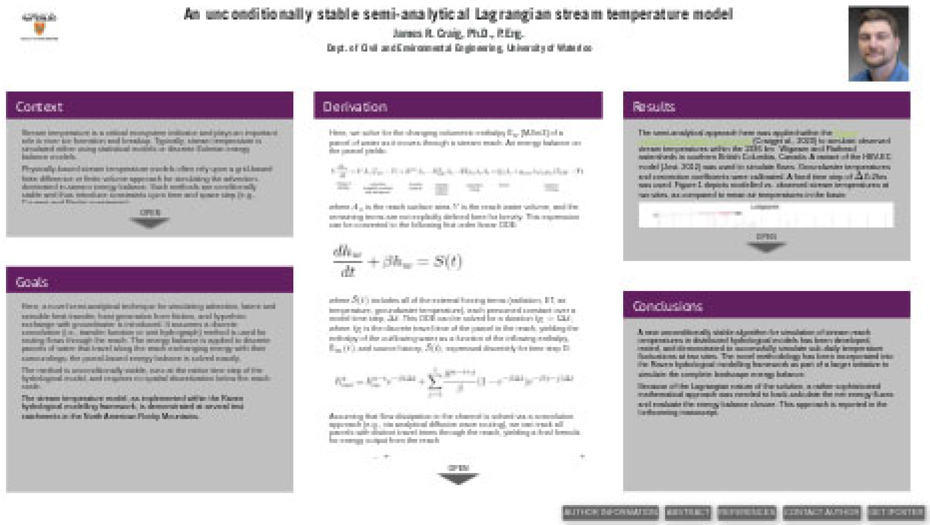
# An unconditionally stable semi-analytical Lagrangian stream temperature model

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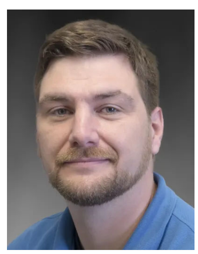

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# CONTEXT

Stream temperature is a critical ecosystem indicator and plays an important role in river ice formation and breakup. Typically, stream temperature is simulated either using statistical models or discrete Eulerian energy balance models.

Physically-based stream temperature models often rely upon a grid-based finite difference or finite volume approach for simulating the advection-dominated in-stream energy balance. Such methods are conditionally stable and thus introduce constraints upon time and space step (e.g., Courant and Peclet constraints).

# GOALS

Here, a novel semi-analytical technique for simulating advection, latent and sensible heat transfer, heat generation from friction, and hyporheic exchange with groundwater is introduced. It assumes a discrete convolution (i.e., transfer function or unit hydrograph) method is used for routing flows through the reach. The energy balance is applied to discrete parcels of water that travel along the reach exchanging energy with their surroundings; the parcel-based energy balance is solved exactly.

The method is unconditionally stable, runs at the native time step of the hydrological model, and requires no spatial discretization below the reach scale.

The stream temperature model, as implemented within the Raven hydrological modelling framework, is demonstrated at several test catchments in the North American Rocky Mountains.

# DERIVATION

Here, we solve for the changing volumetric enthalpy $h_w$ [MJ/m3] of a parcel of water as it moves through a stream reach. An energy balance on the parcel yields:

$$V\frac{dh_w}{dt} = k^* A_s(T_{air} - T) + R^{inc}A_s - R_{LW}^{og}A_s - \mathrm{ET}\rho_w\lambda_v A_s + Q_f A_s + q_{mix}A_b c_w \rho_w(T_{GW} - T)$$

change in enthalpy | convective (sensible) exchange with atmosphere | incoming radiation | outgoing longwave radiation | evaporative cooling | friction | hyporheic exchange

where $A_s$ is the reach surface area, $V$ is the reach water volume, and the remaining terms are not explicitly defined here for brevity. This expression can be converted to the following first order linear ODE:

$$\frac{dh_w}{dt} + \beta h_w = S(t)$$

where $S(t)$ includes all of the external forcing terms (radiation, ET, air temperature, groundwater temperature), each presumed constant over a model time step, $\Delta t$. This ODE can be solved for a duration $t_R = i\Delta t$, where $t_R$ is the discrete travel time of the parcel in the reach, yielding the enthalpy of the outflowing water as a function of the inflowing enthalpy, $h_{in}(t)$, and source history, $S(t)$, expressed discretely for time step $n$:

$$h_{out}^n = h_{in}^{n-i}e^{-\beta i\Delta t} + \sum_{j=1}^{i}\frac{S^{n-i+j}}{\beta}(1 - e^{-\beta\Delta t})e^{-\beta(i-j)\Delta t}$$

Assuming that flow dissipation in the channel is solved via a convolution approach (e.g., via analytical diffusive wave routing), we can track all parcels with distinct travel times through the reach, yielding a final formula for energy output from the reach

$$(Qh_{out})^n = \sum_{i=0}^{N_U}\left[(Qh_{in})^{n-i}\,e^{-\beta i\Delta t} + Q^{n-i}\sum_{j=1}^{i}\frac{S^{n-i+j}}{\beta}\left(1 - e^{-\beta\Delta t}\right)e^{-\beta(i-j)\Delta t}\right]$$

where $N_U$ is the number of streamtubes with different travel times. The resultant flow-weighted enthalpy is readily converted to temperature via a simple relation.

Because this solution is analytical, it is unconditionally stable and can easily simulate edge cases that are exceedingly difficult using discrete methods, such as near-infinite convective exchange coefficient. The only approximation errors are associated with the model time step used to represent the inflow time series, the spatial resolution of the reach network, and the assumption of full mixing at the nodes of the stream network.

# RESULTS

The semi-analytical approach here was applied within the Raven hydrological modelling framework (http://raven.uwaterloo.ca) (Craig et al., 2020) to simulate observed stream temperatures within the 1936 $km^2$ Wigwam and Flathead watersheds in southern British Columbia, Canada. A variant of the HBV-EC model (Jost, 2012) was used to simulate flows. Groundwater temperatures and convection coefficients were calibrated. A fixed time step of $\Delta t$=2hrs was used. Figure 1 depicts modelled vs. observed stream temperatures at two sites, as compared to mean air temperatures in the basin:


The semi-analytical model clearly adequately performs as applied to this real-world test case. However, the key benefits are not accuracy of the physical model, but rather the fact that this accuracy is reached without the need for discretization of each reach and the associated time and space step constraints. The model is therefore exceedingly fast, and incurs little computational overhead atop the flow problem.

Because the problem is posed in terms of enthalpy, the model also can provide some information about partially frozen conditions when it may be assumed that ice doesn't impede flow.

# CONCLUSIONS

A new unconditionally stable algorithm for simulation of stream reach temperatures in distributed hydrological models has been developed, tested, and demonstrated to successfully simulate sub-daily temperature fluctuations at two sites. The novel methodology has been incorporated into the Raven hydrological modelling framework as part of a larger initiative to simulate the complete landscape energy balance.

Because of the Lagrangian nature of the solution, a rather sophisitcated mathematical approach was needed to back-calculate the net energy fluxes and evaluate the energy balance closure. This approach is reported in the forthcoming manuscript.

# AUTHOR INFORMATION

Prof. James Craig is the Canadian Research Chair in Hydrological Modelling and Analysis at the University of Waterloo in Ontario, Canada.

Prof. Craig's interest include the development and applciation fo new modelling approaches for addressing difficult environmental/water resources problems in both surface water and groundwater systems. He and his research group are invested in making these tools and techniques accessible and useful to consultants, educators, and regulators.

website http://www.civil.uwaterloo.ca/jrcraig/ (http://www.civil.uwaterloo.ca/jrcraig/index.html)

# ABSTRACT

Stream temperature is a critical ecosystem indicator and plays an important role in river ice formation and breakup. Typically, stream temperature is simulated either using statistical models or discrete Eulerian energy balance models. These physically-based models often rely upon a grid-based finite difference or finite volume approach for simulating the advection-dominated in-stream energy balance. Such methods are conditionally stable and thus introduce constraints upon time and space step (e.g., Courant and Peclet constraints). They are also not conceptually consistent with commonly applied convolution-based hydrologic routing approaches, which don't require reach discretization. Here, a novel semi-analytical technique for simulating advection, latent and sensible heat transfer, heat generation from friction, and hyporheic exchange with groundwater is introduced. It assumes a discrete convolution (i.e., transfer function or unit hydrograph) method is used for routing flows through the reach. The energy balance is applied to discrete parcels of water that travel along the reach exchanging energy with their surroundings; the parcel-based energy balance is solved exactly. The method is unconditionally stable, runs at the native time step of the hydrological model, and requires no spatial discretization. It can also easily simulate edge cases that are exceedingly difficult using discrete methods, such as a near-infinite convective exchange coefficient. The only approximation errors are associated with the model time step used to represent the inflow time series, the linearization of the Stefan-Boltzmann equation for longwave radiation flux, and the full mixing assumed at nodes of the stream network.

The stream temperature model, as implemented within the Raven hydrological modelling framework, is demonstrated at several test catchments in the North American Rocky Mountains.

# REFERENCES

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