Square integrable surface potentials on non-smooth domains and application to the Laplace equation in L^2

Alexandre Munnier
alexandre.munnier@univ-lorraine.fr

Université de Lorraine, CNRS, Inria, IECL, F-54000 Nancy, France

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Abstract

Motivated by applications in fluid dynamics involving the harmonic Bergman projection, we aim to extend the theory of single and double layer potentials (well documented for functions with $H^1_{\ell oc}$ regularity) to locally square integrable functions. Having in mind numerical simulations for which functions are usually defined on a polygonal mesh, we wish this theory to cover the cases of non-smooth domains (i.e.with Lipschitz continuous or polygonal boundaries).

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1 Introduction

Let Ω be a smooth bounded domain in the plan. The harmonic Bergman projection is the orthogonal projection in $L^2(\Omega)$ onto the closed subspace of harmonic functions (see [5, Chap. 8] and [18]). This operator is known mainly for playing an important role in complex analysis and operator theory but has also applications in the field of partial differential equations ([12], [13, Chap. 4]). In fluid dynamics, it appears in the article [17] and more recently in [14] for the analysis of the Navier-Stokes equations in non-primitive variables (stream function and vorticity). Indeed, for an incompressible fluid flow, the vorticity field is orthogonal in L^2 to the harmonic functions (see [14] and references therein).

The Bergman projection is a kernel operator but this kernel can be explicitly computed only for particular geometries (when Ω is a disk or a half plane for instance). From a numerical point of view, the discretization of the Bergman projection requires the inversion of the mass matrix corresponding to the L^2 scalar product restricted to the subspace of harmonic functions. For this purpose, a discrete basis of harmonic functions in L^2 is needed and an efficient way to construct such a basis consists in using boundary elements and layer potentials. However, while the theory of layer potentials in $H^1_{\ell oc}(\mathbb{R}^2)$ is well documented (see the classical book [4] for instance), little is known on locally square integrable layer potentials. In this paper, we aim to provide a theoretical framework for this notion. Furthermore, in numerical simulations, functions are usually defined on a polygonal mesh, so we want to cover this case, which adds a substantial difficulty.

In its classical meaning, the single layer potential maps the Sobolev space $H^{-1/2}(\Gamma)$ into $H^1_{\ell oc}(\mathbb{R}^2)$ (Γ stands here for a Lipschitz continuous Jordan curve). A natural guess is that the $H^1_{\ell oc}$ regularity could be lowered to $L^2_{\ell oc}$ by extending the single layer potential to the space $H^{-3/2}(\Gamma)$. However the space $H^{3/2}(\Gamma)$, and then also its dual space $H^{-3/2}(\Gamma)$ are ill defined on a Lipschitz continuous boundary, any intrinsic definition of these spaces requiring that the boundary be at least of class $\mathcal{C}^{1,1}$. On the other hand, denoting by γ_d the classical Dirichlet trace operator on Γ, the space $\mathcal{H}^{3/2}(\Gamma) = \gamma_d H^2_{\ell oc}(\mathbb{R}^2)$, although complex to describe in terms of Sobolev regularity, is well defined (and coincides with $H^{3/2}(\Gamma)$ when Γ is smooth). The main idea of the paper is to define the single-layer potentials as Laplacians of biharmonic functions in $\mathbb{R}^2 \setminus \Gamma$, the asymptotic behavior of the functions being taken into account by introducing an appropriate functional framework based on weighted Sobolev spaces. This approach will prove successful and will allow to extend the single layer potential to the space $\mathcal{H}^{-3/2}(\Gamma)$.

Considering the double layer potential, based on similar arguments, it will be extended to $\mathcal{H}^{-1/2}(\Gamma)$, the dual space of $\mathcal{H}^{1/2}(\Gamma) = \gamma_n H^2_{\ell oc}(\mathbb{R}^2)$, where γ_n stands for the Neumann trace operator on Γ. It is worth noticing that $\mathcal{H}^{1/2}(\Gamma)$ is equal to $H^{1/2}(\Gamma)$ when Γ is smooth but this is no longer true as soon as Γ has corners for instance.

Throughout the paper, we will assume without loss of generality that the logarithmic capacity of Γ is lower than 1, using translation and dilatation of the coordinates system if necessary (see [16, Page 263] on this matter). Roughly speaking, we shall prove the following result (that will be rigorously reformulated in Theorem 4.1 thereafter):

Theorem 1.1. *Let Γ be a Lipschitz Jordan curve. Then the single layer potential, considered as an operator defined on $H^{-1/2}(\Gamma)$ valued in $L^2_{\ell oc}(\mathbb{R}^2)$ extends by density to a bounded operator on $\mathcal{H}^{-3/2}(\Gamma)$. The double layer potential, seen as an operator from $H^{1/2}(\Gamma)$ into $L^2_{\ell oc}(\mathbb{R}^2)$ extends by density to a bounded operator on $\mathcal{H}^{-1/2}(\Gamma)$.*

Denote by Ω^- the planar open set enclosed by Γ and by Ω^+ its complement in $\mathbb{R}^2$. Providing that Γ is a polygon, we will be able to reach our initial goal (to represent harmonic functions in $L^2_{\ell oc}$ by surface potentials) by proving (this result is rigorously reformulated later in Corollary 8.1 and Corollary 8.2):

Theorem 1.2. *Any harmonic function in $L^2(\Omega^-)$ can be represented by the restriction to Ω^- of a single or a double layer potential as defined in Theorem 1.1. The same conclusion applies for harmonic functions in $L^2_{\ell oc}(\overline{\Omega^+})$, assuming additional properties on their asymptotic behaviors.*

The remainder of the introduction is devoted to giving the reader an overview of the main steps of the analysis. As with Theorems 1.1 and 1.2, we do not seek to be rigorous at this stage but simply to give a taste of the results. For the sake of brevity, we will focus only on the single layer potential.

The first step of the analysis is to extend the notions of Dirichlet and Neumann traces to functions in $L^2_{\ell oc}(\mathbb{R}^2)$, harmonic in $\mathbb{R}^2 \setminus \Gamma$. This task will be carried out in the case where Γ is a curvilinear $\mathcal{C}^{1,1}$ polygon (i.e. a generalization of the notion of polygon for which the edges are $\mathcal{C}^{1,1}$ curves) and requires the introduction of the spaces:

$$\mathcal{H}^{3/2}_n(\Gamma) = \{\gamma_d u \,:\, u \in H^2_{\ell oc}(\mathbb{R}^2),\, \gamma_n u = 0\} \qquad \text{and} \qquad \mathcal{H}^{1/2}_d(\Gamma) = \{\gamma_n u \,:\, u \in H^2_{\ell oc}(\mathbb{R}^2),\, \gamma_d u = 0\}.$$

When Γ is smooth, we simply have $\mathcal{H}^{3/2}_n(\Gamma) = \mathcal{H}^{3/2}(\Gamma) = H^{3/2}(\Gamma)$ and $\mathcal{H}^{1/2}_d(\Gamma) = \mathcal{H}^{1/2}(\Gamma) = H^{1/2}(\Gamma)$. However, all these equalities turn out to be false when Γ is a $\mathcal{C}^{1,1}$ polygon (this is what makes the analysis tricky). The topologies of which these spaces are provided (and which will be specified thereafter) entail the continuity and the density of the following inclusions:

$$\mathcal{H}^{1/2}_d(\Gamma) \subset \mathcal{H}^{1/2}(\Gamma) \subset L^2(\Gamma) \qquad \text{and} \qquad \mathcal{H}^{3/2}_n(\Gamma) \subset \mathcal{H}^{3/2}(\Gamma) \subset H^{1/2}(\Gamma) \subset L^2(\Gamma).$$

As usual, we denote by $\mathcal{H}^{-3/2}_n(\Gamma)$ the dual space of $\mathcal{H}^{3/2}_n(\Gamma)$ and by $\mathcal{H}^{-1/2}_d(\Gamma)$ the dual space of $\mathcal{H}^{1/2}_d(\Gamma)$, using $L^2(\Gamma)$ as pivot space. More interesting for our purpose, the inclusions between dual spaces are also continuous and dense:

$$L^2(\Gamma) \subset \mathcal{H}^{-1/2}(\Gamma) \subset \mathcal{H}^{-1/2}_d(\Gamma) \qquad \text{and} \qquad L^2(\Gamma) \subset H^{-1/2}(\Gamma) \subset \mathcal{H}^{-3/2}(\Gamma) \subset \mathcal{H}^{-3/2}_n(\Gamma).$$

Theorem 1.3. *Any function u in $L^2_{\ell oc}(\mathbb{R}^2)$, harmonic on both sides of Γ admits one-sided Dirichlet traces (denoted by $\gamma_d^- u$ and $\gamma_d^+ u$) in $\mathcal{H}^{-1/2}_d(\Gamma)$. The function u admits also one-sided Neumann traces (denoted by $\gamma_n^- u$ and $\gamma_n^+ u$) in the space $\mathcal{H}^{-3/2}_n(\Gamma)$. Moreover, the trace operators $\gamma_d^\pm$ and $\gamma_n^\pm$ are the extensions by density of the classical trace operators defined for functions in $H^1(\Omega^-)$ and in $H^1_{\ell oc}(\overline{\Omega^+})$.*

This notion of trace being clarified, we will turn again to the layer potentials and investigate the question of their Dirichlet and Neumann traces on Γ. Let $\mathscr{S}_\Gamma : H^{-1/2}(\Gamma) \longrightarrow H^1_{\ell oc}(\mathbb{R}^2)$ be the classical single layer potential and recall the properties:

$$\gamma_n^\pm \circ \mathscr{S}_\Gamma : H^{-1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma) \qquad \text{and} \qquad \gamma_n^+ \circ \mathscr{S}_\Gamma + \gamma_n^- \circ \mathscr{S}_\Gamma = \mathrm{Id}, \tag{1a}$$

this latter identity being usually called the "jump relation". Let now $\mathscr{S}_\Gamma^\dagger : \mathcal{H}^{-1/2}(\Gamma) \longrightarrow L^2_{\ell oc}(\mathbb{R}^2)$ stands for the extended single layer potential defined in Theorem 1.1. According to Theorem 1.3 we have in this case:

$$\gamma_n^\pm \circ \mathscr{S}_\Gamma^\dagger : \mathcal{H}^{-3/2}(\Gamma) \longrightarrow \mathcal{H}_n^{-3/2}(\Gamma), \tag{1b}$$

where $\mathcal{H}^{-3/2}(\Gamma)$ is continuously and densely embedded in $\mathcal{H}_n^{-3/2}(\Gamma)$ but in general different from $\mathcal{H}_n^{-3/2}(\Gamma)$, which suggests that the jump relation is not likely to apply in this case. Surprisingly enough, the relation is well and truly satisfied. More generally, concerning the traces of the single layer potential, we will establish:

Theorem 1.4. *The two one-sided Dirichlet traces on Γ of a single layer potential (as defined in Theorem 1.1) coincide. The "jump" across Γ of the one-sided Neumann traces of a single layer potential of density $q \in \mathcal{H}^{-3/2}(\Gamma)$ is equal to q.*

Actually, we will show that there do exist single layer potentials for which the one-sided Neumann traces are both in $\mathcal{H}_n^{-3/2}(\Gamma)$ but not in $\mathcal{H}^{-3/2}(\Gamma)$, although their difference is in this latter space. This means that some singular contributions of the normal derivatives cancel out by forming their difference. This notable phenomenon seems to be typical of the single layer potential on non-smooth boundaries.

The next point we shall discuss in the paper is the solvability of the Dirichlet and Neumann Laplace equations with boundary data in $\mathcal{H}_d^{-1/2}(\Gamma)$ and $\mathcal{H}_n^{-3/2}(\Gamma)$. As for the Laplace equation with Dirichlet boundary conditions for example, we will prove:

Theorem 1.5. *Assume that Γ is a (straight) polygon. For every $p \in \mathcal{H}_d^{-1/2}(\Gamma)$ there exists a function $u^- \in L^2(\Omega^-)$ harmonic in Ω^- such that $\gamma_d^- u^- = p$ and there exists a function $u^+ \in L^2_{\ell oc}(\overline{\Omega^+})$ (with a suitable asymptotic behavior), harmonic in Ω^+ such that $\gamma_d^+ u^+ = p$. There is no uniqueness in general.*

The existence of (non zero) harmonic functions in L^2 with vanishing Dirichlet data in a domain with corners has long been known (see for instance [7] where an example of such a function is provided).

At this point, a kind of reciprocal of Theorem 1.4 will still be needed to prove Theorem 1.2. This result can be stated as follows:

Theorem 1.6. *Let u be in $L^2_{\ell oc}(\mathbb{R}^2)$, harmonic in $\mathbb{R}^2 \setminus \Gamma$, with an appropriate asymptotic behavior. If u satisfies $\gamma_d^- u = \gamma_d^+ u$ then $q = \gamma_n^- u + \gamma_n^+ u$ is in $\mathcal{H}^{-3/2}(\Gamma)$ and $u = \mathscr{S}_\Gamma^\dagger q$.*

The proof of Theorem 1.2 now relies on a proper combination of Theorems 1.3, 1.4 and 1.5. Thus, denote by p the one-sided trace $\gamma_d^- u^-$ of a given function u^-, harmonic in $L^2(\Omega^-)$. Theorem 1.5 ensures the existence of a harmonic function u^+ in $L^2_{\ell oc}(\overline{\Omega^+})$ (with an appropriate asymptotic behavior) such that $\gamma_d^+ u^+ = p$. Define $q = \gamma_n^- u^- + \gamma_n^+ u^+$ (the jump of the Neumann trace) and conclude, applying Theorem 1.6 that $\mathscr{S}_\Gamma^\dagger q|_{\Omega^\pm} = u^\pm$.

We shall also provide a negative result, contrasting with what happens for harmonic functions u such that $u|_{\Omega^-} \in H^1(\Omega^-)$ and $u|_{\Omega^+} \in H^1_{\ell oc}(\overline{\Omega^+})$. Indeed, such a function harmonic in $\mathbb{R}^2 \setminus \Gamma$ (and with a suitable asymptotic behavior) can be represented as the sum of a single and double layer potentials. On the contrary:

Theorem 1.7. *There exist functions in $L^2_{\ell oc}(\mathbb{R}^2)$, harmonic in $\mathbb{R}^2 \setminus \Gamma$ (with a suitable asymptotic behavior) that cannot be represented as the sum of a single and a double layer potentials.*

We will end the article by studying the invertibility of the boundary operators introduced in (1). Recall that the logarithmic capacity of Γ is assumed to be lower than 1 and define:

$$\widetilde{\mathcal{H}}_n^{-3/2}(\Gamma) = \big\{ q \in \mathcal{H}_n^{-3/2}(\Gamma) \,:\, \langle\!\langle q, \mathbf{1}_\Gamma \rangle\!\rangle_{-\frac{3}{2},\frac{3}{2},n} = 0 \big\},$$

where $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{-\frac{3}{2},\frac{3}{2},n}$ stands for the duality pairing on $\mathcal{H}_n^{-3/2}(\Gamma) \times \mathcal{H}_n^{3/2}(\Gamma)$ that extends the L^2 inner product.

Theorem 1.8. *The bounded operators $\gamma_d \circ \mathscr{S}_\Gamma^\dagger : \mathcal{H}^{-3/2}(\Gamma) \longrightarrow \mathcal{H}_d^{-1/2}(\Gamma)$, $\gamma_n^- \circ \mathscr{S}_\Gamma^\dagger : \mathcal{H}^{-3/2}(\Gamma) \longrightarrow \widetilde{\mathcal{H}}_n^{-3/2}(\Gamma)$ and $\gamma_n^+ \circ \mathscr{S}_\Gamma^\dagger : \mathcal{H}^{-3/2}(\Gamma) \longrightarrow \mathcal{H}_n^{-3/2}(\Gamma)$ are surjective but not injective in general.*

The paper is organized as follows: The following section is dedicated to the reminder of some basic notions about trace operators and surface potentials. The main function spaces on which the analysis is based when Γ is Lipschitz continuous, are introduced in Section 3. They are used in Section 4 to extend the notion of surface potential to square integrable functions. From Section 5 the boundary Γ is assumed to be a $\mathcal{C}^{1,1}$ (curvilenar) polygon. This additional regularity allows the introduction of new function spaces involved in new trace theorems stated in the next section. The “jump relations” for surface potentials are proved in Section 6. From Section 7, the analysis focuses on the case where Γ is a straight polygon. Section 7 is dedicated to solvability issues for the Laplace equation with Dirichlet and Neumann boundary data. Finally, in section 8, we discuss some transmission problems and address the issue of representing locally square-integrable harmonic functions as surface potentials. We end the paper with the proof of Theorem 1.8.

For the ease of the reader, the appendix contains a list of the main function spaces and operators.

2 Notations and recalls

Geometric settings

Let Ω^- be an open and bounded planar domain whose boundary Γ is a Jordan curve. The (unbounded) complement of $\overline{\Omega^-}$ is denoted by Ω^+. In the sequel, we shall consider four levels of regularity for Γ: It will be either of class $\mathcal{C}^{1,1}$ (referred to as the smooth case), either Lipschitz continuous (see [10, Definition 1.2.1.1] for a precise definition of this notion), either a $\mathcal{C}^{1,1}$ polygon (see [10, Definition 1.4.5.1]), or simply a classical (straight) polygon. In either case, the unit tangent vector field τ (oriented counterclockwise) is a.e. well defined on Γ and the same applies to the outer unit normal vector field $n^- = -\tau^\perp$ and to the inner normal vector field $n^+ = \tau^\perp$ (the superscript $\perp$ meaning that the vector is counterclockwise rotated of an angle $\pi/2$). To lighten the notations, we shall sometimes write simply n instead of n^-.

Traces on the boundary of a Lipschitz domain

In this subsection, we collect some definitions and properties about the Dirichlet and Neumann trace operators in the case where Γ is Lipschitz continuous. On the space

$$\mathscr{D}_{\Omega^\pm}(\mathbb{R}^2) = \big\{u|_{\Omega^\pm} \,:\, u \in \mathscr{D}(\mathbb{R}^2)\big\},$$

the one-sided Dirichlet and Neumann trace operators are classically defined by:

$$\begin{aligned} \gamma_d^\pm : \mathscr{D}_{\Omega^\pm}(\mathbb{R}^2) &\longrightarrow L^2(\Gamma) \\ u &\longmapsto u|_\Gamma \end{aligned} \qquad \text{and} \qquad \begin{aligned} \gamma_n^\pm : \mathscr{D}_{\Omega^\pm}(\mathbb{R}^2) &\longrightarrow L^2(\Gamma) \\ u &\longmapsto \nabla u\cdot n^\pm|_\Gamma. \end{aligned} \tag{2}$$

According to [1, §9.2], when Γ is Lipschitz continuous, the sobolev space $H^s(\Gamma)$ is well (invariantly) defined only for those indices s that belong to $[-1,1]$ and rephrasing [1, Theorem 9.2.1] (or [16, Theorem 3.38]), we have:

Theorem 2.1. *The one-sided Dirichlet trace operators $\gamma_d^\pm$ extend by density to bounded operators from $H^{s+1/2}(\Omega^\pm)$ to $H^s(\Gamma)$ for every $0 < s < 1$.*

According to [15, Theorem 1] we can also state:

Theorem 2.2. *The Dirichlet and Neumann trace operators* (2) *extend by density to bounded operators on $H^2(\Omega^\pm)$ (valued in $L^2(\Gamma)$) and $\ker\gamma_d^\pm \cap \ker\gamma_n^\pm = H^2_0(\Omega^\pm)$, where we recall that $H^2_0(\Omega^\pm)$ is the closure of $\mathscr{D}(\Omega^\pm)$ in $H^2(\Omega^\pm)$.*

The Neumann trace operator can actually be defined on a larger space than $H^2(\Omega^\pm)$, namely on:

$$H^1(\Omega^\pm,\Delta) = \big\{u \in H^1(\Omega^\pm) \,:\, \Delta u \in L^2(\Omega^\pm)\big\}.$$

Thus, according to [16, Lemma 4.3], for every $u \in H^1(\Omega^\pm,\Delta)$, there exists a unique $g_u \in H^{-1/2}(\Gamma)$ such that:

$$\big\langle g_u, \gamma_d^\pm v\big\rangle_{-\frac12,\frac12} = (\Delta u, v)_{L^2(\Omega^\pm)} - (\nabla u,\nabla v)_{L^2(\Omega^\pm;\mathbb{R}^2)} \qquad \text{for all } v \in H^1(\Omega^\pm),$$

where $\langle\cdot,\cdot\rangle_{-\frac12,\frac12}$ stands for the duality bracket between the spaces $H^{-1/2}(\Gamma)$ and $H^{1/2}(\Gamma)$, that extends the L^2 inner product. Since $\mathscr{D}_{\Omega^\pm}(\mathbb{R}^2)$ is dense in $H^1(\Omega^\pm,\Delta)$ (see [10, Lemma 1.5.3.9]), we are allowed to denote $g_u = \gamma_n^\pm u$ and we have (see [16, Theorem 4.4] for the Green's identity):

Proposition 2.1. *The Neumann trace operators $\gamma_n^\pm$ defined in* (2) *extend by density to bounded operators from $H^1(\Omega^\pm, \Delta)$ into $H^{-1/2}(\Gamma)$. Moreover, the second Green's identity holds:*

$$(\Delta u, v)_{L^2(\Omega^\pm)} - (u, \Delta v)_{L^2(\Omega^\pm)} = \big\langle \gamma_n^\pm u, \gamma_0^\pm v \big\rangle_{-\frac{1}{2},\frac{1}{2}} - \big\langle \gamma_n^\pm v, \gamma_0^\pm u \big\rangle_{-\frac{1}{2},\frac{1}{2}} \qquad \text{for all } u, v \in H^1(\Omega^\pm, \Delta). \tag{3}$$

The space $H^{3/2}(\Omega^\pm, \Delta) = \big\{ u \in H^{3/2}(\Omega^\pm) \,:\, \Delta u \in L^2(\Omega^\pm) \big\}$ (provided with the graph norm) is a subspace of $H^1(\Omega^\pm, \Delta)$ and according to [9, Lemma 3.2]:

Proposition 2.2. *The operators $\gamma_n^\pm : H^{3/2}(\Omega^\pm, \Delta) \longrightarrow L^2(\Gamma)$ are bounded and onto.*

Finally, the following density result will be useful in the sequel:

Proposition 2.3. *The spaces $\gamma_d \mathscr{D}_{\Omega^\pm}(\mathbb{R}^2)$ and $\gamma_n \mathscr{D}_{\Omega^\pm}(\mathbb{R}^2)$ are dense in $L^2(\Gamma)$.*

The first assertion is proved in [9, page 88] and the second results from Proposition 2.2 and [6, Lemma 3].

Surface potentials on a Lipschitz boundary

A general presentation of the theory of surface potentials on the boundary of a Lipschitz domain can be found in the book [16], to which we will refer in the following for more details on this subject. For the ease of the reader, let us recall some basics: The fundamental solution of the Laplace's equation is defined by:

$$G(x) = -\frac{1}{2\pi} \ln |x| \qquad \text{for all } x \in \mathbb{R}^2 \setminus \{0\}.$$

The single layer potential is the weakly singular integral operator defined for any $q \in L^2(\Gamma)$ by:

$$\mathscr{S}_\Gamma q(x) = \int_\Gamma G(x-y) q(y) \,\mathrm{d}y \qquad \text{for all } x \in \mathbb{R}^2 \setminus \Gamma,$$

and extended by density to a bounded operator $\mathscr{S}_\Gamma : H^{-1/2}(\Gamma) \longrightarrow H^1_{\ell oc}(\mathbb{R}^2)$. The double layer potential is the singular integral operator: $\mathscr{D}_\Gamma : H^{1/2}(\Gamma) \longrightarrow H^1_{\ell oc}(\mathbb{R}^2)$, defined by:

$$\mathscr{D}_\Gamma p(x) = \int_\Gamma \nabla G(x-y) \cdot n(y) q(y) \,\mathrm{d}y \qquad \text{for all } x \in \mathbb{R}^2 \setminus \Gamma.$$

The single layer potentiel and the double layer potential both admit one-sided Dirichlet and Neumann traces on both sides of Γ. In [16] it is proved that the following operators are well defined and bounded:

$$\begin{aligned} &\gamma_d^\pm \circ \mathscr{S}_\Gamma : H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma) \qquad \text{and} \qquad \gamma_d^\pm \circ \mathscr{D}_\Gamma : H^{1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma), \\ &\gamma_n^\pm \circ \mathscr{S}_\Gamma : H^{-1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma) \qquad \text{and} \qquad \gamma_n^\pm \circ \mathscr{D}_\Gamma : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma). \end{aligned}$$

Moreover $\gamma_d^+ \circ \mathscr{S}_\Gamma = \gamma_d^- \circ \mathscr{S}_\Gamma$ (for the single layer potential, one-sided Dirichlet traces on Γ coincide) and $\gamma_n^+ \circ \mathscr{D}_\Gamma = -\gamma_n^- \circ \mathscr{D}_\Gamma$ (for the double layer potential, one-sided Neumann traces on Γ have opposite signs). To simplify the notation, we shall drop the superscripts + and − when the Dirichlet traces coincide or when the Neumann traces have opposite signs. Thus, we denote $\mathsf{S}_\Gamma = \gamma_d \circ \mathscr{S}_\Gamma$ and $\mathsf{D}_\Gamma = \gamma_n \circ \mathscr{D}_\Gamma$. The operator $\mathsf{S}_\Gamma : H^{-1/2}(\Gamma) \longrightarrow H^{1/2}(\Gamma)$ is an isomorphism (recall that the logarithmic capacity of Γ is assumed to be lower than 1, see [16, Theorem 8.6] about this question). The operator $\mathsf{D}_\Gamma : H^{1/2}(\Gamma) \longrightarrow H^{-1/2}(\Gamma)$ is Fredholm of index 0 with a one dimensional kernel spanned by the function $\mathbf{1}_\Gamma$ (the constant function equal to one on Γ) and with range $\widetilde{H}^{-1/2}(\Gamma) = \big\{ q \in H^{-1/2}(\Gamma) \,:\, \langle q, \mathbf{1}_\Gamma \rangle_{-\frac{1}{2},\frac{1}{2}} = 0 \big\}$. Introducing $\widetilde{H}^{1/2}(\Gamma) = \big\{ p \in H^{1/2}(\Gamma) \,:\, (p, \mathbf{1}_\Gamma)_{L^2(\Gamma)} = 0 \big\}$, we deduce that $\mathsf{D}_\Gamma : \widetilde{H}^{1/2}(\Gamma) \longrightarrow \widetilde{H}^{-1/2}(\Gamma)$ is an isomorphism. The following identities are usually referred to as the "jump relations" on Γ:

$$\gamma_n^+ \circ \mathscr{S}_\Gamma + \gamma_n^- \circ \mathscr{S}_\Gamma = \mathrm{Id} \qquad \text{and} \qquad \gamma_d^+ \circ \mathscr{D}_\Gamma - \gamma_d^- \circ \mathscr{D}_\Gamma = \mathrm{Id}.$$

The space $\mathscr{A}$ of the affine functions in $\mathbb{R}^2$ plays a particular role in the asymptotic behavior of the single layer potential. Indeed, for $|x|$ large, the single layer potential admits the following asymptotic expansion:

$$\mathscr{S}_\Gamma q(x) = -\frac{1}{2\pi} \langle q, \mathbf{1}_\Gamma \rangle_{-\frac{1}{2},\frac{1}{2}} \ln |x| + \frac{1}{2\pi} \frac{x_1}{|x|^2} \langle q, y_1 \rangle_{-\frac{1}{2},\frac{1}{2}} + \frac{1}{2\pi} \frac{x_2}{|x|^2} \langle q, y_2 \rangle_{-\frac{1}{2},\frac{1}{2}} + \mathscr{O}(1/|x|^2). \tag{4a}$$

The three first terms in the right hand side are not in $L^2(\mathbb{R}^2)$ while the remainder is. Let $\mathscr{A}_S^{\frac{1}{2}}$ be the three dimensional subspace of $H^{1/2}(\Gamma)$ spanned by the traces of the affine functions. Let $\mathscr{A}_S^{-\frac{1}{2}} = \mathsf{S}_\Gamma^{-1}\mathscr{A}_S^{\frac{1}{2}}$ (a three dimensional subspace in $H^{-1/2}(\Gamma)$) and define $\{\mathsf{q}_1, \mathsf{q}_2, \mathsf{q}_3\}$ a basis of this space normalized in such a way that $\langle \mathsf{q}_j, \mathsf{S}_\Gamma \mathsf{q}_k\rangle_{-\frac{1}{2},\frac{1}{2}} = \delta_{j,k}$ (the Kronecker symbol) for every indices $j, k \in \{1, 2, 3\}$. Notice that $\mathscr{S}_\Gamma \mathsf{q}_j$ is not an affine function in $\mathbb{R}^2$ but there exist affine functions P_j such that $\mathscr{S}_\Gamma \mathsf{q}_j|_{\Omega^-} = P_j|_{\Omega^-}$ $(j = 1, 2, 3)$.

Considering now the double layer potential, it can be expanded for $|x|$ large as:

$$\mathscr{D}_\Gamma p(x) = -\frac{1}{2\pi}\frac{x_1}{|x|^2}\langle n_1, p\rangle_{-\frac{1}{2},\frac{1}{2}} - \frac{1}{2\pi}\frac{x_2}{|x|^2}\langle n_2, p\rangle_{-\frac{1}{2},\frac{1}{2}} + \mathscr{O}(1/|x|^2), \tag{4b}$$

where we recall that $n = (n_1, n_2)$ is the unit normal vector field on Γ directed toward the exterior of Ω^-. Let $\mathscr{A}_D^{-\frac{1}{2}}$ be the two dimensional subspace of $\widetilde{H}^{-1/2}(\Gamma)$ spanned by n_1 and n_2. Its preimage by D_Γ is a two dimensional subspace of $\widetilde{H}^{1/2}(\Gamma)$ denoted by $\mathscr{A}_D^{\frac{1}{2}}$. Let $\{\mathsf{p}_1, \mathsf{p}_2\}$ be a basis of this space normalized in such a way that $\langle \mathsf{D}_\Gamma \mathsf{p}_j, \mathsf{p}_k\rangle_{-\frac{1}{2},\frac{1}{2}} = \delta_{j,k}$ $(j, k = 1, 2)$. As for the single layer potential, the double layer potential $\mathscr{D}_\Gamma \mathsf{p}_j$ is not an affine function in $\mathbb{R}^2$ but there exists an affine function Q_j such that $\mathscr{D}_\Gamma \mathsf{p}_j|_{\Omega^-} = Q_j|_{\Omega^-}$ (for $j = 1, 2$).

To be complete on the questions of asymptotic behavior of harmonic functions, let us mention a last result borrowed from [5, Chap. 10, Ex. 1]. Any function v harmonic outside a compact set can be expanded in this region as:

$$v(x) = \sum_{j=0}^{+\infty} \mathfrak{p}_m(x) + \mathfrak{q}_0 \ln|x| + \sum_{j=0}^{+\infty} \frac{\mathfrak{q}_m(x)}{|x|^{2m}}, \tag{4c}$$

where $\mathfrak{q}_0 \in \mathbb{R}$ and, for every integer m, $\mathfrak{p}_m, \mathfrak{q}_m$ are harmonic polynomials on $\mathbb{R}^2$ of degree m.

We will mainly rely on the following characterization of the surface potentials in the sequel:

Proposition 2.4. *Assume that Γ is Lipschitz continuous. The single layer potential of density $q \in H^{-1/2}(\Gamma)$ is the unique distribution $u \in \mathscr{D}'(\mathbb{R}^2)$ satisfying:*

$$\langle u, -\Delta\theta\rangle_{\mathscr{D}'(\mathbb{R}^2),\mathscr{D}(\mathbb{R}^2)} = \langle q, \gamma_d\theta\rangle_{-\frac{1}{2},\frac{1}{2}} \qquad \text{for all } \theta \in \mathscr{D}(\mathbb{R}^2); \tag{5a}$$

$$u(x) = \langle q, \mathbf{1}_\Gamma\rangle_{-\frac{1}{2},\frac{1}{2}} G(x) + o(1) \quad \text{as} \quad |x| \longrightarrow +\infty. \tag{5b}$$

The double layer potential of density $p \in H^{1/2}(\Gamma)$ is the unique distribution $v \in \mathscr{D}'(\mathbb{R}^2)$ satisfying:

$$\langle v, -\Delta\theta\rangle_{\mathscr{D}'(\mathbb{R}^2),\mathscr{D}(\mathbb{R}^2)} = \langle \gamma_n\theta, p\rangle_{-\frac{1}{2},\frac{1}{2}} \quad \text{for all } \theta \in \mathscr{D}(\mathbb{R}^2); \tag{6a}$$

$$v(x) = o(1) \qquad \text{as} \quad |x| \longrightarrow +\infty. \tag{6b}$$

Notice that any distribution u satisfying (5a) and any distribution v satisfying (6a) is harmonic in Ω^+ so that, according to the generalization to distributions of Weyl's lemma, they are $\mathcal{C}^\infty$ in Ω^+ and the asymptotic conditions (5b) and (6b) make sens.

Proof. Let q be in $H^{-1/2}(\Gamma)$. Applying the second Green's identity (3), we easily verify that the single layer potential $\mathscr{S}_\Gamma q$ satisfies both conditions (5). On the other hand, if u_1 and u_2 are two distributions satisfying these conditions, then $u = u_1 - u_2$ is a distribution harmonic in the whole plane. According to Weyl's lemma, it is $\mathcal{C}^\infty$ in $\mathbb{R}^2$ and since it tends to 0 at infinity, we conclude with Liouville's theorem that $u = 0$. The same arguments apply for the double layer potential. $\square$

Since in Proposition 2.4, $u = \mathscr{S}_\Gamma q$ and $v = \mathscr{D}_\Gamma p$, it turns out that the distribution u is actually in $H^1_{\ell oc}(\mathbb{R}^2)$ while the distribution v is such that $v|_{\Omega^-} \in H^1(\Omega^-)$ and $v|_{\Omega^+} \in H^1_{\ell oc}(\overline{\Omega^+})$. Our purpose is now to weaken the regularity of q and p and to generalize the definition of the single and double layer potentials in order to represent every function in $L^2_{\ell oc}(\mathbb{R}^2)$, harmonic in $\mathbb{R}^2 \setminus \Gamma$ with asymptotic behaviors as in (5b) or (6b).

3 Main function spaces

Following an idea of [2, §7], we introduce the weight fonctions ρ and lg:

$$\rho(x) = \sqrt{1 + |x|^2} \qquad \text{and} \qquad \mathrm{lg}(x) = \ln(2 + |x|^2) \qquad \text{for all } x \in \mathbb{R}^2, \tag{7}$$

which enter the definition of the weighted Sobolev space:

$$W^2(\mathbb{R}^2) = \Big\{ u \in \mathscr{D}'(\mathbb{R}^2) \,:\, \frac{u}{\rho^2 \lg} \in L^2(\mathbb{R}^2),\ \frac{1}{\rho \lg}\frac{\partial u}{\partial x_j} \in L^2(\mathbb{R}^2) \ \text{ and } \ \frac{\partial^2 u}{\partial x_j \partial x_k} \in L^2(\mathbb{R}^2),\ \forall\, j,k = 1,2 \Big\}.$$

Proposition 3.1. *The space $W^2(\mathbb{R}^2)$, provided with its natural norm, enjoys the following properties (borrowed from [2, Theorem 7.2] for the first and second points and from [2, Theorem 9.6] for the third one):*

1. *The space $\mathscr{D}(\mathbb{R}^2)$ is dense in $W^2(\mathbb{R}^2)$;*
2. *There exists a sequence of cut-off functions $(\phi_k)_{k\geqslant 1}$ in $\mathscr{D}(\mathbb{R}^2)$ such that, for every $u \in W^2(\mathbb{R}^2)$, $\phi_k u \longrightarrow u$ in $W^2(\mathbb{R}^2)$;*
3. *The Laplace operator $\Delta : W^2(\mathbb{R}^2)/\mathscr{A} \longrightarrow L^2(\mathbb{R}^2)$ is an isomorphism (we recall that $\mathscr{A}$ is the space of the affine functions in $\mathbb{R}^2$).*

For $p \in L^2(\mathit{\Gamma})$, we denote by $\mu(p)$ the mean value of p on $\mathit{\Gamma}$, i.e. $\mu(p) = |\mathit{\Gamma}|^{-1}(\mathbf{1}_{\mathit{\Gamma}}, p)_{L^2(\mathit{\Gamma})}$. The original idea at this point is to endow the space $W^2(\mathbb{R}^2)$ with the following inner products (for $u, v \in W^2(\mathbb{R}^2)$):

$$(u,v)_S = (\Delta u, \Delta v)_{L^2(\mathbb{R}^2)} + \sum_{j=1}^{3} \langle \mathsf{q}_j, \gamma_d u\rangle_{-\frac12,\frac12} \langle \mathsf{q}_j, \gamma_d v\rangle_{-\frac12,\frac12} \tag{8a}$$

$$(u,v)_D = (\Delta u, \Delta v)_{L^2(\mathbb{R}^2)} + \sum_{j=1}^{2} (\mathsf{p}_j, \gamma_n u)_{L^2(\mathit{\Gamma})} (\mathsf{p}_j, \gamma_n v)_{L^2(\mathit{\Gamma})} + \mu(\gamma_d u)\mu(\gamma_d v), \tag{8b}$$

where the subscripts S and D refer to "single" (layer) and "double" (layer), as it will become clear in the sequel. The corresponding norms, denoted by $\|\cdot\|_S$ and $\|\cdot\|_D$ are both equivalent to the natural norm of $W^2(\mathbb{R}^2)$, the proof being a straightforward consequence of [2, Corollary 8.4]. It is already worth noting that:

$$\mathscr{A}^\perp = \big\{ u \in W^2(\mathbb{R}^2) \,:\, (\gamma_d u, \gamma_d \theta)_{\frac12} = 0 \quad \forall \theta \in \mathscr{A} \big\} \qquad \text{in } (W^2(\mathbb{R}^2); \|\cdot\|_S),$$
$$\mathscr{A}^\perp = \big\{ u \in W^2(\mathbb{R}^2) \,:\, (\gamma_n u, \gamma_n \theta)_{L^2(\mathit{\Gamma})} + \mu(\gamma_d u)\mu(\gamma_d \theta) = 0 \quad \forall \theta \in \mathscr{A} \big\} \quad \text{in } (W^2(\mathbb{R}^2); \|\cdot\|_D).$$

Next, we introduce the boundary spaces:

$$\mathcal{H}^{3/2}(\mathit{\Gamma}) = \gamma_d W^2(\mathbb{R}^2) \qquad \text{and} \qquad \mathcal{H}^{1/2}(\mathit{\Gamma}) = \gamma_n W^2(\mathbb{R}^2). \tag{9}$$

Since the weight functions (7) do not modify the local properties of the space, we could as well replace the space $W^2(\mathbb{R}^2)$ by the space $H^2_{\ell oc}(\mathbb{R}^2)$ in these definitions. We emphasize that the superscripts 3/2 and 1/2 in (9) have no other meaning than to recall that $\mathcal{H}^{3/2}(\mathit{\Gamma}) = H^{3/2}(\mathit{\Gamma})$ and $\mathcal{H}^{1/2}(\mathit{\Gamma}) = H^{1/2}(\mathit{\Gamma})$ when $\mathit{\Gamma}$ is smooth. We introduce as well the closed subspaces of $W^2(\mathbb{R}^2)$:

$$W^2_d(\mathbb{R}^2) = \{u \in W^2(\mathbb{R}^2) \,:\, \gamma_d u = 0\} \qquad \text{and} \qquad W^2_n(\mathbb{R}^2) = \{u \in W^2(\mathbb{R}^2) \,:\, \gamma_n u = 0\}.$$

The images of $W^2_d(\mathbb{R}^2)$ and $W^2_n(\mathbb{R}^2)$ by γ_n and γ_d respectively are subspaces of $\mathcal{H}^{3/2}(\mathit{\Gamma})$ and $\mathcal{H}^{1/2}(\mathit{\Gamma})$. We denote them by:

$$\mathcal{H}^{3/2}_n(\mathit{\Gamma}) = \gamma_d W^2_n(\mathbb{R}^2) \qquad \text{and} \qquad \mathcal{H}^{1/2}_d(\mathit{\Gamma}) = \gamma_n W^2_d(\mathbb{R}^2). \tag{10}$$

It is well known that when $\mathit{\Gamma}$ is of class $\mathcal{C}^{1,1}$, the spaces $\mathcal{H}^{3/2}_n(\mathit{\Gamma})$ and $\mathcal{H}^{3/2}(\mathit{\Gamma})$ coincide, both being equal to $H^{3/2}(\mathit{\Gamma})$. In the same way, in the smooth case, $\mathcal{H}^{1/2}_d(\mathit{\Gamma}) = \mathcal{H}^{1/2}(\mathit{\Gamma}) = H^{1/2}(\mathit{\Gamma})$. This is no longer true however when $\mathit{\Gamma}$ is a $\mathcal{C}^{1,1}$ (curvilinear) polygon (and a fortiori when $\mathit{\Gamma}$ is only Lipschitz continuous) as explained in [8] where a counterexample is provided. Indeed in this case, a pair of functions $(f,g) \in H^1(\mathit{\Gamma}) \times L^2(\mathit{\Gamma})$ is equal to the Dirichlet and Neumann traces of a function in $H^2_{\ell oc}(\mathbb{R}^2)$ if and only if the vector field $(\partial f/\partial \tau)\tau + gn$ is in $H^{1/2}(\mathit{\Gamma}; \mathbb{R}^2)$. This condition implies in particular that the functions f and g have to satisfy some compatibility conditions at the vortices of the domain (as indicated in [10, Theorem 1.5.2.4]).

For every $p \in \mathcal{H}^{3/2}(\mathit{\Gamma})$, we define $\mathsf{L}^S_d p$ as the unique fonction in $W^2(\mathbb{R}^2)$ achieving:

$$\inf \big\{ \|u\|_S \,:\, u \in W^2(\mathbb{R}^2),\ \gamma_d u = p \big\}. \tag{11}$$

Thus $\mathsf{L}_d^S p$ is the orthogonal projection of any preimage of p by γ_d on the closed subspace $W_d^2(\mathbb{R}^2)^\perp$ of $(W^2(\mathbb{R}^2), \|\cdot\|_S)$. It is not difficult to verify that for every $p \in \mathcal{H}^{3/2}(\mathit{\Gamma})$:

$$\Delta^2(\mathsf{L}_d^S p) = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^2 \setminus \mathit{\Gamma}) \quad \text{and} \quad \gamma_d(\mathsf{L}_d^S p) = p. \tag{12}$$

In the same fashion, we define $\mathsf{L}_d^D p$ by replacing the norm $\|\cdot\|_S$ with the norm $\|\cdot\|_D$ in (11). The function $\mathsf{L}_d^D p$ verifies both identities (12) as well. This allows us to define two scalar products in $\mathcal{H}^{3/2}(\mathit{\Gamma})$:

$$(p_1, p_2)^A_{\frac{3}{2}} = \big(\mathsf{L}_d^A p_1, \mathsf{L}_d^A p_2\big)_A \qquad A \in \{S, D\},$$

whose associated norms, denoted by $\|\cdot\|^A_{\frac{3}{2}}$ are equivalent. The space $\mathcal{H}^{3/2}(\mathit{\Gamma})$ provided with any of these norms is a Hilbert space. We denote by Π_d^A the orthogonal projection onto $W_d^2(\mathbb{R}^2)^\perp$ in $(W^2(\mathbb{R}^2), \|\cdot\|_A)$. The following identities are obvious:

$$\gamma_d \circ \mathsf{L}_d^A = \mathrm{Id} \qquad \text{and} \qquad \mathsf{L}_d^A \circ \gamma_d = \Pi_d^A. \tag{13}$$

The very same procedure can be carried out by replacing the Dirichlet trace operator γ_d with the Neumann trace operator γ_n. This leads us to define for $A \in \{S, D\}$ the operators L_n^A, the projectors Π_n^A, the scalar products $(\cdot,\cdot)^A_{\frac{1}{2}}$ and the norms $\|\cdot\|^A_{\frac{1}{2}}$ in the space $\mathcal{H}^{1/2}(\mathit{\Gamma})$. As in (12), the functions $\mathsf{L}_n^A q$ verify:

$$\Delta^2(\mathsf{L}_n^A q) = 0 \quad \text{in } \mathscr{D}'(\mathbb{R}^2 \setminus \mathit{\Gamma}) \quad \text{and} \quad \gamma_n(\mathsf{L}_n^A q) = q \qquad \text{for all } q \in \mathcal{H}^{\frac{1}{2}}(\mathit{\Gamma}). \tag{14}$$

By construction, the following operators are isometric for any $A \in \{S, D\}$:

$$\mathsf{L}_d^A : \big(\mathcal{H}^{3/2}(\mathit{\Gamma}), \|\cdot\|^A_{\frac{3}{2}}\big) \longrightarrow \big(W_d^2(\mathbb{R}^2)^\perp, \|\cdot\|_A\big), \tag{15a}$$

$$\mathsf{L}_n^A : \big(\mathcal{H}^{1/2}(\mathit{\Gamma}), \|\cdot\|^A_{\frac{1}{2}}\big) \longrightarrow \big(W_n^2(\mathbb{R}^2)^\perp, \|\cdot\|_A\big). \tag{15b}$$

The space $\mathcal{H}^{3/2}(\mathit{\Gamma})$ is continuously embedded in $L^2(\mathit{\Gamma})$ since there exists a constant $C_\mathit{\Gamma} > 0$ such that:

$$\|p\|_{L^2(\mathit{\Gamma})} = \|\gamma_d \circ \mathsf{L}_d^S p\|_{L^2(\mathit{\Gamma})} \leqslant C_\mathit{\Gamma} \|\mathsf{L}_d^S p\|_{W^2(\mathbb{R}^2)} = \|p\|^S_{\frac{3}{2}} \qquad \text{for all } p \in \mathcal{H}^{3/2}(\mathit{\Gamma}).$$

The embedding is also dense (because the space $\gamma_d \mathscr{D}(\mathbb{R}^2)$ is densely embedded in $L^2(\mathit{\Gamma})$ as claimed in Proposition 2.3). Identifying $L^2(\mathit{\Gamma})$ with its dual space by means of Riesz representation theorem, we obtain a so-called Gelfand triple of Hilbert spaces (see [14, Appendix A]):

$$\mathcal{H}^{3/2}(\mathit{\Gamma}) \subset L^2(\mathit{\Gamma}) \subset \mathcal{H}^{-3/2}(\mathit{\Gamma}), \tag{16a}$$

in which $\mathcal{H}^{-3/2}(\mathit{\Gamma})$ is the dual space of $\mathcal{H}^{3/2}(\mathit{\Gamma})$ and $L^2(\mathit{\Gamma})$ is the pivot space. Similarly, we define $\mathcal{H}^{-1/2}(\mathit{\Gamma})$ the dual space of $\mathcal{H}^{1/2}(\mathit{\Gamma})$ and the Gelfand triple:

$$\mathcal{H}^{1/2}(\mathit{\Gamma}) \subset L^2(\mathit{\Gamma}) \subset \mathcal{H}^{-1/2}(\mathit{\Gamma}). \tag{16b}$$

The Gelfand triples (16) justify that the duality brackets $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{-\frac{3}{2},\frac{3}{2}}$ (between the spaces $\mathcal{H}^{-3/2}(\mathit{\Gamma})$ and $\mathcal{H}^{3/2}(\mathit{\Gamma})$) and $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{-\frac{1}{2},\frac{1}{2}}$ (between the spaces $\mathcal{H}^{-1/2}(\mathit{\Gamma})$ and $\mathcal{H}^{1/2}(\mathit{\Gamma})$) "extend" the $L^2(\mathit{\Gamma})$ inner product. Concerning embedding results, we can also state:

Proposition 3.2. *The inclusions $\mathcal{H}^{3/2}(\mathit{\Gamma}) \subset H^{1/2}(\mathit{\Gamma})$ and $\mathcal{H}^{1/2}(\mathit{\Gamma}) \subset H^{-1/2}(\mathit{\Gamma})$ are continuous and dense.*

Proof. The first inclusion is proved the same way as the inclusion $\mathcal{H}^{3/2}(\mathit{\Gamma}) \subset L^2(\mathit{\Gamma})$. The second inclusion results from the continuity and the density of the inclusion $L^2(\mathit{\Gamma}) \subset H^{-1/2}(\mathit{\Gamma})$. ☐

It remains to make precise the topologies of the spaces $\mathcal{H}_n^{3/2}(\mathit{\Gamma})$ and $\mathcal{H}_d^{1/2}(\mathit{\Gamma})$ introduced in (10). For every $p \in \mathcal{H}_n^{3/2}(\mathit{\Gamma})$, we define $\mathcal{L}_n p$ as the unique fonction in $W_n^2(\mathbb{R}^2)$ achieving:

$$\inf\big\{\|u\|_D \,:\, u \in W_n^2(\mathbb{R}^2),\ \gamma_d u = p\big\}. \tag{17}$$

Thus $\mathcal{L}_n p$ is the orthogonal projection of any preimage in $W_n^2(\mathbb{R}^2)$ of p by γ_d on the closed subspace $\big(W_d^2(\mathbb{R}^2) \cap W_n^2(\mathbb{R}^2)\big)^\perp$ in the space $\big(W_n^2(\mathbb{R}^2), \|\cdot\|_D\big)$. The function $\mathcal{L}_n p$ is biharmonic in $\mathbb{R}^2 \setminus \mathit{\Gamma}$ and satisfies $\gamma_d(\mathcal{L}_n)p = p$ and $\gamma_n(\mathcal{L}_n p) = 0$. The space $\mathcal{H}_n^{3/2}(\mathit{\Gamma})$ is endowed with the inner product:

$$(p_1, p_2)_{\frac{3}{2},n} = \big(\mathcal{L}_n p_1, \mathcal{L}_n p_2\big)_D = (\Delta \mathcal{L}_n p_1, \Delta \mathcal{L}_n p_2)_{L^2(\mathbb{R}^2)} + \mu(p_1)\mu(p_2), \qquad \text{for all } p_1, p_2 \in \mathcal{H}_n^{3/2}(\mathit{\Gamma}). \tag{18}$$

We denote by $\|\cdot\|_{\frac{3}{2},n}$ the corresponding norm. Similarly, for any $q \in \mathcal{H}_d^{1/2}(\Gamma)$, $\mathcal{L}_d q$ stands for the unique function in $W_d^2(\mathbb{R}^2)$ achieving:

$$\inf\big\{\|u\|_S \,:\, u \in W_d^2(\mathbb{R}^2),\ \gamma_n u = q\big\}. \tag{19}$$

Thus $\mathcal{L}_d q$ is a function biharmonic in $\mathbb{R}^2 \setminus \Gamma$ that satisfies $\gamma_d(\mathcal{L}_d q) = 0$ and $\gamma_n(\mathcal{L}_d q) = q$. The space $\mathcal{H}_d^{1/2}(\Gamma)$ is provided with the scalar product:

$$(q_1, q_2)_{\frac{1}{2},d} = \big(\mathcal{L}_d q_1, \mathcal{L}_d q_2\big)_S = (\Delta \mathcal{L}_d q_1, \Delta \mathcal{L}_d q_2)_{L^2(\mathbb{R}^2)}, \qquad \text{for all } q_1, q_2 \in \mathcal{H}_d^{1/2}(\Gamma), \tag{20}$$

and the corresponding norm is denoted by $\|\cdot\|_{\frac{1}{2},d}$. The spaces $\big(\mathcal{H}_n^{3/2}(\Gamma), \|\cdot\|_{\frac{3}{2},n}\big)$ and $\big(\mathcal{H}_d^{1/2}(\Gamma), \|\cdot\|_{\frac{1}{2},d}\big)$ are Hilbert spaces and by construction, the following operators are isometric:

$$\mathcal{L}_n : \big(\mathcal{H}_n^{3/2}(\Gamma), \|\cdot\|_{\frac{3}{2},n}\big) \longrightarrow \big(\mathscr{B}_n(\mathbb{R}^2), \|\cdot\|_D\big), \tag{21a}$$

$$\mathcal{L}_d : \big(\mathcal{H}_d^{1/2}(\Gamma), \|\cdot\|_{\frac{1}{2},d}\big) \longrightarrow \big(\mathscr{B}_d(\mathbb{R}^2), \|\cdot\|_S\big), \tag{21b}$$

where $\mathscr{B}_n(\mathbb{R}^2) = \big(W_d^2(\mathbb{R}^2) \cap W_n^2(\mathbb{R}^2)\big)^\perp \cap W_n^2(\mathbb{R}^2)$ and $\mathscr{B}_d(\mathbb{R}^2) = \big(W_d^2(\mathbb{R}^2) \cap W_n^2(\mathbb{R}^2)\big)^\perp \cap W_d^2(\mathbb{R}^2)$. The functions in $\mathscr{B}_n(\mathbb{R}^2)$ are those in $W^2(\mathbb{R}^2)$ which are biharmonic in $\mathbb{R}^2 \setminus \Gamma$ with homogeneous Neumann boundary data and the functions in $\mathscr{B}_d(\mathbb{R}^2)$ are biharmonic in $\mathbb{R}^2 \setminus \Gamma$ with homogeneous Dirichlet boundary data.

4 Square integrable surface potentials

In this section, we still assume that Γ is Lipschitz continuous. To every $q \in \mathcal{H}^{-3/2}(\Gamma)$ (applying Riesz representation Theorem), we can associate a unique $u_q \in W^2(\mathbb{R}^2)$ such that:

$$\big(u_q, \theta\big)_S = \langle\!\langle q, \gamma_d \theta \rangle\!\rangle_{-\frac{3}{2},\frac{3}{2}} \qquad \text{for all } \theta \in W^2(\mathbb{R}^2), \tag{22a}$$

and we define:

$$\mathscr{S}_\Gamma^\dagger q = -\Delta u_q + \sum_{j=1}^{3} \langle \mathsf{q}_j, \gamma_d u_q \rangle_{-\frac{1}{2},\frac{1}{2}} \mathscr{S}_\Gamma \mathsf{q}_j. \tag{22b}$$

Similarly, to every $p \in \mathcal{H}^{-1/2}(\Gamma)$, we can associate a unique $v_p \in W^2(\mathbb{R}^2)$ such that:

$$\big(v_p, \theta\big)_D = \langle\!\langle p, \gamma_n \theta \rangle\!\rangle_{-\frac{1}{2},\frac{1}{2}} \qquad \text{for all } \theta \in W^2(\mathbb{R}^2), \tag{23a}$$

and we define:

$$\mathscr{D}_\Gamma^\dagger p = -\Delta v_p + \sum_{j=1}^{2} (\mathsf{p}_j, \gamma_n v_p)_{L^2(\Gamma)} \mathscr{D}_\Gamma \mathsf{p}_j. \tag{23b}$$

The expressions of the functions u_q and v_q with respect to q and p can be made precise. Considering the Gelfand triple (16a) and (16b), we can classically (see [14, Appendix A]) define the isometric operators

$$\begin{aligned} \mathsf{T}_d : \mathcal{H}^{3/2}(\Gamma) &\longrightarrow \mathcal{H}^{-3/2}(\Gamma) \\ p &\longmapsto (p, \cdot)^S_{\frac{3}{2}} \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathsf{T}_n : \mathcal{H}^{1/2}(\Gamma) &\longrightarrow \mathcal{H}^{-1/2}(\Gamma) \\ q &\longmapsto (q, \cdot)^D_{\frac{1}{2}}. \end{aligned} \tag{24}$$

Lemma 4.1. *For every $q \in \mathcal{H}^{-3/2}(\Gamma)$, the function u_q defined by (22a) is equal to $\mathsf{L}_d^S \circ \mathsf{T}_d^{-1} q$. For every $p \in \mathcal{H}^{-1/2}(\Gamma)$, the function v_p defined by (23a) is equal to $\mathsf{L}_n^D \circ \mathsf{T}_n^{-1} p$. It follows that the applications:*

$$\begin{aligned} \mathcal{H}^{-3/2}(\Gamma) &\longrightarrow \big(W_d^2(\mathbb{R}^2)^\perp, \|\cdot\|_S\big) \\ q &\longmapsto u_q \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{H}^{-1/2}(\Gamma) &\longrightarrow \big(W_n^2(\mathbb{R}^2)^\perp, \|\cdot\|_D\big) \\ p &\longmapsto v_p, \end{aligned}$$

are isometric and that:

$$\mathscr{S}_\Gamma^\dagger q = -\Delta \mathsf{L}_d^S \circ \mathsf{T}_d^{-1} q + \sum_{j=1}^{3} \langle \mathsf{q}_j, \mathsf{T}_d^{-1} q \rangle_{-\frac{1}{2},\frac{1}{2}} \mathscr{S}_\Gamma \mathsf{q}_j \qquad \text{for all } q \in \mathcal{H}^{-3/2}(\Gamma), \tag{25a}$$

$$\mathscr{D}_\Gamma^\dagger p = -\Delta \mathsf{L}_n^D \circ \mathsf{T}_n^{-1} p + \sum_{j=1}^{2} \langle \mathsf{T}_n^{-1} p, \mathsf{p}_j \rangle_{-\frac{1}{2},\frac{1}{2}} \mathscr{D}_\Gamma \mathsf{p}_j \qquad \text{for all } p \in \mathcal{H}^{-1/2}(\Gamma). \tag{25b}$$

Proof. Let $q \in \mathcal{H}^{-3/2}(\Gamma)$ and $\theta \in W^2(\mathbb{R}^2)$. By definition of the operator T_d:

$$\langle\!\langle q, \gamma_d\theta\rangle\!\rangle_{-\frac{3}{2},\frac{3}{2}} = \big(\mathsf{T}_d^{-1}q, \gamma_d\theta\big)^S_{\frac{3}{2}} = \big(\mathsf{L}_d^S \circ \mathsf{T}_d^{-1}q, \mathsf{L}_d^S \circ \gamma_d\theta\big)_S.$$

According to (13):

$$\big(\mathsf{L}_d^S \circ \mathsf{T}_d^{-1}q, \mathsf{L}_d^S \circ \gamma_d\theta\big)_S = \big(\mathsf{L}_d^S \circ \mathsf{T}_d^{-1}q, \Pi_d^S\theta\big)_S = \big(\mathsf{L}_d^S \circ \mathsf{T}_d^{-1}q, \theta\big)_S,$$

which means that $u_q = \mathsf{L}_d^S \circ \mathsf{T}_d^{-1}q$ considering (22a). The result concerning v_p is proved in the same way. ☐

Theorem 4.1. *The linear operators $\mathscr{S}_\Gamma^\dagger : \mathcal{H}^{-3/2}(\Gamma) \longrightarrow L^2_{\ell oc}(\mathbb{R}^2)$ and $\mathscr{D}_\Gamma^\dagger : \mathcal{H}^{-1/2}(\Gamma) \longrightarrow L^2_{\ell oc}(\mathbb{R}^2)$ are bounded and they satisfy:*

- *For every $q \in \mathcal{H}^{-3/2}(\Gamma)$:*

$$\big\langle \mathscr{S}_\Gamma^\dagger q, -\Delta\theta\big\rangle_{\mathscr{D}'(\mathbb{R}^2),\mathscr{D}(\mathbb{R}^2)} = \langle\!\langle q, \gamma_d\theta\rangle\!\rangle_{-\frac{3}{2},\frac{3}{2}} \qquad \textit{for all } \theta \in \mathscr{D}(\mathbb{R}^2); \tag{26a}$$

$$\mathscr{S}_\Gamma^\dagger q(x) = \langle\!\langle q, \mathbf{1}_\Gamma\rangle\!\rangle_{-\frac{3}{2},\frac{3}{2}} G(x) + o(1) \quad \textit{as} \quad |x| \longrightarrow +\infty; \tag{26b}$$

- *For every $p \in \mathcal{H}^{-1/2}(\Gamma)$:*

$$\langle \mathscr{D}_\Gamma^\dagger p, -\Delta\theta\rangle_{\mathscr{D}'(\mathbb{R}^2),\mathscr{D}(\mathbb{R}^2)} = \langle\!\langle p, \gamma_n\theta\rangle\!\rangle_{-\frac{1}{2},\frac{1}{2}} \quad \textit{for all } \theta \in \mathscr{D}(\mathbb{R}^2); \tag{27a}$$

$$\mathscr{D}_\Gamma^\dagger p(x) = o(1) \qquad \textit{as} \quad |x| \longrightarrow +\infty. \tag{27b}$$

The operators $\mathscr{S}_\Gamma^\dagger$ and $\mathscr{D}_\Gamma^\dagger$ are the extensions by density of the classical single and double layer potentials to the spaces $\mathcal{H}^{-3/2}(\Gamma)$ and $\mathcal{H}^{-1/2}(\Gamma)$ respectively.

Proof. For every $q \in \mathcal{H}^{-3/2}(\Gamma)$, we can rewrite (22a):

$$(\Delta u_q, \Delta\theta)_{L^2(\mathbb{R}^2)} + \sum_{j=1}^3 \langle \mathsf{q}_j, \gamma_d u_q\rangle_{-\frac{1}{2},\frac{1}{2}} \langle \mathsf{q}_j, \gamma_d\theta\rangle_{-\frac{1}{2},\frac{1}{2}} = \langle\!\langle q, \gamma_d\theta\rangle\!\rangle_{-\frac{3}{2},\frac{3}{2}} \qquad \text{for all } \theta \in \mathscr{D}(\mathbb{R}^2).$$

According to (5a), we can transform the second term in the left and side to obtain:

$$\Big\langle \Delta u_q - \sum_{j=1}^3 \langle \mathsf{q}_j, \gamma_d u_q\rangle_{-\frac{1}{2},\frac{1}{2}} \mathscr{S}_\Gamma \mathsf{q}_j, \Delta\theta\Big\rangle_{\mathscr{D}'(\mathbb{R}^2),\mathscr{D}(\mathbb{R}^2)} = \langle\!\langle q, \gamma_d\theta\rangle\!\rangle_{-\frac{3}{2},\frac{3}{2}} \qquad \text{for all } \theta \in \mathscr{D}(\mathbb{R}^2),$$

which is (26a). Let now x be a point in Ω^+ and denote by $d(x,\Gamma)$ the distance from x to Γ. On the disk $D(x, d(x,\Gamma))$ of center x and radius $d(x,\Gamma)$, the function Δu_q is harmonic. It follows that:

$$\Delta u_q(x) = -\mathscr{S}_\Gamma^\dagger q(x) + \sum_{k=1}^3 \langle \mathsf{q}_k, \gamma_d u_q\rangle_{-\frac{1}{2},\frac{1}{2}} \mathscr{S}_\Gamma \mathsf{q}_k(x) = \frac{1}{\pi d(x,\Gamma)^2} \int_{D(x,d(x,\Gamma))} \Delta u_q(y)\,\mathrm{d}y,$$

from which we deduce that:

$$\left| \mathscr{S}_\Gamma^\dagger q(x) - \sum_{k=1}^3 \langle \mathsf{q}_k, \gamma_d u_q\rangle_{-\frac{1}{2},\frac{1}{2}} \mathscr{S}_\Gamma \mathsf{q}_k(x) \right| \leqslant \frac{1}{\sqrt{\pi} d(x,\Gamma)} \|\Delta u_q\|_{L^2(\mathbb{R}^2)}. \tag{28}$$

Taking into account the asymptotic expansion (4a), we obtain on the one hand:

$$\sum_{k=1}^3 \langle \mathsf{q}_k, \gamma_d u_q\rangle_{-\frac{1}{2},\frac{1}{2}} \mathscr{S}_\Gamma \mathsf{q}_k(x) = \left(\sum_{k=1}^3 \langle \mathsf{q}_k, \gamma_d u_q\rangle_{-\frac{1}{2},\frac{1}{2}} \langle \mathsf{q}_k, \mathbf{1}_\Gamma\rangle_{-\frac{1}{2},\frac{1}{2}} \right) G(x) + \mathscr{O}(1/|x|). \tag{29a}$$

On the other hand, equality (22a) with $\theta = \mathbf{1}_{\mathbb{R}^2}$ yields:

$$\langle\!\langle q, \mathbf{1}_\Gamma\rangle\!\rangle_{-\frac{3}{2},\frac{3}{2}} = \sum_{k=1}^3 \langle \mathsf{q}_k, \gamma_d u_q\rangle_{-\frac{1}{2},\frac{1}{2}} \langle \mathsf{q}_k, \mathbf{1}_\Gamma\rangle_{-\frac{1}{2},\frac{1}{2}}. \tag{29b}$$

Combining both equations (29) with (28) and letting $|x|$ go to $+\infty$, we obtain (26b). The proof of equalities (27) is similar. The only difficulty consists in noticing that the function v_p in (23a) achieves:

$$\min_{v\in W^2(\mathbb{R}^2)} \frac{1}{2}\|v\|_D^2 - \langle\!\langle p, \gamma_n v\rangle\!\rangle_{-\frac{1}{2},\frac{1}{2}},$$

and therefore that $\mu(v_p) = 0$.

According to (16a) and Propositions 3.2, all the following inclusions are continuous and dense:

$$\mathcal{H}^{3/2}(\Gamma) \subset H^{1/2}(\Gamma) \subset L^2(\Gamma) \subset H^{-1/2}(\Gamma) \subset \mathcal{H}^{-3/2}(\Gamma).$$

It entails that for every $q \in H^{-1/2}(\Gamma)$ and $p \in \mathcal{H}^{3/2}(\Gamma)$, we are allowed to write:

$$\langle\!\langle q, p\rangle\!\rangle_{-\frac{3}{2},\frac{3}{2}} = \langle q, p\rangle_{-\frac{1}{2},\frac{1}{2}}.$$

Comparing (5) and (26), we conclude that $\mathscr{S}_\Gamma^\dagger q = \mathscr{S}_\Gamma q$ for every $q \in H^{-1/2}(\Gamma)$. In the same fashion, we can prove that $\mathscr{D}_\Gamma^\dagger p = \mathscr{D}_\Gamma p$ for every $p \in H^{1/2}(\Gamma)$. It remains only to verify that $\mathscr{S}_\Gamma^\dagger$ and $\mathscr{D}_\Gamma^\dagger$ are bounded but this is a straightforward consequence of the expressions (25) in Lemma 4.1. □

5 Further function spaces

In this section, we assume that Γ is a $\mathcal{C}^{1,1}$ curvilinear polygon and we denote by Γ_j its $\mathcal{C}^{1,1}$ edges and by c_j its vertices ($j = 1, \ldots, N$). In the sequel we will need some particular test functions in $W^2(\mathbb{R}^2)$. Their existence is asserted in the Lemma below:

Lemma 5.1. *1. Any function θ in $W^2(\mathbb{R}^2)$ supported in $\mathbb{R}^2 \setminus \{c_1, \ldots, c_N\}$ can be decomposed into a sum of two functions $\theta_d + \theta_n$ with $\theta_d \in W_d^2(\mathbb{R}^2)$ and $\theta_n \in W_n^2(\mathbb{R}^2)$.*

2. For every index $k \in \{1, \ldots, N\}$, there exists a function θ in $W_n^2(\mathbb{R}^2)$ such that $\theta(c_k) = 1$ and $\theta(c_j) = 0$ when $j \neq k$, $j \in \{1, \ldots, N\}$.

Proof. Addressing the first point of the lemma, we apply [10, Theorem 1.5.2.4] which makes precise the range of the operator (γ_d^-, γ_n^-) defined on the space $H^2(\Omega^-)$. Since, in [1, Theorem 10.4.1], the author proves the existence of a universal extension operator from $H^2(\Omega^-)$ to $H^2(\mathbb{R}^2)$, the range of (γ_d^-, γ_n^-) is the same when we consider this operator as defined on $H^2_{\ell oc}(\mathbb{R}^2)$ or on $W^2(\mathbb{R}^2)$. So let θ be given in $W^2(\mathbb{R}^2)$ and denote respectively by f_j and g_j the restrictions of $\gamma_d^-\theta$ and $\gamma_n^-\theta$ to the edge Γ_j (for j ranging from 1 to N). According to [10, Theorem 1.5.2.8], the pair (f_j, g_j) belongs to the space $H^{3/2}(\Gamma_j) \times H^{1/2}(\Gamma_j)$ for every index $j = 1, \ldots, N$. Considering now the pairs $(f_j, 0)$ in the same space $H^{3/2}(\Gamma_j) \times H^{1/2}(\Gamma_j)$, they trivially satisfy the compatibility conditions at the vertices c_k described in [10, Theorem 1.5.2.8] since every fonction f_j is compactly supported on Γ_j. Therefore they belong to the range of (γ_d^-, γ_n^-) and there exists a function θ_n in $W^2(\mathbb{R}^2)$ such that $\gamma_d^-\theta_n|_{\Gamma_j} = \gamma_d\theta_n|_{\Gamma_j} = f_j$ and $\gamma_n^-\theta_n|_{\Gamma_j} = \gamma_n\theta_n|_{\Gamma_j} = 0$. We define $\theta_d = \theta - \theta_n$ and the former assertion of the lemma is proved.

The proof of the latter rests roughly on the same arguments. Let k be given in $\{1, \ldots, N\}$ and let f be a smooth function defined on Γ that vanishes on a neighborhood of every vertex c_j when $j \neq k$ and is constant in a neighborhood of c_k. Denote by f_j the restriction of f to Γ_j ($j = 1, \ldots, N$). The pairs $(f_j, 0)$ belong to $H^{3/2}(\Gamma_j) \times H^{1/2}(\Gamma_j)$ and they trivially satisfy the compatibility conditions at the vertices described in [10, Theorem 1.5.2.8] (since $\partial f_j/\partial\tau$ vanishes near the vertices), what ensures the existence in $W^2(\mathbb{R}^2)$ of a preimage θ by the operator (γ_d, γ_n). □

Recall that the spaces $\mathcal{H}_n^{3/2}(\Gamma)$ and $\mathcal{H}_d^{1/2}(\Gamma)$ are defined in (10). The following result will play an important role in the rest of the paper:

Theorem 5.1. *The space $\mathcal{H}_n^{3/2}(\Gamma)$ is dense in $\mathcal{H}^{3/2}(\Gamma)$ and the space $\mathcal{H}_d^{1/2}(\Gamma)$ is dense in $\mathcal{H}^{1/2}(\Gamma)$.*

Proof. The proofs of both assertions are similar so let us focus on the latter. Using the isometric operator (15b) and since $\mathsf{L}_n^D \circ \gamma_n = \Pi_n^D$, we are led to prove that $\Pi_n^D W_d^2(\mathbb{R}^2)$ is dense in $\Pi_n^D W^2(\mathbb{R}^2) = W_n^2(\mathbb{R}^2)^\perp$. This is equivalent to showing that $W_d^2(\mathbb{R}^2) \oplus W_n^2(\mathbb{R}^2)$ is dense in $W^2(\mathbb{R}^2)$ or, still equivalently, that $W_d^2(\mathbb{R}^2)^\perp \cap W_n^2(\mathbb{R}^2)^\perp = \{0\}$ (where both superscripts $\perp$ refer to the same scalar product $(\cdot,\cdot)_D$).

So, let u be in $W^2_d(\mathbb{R}^2)^\perp$. Then $u = \Pi^D_d u$ and therefore:

$$(u,\theta)_D = \big(\Pi^D_d u,\theta\big)_D = \big(\Pi^D_d u,\Pi^D_d\theta\big)_D = \big(\mathsf{L}^D_d\circ\gamma_d u,\mathsf{L}^D_d\circ\gamma_d\theta\big)_D = (\gamma_d u,\gamma_d\theta)^D_{\frac{3}{2}} \qquad \text{for all } \theta\in W^2(\mathbb{R}^2). \tag{30a}$$

In the same fashion, assuming that the function u belongs also to $W^2_n(\mathbb{R}^2)^\perp$ we get:

$$(u,\theta)_D = (\gamma_n u,\gamma_n\theta)^D_{\frac{1}{2}} \qquad \text{for all } \theta\in W^2(\mathbb{R}^2). \tag{30b}$$

In addition, u achieves:

$$\inf\big\{\|v\|_D\,:\, v\in W^2(\mathbb{R}^2),\,\gamma_n v=\gamma_n u\big\},$$

and therefore $\mu(u)=0$. Now, recall that $\{c_1,\dots,c_N\}$ are the vertices of the polygon $\varGamma$. According to Lemma 5.1, every function θ in $W^2(\mathbb{R}^2)$ compactly supported in $\mathbb{R}^2\setminus\{c_1,\dots,c_N\}$ can be decomposed into the sum of two functions $\theta_d\in W^2_d(\mathbb{R}^2)$ and $\theta_n\in W^2_n(\mathbb{R}^2)$. It is easy to verify that both functions can be chosen compactly supported in $\mathbb{R}^2\setminus\{c_1,\dots,c_N\}$. If follows that for such a function θ, we have:

$$(u,\theta)_D = (u,\theta_d)_D + (u,\theta_n)_D = 0,$$

where we have used (30a) for the former term in the right hand side and (30b) for the latter. Thus, we have proved in particular that:

$$(\Delta u,\Delta\theta)_{L^2(\mathbb{R}^2)} + \sum_{j=1}^2(\mathsf{p}_j,\gamma_n u)_{L^2(\varGamma)}(\mathsf{p}_j,\gamma_n\theta)_{L^2(\varGamma)} = 0 \qquad \text{for all } \theta\in\mathscr{D}(\mathbb{R}^2\setminus\{c_1,\dots,c_N\}),$$

and this can be rewritten, according to (6a) as:

$$\Big\langle -\Delta u + \sum_{j=1}^2(\mathsf{p}_j,\gamma_n u)_{L^2(\varGamma)}\mathscr{D}_\varGamma\mathsf{p}_j,\Delta\theta\Big\rangle_{\mathscr{D}'(\mathbb{R}^2),\mathscr{D}(\mathbb{R}^2)} = 0 \qquad \text{for all } \theta\in\mathscr{D}(\mathbb{R}^2\setminus\{c_1,\dots,c_N\}).$$

This equality means that the function

$$v = -\Delta u + \sum_{j=1}^2(\mathsf{p}_j,\gamma_n u)_{L^2(\varGamma)}\mathscr{D}_\varGamma\mathsf{p}_j, \tag{31}$$

is harmonic in $\mathbb{R}^2\setminus\{c_1,\dots,c_N\}$ and the distribution Δv is supported in the points $c_1,\dots,c_N$. According to [11, Theorem 1.5.3], we deduce that this distribution is a finite linear combination of Dirac measures and derivatives of Dirac mesures at the points c_j $(j=1,\dots,N)$. Derivatives of Dirac measures must be excluded however since v is in $L^2_{\ell oc}(\mathbb{R}^2)$. Finally, v can only take the form:

$$v = -\sum_{j=1}^n\frac{\varrho_j}{2\pi}\ln|\cdot - c_j| + h, \tag{32}$$

with $\varrho_j\in\mathbb{R}$ and h harmonic in $\mathbb{R}^2$. Proceeding as in the proof of Theorem 4.1, we deduce from identity (31) that $v(x)=o(1)$ as $|x|\longrightarrow+\infty$. It follows that $\sum_{j=1}^n\varrho_j=0$ and $h=0$ with Liouville's theorem. Let $k\in\{1,\dots,N\}$ be given and let θ be a function in $W^2_n(\mathbb{R}^2)$ compactly supported such that $\theta(c_j)=0$ for $j\neq k$ and $\theta(c_k)=1$. Such a function exists according to Lemma 5.1 and yields, applying Green's identity (3):

$$(u,\theta)_D = (\Delta u,\Delta\theta)_{L^2(\mathbb{R}^2)} + \sum_{j=1}^2(\mathsf{p}_j,\gamma_n u)_{L^2(\varGamma)}(\mathsf{p}_j,\gamma_n\theta)_{L^2(\varGamma)} = (v,\Delta\theta)_{L^2(\mathbb{R}^2)}.$$

Using the expression (32) of the function v, we classically obtain that $(u,\theta)_D=\varrho_k$. On the other hand, identity (30b) leads to $(u,\theta)_D=0$, what completes the proof. □

Considering back the Gelfand triples (16), we are allowed to write when $\varGamma$ is a curvilinear $\mathcal{C}^{1,1}$ polygon:

$$\mathcal{H}^{3/2}_n(\varGamma)\subset\mathcal{H}^{3/2}(\varGamma)\subset H^{1/2}(\varGamma)\subset L^2(\varGamma)\subset H^{-1/2}(\varGamma)\subset\mathcal{H}^{-3/2}(\varGamma)\subset\mathcal{H}^{-3/2}_n(\varGamma), \tag{33a}$$

where all the inclusions are continuous and dense, $L^2(\varGamma)$ is the pivot space (i.e. the space identified via Riesz representation theorem with its dual space) and $\mathcal{H}_n^{-3/2}(\varGamma)$ is the dual space of $\mathcal{H}_n^{3/2}(\varGamma)$. In a similar way, we have also:

$$\mathcal{H}_d^{1/2}(\varGamma) \subset \mathcal{H}^{1/2}(\varGamma) \subset L^2(\varGamma) \subset \mathcal{H}^{-1/2}(\varGamma) \subset \mathcal{H}_d^{-1/2}(\varGamma). \tag{33b}$$

We denote respectively by $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{-\frac{3}{2},\frac{3}{2},n}$ and $\langle\!\langle \cdot,\cdot \rangle\!\rangle_{-\frac{1}{2},\frac{1}{2},d}$ the duality pairings on $\mathcal{H}_n^{-3/2}(\varGamma) \times \mathcal{H}_n^{3/2}(\varGamma)$ and on $\mathcal{H}_d^{-1/2}(\varGamma) \times \mathcal{H}_d^{1/2}(\varGamma)$ and we introduce the isometric operators, based on the Gelfand triple structure:

$$\begin{aligned} \mathcal{T}_d : \mathcal{H}_d^{1/2}(\varGamma) &\longrightarrow \mathcal{H}_d^{-1/2}(\varGamma) \\ q &\longmapsto (q,\cdot)_{\frac{1}{2},d}, \end{aligned} \qquad \text{and} \qquad \begin{aligned} \mathcal{T}_n : \mathcal{H}_n^{3/2}(\varGamma) &\longrightarrow \mathcal{H}_n^{-3/2}(\varGamma) \\ p &\longmapsto (p,\cdot)_{\frac{3}{2},n}. \end{aligned} \tag{34}$$

We end this section by defining the closed subspaces of $L^2(\mathbb{R}^2)$ consisting in functions that are harmonic in $\varOmega^+ \cup \varOmega^-$:

$$\mathscr{H}^0(\mathbb{R}^2 \setminus \varGamma) = \big\{ u \in L^2(\mathbb{R}^2) \,:\, (u, \Delta\theta)_{L^2(\mathbb{R}^2)} = 0, \quad \forall\, \theta \in \mathscr{D}(\mathbb{R}^2 \setminus \varGamma) \big\}.$$

Combining Proposition 3.1 with Theorem 2.2, it follows that:

$$\mathscr{H}^0(\mathbb{R}^2 \setminus \varGamma) = \big\{ \Delta u \,:\, u \in \big(W_d^2(\mathbb{R}^2) \cap W_n^2(\mathbb{R}^2)\big)^{\perp} \big\}, \tag{35}$$

where the superscript $\perp$ refers to any of the two scalar products (8) defined on $W^2(\mathbb{R}^2)$ (both leading to the same space). We will also consider more regular harmonic functions:

$$\mathscr{H}^1(\mathbb{R}^2 \setminus \varGamma) = \big\{ u \in L^2(\mathbb{R}^2) \,:\, u|_{\varOmega^+} \in H^1(\varOmega^+) \text{ and } u|_{\varOmega^-} \in H^1(\varOmega^-) \big\}.$$

Proposition 5.1. *The space $\mathscr{H}^1(\mathbb{R}^2 \setminus \varGamma)$ is dense in $\mathscr{H}^0(\mathbb{R}^2 \setminus \varGamma)$.*

Proof. We introduce the closed subspace of $H^{-1/2}(\varGamma) \times H^{1/2}(\varGamma)$:

$$E(\varGamma) = \big\{ (q,p) \in H^{-1/2}(\varGamma) \times H^{1/2}(\varGamma) \,:\, \langle q, \mathbf{1}_\varGamma \rangle_{-\frac{1}{2},\frac{1}{2}} = 0,\ \langle q, y_k \rangle_{-\frac{1}{2},\frac{1}{2}} - \langle n_k, p \rangle_{-\frac{1}{2},\frac{1}{2}} = 0,\ k = 1,2 \big\}.$$

We claim that:

$$\mathscr{H}^1(\mathbb{R}^2 \setminus \varGamma) = \big\{ \mathscr{S}_\varGamma q + \mathscr{D}_\varGamma p \,:\, (q,p) \in E(\varGamma) \big\}. \tag{36}$$

For any $(q,p) \in E(\varGamma)$, the function $u = \mathscr{S}_\varGamma q + \mathscr{D}_\varGamma p$ is in $\mathscr{H}^1(\mathbb{R}^2 \setminus \varGamma)$ according to the asymptotic expansions (4). Reciprocally, let u be in $\mathscr{H}^1(\mathbb{R}^2 \setminus \varGamma)$ and denote $q = \gamma_n^+ u + \gamma_n^- u$ and $p = \gamma_d^+ u - \gamma_d^- u$. The function $v = \mathscr{S}_\varGamma q + \mathscr{D}_\varGamma p - u$ is harmonic on $\mathbb{R}^2$. Since, by hypothesis, $u \in L^2(\mathbb{R}^2)$ we can proceed as in the proof of Proposition 4.1 to show that $u(x) = \mathscr{O}(1/|x|)$ for $|x|$ large. Taking into account (4) again, we deduce that:

$$v(x) = -\frac{1}{2\pi} \langle q, \mathbf{1}_\varGamma \rangle_{-\frac{1}{2},\frac{1}{2}} \ln|x| + \mathscr{O}(1/|x|) \qquad \text{as } |x| \longrightarrow +\infty,$$

and invoking [5, Theorem 9.10], we conclude that the function v vanishes on $\mathbb{R}^2$. It follows that u is equal to $\mathscr{S}_\varGamma q + \mathscr{D}_\varGamma p$ and since this function is in $L^2(\mathbb{R}^2)$, the pair (q,p) is in $E(\varGamma)$ and identity (36) is proved. We can now determine the space $\mathscr{H}^1(\mathbb{R}^2 \setminus \varGamma)^{\perp}$ in $\mathscr{H}^0(\mathbb{R}^2 \setminus \varGamma)$. Let w be in $\mathscr{H}^0(\mathbb{R}^2 \setminus \varGamma)$ such that:

$$(w,u)_{L^2(\mathbb{R}^2)} = 0 \qquad \text{for all } u \in \mathscr{H}^1(\mathbb{R}^2 \setminus \varGamma),$$

or equivalently:

$$\big(w, \mathscr{S}_\varGamma q + \mathscr{D}_\varGamma p\big)_{L^2(\mathbb{R}^2)} = 0 \qquad \text{for all } (q,p) \in E(\varGamma).$$

From (35), we know that there exists a function $v \in \big(W_d^2(\mathbb{R}^2) \cap W_n^2(\mathbb{R}^2)\big)^{\perp}$ such that $w = \Delta v$. Following Proposition 3.1, $\mathscr{D}(\mathbb{R}^2)$ is a dense subspace in $W^2(\mathbb{R}^2)$ so let $(v_k)_{k \geqslant 0}$ be a sequence in $\mathscr{D}(\mathbb{R}^2)$ that converges to v in $W^2(\mathbb{R}^2)$. For every $k \geqslant 0$ we can apply the second Green's formula (3) to obtain:

$$\langle \gamma_n v_k, p \rangle_{-\frac{1}{2},\frac{1}{2}} - \langle q, \gamma_d v_k \rangle_{-\frac{1}{2},\frac{1}{2}} = \big(-\Delta v_k, \mathscr{S}_\varGamma q + \mathscr{D}_\varGamma p \big)_{L^2(\mathbb{R}^2)}.$$

Letting k go to $+\infty$, it comes:

$$\langle \gamma_n v, p \rangle_{-\frac{1}{2},\frac{1}{2}} - \langle q, \gamma_d v \rangle_{-\frac{1}{2},\frac{1}{2}} = \big(-\Delta v, \mathscr{S}_\varGamma q + \mathscr{D}_\varGamma p \big)_{L^2(\mathbb{R}^2)} = 0,$$

and since the inclusions $H^{1/2}(\Gamma) \subset \mathcal{H}^{-1/2}(\Gamma)$ and $H^{-1/2}(\Gamma) \subset \mathcal{H}^{-3/2}(\Gamma)$ are dense (see (33)), we deduce that:

$$\langle\!\langle p, \gamma_n v \rangle\!\rangle_{-\frac{1}{2},\frac{1}{2}} - \langle\!\langle q, \gamma_d v \rangle\!\rangle_{-\frac{3}{2},\frac{3}{2}} = 0, \tag{37a}$$

for every $(q,p) \in \mathcal{H}^{-3/2}(\Gamma) \times \mathcal{H}^{-1/2}(\Gamma)$ such that:

$$\langle\!\langle q, \mathbf{1}_\Gamma \rangle\!\rangle_{-\frac{3}{2},\frac{3}{2}} = 0 \quad \text{and} \quad \langle\!\langle q, y_k \rangle\!\rangle_{-\frac{3}{2},\frac{3}{2}} - \langle\!\langle p, n_k \rangle\!\rangle_{-\frac{1}{2},\frac{1}{2}} = 0 \quad (k=1,2). \tag{37b}$$

Notice that, for $k = 1, 2$, y_k and n_k are the Dirichlet and Neumann traces of affine functions and therefore that they are respectively in $\mathcal{H}^{3/2}(\Gamma)$ and $\mathcal{H}^{1/2}(\Gamma)$. Both equalities (37) mean that there exist three real numbers λ_1, λ_2 and λ_3 such that:

$$\begin{pmatrix} \gamma_d v \\ \gamma_n v \end{pmatrix} = \lambda_1 \begin{pmatrix} y_1 \\ n_1 \end{pmatrix} + \lambda_2 \begin{pmatrix} y_2 \\ n_2 \end{pmatrix} + \lambda_3 \begin{pmatrix} \mathbf{1}_\Gamma \\ 0 \end{pmatrix}.$$

We deduce that v minus a linear combination of affine functions is in $W^2_d(\mathbb{R}^2) \cap W^2_n(\mathbb{R}^2)$. But since $\mathscr{A} \subset \big(W^2_d(\mathbb{R}^2) \cap W^2_n(\mathbb{R}^2)\big)^\perp$ and $v \in \big(W^2_d(\mathbb{R}^2) \cap W^2_n(\mathbb{R}^2)\big)^\perp$, this function is also in $\big(W^2_d(\mathbb{R}^2) \cap W^2_n(\mathbb{R}^2)\big)^\perp$. If follows that $v \in \mathscr{A}$ and $w = \Delta v = 0$, which concludes the proof. □

6 Traces and jump relations

According to Theorem 2.1 and Proposition 2.1, the one-sided trace operators $\gamma_d^\pm$ and $\gamma_n^\pm$ are well defined on $\mathscr{H}^1(\mathbb{R}^2 \setminus \Gamma)$ and bounded. The purpose of this section is to extend these operators to the space $\mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$. Since the single and double layer potentials defined in Theorem 4.1 are equal to the sum of a function in $\mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$ plus a classical single or double layer potential, we will be able to deduce at once traces results for surface potentials. As in Section 5, we continue assuming that Γ is a $\mathcal{C}^{1,1}$ curvilinear polygon. Recall that $\mathscr{B}_n(\mathbb{R}^2) = \big(W^2_d(\mathbb{R}^2) \cap W^2_n(\mathbb{R}^2)\big)^\perp \cap W^2_n(\mathbb{R}^2)$ and $\mathscr{B}_d(\mathbb{R}^2) = \big(W^2_d(\mathbb{R}^2) \cap W^2_n(\mathbb{R}^2)\big)^\perp \cap W^2_d(\mathbb{R}^2)$.

To every $v \in \mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$, we associate $\mathsf{J}_d v$ the element of $\mathcal{H}^{-1/2}_d(\Gamma)$ defined by:

$$\langle\!\langle \mathsf{J}_d v, q \rangle\!\rangle_{-\frac{1}{2},\frac{1}{2},d} = -\big(v, \Delta \mathcal{L}_d q\big)_{L^2(\mathbb{R}^2)} \qquad \text{for all } q \in \mathcal{H}^{1/2}_d(\Gamma), \tag{38}$$

where the operator $\mathcal{L}_d$ is given in (21b). We are going to show that $\mathsf{J}_d v$ is the "jump" of the one-sided Dirichlet traces of v across Γ. We denote by $\mathscr{H}^0_d(\mathbb{R}^2 \setminus \Gamma)$ the image of the space $\mathscr{B}_d(\mathbb{R}^2)$ by the Laplacian. The operator:

$$\begin{aligned} \Delta_d : \mathscr{B}_d(\mathbb{R}^2) &\longrightarrow \mathscr{H}^0_d(\mathbb{R}^2 \setminus \Gamma) \\ u &\longmapsto \Delta u, \end{aligned}$$

being isometric, $\mathscr{H}^0_d(\mathbb{R}^2 \setminus \Gamma)$ is closed and we denote by Π^0_d the orthogonal projection on this space in $L^2(\mathbb{R}^2)$. It can be readily verify that:

$$\mathsf{J}_d v = -\mathcal{T}_d \circ \mathcal{L}_d^{-1} \circ \Delta_d^{-1} \circ \Pi^0_d v \qquad \text{for all } v \in \mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma), \tag{39}$$

where the operator $\mathcal{T}_d$ is defined in (34). Since the operators $\mathcal{T}_d$, $\mathcal{L}_d$ and Δ_d are isometric, it follows that:

$$\|\mathsf{J}_d v\|_{-\frac{1}{2},d} = \|\Pi^0_d v\|_{L^2(\mathbb{R}^2)} \leqslant \|v\|_{L^2(\mathbb{R}^2)} \qquad \text{for all } v \in \mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma). \tag{40}$$

We turn now our attention to the Neumann trace operator. For every $v \in \mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$, we denote by $\mathsf{J}_n v$ the element of $\mathcal{H}^{-3/2}_n(\Gamma)$ defined by:

$$\langle\!\langle \mathsf{J}_n v, p \rangle\!\rangle_{-\frac{3}{2},\frac{3}{2},n} = -\big(v, \Delta \mathcal{L}_n p\big)_{L^2(\mathbb{R}^2)} \qquad \text{for all } p \in \mathcal{H}^{3/2}_n(\Gamma). \tag{41}$$

When $p = \mathbf{1}_\Gamma$ (which is indeed in $\mathcal{H}^{3/2}_n(\Gamma)$), $\mathcal{L}_n p = \mathbf{1}_{\mathbb{R}^2}$ and therefore, the operator J_n is valued in:

$$\widetilde{\mathcal{H}}^{-3/2}_n(\Gamma) = \big\{ q \in \mathcal{H}^{-3/2}_n(\Gamma) \ : \ \langle\!\langle q, \mathbf{1}_\Gamma \rangle\!\rangle_{-\frac{3}{2},\frac{3}{2},n} = 0 \big\}. \tag{42}$$

Looking for an expression like (39) for J_n, we introduce the orthogonal decomposition $\mathscr{B}_n = \widetilde{\mathscr{B}}_n \oplus \langle \mathbf{1}_{\mathbb{R}^2} \rangle$ of the space $\mathscr{B}_n$, where $\widetilde{\mathscr{B}}_n = \{u \in \mathscr{B}_n \ : \ \mu(\gamma_d u) = 0\}$. We introduce as well the space $\widetilde{\mathscr{H}^0_n}(\mathbb{R}^2 \setminus \Gamma) = \{v \in \mathscr{H}^0_n(\mathbb{R}^2 \setminus \Gamma) \ : \ (\mathbf{1}_{\Omega^-}, v)_{L^2(\mathbb{R}^2)} = 0\}$ and the isometric operator:

$$\begin{aligned} \widetilde{\Delta}_n : \big(\widetilde{\mathscr{B}}_n, \|\cdot\|_D\big) &\longrightarrow \widetilde{\mathscr{H}^0_n}(\mathbb{R}^2 \setminus \Gamma) \\ u &\longmapsto \Delta u. \end{aligned}$$

Recalling that the operator $\mathscr{T}_n$ is defined in (34) and denoting by $\widetilde{\Pi}_n^0$ the orthogonal projector onto $\widetilde{\mathscr{H}_n^0}(\mathbb{R}^2 \setminus \Gamma)$ in $L^2(\mathbb{R}^2)$, we establish that:

$$\mathsf{J}_n v = -\mathscr{T}_n \circ \mathcal{L}_n^{-1} \circ \widetilde{\Delta}_n^{-1} \circ \widetilde{\Pi}_n^0 v \qquad \text{for all } v \in \mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma).$$

We deduce that:

$$\|\mathsf{J}_n v\|_{-\frac{3}{2},n} = \|\widetilde{\Pi}_n^0 v\|_{L^2(\mathbb{R}^2)} \leqslant \|v\|_{L^2(\mathbb{R}^2)} \qquad \text{for all } v \in \mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma). \tag{43}$$

On the other hand, for every $v \in \mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$ we can define v^+ and v^- in $\mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$ by:

$$v^+ = \begin{cases} 0 & \text{on } \Omega^- \\ v|_{\Omega^+} & \text{on } \Omega^+ \end{cases} \qquad \text{and} \qquad v^- = \begin{cases} v|_{\Omega^-} & \text{on } \Omega^- \\ 0 & \text{on } \Omega^+. \end{cases}$$

We can now make precise the notion of trace for functions in $\mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$:

Definition 6.1. *For every function $v \in \mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$, we define the one-sided Dirichlet trace operators γ_d^+ and γ_d^- by:*

$$\begin{aligned} \gamma_d^\pm : \mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma) &\longrightarrow \mathcal{H}_d^{-1/2}(\Gamma) \\ v &\longmapsto \pm \mathsf{J}_d v^\pm. \end{aligned} \tag{44a}$$

We define as well the one-sided Neumann trace operators γ_n^+ and γ_n^- by:

$$\begin{aligned} \gamma_n^\pm : \mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma) &\longrightarrow \widetilde{\mathcal{H}}_n^{-3/2}(\Gamma) \\ v &\longmapsto \mathsf{J}_n v^\pm. \end{aligned} \tag{44b}$$

As expected, we have:

Proposition 6.1. *The operators* (44) *are bounded and are the extensions by density of the operators $\gamma_d^\pm$ and $\gamma_n^\pm$ (defined on $\mathscr{H}^1(\mathbb{R}^2 \setminus \Gamma)$) to $\mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$.*

Proof. The boundedness results from (40) and (43). Let v be in $\mathscr{H}^1(\mathbb{R}^2 \setminus \Gamma)$ and q be in $\mathcal{H}_d^{1/2}(\Gamma)$. Green's formula (3) leads to:

$$\big(v^-, \Delta \mathcal{L}_d q\big)_{L^2(\mathbb{R}^2)} = \langle q, \gamma_d^- v\rangle_{-\frac{1}{2},\frac{1}{2}} = (q, \gamma_d^- v)_{L^2(\mathbb{R}^2)} = \langle\!\langle \gamma_d^- v, q\rangle\!\rangle_{-\frac{1}{2},\frac{1}{2},d},$$

the last equality resulting from (33b) (the inclusions allowing $\gamma_d^- v$ to be considered as an element of $\mathcal{H}_d^{-1/2}(\Gamma)$ and asserting that the duality pairing $\langle\!\langle \cdot, \cdot\rangle\!\rangle_{-\frac{1}{2},\frac{1}{2},d}$ extends the L^2 inner product). On the other hand, according to (38):

$$\big(v^-, \Delta \mathcal{L}_d q\big)_{L^2(\mathbb{R}^2)} = -\langle\!\langle \mathsf{J}_d v^-, q\rangle\!\rangle_{-\frac{1}{2},\frac{1}{2},d}.$$

It follows that $\langle\!\langle \mathsf{J}_d v^- + \gamma_d^- v, q\rangle\!\rangle_{-\frac{1}{2},\frac{1}{2},d} = 0$ for every $q \in \mathcal{H}_d^{1/2}(\Gamma)$ and therefore $\mathsf{J}_d v^- = -\gamma_d^- v$ in $\mathcal{H}_d^{-1/2}(\Gamma)$. The proof of the other equalities ($\gamma_d^+ v = \mathsf{J}_d v^+$ and $\gamma_n^\pm v = \mathsf{J}_n v^\pm$) follows from the same arguments. ☐

For every $u \in \mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$, we introduce the classical notations:

$$\big[\gamma_d u\big]_\Gamma = \gamma_d^+ u^+ - \gamma_d^- u^- \qquad \text{and} \qquad \big[\gamma_n u\big]_\Gamma = \gamma_n^+ u^+ + \gamma_n^- u^-,$$

so that $\big[\gamma_d u\big]_\Gamma = \mathsf{J}_d v$ in (38) and $\big[\gamma_n u\big]_\Gamma = \mathsf{J}_n v$ in (41). Before proving the jump relations for the single and the double layer potentials, we need to establish a preliminary technical result:

Lemma 6.1. *1. For every $p \in \mathcal{H}^{3/2}(\Gamma)$, $\big[\gamma_d \Delta \mathsf{L}_d^S p\big]_\Gamma = 0$ and for every $q \in \mathcal{H}^{1/2}(\Gamma)$, $\big[\gamma_n \Delta \mathsf{L}_n^D p\big]_\Gamma = 0$.*

2. Recall the the operator T_d and T_n are defined in (24) *and the operators $\mathscr{T}_d$ and $\mathscr{T}_n$ in* (34)*. The following identities hold:*

$$-\big[\gamma_n \Delta \mathsf{L}_d^S p\big]_\Gamma + \sum_{k=1}^{3} \langle \mathsf{q}_k, p\rangle_{-\frac12,\frac12} \mathsf{q}_k = \mathsf{T}_d p \qquad \textit{for all } p \in \mathcal{H}^{3/2}(\Gamma), \tag{45a}$$

$$-\big[\gamma_d \Delta \mathsf{L}_n^D q\big]_\Gamma + \sum_{j=1}^{2} (\mathsf{p}_j, q)_{L^2(\Gamma)} \mathsf{p}_j = \mathsf{T}_n q \qquad \textit{for all } q \in \mathcal{H}^{1/2}(\Gamma), \tag{45b}$$

$$-\big[\gamma_n \Delta \mathcal{L}_n p\big]_\Gamma + \mu(p)|\Gamma|^{-1}\mathbf{1}_\Gamma = \mathcal{T}_n p \qquad \textit{for all } p \in \mathcal{H}_n^{3/2}(\Gamma) \tag{45c}$$

$$-\big[\gamma_d \Delta \mathcal{L}_d q\big]_\Gamma = \mathcal{T}_d q \qquad \textit{for all } q \in \mathcal{H}_d^{1/2}(\Gamma). \tag{45d}$$

From (45a) *and* (45b) *we deduce in particular that the operators:*

$$\mathsf{T}_d : \mathscr{A}_S^{1/2} \longrightarrow \mathscr{A}_S^{-1/2} \qquad \textit{and} \qquad \mathsf{T}_n : \mathscr{A}_D^{-1/2} \longrightarrow \mathscr{A}_D^{1/2} \tag{45e}$$

are isometric.

Proof. The first assertion of the Lemma results from the combination of (15) and (21) (that make precise the ranges of the operators L_d^S, L_n^D, $\mathcal{L}_d$ and $\mathcal{L}_n$) and the definitions (38) and (41) of the jumps of the Dirichlet and Neumann one-sided traces. For the second assertion, let us verify that for every $p \in \mathcal{H}^{3/2}(\Gamma)$:

$$-\big[\gamma_n \Delta \mathsf{L}_d^S p\big]_\Gamma + \sum_{k=1}^{3} \langle \mathsf{q}_k, p\rangle_{-\frac12,\frac12} \mathsf{q}_k = \mathsf{T}_d p,$$

where T_d is the isometric operator defined in (24). Thus, we have:

$$\langle\!\langle \mathsf{T}_d p, \gamma_d \theta\rangle\!\rangle_{-\frac32,\frac32} = \big(\mathsf{L}_d^S p, \mathsf{L}_d^S \circ \gamma_d \theta\big)_S \qquad \text{for all } \theta \in W^2(\mathbb{R}^2).$$

On the other hand, according to (13):

$$(\mathsf{L}_d^S p, \mathsf{L}_d^S \circ \gamma_d \theta)_S = (\mathsf{L}_d^S p, \Pi_d^S \theta)_S = (\mathsf{L}_d^S p, \theta)_S.$$

Choosing $\theta = \mathcal{L}_n \tilde{p}$ with $\tilde{p}$ any element in the space $\mathcal{H}_n^{3/2}(\Gamma)$, we get:

$$\langle\!\langle \mathsf{T}_d p, \tilde{p}\rangle\!\rangle_{-\frac32,\frac32} = (\mathsf{L}_d^S p, \mathcal{L}_n \tilde{p})_S = (\Delta \mathsf{L}_d^S p, \Delta \mathcal{L}_n \tilde{p})_{L^2(\mathbb{R}^2)} + \sum_{k=1}^{3} \langle \mathsf{q}_k, p\rangle_{-\frac12,\frac12} \langle \mathsf{q}_k, \tilde{p}\rangle_{-\frac12,\frac12},$$

which, once compared with (41), yields the result. The proof of the other equalities are similar.

Regarding (45e), it suffices to notice that $\mathsf{L}_d^S p$ is an affine function when $p \in \mathscr{A}_S^{1/2}$ and the same observation applies to $\mathsf{L}_n^D q$ if q belongs to $\mathscr{A}_D^{-1/2}$. We conclude with equalities (45a) and (45b). □

With the definition of the trace operators given in Definition 6.1:

Proposition 6.2 (Jump relations)**.** *The following equalities hold:*

$$\gamma_d^+ \circ \mathscr{S}_\Gamma^\dagger - \gamma_d^- \circ \mathscr{S}_\Gamma^\dagger = 0 \qquad \gamma_n^+ \circ \mathscr{D}_\Gamma^\dagger + \gamma_n^- \circ \mathscr{D}_\Gamma^\dagger = 0 \tag{46a}$$

$$\gamma_n^+ \circ \mathscr{S}_\Gamma^\dagger + \gamma_n^- \circ \mathscr{S}_\Gamma^\dagger = \mathrm{Id} \qquad \gamma_d^+ \circ \mathscr{D}_\Gamma^\dagger - \gamma_d^- \circ \mathscr{D}_\Gamma^\dagger = \mathrm{Id}. \tag{46b}$$

Notice that the operators $\gamma_n^\pm \circ \mathscr{S}_\Gamma^\dagger$ map $\mathcal{H}^{-3/2}(\Gamma)$ into the larger space $\mathcal{H}_n^{-3/2}(\Gamma)$. The first relation in (46b) means that there is some sort of compensation which makes the sum of both terms $\gamma_n^\pm \circ \mathscr{S}_\Gamma^\dagger$ more regular than each one taken separately. The same remark holds for the second identity in (46b) with the spaces $\mathcal{H}^{-1/2}(\Gamma)$ and $\mathcal{H}_d^{-1/2}(\Gamma)$. This contrasts with what happens when the domain is of class $\mathcal{C}^{1,1}$ where $\mathcal{H}^{-3/2}(\Gamma) = \mathcal{H}_n^{-3/2}(\Gamma) = H^{-3/2}(\Gamma)$ and $\mathcal{H}^{-1/2}(\Gamma) = \mathcal{H}_d^{-1/2}(\Gamma) = H^{-1/2}(\Gamma)$.

Proof. Let q be in $\mathcal{H}^{-3/2}(\Gamma)$. According to (25a):

$$\mathscr{S}_\Gamma^\dagger q = -\Delta \mathsf{L}_d^S \circ \mathsf{T}_d^{-1} q + \sum_{j=1}^{3} \langle \mathsf{q}_j, \mathsf{T}_d^{-1} q\rangle_{-\frac12,\frac12} \mathscr{S}_\Gamma \mathsf{q}_j.$$

For every $\tilde{q} \in \mathcal{H}_d^{1/2}(\Gamma)$:

$$\big(\Delta \mathsf{L}_d^S \circ \mathsf{T}_d^{-1} q, \Delta \mathcal{L}_d \tilde{q}\big)_{L^2(\mathbb{R}^2)} = \big(\mathsf{L}_d^S \circ \mathsf{T}_d^{-1} q, \mathcal{L}_d \tilde{q}\big)_S = 0,$$

because $\mathsf{L}_d^S \circ \mathsf{T}_d^{-1} q \in W_d^2(\mathbb{R}^2)^\perp$ (see (15a)) and $\mathcal{L}_d \tilde{q} \in W_d^2(\mathbb{R}^2)$ (see (21b)). According to (38), we deduce that

$$\mathsf{J}_d\big(\Delta \mathsf{L}_d^S \circ \mathsf{T}_d^{-1} q\big) = \big[\gamma_d \Delta \mathsf{L}_d^S \circ \mathsf{T}_d^{-1} q\big]_\Gamma = 0,$$

and since $\big[\gamma_d \mathscr{S}_\Gamma \mathsf{q}_j\big]_\Gamma = 0$ for $j = 1, 2, 3$, we have proved the first equality in (46a). Continuing with the single layer potential, the first equality in (46b) is a straightforward consequence of (45a). The proofs of the relations related to the double layer potential are similar. □

The rest of this section is devoted to establishing additional properties concerning the traces of harmonic functions in $L^2_{\ell oc}(\mathbb{R}^2)$. To do this, we must first establish a technical lemma.

Lemma 6.2. *For every $u \in \mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$, there exist*

- *$p_1 \in \mathcal{H}^{3/2}(\Gamma)$ and $q_1 \in \mathcal{H}_d^{1/2}$ such that $u = \Delta \mathsf{L}_d^S p_1 + \Delta \mathcal{L}_d q_1$;*
- *$p_2 \in \mathcal{H}_d^{3/2}(\Gamma)$ and $q_2 \in \mathcal{H}^{1/2}$ such that $u = \Delta \mathcal{L}_n p_2 + \Delta \mathsf{L}_n^D q_2$.*

Proof. Let u be in $\mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$. According to (35), there exists $v \in \big(W_d^2(\mathbb{R}^2) \cap W_n^2(\mathbb{R}^2)\big)^\perp$ such that $u = \Delta v$. In line with the orthogonal decomposition in $\big(W^2(\mathbb{R}^2); \|\cdot\|_S\big)$:

$$\big(W_d^2(\mathbb{R}^2) \cap W_n^2(\mathbb{R}^2)\big)^\perp = \underbrace{\Big[\big(W_d^2(\mathbb{R}^2) \cap W_n^2(\mathbb{R}^2)\big)^\perp \cap W_d^2(\mathbb{R}^2)\Big]}_{\mathscr{B}_d(\mathbb{R}^2)} \oplus W_d^2(\mathbb{R}^2)^\perp,$$

we can decompose v as $v = v_1 + v_2$ and there exists $q_1 \in \mathcal{H}_d^{1/2}(\Gamma)$ such that $v_1 = \mathcal{L}_d q_1$ (see (21b)) and $p_1 \in \mathcal{H}^{3/2}(\Gamma)$ such that $v_2 = \mathsf{L}_d^S p_1$ (see (15a)). This proves the first point of the Proposition. For the second, we use the orthogonal decomposition in $\big(W^2(\mathbb{R}^2); \|\cdot\|_D\big)$:

$$\big(W_d^2(\mathbb{R}^2) \cap W_n^2(\mathbb{R}^2)\big)^\perp = \underbrace{\Big[\big(W_d^2(\mathbb{R}^2) \cap W_n^2(\mathbb{R}^2)\big)^\perp \cap W_n^2(\mathbb{R}^2)\Big]}_{\mathscr{B}_n(\mathbb{R}^2)} \oplus W_n^2(\mathbb{R}^2)^\perp,$$

and we conclude the same way. □

Theorem 6.1. *Let u be in $\mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$.*

1. *If $[\gamma_d u]_\Gamma \in \mathcal{H}^{-1/2}(\Gamma)$ or $[\gamma_n u]_\Gamma \in \mathcal{H}^{-3/2}(\Gamma)$ then $\big([\gamma_d u]_\Gamma, [\gamma_n u]_\Gamma\big) \in \mathcal{H}^{-1/2}(\Gamma) \times \mathcal{H}^{-3/2}(\Gamma)$.*
2. *If $[\gamma_d v]_\Gamma = 0$, then $v = \mathscr{S}_\Gamma^\dagger [\gamma_n v]_\Gamma$. If $[\gamma_n v]_\Gamma = 0$, then $v = \mathscr{D}_\Gamma^\dagger [\gamma_d v]_\Gamma$.*
3. *If $[\gamma_d u]_\Gamma = 0$ and $[\gamma_n u]_\Gamma = 0$ then $u = 0$.*

The first point seems particularly noteworthy. It means that as soon as the jump of the one-sided Dirichlet traces or the jump of the one-sided Neumann traces is "regular" (in full generality $[\gamma_d u]_\Gamma$ is only in $\mathcal{H}_d^{-1/2}(\Gamma)$ and $[\gamma_n u]_\Gamma$ in $\mathcal{H}_n^{-3/2}(\Gamma)$), the other jump inherits the same regularity.

Proof. Addressing the first point of the Theorem, let assume that $[\gamma_d u]_\Gamma \in \mathcal{H}^{-1/2}(\Gamma)$. According to (4) (the asymptotic expansions of the single layer potential and of harmonic functions) there exists $\mathsf{q} \in \mathscr{A}_S^{-1/2}$ such that $\mathscr{D}_\Gamma^\dagger [\gamma_d u]_\Gamma - \mathscr{S}_\Gamma \mathsf{q} \in L^2(\mathbb{R}^2)$. Let $v = u - \mathscr{D}_\Gamma^\dagger [\gamma_d u]_\Gamma + \mathscr{S}_\Gamma \mathsf{q}$. This function is in $\mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$ and satisfies $[\gamma_d v]_\Gamma = 0$ and $[\gamma_n v]_\Gamma = [\gamma_n u]_\Gamma + \mathsf{q}$. This entails that for our purpose, up to replacing u by v, we can assume that $[\gamma_d u]_\Gamma = 0$. According to Lemma 6.2, the function u can be decomposed as $u = \Delta \mathsf{L}_d^S p_1 + \Delta \mathcal{L}_d q_1$ with $p_1 \in \mathcal{H}^{3/2}(\Gamma)$ and $q_1 \in \mathcal{H}_d^{1/2}$. Invoking next the first point of Lemma 6.1, we obtain:

$$\big[\gamma_d u\big]_\Gamma = \big[\gamma_d \Delta \mathsf{L}_d^S p_1\big]_\Gamma + \big[\gamma_d \Delta \mathcal{L}_d q_1\big]_\Gamma = \big[\gamma_d \Delta \mathcal{L}_d q_1\big]_\Gamma = 0,$$

which entails that $q_1 = 0$ with (45d). It follows that $\big[\gamma_n u\big]_\Gamma = \big[\gamma_n \Delta \mathsf{L}_d^S p_1\big]_\Gamma$ and therefore $\big[\gamma_n u\big]_\Gamma \in \mathcal{H}^{-3/2}(\Gamma)$ according to (45a).

In a similar fashion, assuming that $[\gamma_n u]_\Gamma \in \mathcal{H}^{-3/2}(\Gamma)$ can be reduced to assuming that $[\gamma_n u]_\Gamma = 0$ up to replacing u by $u - \mathscr{S}_\Gamma^\dagger[\gamma_n u]_\Gamma + \mathscr{D}_\Gamma \mathsf{p}$ for some $\mathsf{p} \in \mathscr{A}_D^{1/2}$. Then we use the latter decomposition provided by Lemma 6.2:

$$u = \Delta\mathcal{L}_n p_2 + \Delta\mathsf{L}_n^D q_2,$$

for some $p_2 \in \mathcal{H}_d^{3/2}(\Gamma)$ and $q_2 \in \mathcal{H}^{1/2}$. Based on Lemma 6.1, we deduce that:

$$[\gamma_n u]_\Gamma = \left[\gamma_n \Delta\mathcal{L}_n p_2\right]_\Gamma + \left[\gamma_n \Delta\mathsf{L}_n^D q_2\right]_\Gamma = \left[\gamma_n \Delta\mathcal{L}_n p_2\right]_\Gamma = 0.$$

This condition means, according to (45c), that $\mathcal{T}_n p_2 = \mu(p_2)|\Gamma|^{-1}\mathbf{1}_\Gamma$ where the operator $\mathcal{T}_n$ is defined in (34). We deduce that $p_2 = \mu(p_2)\mathbf{1}_\Gamma$ and then that $\mathcal{L}_n p_2 = \mu(p_2)\mathbf{1}_{\mathbb{R}^2}$. Finally, $\Delta\mathcal{L}_n p_2 = 0$ and $[\gamma_d u]_\Gamma = \left[\gamma_d \Delta\mathsf{L}_n^D q_2\right]_\Gamma$ which belongs to $\mathcal{H}^{-1/2}(\Gamma)$ according to (45b).

We consider now the second assertion of the Theorem. If $[\gamma_d v]_\Gamma = 0$ then $[\gamma_n v]_\Gamma$ belongs to $\mathcal{H}^{-3/2}(\Gamma)$ according to the first point of the Theorem. Let q be in $\mathscr{A}_S^{-1/2}$ such that $\mathscr{S}_\Gamma^\dagger([\gamma_n v]_\Gamma - \mathsf{q})$ is in $\mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$ and introduce $u = v - \mathscr{S}_\Gamma^\dagger([\gamma_n v]_\Gamma - \mathsf{q})$. This function is in $\mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$ and satisfies $[\gamma_d u]_\Gamma = 0$ and $[\gamma_n u]_\Gamma = \mathsf{q}$. Proceeding as in the proof of the first point of the theorem, this entails that $u = \Delta\mathsf{L}_d^S p$ for some $p \in \mathcal{H}^{3/2}(\Gamma)$, with $[\gamma_n \Delta\mathsf{L}_d^S p]_\Gamma = \mathsf{q}$. This means, with (45a) that

$$\mathsf{T}_d p = \sum_{k=1}^{3} \langle \mathsf{q}_k, p\rangle_{-\frac{1}{2},\frac{1}{2}} \mathsf{q}_k - \mathsf{q},$$

and therefore that $p \in \mathscr{A}_S^{1/2}$ taking into account (45e). It follows that $\mathsf{L}_d^S p \in \mathscr{A}$ and then $u = 0$, $\mathsf{q} = 0$ and finally $v = \mathscr{S}_\Gamma^\dagger[\gamma_n v]_\Gamma$. We proceed in the same manner to prove the other statements involving the double layer potential and since the last point of the theorem is obvious, the proof is complete. □

7 The Laplace equation in L^2

In this section, we assume that Γ is a straight polygon. An important point to keep in mind when looking for solutions in $L^2_{\ell oc}$ to Dirichlet and Neumann problems in a polygonal domain, is the loss of uniqueness. Indeed, there exist non-zero harmonic functions in $L^2(\Omega^+)$ and in $L^2(\Omega^-)$ with zeros Dirichlet data or with zero Neumann data. As explained in [7], in the domain Ω^- on the left of Fig. 1, there exists a square integrable

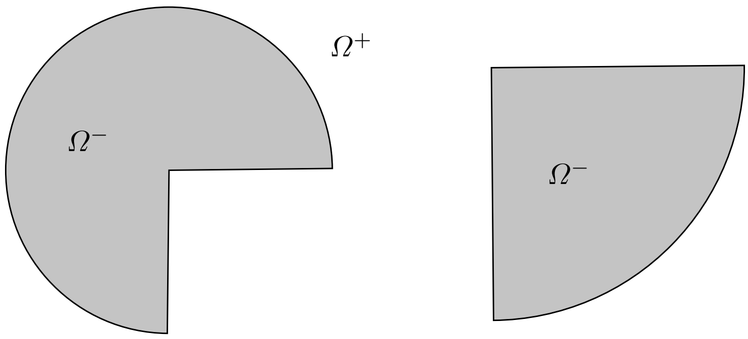

Figure 1: Examples of domains for which there exist square integrable harmonic functions with vanishing boundary data.

harmonic function with zero Dirichlet data. In polar coordinates define the function $U(r,\theta) = r^{-2/3}\sin(2\theta/3)$ and let $\eta \in \mathscr{D}(\mathbb{R}^2)$ be a cut-off function equal to 1 near the corner. Then let X be the variationnal solution in $H_0^1(\Omega^-)$ to the problem $\Delta X = \Delta(\eta U)$ (notice that the right hand side is smooth in Ω^-). The function $\eta U - X$ is in $L^2(\Omega^-)$, non zero (because X belongs to $H^1(\Omega^-)$ and ηU does not), harmonic in Ω^- and equal to zero on the boundary of the domain. There exists also a harmonic function with zero Neumann data which is equal, near the corner to $r^{-2/3}\cos(2\theta/3)$ (this is actually the harmonic conjugate of the preceeding one). The same constructions apply with the domain on the right of Fig. 1 and provide examples of non-zero harmonic functions in Ω^+ with zero Dirichlet or Neumann data.

Theorem 7.1 (Solvability of the interior Dirichlet problem). *For every $p \in \mathcal{H}_d^{-1/2}(\varGamma)$, there exists v_p in $\mathscr{H}^0(\varOmega^-)$ such that $\gamma_d^- v_p = p$. Moreover, the application $p \longmapsto v_p$ is continuous from $\mathcal{H}_d^{-1/2}(\varGamma)$ to $\mathscr{H}^0(\varOmega^-)$.*

Proof. Recall that, for every $q \in \mathcal{H}_d^{1/2}(\varGamma)$:

$$\|q\|_{\frac{1}{2},d} = \inf\left\{\|u\|_S \,:\, u \in W_d^2(\mathbb{R}^2),\, \gamma_n u = q\right\} = \|\mathcal{L}_d q\|_S.$$

According to [1, Theorem 10.4.1], there exists an extension operator from $H^2(\varOmega^-)$ to $H^2(\mathbb{R}^2)$ and since $H^2(\mathbb{R}^2)$ is continuously embedded in $W^2(\mathbb{R}^2)$, we can assume that this operator is valued in this latter space. This yields the existence of a constant $C > 0$ such that:

$$\|q\|_{\frac{1}{2},d} \leqslant C\|\mathcal{L}_d q|_{\varOmega^-}\|_{H^2(\varOmega^-)} \qquad \text{for all } q \in \mathcal{H}_d^{1/2}(\varGamma). \tag{47a}$$

On the other hand, according to [10, Theorem 4.3.1.4] and since $\varOmega^-$ is assumed to be a straight polygonal domain, there exists a constant $C > 0$ such that:

$$\|\mathcal{L}_d q|_{\varOmega^-}\|_{H^2(\varOmega^-)} \leqslant C\big(\|\Delta\mathcal{L}_d q\|_{L^2(\varOmega^-)} + \|\mathcal{L}_d q\|_{L^2(\varOmega^-)}\big) \qquad \text{for all } q \in \mathcal{H}_d^{1/2}(\varGamma). \tag{47b}$$

We deduce from both estimates (47) that, on the space $\mathcal{H}_d^{1/2}(\varGamma)$, the norm deriving from the scalar product:

$$\big(\Delta\mathcal{L}_d q_1, \Delta\mathcal{L}_d q_2\big)_{L^2(\varOmega^-)} + \big(\mathcal{L}_d q_1, \mathcal{L}_d q_2\big)_{L^2(\varOmega^-)} \qquad \text{for all } q_1, q_2 \in \mathcal{H}_d^{1/2}(\varGamma),$$

is equivalent to the norm $\|\cdot\|_{\frac{1}{2},d}$ associated to the scalar product (20). Riesz representation theorem asserts that for every $p \in \mathcal{H}_d^{-1/2}(\varGamma)$, there exists $q_p \in \mathcal{H}_d^{1/2}(\varGamma)$ such that:

$$\langle\!\langle p, q\rangle\!\rangle_{-\frac{1}{2},\frac{1}{2},d} = \big(\Delta\mathcal{L}_d q_p, \Delta\mathcal{L}_d q\big)_{L^2(\varOmega^-)} + \big(\mathcal{L}_d q_p, \mathcal{L}_d q\big)_{L^2(\varOmega^-)} \qquad \text{for all } q \in \mathcal{H}_d^{1/2}(\varGamma),$$

and that the mapping $p \longmapsto q_p$ is continuous from $\mathcal{H}_d^{-1/2}(\varGamma)$ to $\mathcal{H}_d^{1/2}(\varGamma)$. Let now w_p be the unique solution in $H_0^1(\varOmega^-)$ of the variational Dirichlet problem:

$$(\nabla w_p, \nabla\theta)_{L^2(\varOmega^-;\mathbb{R}^2)} = -\big(\mathcal{L}_d q_p, \theta\big)_{L^2(\varOmega^-)} \qquad \text{for all } \theta \in H_0^1(\varOmega^-).$$

Since $p \longmapsto \mathcal{L}_d q_p|_{\varOmega^-}$ is continuous from $\mathcal{H}_d^{-1/2}(\varGamma)$ to $H^2(\varOmega^-)$, we deduce that $p \longmapsto w_p$ is also continuous from $\mathcal{H}_d^{-1/2}(\varGamma)$ to $H_0^1(\varOmega^-)$. Applying Green's formula (3), we then obtain that:

$$\big(\mathcal{L}_d q_p, \mathcal{L}_d q\big)_{L^2(\varOmega^-)} = \big(w_p, \Delta\mathcal{L}_d q\big)_{L^2(\varOmega^-)} \qquad \text{for all } q \in \mathcal{H}_d^{1/2}(\varGamma).$$

Let $\Pi_{\varOmega^-} : L^2(\varOmega^-) \longrightarrow \mathscr{H}^0(\varOmega^-)$ be the Bergman projection i.e. the orthogonal projection in $L^2(\varOmega^-)$ onto the closed subspace $\mathscr{H}^0(\varOmega^-)$ of the harmonic functions and define the function $v_p = \Delta\mathcal{L}_d q_p + \Pi_{\varOmega^-} w_p$. It is clear that the mapping $p \longmapsto v_p$ is continuous from $\mathcal{H}_d^{-1/2}(\varGamma)$ to $\mathscr{H}^0(\varOmega^-)$ and since:

$$\langle\!\langle p, q\rangle\!\rangle_{-\frac{1}{2},\frac{1}{2},d} = \big(v_p, \Delta\mathcal{L}_d q\big)_{L^2(\varOmega^-)} \qquad \text{for all } q \in \mathcal{H}_d^{1/2}(\varGamma),$$

the proof is completed. □

Recall that $\mathscr{A}_S^{\frac{1}{2}}$ is the three dimensional subspace of $H^{1/2}(\varGamma)$ spanned by the traces of the affine functions. Since $H^{1/2}(\varGamma) \subset L^2(\varGamma) \subset \mathcal{H}_d^{-1/2}(\varGamma)$, the space $\mathscr{A}_S^{\frac{1}{2}}$ can also be seen as a subspace of $\mathcal{H}_d^{-1/2}(\varGamma)$ and we denote by $\mathscr{A}_{S,d}^{1/2}$ the space $\mathcal{T}_d^{-1}\mathscr{A}_S^{\frac{1}{2}}$, where the operator $\mathcal{T}_d$ is defined in (34). Therefore $\mathscr{A}_{S,d}^{1/2}$ is the subspace of $\mathcal{H}_d^{1/2}(\varGamma)$ such that $[\gamma_d \Delta\mathcal{L}_d q]_\varGamma \in \mathscr{A}_S^{1/2}$ for every $q \in \mathscr{A}_{S,d}^{1/2}$. It follows that

$$\big(\mathscr{A}_{S,d}^{1/2}\big)^\perp = \big\{q \in \mathcal{H}_d^{1/2}(\varGamma) \,:\, (q,\theta)_{L^2(\varGamma)} = 0 \quad \text{for all } \theta \in \mathscr{A}_S^{1/2}\big\}.$$

The spaces $\mathscr{A}_{S,d}^{1/2}$ and $\big(\mathscr{A}_{S,d}^{1/2}\big)^\perp$ will be used in the proof of the next theorem.

Theorem 7.2 (Solvability of the exterior Dirichlet problem)**.** *For every $p \in \mathcal{H}_d^{-1/2}(\Gamma)$, there exists v_p in $\mathscr{H}^0(\Omega^+)$ and $\mathsf{q}_p \in \mathscr{A}_S^{-1/2}$ such that $\gamma_d^+(v_p + \mathscr{S}_\Gamma \mathsf{q}_p) = p$.*

The continuity of the solution with respect to the boundary data is not clear so far. The proof relies on the following Lemma:

Lemma 7.1. *Let ξ be a distribution in $H^{-1}(\Omega^+)$ compactly supported in $\overline{\Omega^+}$. Then there exists $p \in \mathscr{A}_S^{\frac{1}{2}}$ such that the Dirichlet problem*

$$-\Delta u = \xi \quad \text{in } \Omega^+ \qquad \text{and} \qquad \gamma_d^+ u = p \quad \text{on } \Gamma, \tag{48}$$

admits a solution in $H^1(\Omega^+)$. If $\xi \in L^2(\Omega^+)$, this solution is in $H^1(\Omega^+, \Delta)$.

Proof. Following the method described in [3], we introduce the weighted Sobolev space (remind that the functions ρ and lg were defined earlier in (7)):

$$W_d^1(\Omega^+) = \Big\{ u \in \mathscr{D}'(\Omega^+) \,:\, \frac{u}{\rho \lg} \in L^2(\Omega^+), \, \frac{\partial u}{\partial x_j} \in L^2(\Omega^+), \, (j = 1, 2) \text{ and } \gamma_d^+ u = 0 \Big\}.$$

This space is strictly bigger than $H_0^1(\Omega^+)$ and its purpose is that the norm deriving from the scalar product:

$$(\nabla u_1, \nabla u_2)_{L^2(\Omega^+;\mathbb{R}^2)} \qquad \text{for all } u_1, u_2 \in W_d^1(\Omega^+),$$

is equivalent to the natural norm (i.e. Poincaré inequality holds true). Furthermore, the space $\mathscr{D}(\Omega^+)$ is dense in $W_d^1(\Omega^+)$ so that $W_d^1(\Omega^+)$ is a distribution space. Let v be the solution in $W_d^1(\Omega^+)$ of the variational problem:

$$(\nabla v, \nabla \theta)_{L^2(\Omega^+;\mathbb{R}^2)} = \langle \xi, \theta \rangle_{H^{-1}(\Omega^+), H^1(\Omega^+)} \qquad \text{for all } \theta \in W_d^1(\Omega^+). \tag{49}$$

Since $W_d^1(\Omega^+) \subset H_{\ell oc}^1(\Omega^+)$ the term in the right hand side makes sens, recalling that ξ is assumed to be compactly supported. The function v being harmonic outside a compact set, according to (4c), it can be expanded in this region as:

$$v(x) = \sum_{j=0}^{+\infty} \mathfrak{p}_m(x) + \mathfrak{q}_0 \ln |x| + \sum_{j=0}^{+\infty} \frac{\mathfrak{q}_m(x)}{|x|^{2m}},$$

where $\mathfrak{q}_0 \in \mathbb{R}$ and, for every integer m, $\mathfrak{p}_m, \mathfrak{q}_m$ are harmonic polynomials on $\mathbb{R}^2$ of degree m. By definition of $W_d^1(\Omega^+)$, the function $v/(\rho \lg)$ is in $L^2(\Omega^+)$, which entails that $\mathfrak{p}_m = 0$ for every $m \geqslant 1$ and $\mathfrak{q}_0 = 0$. On the other hand, according to (4a), there exists $\mathsf{q} \in \mathscr{A}_S^{-1/2}(\Gamma)$ (the subspace of $H^{-1/2}(\Gamma)$ spanned by $\mathsf{q}_1, \mathsf{q}_2$ and q_2) such that, for $|x|$ large:

$$\mathscr{S}_\Gamma \mathsf{q}(x) = \frac{\mathfrak{q}_1(x)}{|x|^2} + \mathscr{O}(1/|x|^2).$$

The function $u = v - \mathscr{S}_\Gamma \mathsf{q}|_{\Omega^+} - \mathfrak{p}_0 \mathbf{1}_{\Omega^+}$ is the solution we are looking for (in particular it is in $L^2(\Omega^+)$). □

We can now carry out the

Proof of Theorem 7.2. Let D^- and D^+ be two large open disks containing Ω^- such that $\overline{\Omega^-} \subset D^-$ and $\overline{D^-} \subset D^+$. Let χ be a smooth cut-off function defined in $\mathbb{R}^2$ such that $0 \leqslant \chi \leqslant 1$, $\chi = 1$ in D^- and $\chi = 0$ in $\mathbb{R}^2 \setminus \overline{D^+}$. Following the lines of the proof of Theorem 7.1, the norm deriving from the scalar product:

$$\big(\Delta \chi \mathcal{L}_d q_1, \Delta \chi \mathcal{L}_d q_2\big)_{L^2(\Omega^+)} + \big(\chi \mathcal{L}_d q_1, \chi \mathcal{L}_d q_2\big)_{L^2(\Omega^+)} \qquad \text{for all } q_1, q_2 \in \mathcal{H}_d^{1/2}(\Gamma),$$

is equivalent to the norm $\| \cdot \|_{\frac{1}{2},d}$ associated to the scalar product (20) in the space $\mathcal{H}_d^{1/2}(\Gamma)$. Applying Riesz representation Theorem we deduce that for every $p \in \mathcal{H}_d^{-1/2}(\Gamma)$, there exists $q_p \in \mathcal{H}_d^{1/2}(\Gamma)$ such that:

$$\langle\!\langle p, q \rangle\!\rangle_{-\frac{1}{2},\frac{1}{2},d} = \big(\Delta \chi \mathcal{L}_d q_p, \Delta \chi \mathcal{L}_d q\big)_{L^2(\Omega^+)} + \big(\chi \mathcal{L}_d q_p, \chi \mathcal{L}_d q\big)_{L^2(\Omega^+)} \qquad \text{for all } q \in \mathcal{H}_d^{1/2}(\Gamma). \tag{50}$$

The first term in the right hand side is next expanded as follows:

$$\big(\Delta\chi\mathcal{L}_d q_p, \Delta\chi\mathcal{L}_d q\big)_{L^2(\Omega^+)} = \big(\chi\Delta\chi\mathcal{L}_d q_p, \Delta\mathcal{L}_d q\big)_{L^2(\Omega^+)} + 2\big(\Delta\chi\mathcal{L}_d q_p\nabla\chi, \nabla\mathcal{L}_d q\big)_{L^2(\Omega^+,\mathbb{R}^2)} + \big(\Delta\chi\Delta\chi\mathcal{L}_d q_p, \mathcal{L}_d q\big)_{L^2(\Omega^+)}. \quad (51)$$

Focusing on the second term of this expansion, Lemma 7.1 ensures the existence of a function $u_1 \in H^1(\Omega^+)$ such that $\gamma_d^+ u_1 \in \mathscr{A}_S^{1/2}$ and:

$$(\nabla u_1, \nabla\theta)_{L^2(\Omega^+,\mathbb{R}^2)} = \big(\Delta\chi\mathcal{L}_d q_p\nabla\chi, \nabla\theta\big)_{L^2(\Omega^+,\mathbb{R}^2)} \qquad \text{for all } \theta \in W_d^1(\Omega^+).$$

For every $k \geqslant 1$ denote by $\mathcal{L}_d^k q$ the function $\phi_k\mathcal{L}_d q$ where ϕ_k is the cut-off function mentioned in Proposition 3.1. For k large enough, since the function $\Delta\chi\mathcal{L}_d q_p\nabla\chi$ is compactly supported, we are allowed to write:

$$\begin{aligned}\big(\Delta\chi\mathcal{L}_d q_p\nabla\chi, \nabla\mathcal{L}_d q\big)_{L^2(\Omega^+,\mathbb{R}^2)} &= \big(\nabla u_1, \nabla\mathcal{L}_d^k q\big)_{L^2(\Omega^+,\mathbb{R}^2)}\\ &= -\big(\gamma_d^+ u_1, q\big)_{L^2(\Gamma)} - \big(u_1, \Delta\mathcal{L}_d^k q\big)_{L^2(\Omega^+)}.\end{aligned}$$

Assume from now on that q is in $\big(\mathscr{A}_{S,d}^{1/2}\big)^\perp$. If follows that $(\gamma_d^+ u_1, q)_{L^2(\Gamma)} = 0$ and letting k go to $+\infty$ we obtain:

$$\big(\Delta\chi\mathcal{L}_d q_p\nabla\chi, \nabla\mathcal{L}_d q\big)_{L^2(\Omega^+,\mathbb{R}^2)} = -\big(u_1, \Delta\mathcal{L}_d q\big)_{L^2(\Omega^+)}. \quad (52a)$$

Considering now the last term in (51), we denote by u_2 the function in $H^1(\Omega^+)$, provided by Lemma 7.1, satisfying $-\Delta u_2 = \Delta\chi\Delta\chi\mathcal{L}_d q_p$ and $\gamma_d^+ u_2 \in \mathscr{A}_S^{\frac{1}{2}}$. For k large enough, we can write that:

$$\begin{aligned}\big(\Delta\chi\Delta\chi\mathcal{L}_d q_p, \mathcal{L}_d q\big)_{L^2(\Omega^+)} &= -\big(\Delta u_2, \mathcal{L}_d^k q\big)_{L^2(\Omega^+)}\\ &= -\big(q, \gamma_d^+ u_2\big)_{L^2(\Gamma)} - \big(u_2, \Delta\mathcal{L}_d^k q\big)_{L^2(\Omega^+)}.\end{aligned}$$

Again, since q is assumed to be in $\big(\mathscr{A}_{S,d}^{1/2}\big)^\perp$ the boundary integral vanishes and letting k go to $+\infty$ we are left with:

$$\big(\Delta\chi\Delta\chi\mathcal{L}_d q_p, \mathcal{L}_d q\big)_{L^2(\Omega^+)} = -\big(u_2, \Delta\mathcal{L}_d q\big)_{L^2(\Omega^+)}. \quad (52b)$$

In the same manner, for the second term in the right hand side of (50), there exists a function u_3 in $H^1(\Omega^+)$ such that:

$$\big(\chi\mathcal{L}_d q_p, \chi\mathcal{L}_d q\big)_{L^2(\Omega^+)} = \big(\chi^2\mathcal{L}_d q_p, \mathcal{L}_d q\big)_{L^2(\Omega^+)} = -\big(u_3, \Delta\mathcal{L}_d q\big)_{L^2(\Omega^+)}. \quad (52c)$$

Using the expressions (52) in (51) and (50), we obtain eventually:

$$\langle\!\langle p, q\rangle\!\rangle_{-\frac{1}{2},\frac{1}{2},d} = -\big(v_p, \Delta\mathcal{L}_d q\big)_{L^2(\mathbb{R}^2)} \qquad \text{for all } q \in \big(\mathscr{A}_{S,d}^{1/2}\big)^\perp,$$

where $v_p = \Pi_{\Omega^+}\big(2u_1 + u_2 + u_3 - \chi\Delta\chi\mathcal{L}_d q_p\big)$ and Π_{Ω^+} stands for the Bergman projection in $L^2(\Omega^+)$.

It remains to construct $\mathsf{q}_p \in \mathscr{A}_S^{-1/2}$ as announced in the statement of the theorem. Let $\{P_1, P_2, P_3\}$ be a basis of $\mathscr{A}$ such that $(P_k, P_j)_{L^2(\Omega^-)} = \delta_{j,k}$. For every $j = 1, 2, 3$, let $\tilde{\mathsf{q}}_j \in \mathscr{A}_S^{-1/2}$ and $\hat{q}_j \in \mathscr{A}_{S,d}^{1/2}$ be such that $\mathscr{S}_\Gamma\tilde{\mathsf{q}}_j|_{\Omega^-} = \Delta\mathcal{L}_d\hat{q}_j|_{\Omega^-} = P_j|_{\Omega^-}$. If follows that

$$\big(\mathsf{S}_\Gamma\tilde{\mathsf{q}}_j, \hat{q}_k\big)_{L^2(\Gamma)} = (P_j, P_k)_{L^2(\Omega^-)} = \delta_{j,k} \qquad \text{for all } j, k = 1, 2, 3.$$

This proves that we can always define $\mathsf{q}_p \in \mathscr{A}_S^{-1/2}$ such that:

$$\big(\mathsf{S}_\Gamma\mathsf{q}_p, q\big)_{L^2(\Gamma)} = -\langle\!\langle p, q\rangle\!\rangle_{-\frac{1}{2},\frac{1}{2},d} - \big(v_p, \Delta\mathcal{L}_d q\big)_{L^2(\Omega^+)} \qquad \text{for all } q \in \mathscr{A}_{S,d}^{1/2},$$

and completes the proof. □

Let us address now the Neumann problems. Recall that:

$$\widetilde{\mathcal{H}}_n^{-3/2}(\Gamma) = \big\{q \in \mathcal{H}_n^{-3/2}(\Gamma) \,:\, \langle\!\langle q, \mathbf{1}_\Gamma\rangle\!\rangle_{-\frac{3}{2},\frac{3}{2},n} = 0\big\}.$$

Theorem 7.3 (Solvability of the interior Neumann problem)**.** *For every $q \in \widetilde{\mathcal{H}}_n^{-3/2}(\Gamma)$ there exists v_q in $\mathscr{H}^0(\Omega^-)$ such that $\gamma_n^- v_q = q$. Moreover, the application $q \longmapsto v_q$ is continuous from $\widetilde{\mathcal{H}}_n^{-3/2}(\Gamma)$ to $\mathscr{H}^0(\Omega^-)$.*

We omit the proof which is similar to that of Theorem 7.1.

Theorem 7.4 (Solvability of the exterior Neumann problem)**.** *For every $q \in \widetilde{\mathcal{H}}_n^{-3/2}(\Gamma)$ there exists v_q in $\mathscr{H}^0(\Omega^+)$ and $\mathsf{p}_q \in \mathscr{A}_D^{1/2}$ such that $\gamma_n^+\big(v_q + \mathscr{D}_\Gamma \mathsf{p}_q\big) = q$.*

The proof is roughly the same as the one of Theorem 7.2 and rests on the following lemma:

Lemma 7.2. *Let ξ be a distribution in $H^{-1}(\Omega^+)$ either compactly supported in Ω^+ or in $L^2(\Omega^+)$ and compactly supported in $\overline{\Omega^+}$ and define the constant $\alpha = |\Gamma|^{-1}\langle \xi, \mathbf{1}_{\Omega^+}\rangle_{H^{-1}(\Omega^+),H^1(\Omega^+)}$. Then there exists $\mathsf{q} \in \mathscr{A}_D^{-1/2}$ such that the Dirichlet problem*

$$-\Delta u = \xi \quad \text{in } \Omega^+ \qquad \text{and} \qquad \gamma_n^+ u = \alpha \mathbf{1}_\Gamma + \mathsf{q} \quad \text{on } \Gamma, \tag{53}$$

admits a solution in $H^1(\Omega^+)$. The solution is in $H^1(\Omega^+, \Delta)$ if ξ is in $L^2(\Omega^+)$.

If ξ is compactly supported in Ω^+, the solution u is harmonic near the boundary Γ and the Neumann trace is well defined. If ξ is in $L^2(\Omega^+)$, then u is in $H^1(\Omega^+, \Delta)$ and again the boundary condition makes sens.

Proof. The only notable difference with the proof of Lemma 7.1 is that the space $W_0^1(\Omega^+)$ must be replaced with the space:

$$W^1(\Omega^+) = \Big\{ u \in \mathscr{D}'(\Omega^+) \,:\, \frac{u}{\rho \lg} \in L^2(\Omega^+), \, \frac{\partial u}{\partial x_j} \in L^2(\Omega^+), \, (j = 1, 2) \Big\},$$

provided with the scalar product:

$$(\nabla u_1, \nabla u_2)_{L^2(\Omega^+, \mathbb{R}^2)} + \mu(\gamma_d u_1)\mu(\gamma_d u_2) \qquad \text{for all } u_1, u_2 \in W^1(\Omega^+),$$

whose corresponding norm is equivalent to the natural norm. $\square$

8 Transmission problems

We continue assuming that Γ is a straight polygon. We are interested in the following transmission problems:

<u>Problem 1</u>: Let p be in $\mathcal{H}_d^{-1/2}(\Gamma)$ and q be in $\mathcal{H}^{-3/2}(\Gamma)$. Find $u \in L^2_{\ell oc}(\mathbb{R}^2)$ such that, for some $a \in \mathbb{R}$:

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega^- \cup \Omega^+ & \text{(54a)} \\ [\gamma_d u]_\Gamma = 0, & \text{(54b)} \\ \gamma_d u = p \quad \text{or} \quad [\gamma_n u]_\Gamma = q, & \text{(54c)} \\ u(x) = a \ln|x| + \mathscr{O}(1/|x|) \quad \text{as} \quad |x| \longrightarrow +\infty. & \text{(54d)} \end{cases}$$

Theorem 8.1. *Problem 1 admits always a solution. Any solution u is a single layer potential $\mathscr{S}_\Gamma^\dagger \bar{q}$ for some $\bar{q} \in \mathcal{H}^{-3/2}(\Gamma)$. This solution is unique if condition* (54c) *is $[\gamma_n u]_\Gamma = q$, in which case $\bar{q} = q$. If condition* (54c) *is $\gamma_d u = p$, the solution is not unique in general.*

Proof. Let u be a solution to Problem 1. Then, according to (4a), there exists $\mathsf{q} \in \mathscr{A}_S^{-1/2}$ such that the function $v = u - \mathscr{S}_\Gamma \mathsf{q}$ belongs to $\mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$. This functions satisfies furthermore $[\gamma_d v]_\Gamma = 0$ which means, applying point 2 of Theorem 6.1, that v and hence also u is a single layer potential.

If condition (54c) is $[\gamma_n u]_\Gamma = q$, the function $u = \mathscr{S}_\Gamma^\dagger q$ is indeed a solution of the transmission problem. To prove uniqueness, assume that u is a solution to the problem with $q = 0$. According to (4a), there exists $\mathsf{q} \in \mathscr{A}_S^{-1/2}$ such that $v = u + \mathscr{S}_\Gamma \mathsf{q}$ is in $L^2(\mathbb{R}^2)$. This function is in $\mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$ and satisfies $[\gamma_d v]_\Gamma = 0$ and $[\gamma_n v]_\Gamma = \mathsf{q}$. The second point of Theorem 6.1 asserts that $v = \mathscr{S}_\Gamma \mathsf{q}$ whence we deduce with (4a) again that $\mathsf{q} = 0$, and then $u = 0$.

Assume now that condition (54c) is $\gamma_d u = p$ for some given p in $\mathcal{H}_d^{-1/2}(\Gamma)$. Applying Theorem 7.2, there exists v_p^+ in $\mathscr{H}^0(\Omega^+)$ and $\mathsf{q}_p \in \mathscr{A}_S^{-1/2}$ such that $\gamma_d^+(v_p^+ + \mathscr{S}_\Gamma \mathsf{q}_p) = p$. On the other hand, Theorem 7.1 provides us with a function $v_p^- \in \mathscr{H}^0(\Omega^-)$ such that $\gamma_d^- v_p^- = p - \gamma_d^- \mathscr{S}_\Gamma \mathsf{q}_p$. Define now $v_p \in \mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$ by setting $v_p|_{\Omega^+} = v_p^+$ and $v_p|_{\Omega^-} = v_p^-$. Since $[\gamma_d v_p]_\Gamma = 0$ we are entitled to apply Theorem 6.1 which ensures us that $v_p = \mathscr{S}_\Gamma^\dagger [\gamma_n v_p]_\Gamma$. It follows that $u = \mathscr{S}_\Gamma^\dagger \bar{q}$ with $\bar{q} = [\gamma_n v_p]_\Gamma + \mathsf{q}_p$. $\square$

From this proof, we easily deduce:

Corollary 8.1. *For every u in $\mathscr{H}^0(\Omega^-)$, there exists $q \in \mathcal{H}^{-3/2}(\Gamma)$ such that $\mathscr{S}_\Gamma^\dagger q|_{\Omega^-} = u$. For every u in $L^2_{\ell oc}(\overline{\Omega^+})$, harmonic and such that $u(x) = a\ln|x| + \mathscr{O}(1/|x|)$ as $|x| \longrightarrow +\infty$ (for some $a \in \mathbb{R}$) there exists $q \in \mathcal{H}^{-3/2}(\Gamma)$ such that $\mathscr{S}_\Gamma^\dagger q|_{\Omega^+} = u$.*

__**Problem 2**__: Let p be in $\mathcal{H}^{-1/2}(\Gamma)$ and q be in $\widetilde{\mathcal{H}}_n^{-3/2}(\Gamma)$. Find $u \in L^2_{\ell oc}(\mathbb{R}^2)$ such that:

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega^- \cup \Omega^+ & (55a)\\ [\gamma_n u]_\Gamma = 0, & (55b)\\ [\gamma_d u]_\Gamma = p \quad \text{or} \quad \gamma_n u = q, & (55c)\\ u(x) = \mathscr{O}(1/|x|) \quad \text{as} \quad |x| \longrightarrow +\infty. & (55d)\end{cases}$$

The proofs of the following Theorem and Corollary are omitted because they are similar to those of Theorem 8.1 and Corollary 8.1. Introducing the space:

$$\widetilde{\mathcal{H}}^{-1/2}(\Gamma) = \big\{p \in \mathcal{H}^{-1/2}(\Gamma) \,:\, \langle\!\langle p, \mathbf{1}_\Gamma\rangle\!\rangle_{-\frac12,\frac12} = 0\big\},$$

they are stated as follows:

Theorem 8.2. *Problem 2 admits always a solution. Any solution u is a double layer potential $\mathscr{D}_\Gamma^\dagger \bar p$ for some $\bar p \in \mathcal{H}^{-1/2}(\Gamma)$. This solution is unique if condition (55c) is $[\gamma_d u]_\Gamma = p$, in which case $\bar p = p$. If condition (55c) is $\gamma_n u = q$, $\bar p$ can be chosen in $\widetilde{\mathcal{H}}^{-1/2}(\Gamma)$ and the solution is not unique in general.*

Corollary 8.2. *For every u in $\mathscr{H}^0(\Omega^-)$, there exists $p \in \mathcal{H}^{-1/2}(\Gamma)$ such that $\mathscr{D}_\Gamma^\dagger p|_{\Omega^-} = u$. For every u in $L^2_{\ell oc}(\overline{\Omega^+})$, harmonic and such that $u(x) = \mathscr{O}(1/|x|)$ as $|x| \longrightarrow +\infty$ there exists $p \in \widetilde{\mathcal{H}}^{-1/2}(\Gamma)$ such that $\mathscr{D}_\Gamma^\dagger p|_{\Omega^+} = u$.*

__**Problem 3**__: Let $p \in \mathcal{H}_d^{-1/2}(\Gamma)$ and $q \in \mathcal{H}_n^{-3/2}(\Gamma)$ be such that $p \in \mathcal{H}^{1/2}(\Gamma)$ or $q \in \mathcal{H}^{-3/2}(\Gamma)$. Find $u \in L^2_{\ell oc}(\mathbb{R}^2)$ such that, for some $a \in \mathbb{R}$:

$$\begin{cases} \Delta u = 0 \quad \text{in } \Omega^- \cup \Omega^+ & (56a)\\ [\gamma_d u]_\Gamma = p \quad \text{and} \quad [\gamma_n u]_\Gamma = q, & (56b)\\ u(x) = a\ln|x| + \mathscr{O}(1/|x|) \quad \text{as} \quad |x| \longrightarrow +\infty. & (56c)\end{cases}$$

Theorem 8.3. *Problem 3 admits a unique solution given by $u = \mathscr{S}_\Gamma^\dagger q + \mathscr{D}_\Gamma^\dagger p$.*

Proof. According to the first point of Theorem 6.1, $(p,q) \in \mathcal{H}^{-1/2}(\Gamma) \times \mathcal{H}^{-3/2}(\Gamma)$ and $u = \mathscr{S}_\Gamma^\dagger q + \mathscr{D}_\Gamma^\dagger p$ solves System (56). The uniqueness is proved in the same way as in the proof of Theorem 8.1. ☐

Proposition 8.1. *On the contrary to what happens for functions in $\mathscr{H}^1(\mathbb{R}^2\setminus\Gamma)$ (see (36)), there exist functions $u \in \mathscr{H}^0(\mathbb{R}^2 \setminus \Gamma)$ that cannot be achieved as the sum of a single and a double layer potential.*

Proof. Let u^- be in $\mathscr{H}(\Omega^-)$ such that $\gamma_d^- u^- = p$ with $p \in \mathcal{H}_d^{-1/2}(\Gamma)$ but $p \notin \mathcal{H}^{-1/2}(\Gamma)$. Define u in $\mathscr{H}^0(\mathbb{R}^2\setminus\Gamma)$ by setting $u|_{\Omega^-} = u^-$ and $u|_{\Omega^+} = 0$. Then $[\gamma_d u]_\Gamma = p \notin \mathcal{H}^{-1/2}(\Gamma)$ and therefore u cannot be the sum of a single and a double layer potential. ☐

We end this section with the question of representing the harmonic functions defined in Theorems 7.1, 7.2, 7.3 and 7.4 as layer potentials. We need to define first:

$$\widetilde{\mathcal{H}}^{-1/2}(\Gamma) = \big\{p \in \mathcal{H}^{-1/2}(\Gamma) \,:\, \langle\!\langle p, \mathbf{1}_\Gamma\rangle\!\rangle_{-\frac12,\frac12} = 0\big\}.$$

Theorem 8.4. *The bounded operators $\gamma_d \circ \mathscr{S}_\Gamma^\dagger : \mathcal{H}^{-3/2}(\Gamma) \longrightarrow \mathcal{H}_d^{-1/2}(\Gamma)$, $\gamma_n^- \circ \mathscr{S}_\Gamma^\dagger : \mathcal{H}^{-3/2}(\Gamma) \longrightarrow \widetilde{\mathcal{H}}_n^{-3/2}(\Gamma)$ and $\gamma_n^+ \circ \mathscr{S}_\Gamma^\dagger : \mathcal{H}^{-3/2}(\Gamma) \longrightarrow \mathcal{H}_n^{-3/2}(\Gamma)$ are surjective but not injective in general. The same conclusion applies for $\gamma_n \circ \mathscr{D}_\Gamma^\dagger : \widetilde{\mathcal{H}}^{-1/2}(\Gamma) \longrightarrow \widetilde{\mathcal{H}}_n^{-3/2}(\Gamma)$ and $\gamma_d^- \circ \mathscr{D}_\Gamma^\dagger : \mathcal{H}^{-1/2}(\Gamma) \longmapsto \mathcal{H}_d^{-1/2}(\Gamma)$.*

Proof. The subjectivity of $\gamma_d \circ \mathscr{S}_\Gamma^\dagger : \mathcal{H}^{-3/2}(\Gamma) \longrightarrow \mathcal{H}_d^{-1/2}(\Gamma)$ results from Theorem 8.1.

Let now q be given in $\widetilde{\mathcal{H}}_n^{-3/2}(\Gamma)$. According to Theorem 7.3, there exists a function v^- in $\mathscr{H}(\Omega^-)$ whose normal trace $\gamma_n^- v^-$ is equal to q. We apply next Theorem 7.2 which asserts the existence of a function v^+ (the sum of a function in $\mathscr{H}^0(\Omega^+)$ and a single layer potential) such that $\gamma_d^+ v^+ = \gamma_d^- v^-$. The function v defined by $v|_{\Omega^+} = v^+$ and $v|_{\Omega^-} = v^-$. This function v is a solution to the transmission problem 1 and therefore, according to Theorem 8.1, it is a single layer potential, what proves that $\gamma_n^- \circ \mathscr{S}_\Gamma^\dagger$ is surjective.

Let us verify that the range of $\gamma_n^+ \circ \mathscr{S}_\Gamma^\dagger$ is $\mathcal{H}^{-3/2}(\Gamma)$. Any q in $\mathcal{H}^{-3/2}(\Gamma)$ can be decomposed as $\bar{q} + \alpha \mathbf{1}_\Gamma$ with $\bar{q} \in \widetilde{\mathcal{H}}^{-3/2}(\Gamma)$ and $\alpha = |\Gamma|^{-1}\langle q, \mathbf{1}_\Gamma\rangle_{-\frac{1}{2},\frac{1}{2}}$. According to Theorem 7.4, there exists $v_{\bar{q}} \in \mathscr{H}^0(\Omega^+)$ and $\mathsf{p}_{\bar{q}}$ in $\mathscr{A}_D^{1/2}$ such that $\gamma_n^+(v_{\bar{q}} + \mathscr{D}_\Gamma^\dagger \mathsf{p}_{\bar{q}}) = \bar{q}$. Denote by p the external one-sided Dirichlet trace $\gamma_d^+(v_{\bar{q}} + \mathscr{D}_\Gamma^\dagger \mathsf{p}_{\bar{q}})$ which belongs to $\mathcal{H}_d^{-1/2}(\Gamma)$ and apply Theorem 7.1: There exists $u^- \in \mathscr{H}^0(\Omega^-)$ such that $\gamma_d^- u^- = p$. Define now a function v by setting $v|_{\Omega^+} = \big(v_{\bar{q}} + \mathscr{D}_\Gamma^\dagger \mathsf{p}_{\bar{q}}\big)|_{\Omega^+}$ and $v|_{\Omega^-} = u^-$. Since $[\gamma_d v]_\Gamma = 0$, we can apply Theorem 8.1 and conclude that v is a single layer potential. Denote by e_Γ the equilibrium density of Γ i.e. the unique element in $H^{-1/2}(\Gamma)$ such that $\gamma_d \circ \mathscr{S}_\Gamma \mathsf{e}_\Gamma$ is a constat function on Γ normalized in such a way that $\langle \mathsf{e}_\Gamma, \mathbf{1}_\Gamma\rangle_{-\frac{1}{2},\frac{1}{2}} = 1$ (see [16, page 263]). The function $v + (\alpha/c_\Gamma)\mathscr{S}_\Gamma \mathsf{e}_\Gamma$ (c_Γ is the constant value taken by $\mathscr{S}_\Gamma \mathsf{e}_\Gamma$ on Γ) is a preimage of q by γ_n^+.

The remaining two results are proved in the same way, so the proof is omitted. □

The last operator however deserves a special treatment:

Theorem 8.5. *Let p be given in $\mathcal{H}_d^{-1/2}(\Gamma)$. There exists a constant c and $\bar{p} \in \widetilde{\mathcal{H}}^{-1/2}(\Gamma)$ such that $\gamma_d^+ \circ \mathscr{D}_\Gamma^\dagger \bar{p} = p + c$.*

Proof. Let p be given in $\mathcal{H}_d^{-1/2}(\Gamma)$. According to Theorem 8.1, there exists $\bar{q} \in \mathcal{H}^{-3/2}(\Gamma)$ such that $\gamma_d \circ \mathscr{S}_\Gamma^\dagger \bar{q} = p$. Next, denote by $\tilde{q}$ the external one-sided Neumann trace $\gamma_n^+ \circ \mathscr{S}_\Gamma^\dagger \bar{q}$ and let $\alpha \in \mathbb{R}$ be such that $\langle\!\langle \tilde{q} + \alpha \mathsf{e}_\Gamma, \mathbf{1}_\Gamma \rangle\!\rangle_{-\frac{3}{2},\frac{3}{2},n} = 0$. It holds $\gamma_d \circ \mathscr{S}_\Gamma^\dagger(\bar{q} + \alpha \mathsf{e}_\Gamma) = p + \alpha c_\Gamma$ (recall that c_Γ is the constant value taken by the function $\gamma_d \circ \mathscr{S}_\Gamma \mathsf{e}_\Gamma$) and $\gamma_n^+ \circ \mathscr{S}_\Gamma^\dagger(\bar{q} + \alpha \mathsf{e}_\Gamma) = \tilde{q} + \alpha \mathsf{e}_\Gamma$ since $\gamma_n^- \circ \mathscr{S}_\Gamma \mathsf{e}_\Gamma = 0$. We define now a function v be setting $v|_{\Omega^+} = \big(\mathscr{S}_\Gamma^\dagger(\bar{q} + \alpha \mathsf{e}_\Gamma)\big)|_{\Omega^+}$ and $v|_{\Omega^-}$ is the solution, provided by Theorem 7.3, to the interior Neumann problem with boundary data $\tilde{q} + \alpha \mathsf{e}_\Gamma$. The function v is a solution to Problem 2 and therefore, according to Theorem 8.2, it is a double layer potential. □

A List of the main function spaces and operators

Weigthed Sobolev spaces

The space

$$W^2(\mathbb{R}^2) = \Big\{u \in \mathscr{D}'(\mathbb{R}^2) \,:\, \frac{u}{\rho^2 \lg} \in L^2(\mathbb{R}^2),\ \frac{1}{\rho \lg}\frac{\partial u}{\partial x_j} \in L^2(\mathbb{R}^2) \ \text{ and } \ \frac{\partial^2 u}{\partial x_j \partial x_k} \in L^2(\mathbb{R}^2),\ \forall\, j,k = 1,2\Big\},$$

and its subspaces

$$\begin{aligned}
W_d^2(\mathbb{R}^2) &= \big\{u \in W^2(\mathbb{R}^2) \,:\, \gamma_d u = 0\big\},\\
W_n^2(\mathbb{R}^2) &= \big\{u \in W^2(\mathbb{R}^2) \,:\, \gamma_n u = 0\big\},\\
\mathscr{B}_n(\mathbb{R}^2) &= \big(W_d^2(\mathbb{R}^2) \cap W_n^2(\mathbb{R}^2)\big)^\perp \cap W_n^2(\mathbb{R}^2),\\
\mathscr{B}_d(\mathbb{R}^2) &= \big(W_d^2(\mathbb{R}^2) \cap W_n^2(\mathbb{R}^2)\big)^\perp \cap W_d^2(\mathbb{R}^2),\\
\mathscr{A} &= \{(x_1,x_2) \longmapsto a + b_1 x_1 + b_2 x_2 \,:\, a, b_1, b_2 \in \mathbb{R}\} \quad \text{(the affine functions)}
\end{aligned}$$

are provided with either one of the scalar products:

$$\begin{aligned}
(\cdot,\cdot)_S &= (\Delta\,\cdot, \Delta\,\cdot)_{L^2(\mathbb{R}^2)} + \sum_{j=1}^{3} \langle \mathsf{q}_j, \gamma_d\,\cdot\rangle_{-\frac{1}{2},\frac{1}{2}} \langle \mathsf{q}_j, \gamma_d\,\cdot\rangle_{-\frac{1}{2},\frac{1}{2}}\\
(\cdot,\cdot)_D &= (\Delta\,\cdot, \Delta\,\cdot)_{L^2(\mathbb{R}^2)} + \sum_{j=1}^{2} (\mathsf{p}_j, \gamma_n\,\cdot)_{L^2(\Gamma)} (\mathsf{p}_j, \gamma_n\,\cdot)_{L^2(\Gamma)} + \mu(\gamma_d\,\cdot)\mu(\gamma_d\,\cdot).
\end{aligned}$$

Boundary spaces

$$\begin{aligned}
\mathcal{H}^{3/2}(\mathit{\Gamma}) &= \gamma_d W^2(\mathbb{R}^2) &&\text{and}&& \mathcal{H}^{1/2}(\mathit{\Gamma}) = \gamma_n W^2(\mathbb{R}^2).\\
\mathcal{H}_n^{3/2}(\mathit{\Gamma}) &= \gamma_d W_n^2(\mathbb{R}^2) &&\text{and}&& \mathcal{H}_d^{1/2}(\mathit{\Gamma}) = \gamma_n W_d^2(\mathbb{R}^2),\\
\mathscr{A}_S^{1/2} &= \gamma_d \mathscr{A} &&\text{and}&& \mathscr{A}_S^{-1/2} = \mathsf{S}_\mathit{\Gamma}^{-1}\mathscr{A}_S^{1/2},\\
\mathscr{A}_D^{-1/2} &= \gamma_n \mathscr{A} &&\text{and}&& \mathscr{A}_D^{1/2} = \mathsf{D}_\mathit{\Gamma}^{-1}\mathscr{A}_D^{-1/2}.
\end{aligned}$$

Space and dual space	Duality bracket	Scalar product
$H^{1/2}(\mathit{\Gamma}),\quad H^{-1/2}(\mathit{\Gamma})$	$\langle\cdot,\cdot\rangle_{-\frac12,\frac12}$	$(\cdot,\cdot)_{\frac12} = \langle \mathsf{S}_\mathit{\Gamma}^{-1}\cdot,\cdot\rangle_{-\frac12,\frac12}$
$\mathcal{H}^{1/2}(\mathit{\Gamma}),\quad \mathcal{H}^{-1/2}(\mathit{\Gamma})$	$\langle\!\langle\cdot,\cdot\rangle\!\rangle_{-\frac12,\frac12}$	$(\cdot,\cdot)_{\frac12}^A = (\mathsf{L}_n^A\cdot,\mathsf{L}_n^A\cdot)_A,\quad A\in\{S,D\}$
$\mathcal{H}^{3/2}(\mathit{\Gamma}),\quad \mathcal{H}^{-3/2}(\mathit{\Gamma})$	$\langle\!\langle\cdot,\cdot\rangle\!\rangle_{-\frac32,\frac32}$	$(\cdot,\cdot)_{\frac32}^A = (\mathsf{L}_d^A\cdot,\mathsf{L}_d^A\cdot)_A,\quad A\in\{S,D\}$
$\mathcal{H}_d^{1/2}(\mathit{\Gamma}),\quad \mathcal{H}_d^{-1/2}(\mathit{\Gamma})$	$\langle\!\langle\cdot,\cdot\rangle\!\rangle_{-\frac12,\frac12,d}$	$(\cdot,\cdot)_{\frac12,d} = (\mathcal{L}_d\cdot,\mathcal{L}_d\cdot)_S$
$\mathcal{H}_n^{3/2}(\mathit{\Gamma}),\quad \mathcal{H}_n^{-3/2}(\mathit{\Gamma})$	$\langle\!\langle\cdot,\cdot\rangle\!\rangle_{-\frac32,\frac32,n}$	$(\cdot,\cdot)_{\frac32,n} = (\mathcal{L}_n\cdot,\mathcal{L}_n\cdot)_D$

$$\begin{aligned}
\widetilde{\mathcal{H}}^{-1/2}(\mathit{\Gamma}) &= \big\{p\in\mathcal{H}^{-1/2}(\mathit{\Gamma})\ :\ \langle\!\langle p,\mathbf{1}_\mathit{\Gamma}\rangle\!\rangle_{-\frac12,\frac12}=0\big\},\\
\widetilde{\mathcal{H}}_n^{-3/2}(\mathit{\Gamma}) &= \big\{q\in\mathcal{H}_n^{-3/2}(\mathit{\Gamma})\ :\ \langle\!\langle q,\mathbf{1}_\mathit{\Gamma}\rangle\!\rangle_{-\frac32,\frac32,n}=0\big\}.
\end{aligned}$$

Some isometric operators

$A=S$ or $A=D$ in the definitions below:

$$\begin{aligned}
\mathsf{L}_d^A : \big(\mathcal{H}^{3/2}(\mathit{\Gamma}),\|\cdot\|_{\frac32}^A\big) &\longrightarrow \big(W_d^2(\mathbb{R}^2)^\perp,\|\cdot\|_A\big)\\
p &\longmapsto \inf\big\{\|u\|_A\ :\ u\in W^2(\mathbb{R}^2),\ \gamma_d u = p\big\},\\
\mathsf{L}_n^A : \big(\mathcal{H}^{1/2}(\mathit{\Gamma}),\|\cdot\|_{\frac12}^A\big) &\longrightarrow \big(W_n^2(\mathbb{R}^2)^\perp,\|\cdot\|_A\big)\\
q &\longmapsto \inf\big\{\|u\|_A\ :\ u\in W^2(\mathbb{R}^2),\ \gamma_n u = q\big\},\\
\mathcal{L}_n : \big(\mathcal{H}_n^{3/2}(\mathit{\Gamma}),\|\cdot\|_{\frac32,n}\big) &\longrightarrow \big(\mathscr{B}_n(\mathbb{R}^2),\|\cdot\|_D\big)\\
p &\longmapsto \inf\big\{\|u\|_S\ :\ u\in W_n^2(\mathbb{R}^2),\ \gamma_d u = p\big\},\\
\mathcal{L}_d : \big(\mathcal{H}_d^{1/2}(\mathit{\Gamma}),\|\cdot\|_{\frac12,d}\big) &\longrightarrow \big(\mathscr{B}_d(\mathbb{R}^2),\|\cdot\|_S\big)\\
q &\longmapsto \inf\big\{\|u\|_S\ :\ u\in W_d^2(\mathbb{R}^2),\ \gamma_n u = q\big\}.
\end{aligned}$$

Continuous and dense inclusions

$$\mathcal{H}_n^{3/2}(\mathit{\Gamma})\subset\mathcal{H}^{3/2}(\mathit{\Gamma})\subset H^{1/2}(\mathit{\Gamma})\subset L^2(\mathit{\Gamma})\subset H^{-1/2}(\mathit{\Gamma})\subset\mathcal{H}^{-3/2}(\mathit{\Gamma})\subset\mathcal{H}_n^{-3/2}(\mathit{\Gamma}),$$
$$\mathcal{H}_d^{1/2}(\mathit{\Gamma})\subset\mathcal{H}^{1/2}(\mathit{\Gamma})\subset L^2(\mathit{\Gamma})\subset\mathcal{H}^{-1/2}(\mathit{\Gamma})\subset\mathcal{H}_d^{-1/2}(\mathit{\Gamma}).$$

Further isometric operators

$$\begin{aligned}
\mathsf{T}_d : \mathcal{H}^{3/2}(\mathit{\Gamma}) &\longrightarrow \mathcal{H}^{-3/2}(\mathit{\Gamma}) &&\text{and}& \mathsf{T}_n : \mathcal{H}^{1/2}(\mathit{\Gamma}) &\longrightarrow \mathcal{H}^{-1/2}(\mathit{\Gamma})\\
p &\longmapsto (p,\cdot)_{\frac32}^S &&& q &\longmapsto (q,\cdot)_{\frac12}^D,\\
\mathcal{T}_d : \mathcal{H}_d^{1/2}(\mathit{\Gamma}) &\longrightarrow \mathcal{H}_d^{-1/2}(\mathit{\Gamma}) &&\text{and}& \mathcal{T}_n : \mathcal{H}_n^{3/2}(\mathit{\Gamma}) &\longrightarrow \mathcal{H}_n^{-3/2}(\mathit{\Gamma})\\
q &\longmapsto (q,\cdot)_{\frac12,d}, &&& p &\longmapsto (p,\cdot)_{\frac32,n}.
\end{aligned}$$

We use $L^2(\mathit{\Gamma})$ as pivot space, so none of these operators reduce to the identity.

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