEx distribution in equation (9) and equation (10) will reduces to ***Generalized Alpha Power Inverse Exponential (GAPIEx) distribution***.

$$F_{GAPIEx}(x) = (\alpha-1)^{-b}\left(\alpha^{e^{-\frac{k}{x}}}-1\right)^b$$

$$f_{GAPIEx}(x) = bk\log(\alpha)(\alpha-1)^{-b}x^{-2}e^{-\frac{k}{x}}\alpha^{e^{-\frac{k}{x}}}\left(\alpha^{e^{-\frac{k}{x}}}-1\right)^{b-1} \quad ;\text{if } \ x, \alpha, b, k > 0, \alpha \neq 1$$

TABLE 1 Summary of submodels of the GAPEIEx distribution.

Submodels	**Parameters**			
	α	a	b	k
APIEx Distribution	α	1	1	k
APEIEx Distribution	α	a	1	k
GAPIEx Distribution	α	1	b	k

A reversed J-shape, a J-shape, a right-skewed with a heavy tail and high kurtosis, a left-skewed, a unimodal, decreasing, decrease-increase-decrease, a bathtub, and an inverted bathtub shape are just a few examples of the different GAPEIEx distribution PDF and hazard function shapes shown in Figures 1 and 2. The GAPEIEx PDF distribution and hazard function have shapes that can model a range of data, thanks to their unique characteristics.

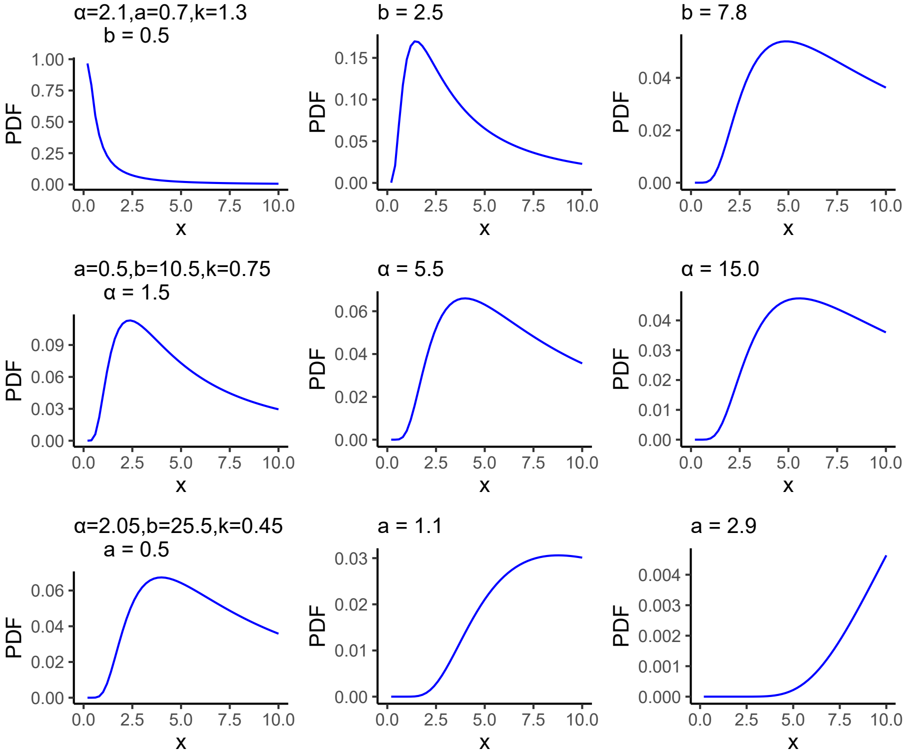


FIGURE 1 The PDF of the GAPEIEx distribution.

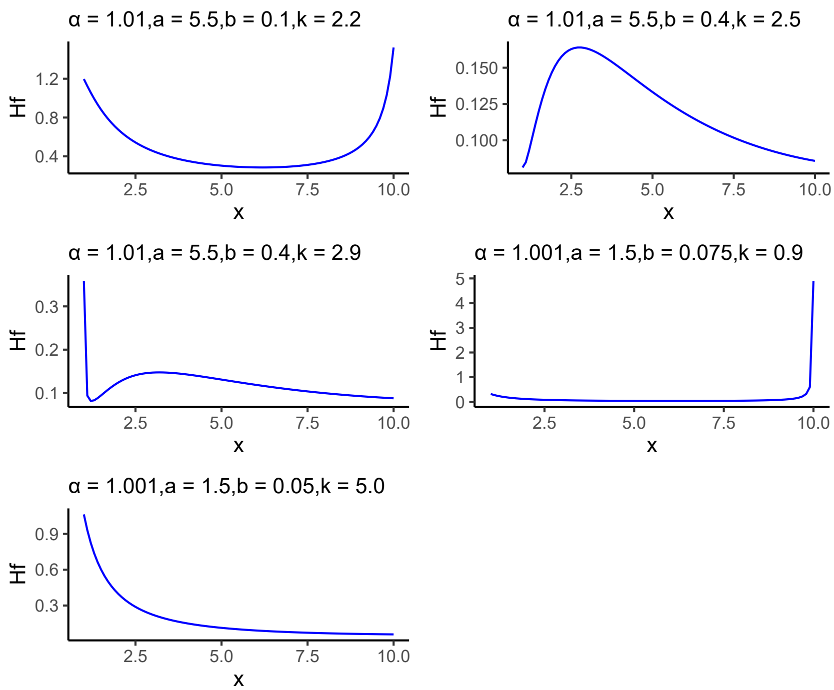


FIGURE 2 The Hazard function of the GAPEIEx distribution.

3 | STATISTICAL PROPERTIES

The derived statistical characteristics, graphs, and numerical values of the GAPEIEx distribution will be displayed in this section.

3.1 | Quantile function [$Q(\eta)$]

Since $F_{GAPEIEx}(x)$ is the CDF of the GAPEIEx distribution, it $Q(\eta)$ is the inverse of equation (9), and it is defined on the unit interval $\eta \in (0,1)$.

Hence the $Q(\eta)$ of the GAPEIEx distribution is;

$$Q(\eta) = \frac{-ak}{\log\left\{\frac{\log\left(\eta^{\frac{1}{b}}(\alpha-1)+1\right)}{\log(\alpha)}\right\}} \quad ; \alpha, a, b, k > 0 \tag{13}$$

Proof:

Let $F_{GAPEIEx}(x) = \eta$, where $\eta \in (0,1)$ is the probability value, now we solve for x;

$$\eta = (\alpha-1)^{-b}\left(\alpha^{e^{-\frac{ak}{x}}} - 1\right)^{b}$$

$$\alpha^{e^{-\frac{ak}{x}}} - 1 = \eta^{\frac{1}{b}}(\alpha-1)$$

$$\alpha^{e^{-\frac{ak}{x}}} = \eta^{\frac{1}{b}}(\alpha-1)+1$$

$$e^{-\frac{ak}{x}}\log(\alpha) = \log\left(\eta^{\frac{1}{b}}(\alpha-1)+1\right)$$

$$e^{-\frac{ak}{x}} = \frac{\log\left(\eta^{\frac{1}{b}}(\alpha-1)+1\right)}{\log(\alpha)}$$

$$-\frac{ak}{x} = \log\left\{\frac{\log\left(\eta^{\frac{1}{b}}(\alpha-1)+1\right)}{\log(\alpha)}\right\}$$

$$x = \frac{-ak}{\log\left\{\frac{\log\left(\eta^{\frac{1}{b}}(\alpha-1)+1\right)}{\log(\alpha)}\right\}}$$

The Median of the GAPEIEx distribution is at the point where $\eta = \frac{1}{2}$

$$Q\left(\frac{1}{2}\right) = \frac{-ak}{\log\left\{\frac{\log\left(\left(\frac{1}{2}\right)^{\frac{1}{b}}(\alpha-1)+1\right)}{\log(\alpha)}\right\}} \quad ; \alpha, a, b, k > 0 \tag{14}$$

Using equation (13) the random variables/numbers of the GAPEIEx distribution can be simulated using;

$$x_{\eta} = \frac{-ak}{\log\left\{\frac{\log\left(\eta^{\frac{1}{b}}(\alpha-1)+1\right)}{\log(\alpha)}\right\}} \quad ; \text{where} \quad \eta \in (0,1), \alpha, a, b, k > 0 \tag{15}$$

Table 2, displays the quantile values for selected parameter values. It is obvious that the quantile values rise in proportion to the probability value (η).

TABLE 2 Quantile Values of the GAPEIEx distribution.

η	$(\alpha, a, b, k = 1.05, 0.05, 5.5, 0.9)$	$(\alpha, a, b, k = 1.05, 0.05, 3.5, 0.8)$
0.1	0.10965	0.06190
0.2	0.15707	0.08871
0.3	0.21012	0.11871
0.4	0.27624	0.15610
0.5	**0.36533**	**0.20649**
0.6	0.49590	0.28034
0.7	0.71044	0.40170
0.8	1.13589	0.64235
0.9	2.40631	1.36097

3.2 | r^{th} moments

The r^{th} moments of the GAPEIEx distribution is define as;

$$\mu^r = E(X^r) = \int_0^\infty x^r f_{GAPEIEx}(x)dx \tag{16}$$

By replacing $f_{GAPEIEx}(x)$ in equation (16) with equation (10), the r^{th} moments is given as;

$$\mu^r = \phi \sum_{j=0}^{b-1} (-1)^j \binom{b-1}{j} \left\{ \int_0^\infty x^{r-2} e^{-\frac{ak}{x}} \left(\alpha^{e^{-\frac{ak}{x}}}\right)^{b-j} dx \right\} \tag{17}$$

where $\phi = abk(\alpha - 1)^{-b} \log(\alpha)$

Proof;

$$\begin{aligned}
\mu^r = E(X^r) &= \int_0^\infty x^r f_{GAPEIEx}(x)dx \\
&= \int_0^\infty x^r \phi x^{-2} e^{-\frac{ak}{x}} \alpha^{e^{-\frac{ak}{x}}} \left(\alpha^{e^{-\frac{ak}{x}}} - 1\right)^{b-1} dx \\
&= \phi \int_0^\infty x^{r-2} e^{-\frac{ak}{x}} \alpha^{e^{-\frac{ak}{x}}} \left(\alpha^{e^{-\frac{ak}{x}}} - 1\right)^{b-1} dx
\end{aligned}$$

Using the generalized binomial expansion on $\left(\alpha^{e^{-\frac{ak}{x}}} - 1\right)^{b-1}$ where $b > 0$;

$$\left(\alpha^{e^{-\frac{ak}{x}}} - 1\right)^{b-1} = \sum_{j=0}^{b-1} (-1)^j \binom{b-1}{j} \left(\alpha^{e^{-\frac{ak}{x}}}\right)^{b-1-j}$$

$$\begin{aligned}
\mu^r &= \phi \int_0^\infty x^{r-2} e^{-\frac{ak}{x}} \alpha^{e^{-\frac{ak}{x}}} \sum_{j=0}^{b-1} (-1)^j \binom{b-1}{j} \left(\alpha^{e^{-\frac{ak}{x}}}\right)^{b-1-j} dx \\
&= \phi \sum_{j=0}^{b-1} (-1)^j \binom{b-1}{j} \left\{ \int_0^\infty x^{r-2} e^{-\frac{ak}{x}} \left(\alpha^{e^{-\frac{ak}{x}}}\right)^{b-j} dx \right\}
\end{aligned}$$

where $\phi = abk(\alpha - 1)^{-b} \log(\alpha)$

3.2.1 | Mean $[E(X)]$

The Mean of equation (17) is at $r = 1$;

$$E(X) = \phi \sum_{j=0}^{b-1} (-1)^j \binom{b-1}{j} \left\{ \int_0^\infty x^{-1} e^{-\frac{ak}{x}} \left(\alpha^{e^{-\frac{ak}{x}}} \right)^{b-j} dx \right\}$$

3.2.2 | Variance $[Var(X)]$

Form the traditional definition of variance of a random variable X;

$$Var(X) = E(X^2) - (E(X))^2$$

From equation (17),

$$Var(X) = \phi \sum_{j=0}^{b-1} (-1)^j \binom{b-1}{j} \left\{ \int_0^\infty e^{-\frac{ak}{x}} \left(\alpha^{e^{-\frac{ak}{x}}} \right)^{b-j} dx \right\} - \left\{ \phi \sum_{j=0}^{b-1} (-1)^j \binom{b-1}{j} \left(\int_0^\infty x^{-1} e^{-\frac{ak}{x}} \left(\alpha^{e^{-\frac{ak}{x}}} \right)^{b-j} dx \right) \right\}^2$$

3.2.3 | Coefficient of Variation $[CV]$

The CV is define as the Standard Deviation (SD) divided by the Mean $[E(X)]$.

$$CV = \frac{\sqrt{Var(X)}}{E(X)} = \frac{\sqrt{\phi \sum_{j=0}^{b-1} (-1)^j \binom{b-1}{j} \left(\int_0^\infty e^{-\frac{ak}{x}} \left(\alpha^{e^{-\frac{ak}{x}}} \right)^{b-j} dx \right) - \left(\prod(x) \right)^2}}{\phi \sum_{j=0}^{b-1} (-1)^j \binom{b-1}{j} \left(\int_0^\infty x^{-1} e^{-\frac{ak}{x}} \left(\alpha^{e^{-\frac{ak}{x}}} \right)^{b-j} dx \right)}$$

$$\text{where, } \prod(x) = \phi \sum_{j=0}^{b-1} (-1)^j \binom{b-1}{j} \left(\int_0^\infty x^{-1} e^{-\frac{ak}{x}} \left(\alpha^{e^{-\frac{ak}{x}}} \right)^{b-j} dx \right)$$

3.2.4 | Bowley (Galton) Skewness (B_s)

$$B_s = \frac{Q_1 + Q_3 - 2Q_2}{Q_3 - Q_1}$$

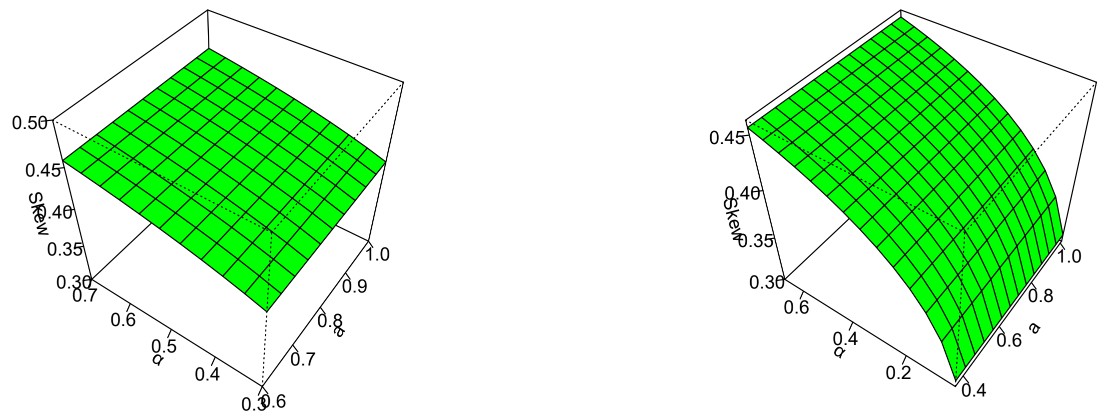
where Q is the quantile function.

3.2.5 | Moors Kurtosis (M_k)

$$M_k = \frac{Q(0.875) + Q(0.375) - [Q(0.625) + Q(0.125)]}{Q(0.75) - Q(0.25)}$$

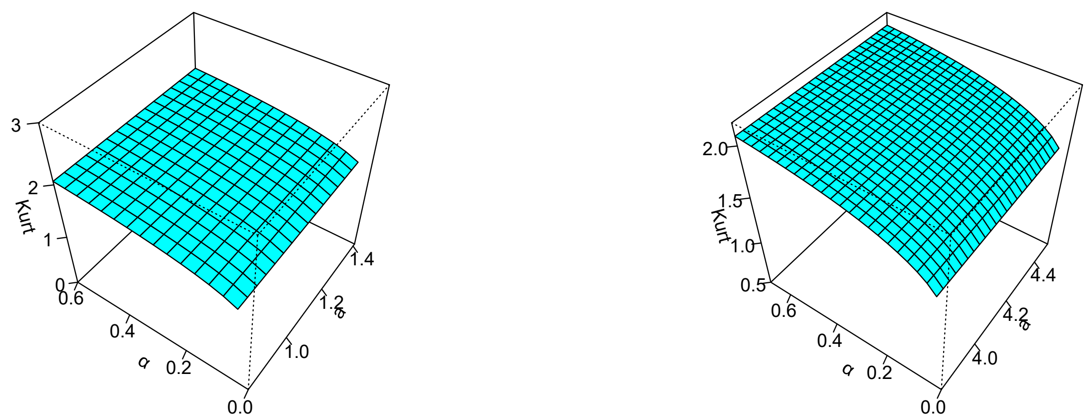
where Q is the quantile function

The additional parameters (α, a) clearly have an impact on the Skewness and Kurtosis of the GAPEIEx distribution as seen in Figures 3 & 4. This demonstrates the flexibility of the GAPEIEx distribution and the importance of the extra parameters.



(a) (b)

FIGURE 3 Plot for the GAPEIEx Bowley Skewness with fixed $(b, k = 0.3, 0.2)$



(a) (b)

FIGURE 4 Plot for the GAPEIEx Moors Kurtosis with fixed $(b, k = 0.3, 0.2)$

3.3 | Rényi Entropy

The Rényi Entropy of the GAPEIEx distribution for a random variable X is define as;

$$R_\delta(X) = \frac{1}{1-\delta}\log\left\{ \phi^\delta \sum_{j=0}^{b-1}\left((-1)^j\binom{b-1}{j}\right)^\delta \int_0^\infty \left[x^{-2}e^{-\frac{ak}{x}}\left(\alpha^{e^{-\frac{ak}{x}}}\right)^{b-j}\right]^\delta dx \right\} \tag{18}$$

where $\phi = abk(\alpha-1)^{-b}\log(\alpha)$

Proof;

$$\text{From;}\quad R_\delta(X) = \frac{1}{1-\delta}\log\left(\int_0^\infty f^\delta_{GAPEIEx}(x)dx\right)\quad ;\text{if}\quad \delta(\neq 1) > 0 \tag{19}$$

where δ is the order

By replacing $f_{GAPEIEx}(x)$ in equation (19) with equation (10), the Rényi Entropy is given as;

$$R_\delta(X) = \frac{1}{1-\delta}\log\left\{\int_0^\infty\left(\phi x^{-2}e^{-\frac{ak}{x}}\alpha^{e^{-\frac{ak}{x}}}\left(\alpha^{e^{-\frac{ak}{x}}}-1\right)^{b-1}\right)^\delta dx\right\}$$

$$\text{From,}\quad \left(\alpha^{e^{-\frac{ak}{x}}}-1\right)^{b-1} = \sum_{j=0}^{b-1}(-1)^j\binom{b-1}{j}\left(\alpha^{e^{-\frac{ak}{x}}}\right)^{b-1-j}$$

$$R_\delta(X) = \frac{1}{1-\delta}\log\left\{\int_0^\infty\left(\phi x^{-2}e^{-\frac{ak}{x}}\alpha^{e^{-\frac{ak}{x}}}\sum_{j=0}^{b-1}(-1)^j\binom{b-1}{j}\left(\alpha^{e^{-\frac{ak}{x}}}\right)^{b-1-j}\right)^\delta dx\right\}$$

$$= \frac{1}{1-\delta}\log\left\{\phi^\delta\sum_{j=0}^{b-1}\left((-1)^j\binom{b-1}{j}\right)^\delta\int_0^\infty\left[x^{-2}e^{-\frac{ak}{x}}\left(\alpha^{e^{-\frac{ak}{x}}}\right)^{b-j}\right]^\delta dx\right\}$$

3.4 | Order Statistics

If $X_1, X_2, X_3, ..., X_n$ denote the random sample of size n from the GAPEIEx distribution, hence $X_{(1)} \leqslant X_{(2)} \leqslant X_{(3)} \leqslant ... \leqslant X_{(n)}$ will be the order statistics of a distribution.

The l^{th} order statistics is define as;

$$f_{X_{(l)}}(x) = \frac{n!}{(l-1)!(n-l)!}f(x)\Big(F(x)\Big)^{l-1}\Big(1-F(x)\Big)^{n-l}$$

Using the generalized binomial expansion for $\Big(1-F(x)\Big)^{n-l}$,

$$\text{Hence,}\quad \Big(1-F(x)\Big)^{n-l} = \sum_{t=0}^{n-l}\binom{n-l}{t}(1)^{n-l-t}\Big(-F(x)\Big)^t$$
$$= \sum_{t=0}^{n-l}(-1)^t\binom{n-l}{t}\Big(F(x)\Big)^t$$

Hence the l^{th} order statistics is;

$$\begin{aligned} f_{X_{(l)}}(x) &= \frac{n!}{(l-1)!(n-l)!}f(x)\Big(F(x)\Big)^{l-1}\sum_{t=0}^{n-l}(-1)^t\binom{n-l}{t}\Big(F(x)\Big)^t \\ &= \frac{n!}{(l-1)!(n-l)!}\sum_{t=0}^{n-l}(-1)^t\binom{n-l}{t}f(x)\Big(F(x)\Big)^{l-1+t} \end{aligned} \tag{20}$$

where $l = 1$ is the l^{th} minimum, and $l = n$ is the l^{th} maximum order statistics of the distribution.

3.4.1 | l^{th} minimum order statistics

$$f_{X_{(1)}}(x) = n\phi x^{-2}e^{-\frac{ak}{x}}\alpha^{e^{-\frac{ak}{x}}}\left(\alpha^{e^{-\frac{ak}{x}}}-1\right)^{b-1}\sum_{t=0}^{n-1}(-1)^t\binom{n-1}{t}\left\{(\alpha-1)^{-b}\left(\alpha^{e^{-\frac{ak}{x}}}-1\right)^b\right\}^t$$

Using binomial expansion;

$$\left(\alpha^{e^{-\frac{ak}{x}}}-1\right)^{b-1} = \sum_{h=0}^{b-1}(-1)^h\binom{b-1}{h}\left(\alpha^{e^{-\frac{ak}{x}}}\right)^{b-1-h}$$

$$\left(\alpha^{e^{-\frac{ak}{x}}}-1\right)^{b} = \sum_{g=0}^{b}(-1)^g\binom{b}{g}\left(\alpha^{e^{-\frac{ak}{x}}}\right)^{b-g} \tag{21}$$

$$f_{X_{(1)}}(x) = n\phi x^{-2}e^{-\frac{ak}{x}}\sum_{t=0}^{n-1}\sum_{h=0}^{b-1}(-1)^{t+h}\binom{n-1}{t}\binom{b-1}{h}\left(\alpha^{e^{-\frac{ak}{x}}}\right)^{b-h}\left\{\sum_{g=0}^{b}(-1)^g\binom{b}{g}(\alpha-1)^{-b}\left(\alpha^{e^{-\frac{ak}{x}}}\right)^{b-g}\right\}^{t} \tag{22}$$

3.4.2 | l^{th} maximum order statistics

$$f_{X_{(n)}}(x) = n\phi x^{-2}e^{-\frac{ak}{x}}\alpha^{e^{-\frac{ak}{x}}}\left(\alpha^{e^{-\frac{ak}{x}}}-1\right)^{b-1}\left((\alpha-1)^{-b}\left(\alpha^{e^{-\frac{ak}{x}}}-1\right)^{b}\right)^{n-1}$$

from equation (21);

$$f_{X_{(n)}}(x) = n\phi x^{-2}e^{-\frac{ak}{x}}\sum_{h=0}^{b-1}(-1)^h\binom{b-1}{h}\left(\alpha^{e^{-\frac{ak}{x}}}\right)^{b-h}\left\{\sum_{g=0}^{b}(-1)^g\binom{b}{g}(\alpha-1)^{-b}\left(\alpha^{e^{-\frac{ak}{x}}}\right)^{b-g}\right\}^{n-1} \tag{23}$$

where $\phi = abk(\alpha-1)^{-b}\log(\alpha)$

4 | PARAMETERS ESTIMATION

To find the MLEs of the GAPEIEx distribution parameters, we make use of the log-likelihood function of the distribution.

To find the MLEs of the GAPEIEx distribution use equation (10)

$$L = \ln L\left(\underline{x};\lambda\right) = \ln\left(\prod_{i=1}^{n} f(x_i;\lambda)\right) \quad ;\text{where} \quad \lambda \in (\alpha,a,b,k) > 0$$

$$= n\ln(a)+n\ln(b)+n\ln(k)-nb\ln(\alpha-1)+n\ln\left(\log(\alpha)\right)-2\sum_{i=1}^{n}\ln(x_i)-ak\sum_{i=1}^{n}x_i^{-1} \tag{24}$$

$$+\ln(\alpha)\sum_{i=1}^{n}e^{-\frac{ak}{x_i}}+(b-1)\ln\left(\alpha^{\sum_{i=1}^{n}e^{-\frac{ak}{x_i}}}-1\right)$$

The parameters $a, b, k, \&\ \alpha$, the MLEs $\widehat{a}, \widehat{b}, \widehat{k}, \&\ \widehat{\alpha}$ are values that maximize the log-likelihood function of (24). By finding the partial derivatives of the log-likelihood function of in equation (24) with respect to (wrt) each parameter (i.e $a, b, k, \&\ \alpha$) and equate to zero, it is given as;

From equation (24), Let

$$\Psi = (b-1)\ln\left(\alpha^{\sum_{i=1}^{n}e^{-\frac{ak}{x_i}}}-1\right)$$

Now let find the derivatives of Ψ wrt to each parameter.

$$\Psi'_a = \frac{\partial\Psi}{\partial a} = \frac{\partial}{\partial a}\left[(b-1)\ln\left(\alpha^{\sum_{i=1}^{n}e^{-\frac{ak}{x_i}}}-1\right)\right]$$

$$= \frac{b-1}{\alpha^{\sum_{i=1}^{n}e^{-\frac{ak}{x_i}}}-1}\left[-k\ln(\alpha)\sum_{i=1}^{n}\left(x_i^{-1}e^{-\frac{ak}{x_i}}\right)\alpha^{\sum_{i=1}^{n}e^{-\frac{ak}{x_i}}}\right]$$

$$\Psi_b' = \frac{\partial \Psi}{\partial b} = \frac{\partial}{\partial b}\left[(b-1)\ln\left(\alpha^{\sum_{i=1}^n e^{-\frac{ak}{x_i}}} - 1\right)\right]$$
$$= \ln\left(\alpha^{\sum_{i=1}^n e^{-\frac{ak}{x_i}}} - 1\right)$$

$$\Psi_k' = \frac{\partial \Psi}{\partial k} = \frac{\partial}{\partial k}\left[(b-1)\ln\left(\alpha^{\sum_{i=1}^n e^{-\frac{ak}{x_i}}} - 1\right)\right]$$
$$= \frac{b-1}{\alpha^{\sum_{i=1}^n e^{-\frac{ak}{x_i}}} - 1}\left[-a\ln(\alpha)\sum_{i=1}^n\left(x_i^{-1}e^{-\frac{ak}{x_i}}\right)\alpha^{\sum_{i=1}^n e^{-\frac{ak}{x_i}}}\right]$$

$$\Psi_\alpha' = \frac{\partial \Psi}{\partial \alpha} = \frac{\partial}{\partial \alpha}\left[(b-1)\ln\left(\alpha^{\sum_{i=1}^n e^{-\frac{ak}{x_i}}} - 1\right)\right]$$
$$= \frac{b-1}{\alpha^{\sum_{i=1}^n e^{-\frac{ak}{x_i}}} - 1}\left[\frac{1}{\alpha}\left(\sum_{i=1}^n e^{-\frac{ak}{x_i}}\right)\alpha^{\sum_{i=1}^n e^{-\frac{ak}{x_i}}}\right]$$

The partial derivatives of L wrt the parameters (i.e. α, a, b, k) and equating each partial derivative to zero are given as:

$$\frac{\partial L}{\partial a} = \frac{n}{a} - k\sum_{i=1}^n x_i^{-1} - k\ln(\alpha)\sum_{i=1}^n\left(x_i^{-1}e^{-\frac{ak}{x_i}}\right) + \Psi_a' = 0 \tag{25}$$

$$\frac{\partial L}{\partial b} = \frac{n}{b} - n\ln(\alpha-1) + \Psi_b' = 0 \tag{26}$$

$$\frac{\partial L}{\partial k} = \frac{n}{k} - a\sum_{i=1}^n x_i^{-1} - a\ln(\alpha)\sum_{i=1}^n\left(x_i^{-1}e^{-\frac{ak}{x_i}}\right) + \Psi_k' = 0 \tag{27}$$

$$\frac{\partial L}{\partial \alpha} = -\frac{nb}{\alpha-1} + \frac{n}{\alpha\log(\alpha)} + \frac{1}{\alpha}\sum_{i=1}^n e^{-\frac{ak}{x_i}} + \Psi_\alpha' = 0 \tag{28}$$

Since equations (25) to (28) are not closed-form solutions, a numerical optimization method is required to find their solutions. In this study, a 2^{nd} - order optimization algorithm called Broyden-Fletcher-Goldfarb-Shannon (BFGS) will be used to solve unconstrained nonlinear optimization problems.

This is always the case when solving real-valued optimization problems and an expectation when using many 2^{nd} - order methods.

The Hessian matrix of the GAPEIEx distribution is define as;

$$J^{-1}(\lambda) = \begin{Bmatrix} \frac{\partial^2 L}{\partial a^2} & \frac{\partial^2 L}{\partial a \partial b} & \frac{\partial^2 L}{\partial a \partial k} & \frac{\partial^2 L}{\partial a \partial \alpha} \\ & \frac{\partial^2 L}{\partial b^2} & \frac{\partial^2 L}{\partial b \partial k} & \frac{\partial^2 L}{\partial b \partial \alpha} \\ & & \frac{\partial^2 L}{\partial k^2} & \frac{\partial^2 L}{\partial k \partial \alpha} \\ & & & \frac{\partial^2 L}{\partial \alpha^2} \end{Bmatrix} = \begin{Bmatrix} Var(\hat{a}) & Cov(\hat{a},\hat{b}) & Cov(\hat{a},\hat{k}) & Cov(\hat{a},\hat{\alpha}) \\ & Var(\hat{b}) & Cov(\hat{b},\hat{k}) & Cov(\hat{b},\hat{\alpha}) \\ & & Var(\hat{k}) & Cov(\hat{k},\hat{\alpha}) \\ & & & Var(\hat{\alpha}) \end{Bmatrix}$$

where $\lambda = (a, b, k, \alpha)'$.

The expressions for terms in the Hessian matrix are available if need arises.

5 | SIMULATION STUDY

In this section, a Monte Carlo Simulation Study was conducted to check the behavior of the estimates and Root Mean Square Errors (RMSEs) of the MLEs for the parameters of the GAPEIEx distribution.

In order to carry out the simulation investigation, equation (15) is used to produce random samples of various lengths using various parameter values. The simulation study made **1500** replications for the following selected sample sizes: **n = 50, 100,**

150,..., 550.

The equation Average Bias (AB), and Root Mean Square Error (RMSE) are define as;

$$\text{AB} = \frac{1}{M} \sum_{i=1}^{M} (\hat{\varphi}_i - \varphi) \tag{29}$$

and

$$\text{RMSE} = \sqrt{\frac{1}{M} \sum_{i=1}^{M} (\hat{\varphi}_i - \varphi)^2} \tag{30}$$

where $\hat{\varphi}_i$ is the estimator of $\varphi(\alpha, a, b, k)$, $\varphi(\alpha, a, b, k)$ is the true value of the parameter being estimated and M is the number of iterations.

The numerical values of the simulation results for the estimations, ABs, and RMSEs are shown in Tables 3 & 4. The estimates approach the starting parameter values as the sample size increases, and the related ABs and RMSEs typically decrease as the sample size increases, as seen in Table 3. It is evident that variations in sample size have an impact on $alpha$'s sensitivity.

Similar to Table 3, it is clear in Table 4 that estimates get closer to the initial parameter values as sample size (n) increases, and the corresponding ABs and RMSEs for decreasing values of $\alpha, a, \&b$ and increasing k generally decrease as sample size (n) increases.

This provides a good illustration of how well the maximum likelihood method performs when predicting the parameters from Tables 3 & 4, respectively.

TABLE 3 The estimates, corresponding ABs, and RMSEs result.

Initial Parameters: $\alpha = 1.65, a = 0.02, b = 1.11, k = 0.01$								
n	**Estimates**				**ABs**			
	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{k}$	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{k}$
50	7.01845	0.02142	1.09765	0.01122	5.36845	0.00142	-0.01235	0.00122
100	3.31667	0.02023	1.16093	0.01039	1.66667	0.00023	0.05093	0.00039
150	2.47480	0.02003	1.14010	0.01076	0.82480	0.000029	0.03010	0.00076
200	2.10224	0.01997	1.11911	0.01005	0.45224	-0.000028	0.00911	0.000051
250	1.93604	0.01977	1.13753	0.00992	0.28604	-0.00023	0.02753	-0.000075
300	1.84843	0.01980	1.13343	0.00994	0.19843	-0.00020	0.02343	-0.000055
350	1.77313	0.01999	1.11884	0.01003	0.12313	-0.0000044	0.00884	0.000032
400	1.71418	0.01993	1.11550	0.00996	0.06418	-0.000068	0.00550	-0.000036
450	1.70715	0.01997	1.11161	0.00998	0.05715	-0.000031	0.00161	-0.000021
500	1.68005	0.02001	1.11156	0.00999	0.03005	0.0000078	0.00156	-0.0000085
550	1.67336	0.01998	1.11342	0.01000	0.02336	-0.000022	0.00342	-0.0000044

n	**RMSEs**			
	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{k}$
50	33.82066	0.00987	0.63368	0.00784
100	4.95962	0.00705	0.64737	0.00542
150	2.54031	0.00532	0.48980	0.02288
200	1.47891	0.00317	0.35092	0.00157
250	1.04402	0.00268	0.30870	0.00115
300	0.78806	0.00219	0.31941	0.00092
350	0.56682	0.00229	0.22597	0.00238
400	0.38292	0.00105	0.12526	0.00051
450	0.32580	0.00105	0.12268	0.00050
500	0.24293	0.00118	0.10024	0.00041
550	0.23289	0.00124	0.09324	0.00042

TABLE 4 The estimates, corresponding ABs, and RMSEs result.

Initial Parameters: $\alpha = 1.60, a = 0.01, b = 1.10, k = 0.02$								
n	Estimates				ABs			
	$\widehat{\alpha}$	$\widehat{a}$	$\widehat{b}$	$\widehat{k}$	$\widehat{\alpha}$	$\widehat{a}$	$\widehat{b}$	$\widehat{k}$
50	7.07877	0.01097	1.10594	0.02168	5.47877	0.00097	0.00594	0.00168
100	3.35248	0.01053	1.12547	0.02049	1.75248	0.00053	0.02547	0.00049
150	2.44775	0.01012	1.11968	0.02004	0.84775	0.00012	0.01968	0.000042
200	2.01743	0.01013	1.10845	0.01999	0.41743	0.00013	0.00845	-0.0000024
250	1.85122	0.00995	1.11859	0.01981	0.25122	-0.000051	0.01859	-0.00019
300	1.75575	0.01019	1.10940	0.01996	0.15575	0.00019	0.00940	-0.000042
350	1.71410	0.00998	1.10602	0.01992	0.11410	-0.000016	0.00602	-0.000080
400	1.65335	0.00999	1.10228	0.01994	0.05335	-0.0000052	0.00228	-0.000056
450	1.64086	0.00999	1.09953	0.01998	0.04086	-0.0000098	-0.00047	-0.000025
500	1.63361	0.00998	1.10430	0.01994	0.03361	-0.000018	0.00430	-0.000060
550	1.61574	0.01000	1.10070	0.02002	0.01573	0.00000085	0.00070	0.000017

n	RMSEs			
	$\widehat{\alpha}$	$\widehat{a}$	$\widehat{b}$	$\widehat{k}$
50	56.83963	0.00600	0.69483	0.02409
100	10.06289	0.00727	0.56179	0.00681
150	5.75429	0.00338	0.46019	0.00429
200	1.59434	0.00257	0.33198	0.00362
250	0.89836	0.00113	0.26692	0.00238
300	0.63495	0.00539	0.21434	0.00306
350	0.53876	0.00066	0.21079	0.00164
400	0.33757	0.00046	0.11150	0.00111
450	0.28209	0.00029	0.09249	0.00081
500	0.26682	0.00035	0.10071	0.00083
550	0.16464	0.00045	0.05996	0.00113

6 | DATA APPLICATION

In this section, the adaptability of the GAPEIEx distribution is accessed by comparing the distribution and its well-known submodels using real data sets. Such submodels are the IEx distribution by [18], the Exponentiated Inverse Exponential (EIEx) distribution, the Alpha Power Inverse Exponential (APIEx) distribution by [19], the Alpha Power Exponentiated Inverse Exponential (APEIEx) distribution (New distribution), and the Generalized Alpha Power Inverse Exponential (GAPIEx) distribution (New distribution).

6.1 | Data Set 1: Sierra Leone Covid-19 Daily Reported New Cases

Table 5, consist of Covid-19 daily reported new cases data set in Sierra Leone retrived from [20].

TABLE 5 Sierra Leone Covid-19 Daily Reported New Cases.

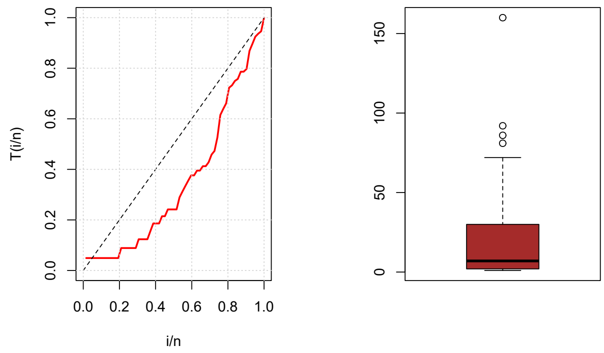
3	15	1	1	1	11	18	19	34	5	15	48	86	160	72	92
81	50	65	41	48	40	44	30	23	43	6	7	32	7	13	16
13	2	14	4	5	5	12	6	3	14	1	3	1	3	7	1
2	2	1	9	1	10	2	2	7	2	1	1	1	1		

Table 6, consist of the summary statistics of Sierra Leone Covid-19 daily reported new cases. The value of skewness is positive (i.e. data set 1 is skewed to the right), the value of the kurtosis is > 3 (i.e. data set 1 is Leptokurtic), and data set 1 is unimodal (i.e. one mode).

TABLE 6 Summary Statistics of Sierra Leone Covid-19 Data.

N	Max.	Min.	Mean	Median	Mode	SD	CV	Skewness	Kurtosis
62	160	1	20.37	7	1	29.33	1.44	2.52	7.98

For data set 1, the TTT-transform plot displays a convex below the 45^0 line and four outliers in the boxplot as illustrated in Figure 5.



(a) TTT-transform (b) Boxplot

FIGURE 5 The TTT-transform plot and the Boxplot of Data Set 1.

The MLEs for the model parameters and related standard errors, which are displayed in brackets, are found in Table 7. The bulk of the fitted distributions were significant at the 5% level because the standard error test reveals that the values of the parameter standard errors are $<\frac{1}{2}$ the parameter value.

TABLE 7 The MLEs and the Standard Error (in parentheses) for Data Set 1.

Distribution	$\hat{\alpha}$	$\hat{a}$	$\hat{b}$	$\hat{k}$
GAPEIEx	9.8665(6.2659)	1.7803(0.8715)	0.3173(0.1121)	4.0090(1.9625)
GAPIEx	4.2149(2.5693)	–	1.0625(0.4568)	1.9704(0.9098)
APEIEx	5.8921(5.2330)	4.5930(6.9039)	–	0.4598(0.6911)
APIEx	5.9335(5.0808)	–	–	2.0903(0.5887)
EIEx	–	10.5152(87.8503)	–	0.3033(2.5338)
IEx	–	–	–	3.1894(0.4051)

Table 8, displays the goodness-of-fit and $p-value$ results of data set 1 for the GAPEIEx distribution, its submodels, and some well-known distributions. It's clear that the GAPEIEx distribution has the lowest $K-S$, CVM, A^*, and the largest $p-value$. This tells us that the GAPEIEx distribution matched data set 1 better than its submodels, but some of its submodels also fitted the data.

TABLE 8 The Goodness-of-fit and the $p-values$ results for Data Set 1.

Distribution	$K-S$	CVM	A^*	$p-value$
GAPEIEx	**0.1260**	**0.1514**	**1.2922**	**0.2789**
GAPIEx	0.1663	0.1909	1.5164	0.0649
APEIEx	0.1446	0.1859	1.4817	0.1495
APIEx	0.1436	0.1858	1.4811	0.1548
EIEx	0.2021	0.2324	1.7918	0.0126<5%
IEx	0.2021	0.2324	1.7918	0.0126<5%

Table 9, consists of the negative log-likelihood and the information criteria results of data set 1 for the GAPEIEx distribution, its submodels, and some well-known distributions. It's clear that the GAPEIEx distribution has the highest $-L$, and the lowest AIC, $AICc$, & $HQIC$. This tells us that the GAPEIEx distribution is a better model for modeling data set 1 than its submodels.

TABLE 9 The $-L$ and the Information Criteria results for Data Set 1.

Distribution	$-L$	AIC	$AICc$	$HQIC$
GAPEIEx	**237.7817**	**483.5633**	**484.2651**	**486.9040**
GAPIEx	241.4361	488.8721	489.2859	491.3776
APEIEx	241.0246	488.0493	488.4631	490.5548
APIEx	241.0280	486.0560	486.2594	487.7263
EIEx	244.0435	492.0869	492.2903	493.7573
IEx	244.0435	490.0869	490.1536	490.9221

From Figure 6, it's clear that the GAPEIEx distribution provides a better fit to the COVID-19 daily reported new cases data set in Sierra Leone (i.e., data set 1) than its submodels.

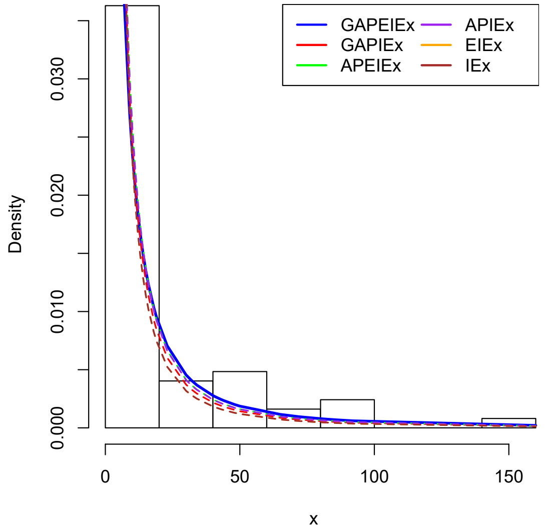


FIGURE 6 The Fitted densities plot of Data Set 1.

6.2 | Data Set 2: United Kingdom Covid-19 Mortality Rates Data

Table 10, consist of Covid-19 mortality rates data set in the United Kingdom used by[21].

TABLE 10 United Kingdom Covid-19 Mortality Rates Data.

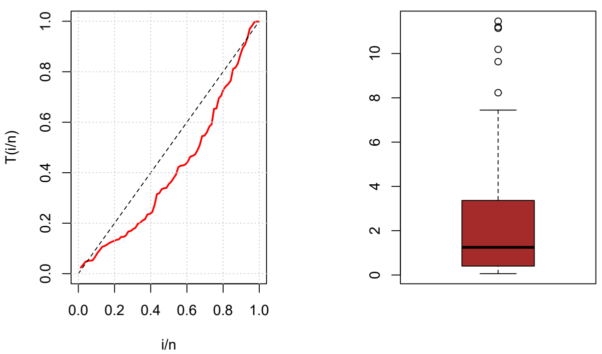
0.0587	0.0863	0.1165	0.1247	0.1277	0.1303	0.1652	0.2079
0.2395	0.2845	0.2992	0.3188	0.3317	0.3446	0.3553	0.3622
0.3926	0.3926	0.4633	0.4690	0.4954	0.5139	0.5696	0.5837
0.6197	0.6365	0.7096	0.7444	0.8590	1.0438	1.0602	1.1305
1.1468	1.1533	1.2260	1.2707	1.4149	1.5709	1.6017	1.6083
1.6324	1.6998	1.8164	1.8392	1.8721	2.1360	2.3987	2.4153
2.5225	2.7087	2.7946	3.3609	3.3715	3.7840	4.1969	4.3451
4.4627	4.6477	5.3664	5.4500	5.7522	6.4241	7.0657	8.2307
9.6315	10.1870	11.1429	11.2019	11.4584	0.2751	0.4110	0.7193
1.3423	1.9844	3.9042	7.4456				

Table 11, comprises of the summary statistics of UK COVID-19 mortality rates data. The value of skewness is positive (i.e., data set 2 is right-skewed), the value of the kurtosis is <3 (i.e., data set 2 is Platykurtic), and data set 2 is unimodal (i.e., one mode).

TABLE 11 Summary Statistics of UK Covid-19 Mortality Rates Data.

N	Max.	Min.	Mean	Median	Mode	SD	CV	Skewness	Kurtosis
76	11.46	0.06	2.44	1.25	0.39	2.94	1.21	1.74	2.32

The failure rate of data set 2 is bathtub shape since the TTT-transform plot is first convex below the 45^0 line and then concave above the 45^0 line and have few outliers n the boxplot as shown in Figure 7.



(a) TTT-transform (b) Boxplot

FIGURE 7 The TTT-transform plot and the Boxplot of Data Set 2.

Table 12, displays the MLEs for the model parameters and associated standard errors (given in brackets). At 5% level of significance, the standard error test shows that the majority of the fitted distribution values are less than half ($< \frac{1}{2}$) the parameter value.

TABLE 12 The MLEs and the Standard Error (in parentheses) for Data Set 2.

Distribution	$\widehat{\alpha}$	$\widehat{a}$	$\widehat{b}$	$\widehat{k}$
GAPEIEx	12.4305(11.2936)	0.1340(0.0643)	0.5453(0.6890)	4.4385(2.1099)
GAPIEx	3.6376(1.7413)	-	1.0218(0.4568)	0.3614(0.1733)
APEIEx	1.3176(0.4723)	28.6423(16.2083)	-	0.0166(0.0094)
APIEx	7.0379(4.0201)	-	-	0.3112(0.0625)
EIEx	-	0.0882(0.4514)	-	5.8455(29.9281)
IEx	-	-	-	0.5154(0.0591)

The goodness-of-fit and p-value results for data set 2 for the GAPEIEx distribution, its submodels, and other well-known distributions (i.e., the EIEx and IEx distributions) are shown in Table 13. The GAPEIEx distribution clearly has the highest p-value and the lowest $K-S$, CVM, and A^* values. This suggests that the GAPEIEx distribution fits data set 2 better than its submodels, despite the fact that some of them also do, with the exception of the APEIEx, EIEx, and IEx distributions.

TABLE 13 The Goodness-of-fit and the $p-values$ results for Data Set 2.

Distribution	$K-S$	CVM	A^*	$p-value$
GAPEIEx	**0.0749**	**0.0917**	**0.6458**	**0.7868**
GAPIEx	0.1379	0.1361	0.9199	0.1110
APEIEx	0.1799	0.1803	1.2102	0.0146 <5%
APIEx	0.1181	0.1215	0.8150	0.2393
EIEx	0.1893	0.1976	1.3183	0.0086 <5%
IEx	0.1893	0.1976	1.3183	0.0086 <5%

Table 14, displays the negative log-likelihood and the information criteria results of data set 2 for the GAPEIEx distribution, its submodels, and some well-known distributions. It's clear that the GAPEIEx distribution is the best model for modeling data set 2 than its submodels since it has the highest $-L$ and the lowest AIC, $AICc$, & $HQIC$.

TABLE 14 The $-L$ and the Information Criteria results for Data Set 2.

Distribution	$-L$	AIC	$AICc$	$HQIC$
GAPEIEx	**140.0126**	**288.0252**	**288.7886**	**291.9511**
GAPIEx	144.8443	295.6886	296.0220	298.4830
APEIEx	148.2166	302.4331	302.7665	305.2276
APIEx	143.7749	291.5498	291.7141	293.4127
EIEx	149.6024	303.2048	303.3692	305.0678
IEx	149.6024	301.2048	301.2589	302.1363

From Figure 8, it's clear that the GAPEIEx distribution provides a better fit to the COVID-19 mortality rates data set in the United Kingdom (i.e., data set 2) than its submodels.

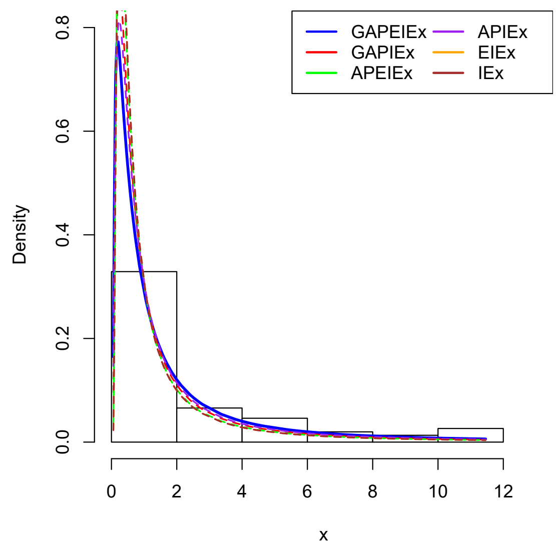


FIGURE 8 The Fitted densities plot of Data Set 2.

6.3 | Concluding Remarks in Relation to Data Application

1. In relation to the data sets in Tables 5 & 10, we can see that the GAPEIEx distribution provides the highest p-values, lowest Goodness-of-fit, Negative log-likelihood, and Information Criteria compared to its submodels and some well-known distributions.

2. When data set 1 is fitted into Figure 6 and data set 2 is fitted into Figure 8, the GAPEIEx distribution seems to outperform its submodels.

3. The EIEx and the IEx distributions fit data set 1 poorly, as shown in Table 8, because their p-values are <5%.

4. From Table 13, the APEIEx, the EIEx, and the IEx distribution give a poor fitting as their p-values are <5% for data set 2..

5. Last but not least, it is clear from the data application that the GAPEIEx distribution outperformed its submodels.

7 | CONCLUSIONS

The Inverse Exponential distribution has no shape parameter, and it's exponentially bounded for not modeling data sets that are characterized by either a bathtub or an inverted bathtub failure rate shape.
This study introduced the Generalized Alpha Power Exponentiated Inverse Exponential distribution, a four-parameter distribution with three shape parameters and one scale parameter. Several Statistical properties of the distribution (Survival, Hazard, Quantile function, r^{th} moment, Rényi Entropy, and other statistics) are derived.
The estimates, the Average Bias, and the Root Mean Square Error of the parameters of the GAPEIEx distribution were computed, and their performance was assessed through simulations, which clearly shows that the maximum likelihood method works well in estimating the parameters (i.e., as the sample size (n) goes higher, the estimates go closer to their initial values).
The adaptability of the GAPEIEx distribution is assessed by comparing it to its submodels using two real data sets. At the end of the application, we found out that the GAPEIEx distribution outperformed its submodels and other well-known distributions. For further study, the scholar(s) may develop a more adaptable distribution by adding a shape parameter or using the Transmuted method to the GAPEIEx distribution.

AUTHOR CONTRIBUTION

Conceptualization: Moses Kargbo
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Conflicts of Interest

The writers guarantee that they have no conflicts of interest requiring disclosure.

Data availability

The real data sets used for application of this proposed distribution are within the manuscript.

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