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Delayed analogue of pseudo-Mittag-Leffler functions and their applications to pseudo-Hilfer fractional time retarded differential equations

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Summary

In this write-up, we take account of pseudo-Hilfer-type fractional order delayed differential equations that provide a bounded definite integral initial condition on $[0, T]$. After proving certain lemmas at the beginning of the article, we found the solution of the homogeneous pseudo-Hilfer-type fractional order retarded differential equation, which satisfies the appropriate initial condition using classical methods. Then, we found the explicit formula of solutions to linear inhomogeneous pseudo-Hilfer-type fractional time retarded differential equations with constant coefficients by imposing classical conceptions. Furthermore, the existence and uniqueness of the solution of the pseudo-Hilfer-type fractional order delayed differential equation was investigated in the article, and the stability of the given differential equation was demonstrated in the Ulam-Hyers sense on $[0, T]$.

KEYWORDS:
Pseudo-fractional operator, Existence and uniqueness, Delayed analogue pseudo-Mittag-Leffler type function

1 | INTRODUCTION

In the eventual times, the field of fractional differential equations (FDEs) has allured increasing interest according to their comprehensive implementations in mechanics electrical circuits and time-delay systems stability analysis. FDEs are differential equations with derivatives of arbitrary fractional order. This field is a generalization of classical differential equations that has allured increasing interest owing to their applications for plenty of issues in engineering and science at the same time the mathematical theory and foundations can be found in lots of books dedicated to fractional calculus and fractional differential equations the use of fractional-order derivatives in return for integer-order derivatives permits for modeling more diverse behaviors.

There exists also a mathematical theory called pseudo-analysis, which in its own way is a generalization of classical analysis. In exchange for real numbers, this theory relies on semiconductors appointed by pseudo-addition and pseudo-multiplication in the real range. This is a curious topic for numerous explorers from different fields of knowledge such as functionality analysis, functional equations, and variational calculus. In eventual times, many scholars have worked on new formulations of inequalities involving fractional integrals, investigating in this context the properties of pseudo-fractional operators; in particular, they have presented the definition of the Riemann-Liouville pseudo-fractional derivative of another function.

Generally, the existence and uniqueness problems of the fractional differential equation with constant delay and stability of the solution have an essential role in Fractional differential equations. In thisfield, a lot of scientists engaged in these problems like Khusainov, Shuklin, Mahmudov, Huseynov, Hal L. Smith, R. D. Driver, Wang JR, Zhang Y, D.S.Oliveira, J.Vanterler da C.Sousa and so on.

For instance: J. Vanterler da C. Sousa, Rubens F. Camargo, E. Capelas de Oliveira GastÃ¡o S. F. Frederico have looked following pseudo-Hilfer-type FDE([2]).

$$\begin{cases} H^{\alpha,\beta:\psi}_{\oplus,\odot,t_0+}y(t) = Ay(t) \oplus f(t,y(t)), t \in J, \\ I^{1-\gamma}_{\oplus,\odot,t_0+}y(t) = y_0. \end{cases} \tag{1}$$

where $H^{\alpha,\beta:\psi}_{\oplus,\odot,t_0+}(\cdot)$ is the pseudo-fractional ψ-Hilfer derivative with order $0 < \alpha \leq 1$ and type $0 \leq \beta \leq 1$, $\gamma = \alpha - \beta(1-\alpha)$, $n \times n$ matrix A and $f : [t_0, +\infty) \times R^n \times R^n \to R^n$ be a continuous function. In that research paper, they explore the presence and uniqueness of the solution of the equation.

Furthermore, in 2020 Sousa et.al.[35] researched the existence and uniqueness of global solution of the initial issue associated with data (t_0, y_0) any solution $(I := [a,b], y)$ is given by

$$\begin{cases} \frac{d^{\oplus}}{dt}y(t) = F(t,y(t)), \\ y(t_0) = y_0. \end{cases} \tag{2}$$

with $t_0 \in I$. Afterwards, in 2020, Sosa et al. [36],discussed the reachability of linear and non-linear systems in the sense of the ψ-Hilfer pseudo-fractional derivative in g-calculus by means of the Mittag-Leffler functions with the form

$$\begin{cases} H^{\alpha,\beta;\psi}_{\oplus,\odot,0+}y(t) = Ay(t) \oplus Bu(t), t \in [t_0, t_1], \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t_0) = 0. \end{cases} \tag{3}$$

and

$$\begin{cases} H^{\alpha,\beta;\psi}_{\oplus,\odot,0+}y(t) = Ay(t) \oplus Bu(t) \oplus f(t,y(t),u(t)), t \in [t_0, t_1], \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t_0) = 0. \end{cases} \tag{4}$$

where $H^{\alpha,\beta;\psi}_{\oplus,\odot,0+}(\cdot)$ is the pseudo-fractional ψ-Hilfer derivative with order $0 < \alpha \leq 1$ and type $0 \leq \beta \leq 1$, $\gamma = \alpha - \beta(1-\alpha)$, $I^{1-\gamma}_{\oplus,\odot,0+}(\cdot)$ is the Riemann-Liouvile pseudo-fractional integral with respect to another function $1-\gamma$, the state vector $y \in R^n$,the control vector $u \in R^m$ and A and B are the constant matrices of dimension $n \times n$ and $n \times m$ accordingly and the non-linear function $f : J \times R^n \times R^m \to R^n$ is continuous, accordingly.

But in this research work, we explore the following pseudo-Hilfer-type fractional delay differential equation.

$$\begin{cases} H^{\alpha,\beta}_{\oplus,\odot,0+}y(t) = A \odot y(t) \oplus B \odot y(t-\tau) \oplus f(t), t \in (0;T], \tau > 0, \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t) = \phi(t), t \in [-\tau, 0]. \end{cases} \tag{5}$$

where $m-1 < \alpha < m$, $0 \leq \beta \leq 1$, $\gamma = (\beta-1)(m-\alpha)+k+1, k = 0, ..., m-1$.

Our main target is to find an analytical solution of the pseudo-Hilfer-type fractional differential equation (1.5) with a constant delay imposing classical methods. For this, first of all, we get the solution of the homogeneous pseudo-Hilfer-type fractional delay equations (1.6).

$$\begin{cases} H^{\alpha,\beta}_{\oplus,\odot,0+}y(t) = A \odot y(t) \oplus B \odot y(t-\tau), t \in (0;T], \tau > 0, \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t) = \phi(t), t \in [-\tau, 0]. \end{cases} \tag{6}$$

where $m-1 < \alpha < m$, $0 \leq \beta \leq 1$, $\gamma = (\beta-1)(m-\alpha)+k+1, k = 0, ..., m-1$. Next, we guess the explicit solution formula for linear inhomogeneous differential equations of the pseudo-Hilfer kind with fractional time delay and invariable coefficients by imposing the well-known ideas to obtain the solution of (1.5). To do this, we look at equation (1.7)

$$\begin{cases} H^{\alpha,\beta}_{\oplus,\odot,0+}y(t) = A \odot y(t) \oplus B \odot y(t-\tau) \oplus f(t), t \in (0;T], \tau > 0, \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t) = 0, t \in [-\tau, 0]. \end{cases} \tag{7}$$

where $m-1 < \alpha < m$, $0 \leq \beta \leq 1$, $\gamma = (\beta-1)(m-\alpha)+k+1, k = 0, ..., m-1$. Using the solution of equation (1.7) that is a particular solution to equation (1.5), we receive the analytic solution of equation (1.5). On the other hand, we demonstrated the presence and uniqueness of the solution in this article. In addition, we discuss the stability of the pseudo-Hilfer-type DDE(1.5) in the Ulam-Hyers sense on $[0, T]$.

2 | PRELIMINARIES

In this part, we mention that important information which it deals with pseudo-analysis, the elements of the fractional analysis and some necessary lemmas which will use the proof of the theorem.

- Gamma function:
$$\Gamma(\alpha) = \int\limits_0^\infty \tau^{\alpha-1} e^{-\tau} d\tau, \quad \alpha > 0.$$

- Beta function:
$$B(t,s) = \int\limits_0^1 z^{t-1}(1-z)^{s-1} dz, \quad t,s > 0.$$
Let $g : J \to R_+$ be a monotone and continuous function, where $J = [a,b]$ and $R_+ = [0,+\infty]$. Then we will defined Mittag-Leffler function as follow.

- The tree parametr Mittag-Leffler function:
$$E^{\delta}_{\alpha,\beta} g(z) = \sum_{s=0}^{\infty} \frac{(\delta)_s}{\Gamma(\alpha s+\beta)} \frac{(g(z))^s}{s!} = \sum_{s=0}^{\infty} \frac{(\delta)_s}{\Gamma(\alpha s+\beta)} \frac{(g(z))^s}{s!}$$

- Delayed analogue of Mittag-Leffler type function generated by $A, B \in R$ of three parameters:
$$E^{\tau}_{\alpha,\beta,\gamma}(g(A), g(B); t) = \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} \binom{n+q}{q} \frac{(g(A))^n (g(B))^q}{\Gamma(n\alpha+q\beta+\gamma)} (t-n\tau)^{n\alpha+q\beta+\gamma-1} H(t-n\tau)$$

- Exponentially bounded $f : [0,\infty) \to R$ holds an inequality of the form
$$||f(t)|| \le Le^{\sigma t}, \quad t > T,$$
for the real constants $\sigma, L > 0$ and $T > 0$.

- Laplace transform $\mathfrak{L}\{f(t)\}(s)$:
$$F(s) = \mathfrak{L}\{f(t)\}(s) = \int\limits_0^\infty e^{-st} f(t) dt, \quad s \in C,$$
where $f : [0,\infty) \to R$ is measurable and exponentially bounded on $[0,\infty)$, then the appointed by exists and is an analytic function of s for $Re(s) > 0$.

- Time shift feature of the Laplace transform:
$$\mathfrak{L}\{f(t-a)H(t-a)\}(s) = e^{-as}F(s).$$

- Convolution feature of Laplace transform:
$$\mathfrak{L}\{(f * h)(t)\} = \mathfrak{L}\{f(t)\}(s)\mathfrak{L}\{h(t)\}(s),$$
where $f, h : [0,\infty) \to R$ are exponetially bounded functions.

- Riemann-Liouville fractional integral:
$$I^{\alpha}_{a+} f(x) = \frac{1}{\Gamma(\alpha)} \int\limits_a^x (x-t)^{\alpha-1} f(t) dt.$$

- Hilfer fractional derivative Let $m-1 < \alpha < m$, with $m \in N$. The right-sided Hilfer fractional derivatives, denoted by ${}^{H}D^{\alpha,\beta}_{a+}(\cdot)$ of a function f of order α and kind $0 \le \beta \le 1$, are appointed by
$${}^{H}D^{\alpha,\beta}_{a+} f(x) = I^{\beta(m-\alpha)}_{a+} \frac{d^m}{dx^m} I^{(1-\beta)(m-\alpha)}_{a+} f(x). \tag{8}$$

Taking the limit $\beta \to 0$ in Eq.(8), we have the Rieman-Liouville derivative, given by:

$$^{RL}D^{\alpha}_{a+}f(x) = \frac{d^m}{dx^m} I^{(m-\alpha)}_{a+} f(x)$$

Taking the limit $\beta \to 1$ in Eq.(8), we have the Caputo derivative, given by:

$$^{C}D^{\alpha}_{a+}f(x) = I^{(m-\alpha)}_{a+} \frac{d^m}{dx^m} f(x).$$

- For any linear and bounded operator Ω appointed on a Banach space with $||\Omega|| < 1$, the operator $(I-\Omega)^{-1}$ is linear and bounded with property

$$(I-\Omega)^{-1} = \sum_{k=0}^{\infty} \Omega^k. \tag{9}$$

Lemma 1. Let $g : J \to R_+$ be a monotone and continious function , where $J = [a, b]$ and $R_+ = [0, +\infty]$.Then, for $\alpha > 0, A \in R$, $n \in N_0 = 0, 1, 2, ...,$ we have

$$\mathfrak{L}^{-1}\left\{\frac{1}{(s^{\alpha}-g(A))^{n+1}}\right\}(t) = \sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q \frac{t^{\alpha(n+q+1)-1}}{\Gamma(\alpha(n+q+1)} = t^{(n+1)\alpha-1}E^{n+1}_{\alpha,(n+1)\alpha}(g(A)t^{\alpha}), \quad Re(s) > 0.$$

Proof. Using the expansion

$$\frac{1}{(1-t)^{n+1}} = \sum_{q=0}^{\infty}\binom{n+q}{q} t^q, \quad |t| < 1,$$

for $|t| = \left|\frac{g(A)}{s^{\alpha}}\right| < 1$, we find that

$$\frac{1}{(s^{\alpha}-g(A))^{n+1}} = \frac{1}{s^{\alpha(n+1)}} \times \frac{1}{\left(1-\frac{g(A)}{s^{\alpha}}\right)^{n+1}} = \frac{1}{s^{\alpha(n+1)}}\sum_{q=0}^{\infty}\binom{n+q}{q}\left(\frac{g(A)}{s^{\alpha}}\right)^q = \sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^q}{s^{q\alpha+\alpha(n+1)}}$$

Taking inverse-Laplace transform of the above, we obtain that

$$\mathfrak{L}^{-1}\left\{\frac{1}{(s^{\alpha}-g(A))^{n+1}}\right\}(t) = \mathfrak{L}^{-1}\left\{\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^q}{s^{q\alpha+\alpha(n+1)}}\right\}(t) = \sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q$$

$$\times \mathfrak{L}^{-1}\left\{\frac{1}{s^{q\alpha+\alpha(n+1)}}\right\}(t) = \sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^q t^{\alpha(n+q+1)-1}}{\Gamma(\alpha(n+q+1))} = t^{(n+1)\alpha-1}E^{n+1}_{\alpha,(n+1)\alpha}(g(A)t^{\alpha}).$$

□

Lemma 2. Let $g : J \to R_+$ be a monotone and continuous function, where $J = [a, b]$ and $R_+ = [0, +\infty]$.Then, for $\alpha > 0, \alpha > \gamma$, we obtain.

$$\mathfrak{L}^{-1}\left\{\frac{s^{\gamma}}{s^{\alpha}-g(A)-g(B)e^{-s\tau}}\right\}(t) = E^{\tau}_{\alpha,\alpha,\alpha-\gamma}(g(A), g(B); t)$$

Proof. According to the well-known Neumann series, $\frac{s^{\gamma}}{s^{\alpha}-g(A)-g(B)e^{-s\tau}}$ can be written through a series expansion as below:

$$\frac{s^{\gamma}}{s^{\alpha}-g(A)-g(B)e^{-s\tau}} = \frac{s^{\gamma}}{s^{\alpha}-g(A)}\frac{1}{1-\frac{g(B)e^{-s\tau}}{s^{\alpha}-g(A)}} = \frac{s^{\gamma}}{s^{\alpha}-g(A)}\sum_{n=0}^{\infty}\frac{(g(B))^n e^{-sn\tau}}{(s^{\alpha}-g(A))^n} = \sum_{n=0}^{\infty}\frac{(g(B))^n e^{-sn\tau}s^{\gamma}}{(s^{\alpha}-g(A))^{n+1}}$$

Then imposing Lemma 2.2 to the final consideration we get:

$$\frac{s^{\gamma}}{s^{\alpha}-g(A)-g(B)e^{-s\tau}} = \sum_{n=0}^{\infty}\frac{(g(B))^n e^{-sn\tau}s^{\gamma}}{s^{\alpha(n+1)}(1-\frac{g(A)}{s^{\alpha}})^{n+1}} = \sum_{n=0}^{\infty}\frac{(g(B))^n e^{-sn\tau}s^{\gamma}}{s^{\alpha(n+1)}}\sum_{q=0}^{\infty}\binom{n+q}{q}\left(\frac{g(A)}{s^{\alpha}}\right)^q$$

$$= \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{g(A)^q(g(B))^n e^{-sn\tau}}{s^{\alpha(n+1)+q\alpha-\gamma}}$$

From the time delay feature of the Laplace integral transform (2.6), we have

$$\mathfrak{L}\{g(t-\tau)\}(s)(H(t-\tau) = e^{-s\tau}\mathfrak{L}\{g(t)\}(s)$$

Then, by taking the Inverse Laplace transform of the aforementioned function, we get

$$\mathfrak{L}^{-1}\left\{\frac{s^{\gamma}}{s^{\alpha}-g(A)-g(B)e^{-s\tau}}\right\}(t)=\mathfrak{L}^{-1}\left\{\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{g(A)^{q}(g(B))^{n}e^{-sn\tau}}{s^{\alpha(n+1)+q\alpha-\gamma}}\right\}(t)$$
$$=\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n}\mathfrak{L}^{-1}\left(\frac{e^{-sn\tau}}{s^{\alpha(n+1)+q\alpha-\gamma}}\right)(t)$$
$$=\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n}\frac{(t-n\tau)^{\alpha(n+1)+q\alpha-\gamma-1}H(t-n\tau)}{\Gamma(\alpha(n+1)+q\alpha-\gamma)}=E^{\tau}_{\alpha,\alpha,\alpha-\gamma}(g(A),g(B);t).$$

We need additional conditions on s, namely: $s^{\alpha}>|A|$ and $|s^{\alpha}-g(A)|>|B|e^{-s\tau}$ for convergence of the series. But, these conditions can be removed at the end of the evaluation with analytical continuation, to obtain the desired conclusion for all $s\in C$ with $Re(s)>0$. □

2.1 | PSEUDO-ANALYSIS

Assume $g:[\alpha,\beta]\to[0,\infty]$ be monotone and continuous function. We will define pseudo operators as follow.

- Pseudo operators:

$$\alpha\oplus\beta=g^{-1}(g(\alpha)+g(\beta))\quad and\quad \alpha\odot\beta=g^{-1}(g(\alpha)g(\beta)),$$
$$\alpha\ominus\beta=g^{-1}(g(\alpha)-g(\beta))\quad and\quad \alpha\oslash\beta=g^{-1}\left(\frac{g(\alpha)}{g(\beta)}\right).$$

  Suppose that $f:[c,d]\to[a,b]$ is measurable function.

- g-integral:

$$\int_{[c,d]}^{\oplus}f\odot dx=g^{-1}\left(\int_{c}^{d}g(f(x))dx\right).$$

- g-Laplace transform:

$$\mathfrak{L}^{\oplus}\{f(x)\}(s)=g^{-1}\big(\mathfrak{L}\{g(f(x))\}(s)\big).$$

  Suppose that g is the additive generator of the strict pseudo-addition $\oplus$ on $[a,b]$ so that g is continuously differentiable on (a,b).The corresponding pseudo-multiplication $\odot$ is appointed as $x\odot y=g^{-1}(g(x)g(y))$. If a function f is differentiable on (c,d) and has the same monotonicity as the function g, then the g-derivative of f at the point $x\in(c,d)$ is appointed by:

- g-derivative:

$$\frac{d^{\oplus}f(x)}{dx}=g^{-1}\left(\frac{d}{dx}g(f(x))\right).$$

- n^{th}-g-derivative:

$$\frac{d^{(n)\oplus}f(x)}{dx}=g^{-1}\left(\frac{d^{n}}{dx^{n}}g(f(x))\right).$$

  Now we will give some essential information about Hilfer operator and Hilfer-type fractional derivative .

- Riemann-Liouville pseudo-fractional integral.

  Suppose that a generator $g:[a,b]\to[0,+\infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. The right-sided and left-sided Riemann-Liouville pseudo-fractional integrals of order $\alpha>0$ of a measurable function $f:[a,b]\to[a,b]$ are appointed by

$$I^{\alpha}_{\oplus,\odot,a+}f(x)=g^{-1}\left(I^{\alpha}_{a+}g(f(x))\right)=\int_{[a,x]}^{\oplus}\left[g^{-1}\left(\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\right)\odot f(t)\right]\odot dt$$

  and

$$I^{\alpha}_{\oplus,\odot,b-}f(x)=g^{-1}\left(I^{\alpha}_{b-}g(f(x))\right)=\int_{[x,b]}^{\oplus}\left[g^{-1}\left(\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\right)\odot f(t)\right]\odot dt$$

- Hilfer pseudo-fractional derivatives. Assume that a generator $g : [a,b] \to [0,\infty]$ of the pseudo-addition $\oplus$ and the pseudo-multiplication $\odot$ be an increasing function. The right-sided and left-sided Hilfer pseudo-fractional derivatives of order $m-1<\alpha<m$ and type $0\le\beta\le 1$ of a measurable function $f : [a,b] \to [a,b]$ are appointed by

$$H^{\alpha,\beta}_{\oplus,\odot,a+}f(x) = g^{-1}\left({}^{H}D^{\alpha,\beta}_{a+}g(f(x))\right) = I^{\beta(m-\alpha)}_{\alpha,\beta,a+}g^{-1}\left(\frac{d^m}{dx^m}\right)\odot I^{1-\gamma}_{\oplus,\odot,a+}f(x)$$

and

$$H^{\alpha,\beta}_{\oplus,\odot,b-}f(x) = g^{-1}\left({}^{H}D^{\alpha,\beta}_{b-}g(f(x))\right) = I^{\beta(m-\alpha)}_{\alpha,\beta,b-}g^{-1}\left(\frac{d^m}{dx^m}\right)\odot I^{1-\gamma}_{\oplus,\odot,b-}f(x)$$

Note that

$$H^{\alpha,\beta}_{\oplus,\odot,a+}f(x) = g^{-1}\left(I^{\gamma-\alpha}_{a+}{}^{RL}D^{\gamma}_{a+}g(f(x))\right) = I^{\gamma-\alpha}_{\oplus,\odot,a+}{}^{RL}D^{\gamma}_{\oplus,\odot,a+}f(x)$$

and

$$H^{\alpha,\beta}_{\oplus,\odot,b-}f(x) = g^{-1}\left(I^{\gamma-\alpha}_{b-}{}^{RL}D^{\gamma}_{b-}g(f(x))\right) = I^{\gamma-\alpha}_{\oplus,\odot,b-}{}^{RL}D^{\gamma}_{\oplus,\odot,b-}f(x)$$

where $\gamma = \alpha+\beta(m-\alpha)$.

For extra information about pseudo-analysis, see [29, 39, 40, 41].

In the following, we will first discuss the derivation of the formulas of the pseudo-Mittag-Leffler functions and their definitions based on these calculations.

- The one parameter pseudo-Mittag-Leffler function::

$$E^{\oplus}_{\alpha}(z) = g^{-1}\left(E_{\alpha}g(z)\right) = g^{-1}\left(\sum_{s=0}^{\infty}\frac{(g(z))^s}{\Gamma(\alpha s+1)}\right) = \bigoplus_{s=0}^{\infty}g^{-1}\left(\frac{(g(z))^s}{\Gamma(\alpha s+1)}\right) = \bigoplus_{s=0}^{\infty}\left[g^{-1}\left((g(z))^s\right)\oslash g^{-1}\left(\Gamma(\alpha s+1)\right)\right]$$

Where $(\delta)_s$ is the famous Pochhammer symbol denoting $\frac{\Gamma(\delta+s)}{\Gamma(\delta)}$.

- The two parameter pseudo-Mittag-Leffler function:

$$E^{\oplus}_{\alpha,\beta}(z) = g^{-1}\left(E_{\alpha,\beta}g(z)\right) = g^{-1}\left(\sum_{s=0}^{\infty}\frac{(g(z))^s}{\Gamma(\alpha s+\beta)}\right) = \bigoplus_{s=0}^{\infty}g^{-1}\left(\frac{(g(z))^s}{\Gamma(\alpha s+\beta)}\right) = \bigoplus_{s=0}^{\infty}\left[g^{-1}\left((g(z))^s\right)\oslash g^{-1}\left(\Gamma(\alpha s+\beta)\right)\right]$$

- The three parameter pseudo-Mittag-Leffler function:

$$E^{\delta,\oplus}_{\alpha,\beta}(z) = g^{-1}\left(E^{\delta}_{\alpha,\beta}g(z)\right) = g^{-1}\left(\sum_{s=0}^{\infty}\frac{(\delta)_s}{\Gamma(\alpha s+\beta)}\frac{(g(z))^s}{s!}\right) = \bigoplus_{s=0}^{\infty}g^{-1}\left(\frac{(\delta)_s}{\Gamma(\alpha s+\beta)}\frac{(g(z))^s}{s!}\right)$$
$$= \bigoplus_{s=0}^{\infty}g^{-1}\left[g\left(g^{-1}\left(\frac{(\delta)_s}{\Gamma(\alpha s+\beta)}\right)\right)g\left(g^{-1}\left(\frac{(g(z))^s}{s!}\right)\right)\right] = \bigoplus_{s=0}^{\infty}\left[g^{-1}\left(\frac{(\delta)_s}{\Gamma(\alpha s+\beta)}\right)\odot g^{-1}\left(\frac{(g(z))^s}{s!}\right)\right]$$
$$= \bigoplus_{s=0}^{\infty}\left[\left(g^{-1}((\delta)_s)\oslash g^{-1}(\Gamma(\alpha s+\beta))\right)\odot\left(g^{-1}((g(z))^s)\oslash g^{-1}(s!)\right)\right]$$

- The pseudo-bivariate Mittag-Leffler function:

$$E^{\delta,\oplus}_{\alpha,\beta,\gamma}(a,b) = g^{-1}\left(E^{\delta}_{\alpha,\beta,\gamma}(g(a),g(b))\right) = g^{-1}\left(\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\delta)_{l+s}}{\Gamma(l\alpha+s\beta+\gamma)}\frac{(g(a))^l(g(b))^s}{l!s!}\right)$$
$$= \bigoplus_{l=0}^{\infty}\bigoplus_{s=0}^{\infty}g^{-1}\left[\frac{(\delta)_{l+s}}{\Gamma(l\alpha+s\beta+\gamma)}\frac{(g(a))^l(g(b))^s}{l!s!}\right] = \bigoplus_{l=0}^{\infty}\bigoplus_{s=0}^{\infty}\left[g^{-1}\left(\frac{(\delta)_{l+s}}{\Gamma(l\alpha+s\beta+\gamma)}\right)\odot g^{-1}\left(\frac{(g(a))^l(g(b))^s}{l!s!}\right)\right]$$
$$= \bigoplus_{l=0}^{\infty}\bigoplus_{s=0}^{\infty}\left[\left(g^{-1}((\delta)_{l+s})\oslash g^{-1}(\Gamma(l\alpha+s\beta+\gamma))\right)\odot\left(\left(g^{-1}((g(a))^l(g(b))^s)\right)\oslash\left(g^{-1}(l!\times s!)\right)\right)\right]$$
$$= \bigoplus_{l=0}^{\infty}\bigoplus_{s=0}^{\infty}\left[\left(g^{-1}((\delta)_{l+s})\oslash g^{-1}(\Gamma(l\alpha+s\beta+\gamma))\right)\odot\left(\left(g^{-1}((g(a))^l)\odot g^{-1}((g(b))^s)\right)\oslash\left(g^{-1}(l!)\odot g^{-1}(s!)\right)\right)\right]$$

- Delayed analogue of pseudo-Mittag-Leffler type function generated by $A, B \in R$ of three parameters:

$$E^{\tau,\oplus}_{\alpha,\beta,\gamma}(A,B;t) = g^{-1}\Big(E^{\tau}_{\alpha,\beta,\gamma}(g(A),g(B);g(t))\Big)$$

$$= g^{-1}\Bigg(\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^n(g(B))^q}{\Gamma(n\alpha+q\beta+\gamma)}(g(t-n\tau))^{n\alpha+q\beta+\gamma-1}H(g(t-n\tau))\Bigg)$$

$$= \bigoplus_{n=0}^{\infty}\bigoplus_{q=0}^{\infty} g^{-1}\Bigg(\binom{n+q}{q}(g(A))^n(g(B))^n\frac{(g(t-n\tau))^{n\alpha+q\beta+\gamma-1}H(g(t-n\tau))}{\Gamma(n\alpha+q\beta+\gamma)}\Bigg)$$

$$= \bigoplus_{n=0}^{\infty}\bigoplus_{q=0}^{\infty}\Bigg[g^{-1}\Bigg(\binom{n+q}{q}\Bigg)\odot g^{-1}\Big((g(A))^n\Big)\odot g^{-1}\Big((g(B))^q\Big)$$

$$\odot g^{-1}\Big((g(t-n\tau))^{n\alpha+q\beta+\gamma-1}\Big)\odot g^{-1}\Big(H(g(t-n\tau))\Big)\oslash g^{-1}\Big(\Gamma(n\alpha+q\beta+\gamma)\Big)\Bigg]$$

where $H(\cdot) : R \to R$ is the Heaviside function appointed as follows

$$H(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0. \end{cases}$$

Theorem 1. ([1], p.254, theorem 27.)

Assume that g is the additive generator of the strict-pseudo-addition $\oplus$ on [a,b], so that g is continiously differentiable on(a,b), $0 < m-1 \leq \alpha < m, \quad 0 \leq \beta \leq 1$ and $s \in R$ Then, the g-Laplace transform of the pseudo-Hilfer pseudo-fractional derivative of order α is given by:

$$\mathscr{L}^{\oplus}\left\{{}^{H}D^{\alpha,\beta}_{\oplus,\odot,0+}f(x)\right\} = [g^{-1}(s^{\alpha})\odot\mathfrak{L}^{\oplus}\{f(x)\}]\ominus\bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1})\odot I^{(1-\beta)(m-\alpha)-k}f(0)\right] \tag{10}$$

3 | EXPLICIT SOLUTIONS OF HOMOGENEOUS PSEUDO-HILFER-TYPE FRACTIONAL DIFERENTIONAL EQUATION

In this part, we have proved the explicit solution given by following (3.1) pseudo-Hilfer-type fractional differential equation system.

$$\begin{cases} H^{\alpha,\beta}_{\oplus,\odot,0+}y(t) = A\odot y(t)\oplus B\odot y(t-\tau), t\in(0;T], \tau>0, \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t) = \phi(t), t\in[-\tau,0]. \end{cases} \tag{11}$$

where $m-1<\alpha<m, 0\leq\beta\leq 1, \gamma=(\beta-1)(m-\alpha)+k+1, k=0,...,m-1.$

Theorem 2. A unique analytical solution $y \in C^m([-\tau,T],R)$ of the initial problem (3.1) has as shown below:

$$y(t) = \bigoplus_{k=0}^{m-2}\Bigg(g^{-1}\Bigg(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\Bigg)\oplus(A\oplus B)\odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;t-\tau)\Bigg)\odot\phi_0^{(k)}$$

$$\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;t)\odot\phi_0^{(m-1)}\oplus B\odot\int^{\oplus}_{[-\tau,\min(t-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;t-\tau-s)\odot\phi(s)\odot ds$$

Proof. Suppose that $T=\infty$. Assume that (1.5) has a unique m times continuously differentiable solution y and f are continuous and exponentially bounded, and $H^{\alpha,\beta}_{\oplus,\odot,0+}y$ is exponentially bounded on $[0,\infty)$, then Laplace transforms of them exist. And we are going to receive an integral representation of the solution to the linear homogeneous pseudo-Hilfer-type fractional differential equation.

First of all, we are imposing Laplace integral transform to both sides of (3.1) with the help of theorem (2.1).

$$\mathfrak{L}^{\oplus}\left\{H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)\right\}(s)=g^{-1}\left[\mathfrak{L}\left\{g\left(H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)\right)\right\}(s)\right]=g^{-1}\left[\mathfrak{L}\left\{{}^{H}D^{\alpha,\beta}_{0+}g(y(t)\right\}\right]$$
$$=g^{-1}\left[s^{\alpha}\mathfrak{L}\left\{g(y(t))\right\}(s)-\sum_{k=0}^{m-1}s^{m(1-\beta)+\alpha\beta-k-1}\left(I^{(1-\beta)(m-\alpha)-k}_{0+}g(y)\right)(0)\right]$$
$$=g^{-1}(s^{\alpha})\odot\mathfrak{L}\left\{y(t)\right\}(s)\ominus\bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1})\odot I^{(1-\beta)(m-\alpha)-k}_{\oplus,\odot,0+}y(0)\right]$$
$$=g^{-1}(s^{\alpha})\odot Y(s)\ominus\bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1})\odot\phi^{k}_{0}\right]$$

$$\mathfrak{L}^{\oplus}[H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)](s)=g^{-1}(s^{\alpha})\odot Y(s)\ominus\bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1})\odot\phi^{k}_{0}\right].\tag{12}$$

where, $\mathfrak{L}^{\oplus}\left\{y(t)\right\}(s)=Y(s)$

$$\mathfrak{L}^{\oplus}\left\{A\odot y(t)\oplus B\odot y(t-\tau)\right\}(s)=g^{-1}\left(\mathfrak{L}\left\{g(A\odot y(t)\oplus B\odot y(t-\tau)\right\}\right)$$
$$=g^{-1}\left(\mathfrak{L}\left\{g(A)g(y(t))+g(B)g\left(y(t-\tau)\right)\right\}\right)=A\odot\mathfrak{L}^{\oplus}(y(t))\oplus B\odot\mathfrak{L}^{\oplus}(y(t-\tau))$$
$$=A\odot Y(s)\oplus B\odot\mathfrak{L}^{\oplus}(y(t-\tau))$$

we get

$$\mathfrak{L}^{\oplus}\left\{A\odot y(t)\oplus B\odot y(t-\tau)\right\}(s)=A\odot Y(s)\oplus B\odot\mathfrak{L}^{\oplus}(y(t-\tau))\tag{13}$$

$$\mathfrak{L}^{\oplus}(y(t-\tau))(s)=g^{-1}(\mathfrak{L}(g(t-\tau))(s))$$

and by using substitution of $t-\tau=\theta$, we receive that

$$\mathfrak{L}\left\{g(t-\tau)\right\}(s)=\int_{0}^{\infty}g(t-\tau)e^{-st}dt=\int_{-\tau}^{\infty}g(y(\theta))e^{-s(\tau+\theta)}d\theta=e^{-s\tau}\int_{-\tau}^{\infty}g(y(\theta))e^{-s(\theta)}d\theta$$
$$=e^{-s\tau}\left[\int_{-\tau}^{0}g(y(\theta))e^{-s(\theta)}d\theta+\int_{0}^{\infty}g(y(\theta))e^{-s(\theta)}d\theta\right]=\int_{-\tau}^{0}g(y(\theta))e^{-s(\tau+\theta)}d\theta$$
$$+e^{-s\tau}\mathfrak{L}(g(y(\theta))(s))=\int_{0}^{\tau}g(y(t-\tau))e^{-st}dt+e^{-s\tau}\mathfrak{L}(g(y(\theta))(s))$$

On the other hand, due to the integral property of the pseudo-Riemann-Liouville-fraction, we obtain the following results. Let's also note that the initial condition of the issue we are reviewing is manifested in the following case.

$$I^{0}_{\oplus,\odot,0+}y(t)=y(t)\ \Rightarrow\ y(t)=\phi(t),t\in[-\tau,0]$$

in there $\tilde{\phi}(\cdot):R\to R$ is the unit-step function,which it has defined as bellow:

$$\tilde{\phi}(t)=\begin{cases}\phi(t) & \text{if }-\tau\le t\le 0\\ 0 & \text{if } t>0\end{cases}$$

Therefore we get following relations:

$$\mathfrak{L}\left\{g(t-\tau)\right\}(s) = \int_0^\tau g\big(y(t-\tau)\big)e^{-st}dt + e^{-s\tau}\mathfrak{L}\left\{g(y(\theta))\right\}(s) = \int_0^\infty g(\tilde{\phi}(t-\tau))e^{-st}dt + e^{-s\tau}\mathfrak{L}\left\{g(y(\theta))\right\}(s)$$

$$\mathfrak{L}^{\oplus}(y(t-\tau))(s) = g^{-1}(e^{-s\tau}) \odot Y(s) \oplus \mathfrak{L}^{\oplus}(\tilde{\phi}(t-\tau))(s) \tag{14}$$

By using formula (3.2), (3.3), (3.4) we get the following results.

$$g^{-1}(s^\alpha)\odot Y(s) \ominus \bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1})\odot \phi_0^{(k)}\right] = A\odot Y(s)\oplus B\odot\left[g^{-1}(e^{-s\tau})\odot Y(s)\oplus \mathfrak{L}^{\oplus}\left\{\tilde{\phi}(t-\tau)\right\}(s)\right]$$

Afterward, we write the above relation in the following explicit form

$$\left[g^{-1}(s^\alpha)\ominus A\ominus B\odot g^{-1}(e^{-s\tau})\right]\odot Y(s) = \bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1})\odot \phi_0^{k}\right]\oplus B\odot \mathfrak{L}^{\oplus}\left\{\tilde{\phi}(t-\tau)\right\}(s) \tag{15}$$

Then, we solve (3.5) with respect to Y(s),

$$\begin{aligned}
Y(s) &= \left[\bigoplus_{k=0}^{m-1}\left(g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1})\odot\phi_0^{(k)}\right)\oplus B\odot\mathfrak{L}^{\oplus}\left\{\tilde{\phi}(t-\tau)\right\}(s)\right]\oslash\left[g^{-1}(s^\alpha)\ominus A\ominus B\odot g^{-1}(e^{-s\tau})\right]\\
&= g^{-1}\left(\frac{\sum_{k=0}^{m-1}[s^{m(1-\beta)+\alpha\beta-k-1}g(\phi_0^k)] + g(B)g(\mathfrak{L}^{\oplus}(\tilde{\phi}(t-\tau))(s))}{s^\alpha - g(A) - g(B)e^{-s\tau}}\right)\\
&= g^{-1}\left(\sum_{k=0}^{m-2}\frac{s^{m(1-\beta)+\alpha\beta-k-1}g(\phi_0^{(k)})}{s^\alpha-g(A)-g(B)e^{-s\tau}} + \frac{s^{-m\beta+\alpha\beta}}{s^\alpha-g(A)-g(B)e^{-s\tau}}g(\phi_0^{(m-1)}) + \frac{g(B)}{s^\alpha-g(A)-g(B)e^{-s\tau}}\mathfrak{L}\left\{g(\tilde{\phi}(t-\tau))\right\}\right)\\
&= g^{-1}\left[\left(1+\frac{g(A)+g(B)e^{-s\tau}}{s^\alpha-g(A)-g(B)}\right)\sum_{k=0}^{m-2}\frac{s^{m(1-\beta)+\alpha\beta-k-1}g(\phi_0^{(k)})}{s^\alpha-g(A)-g(B)e^{-s\tau}} + \frac{g(B)}{s^\alpha-g(A)-g(B)e^{-s\tau}}\mathfrak{L}\left\{g(\tilde{\phi}(t-\tau))\right\}\right]
\end{aligned}$$

In accordance with relation (9), we have

$$\begin{aligned}
\left[s^\alpha - g(A) - g(B)e^{-s\tau}\right]^{-1} &= (s^\alpha - g(A))^{-1}\left[1-(s^\alpha-g(A))^{-1}g(B)e^{-s\tau}\right]^{-1}\\
&= (s^\alpha - g(A))^{-1}\sum_{n=0}^{\infty}\left[(s^\alpha-g(A))^{-n}(g(B))^n e^{-sn\tau}\right]
\end{aligned}$$

$$\left[s^\alpha - g(A) - g(B)e^{-s\tau}\right]^{-1} = \sum_{n=0}^{\infty}\left[(s^\alpha-g(A))^{-(n+1)}(g(B))^n e^{-sn\tau}\right] \tag{16}$$

If we replace the expression (3.6) in the Y(s) formula obtained above, we get the following results.

$$\begin{aligned}
Y(s) = g^{-1}\Bigg(&\left(\sum_{k=0}^{m-2}s^{m(1-\beta)+\alpha\beta-k-1}g(\phi_0^{(k)}) + (g(A)+g(B)e^{-s\tau})\sum_{k=0}^{m-2}s^{m(1-\beta)+\alpha\beta-k-1}g(\phi_0^{(k)})\right)\\
&\times\sum_{n=0}^{\infty}\left[(s^\alpha-g(A))^{-n}(g(B))^n e^{-sn\tau}\right] + g(\phi_0^{(m-1)})s^{-m\beta+\alpha\beta}\sum_{n=0}^{\infty}\left[(s^\alpha-g(A))^{-(n+1)}(g(B))^n e^{-sn\tau}\right]\\
&+g(B)\mathfrak{L}\left\{g(\tilde{\phi}[t-\tau])\right\}(s)\sum_{n=0}^{\infty}\left[(s^\alpha-g(A))^{-(n+1)}(g(B))^n e^{-sn\tau}\right]\Bigg)
\end{aligned}$$

Imposing the inverse g-Laplace transform to the above result, we get:

$$y(t) = g^{-1}\Bigg(\mathfrak{L}^{-1}\Bigg[\sum_{k=0}^{m-2} s^{m(1-\beta)+\alpha\beta-k-1} g(\phi_0^{(k)}) + (g(A)+g(B)e^{-s\tau})\sum_{k=0}^{m-2} s^{m(1-\beta)+\alpha\beta-k-1} g(\phi_0^{(k)})$$
$$\times\sum_{n=0}^{\infty}\Big[(s^\alpha - g(A))^{-n}(g(B))^n e^{-sn\tau}\Big] + g(\phi_0^{(m-1)})s^{-m\beta+\alpha\beta}\sum_{n=0}^{\infty}\Big[(s^\alpha-g(A))^{-(n+1)}(g(B))^n e^{-sn\tau}\Big]$$
$$+g(B)\sum_{n=0}^{\infty}\Big[(s^\alpha-g(A))^{-(n+1)}(g(B))^n e^{-sn\tau}\Big]\mathfrak{L}\left\{g(\tilde{\phi}[t-\tau])\right\}(s)\Bigg](t)\Bigg)$$

Taking inverse Laplace transform of the statement above and by using Lemma2.1, Lemma 2.2 and time shift and convolution property of the Laplace transform, we gain an explict representation of solution for a initial issue (3.1)

$$y(t) = g^{-1}\Bigg(\mathfrak{L}^{-1}\Bigg[\sum_{k=0}^{m-2} s^{m(1-\beta)+\alpha\beta-k-1} g(\phi_0^{(k)})$$
$$+\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\binom{n+q}{q}(g(A))^{q+1}(g(B))^n\frac{s^{(1-\beta)(m-\alpha)-1}}{s^{\alpha(n+q+1)}}e^{-sn\tau}g(\phi_0)$$
$$+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n+1}\frac{s^{(1-\beta)(m-\alpha)-1}e^{-sn\tau}}{s^{\alpha(n+q+1)}}g(\phi_0)+\cdots+$$
$$+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q+1}(g(B))^n\frac{s^{(1-\beta)(m-\alpha)-m+1e^{-sn\tau}}}{s^{\alpha(n+q+1)}}g(\phi_0^{(m-2)})$$
$$+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n+1}\frac{s^{(1-\beta)(m-\alpha)-m+1}e^{-sn\tau}}{s^{\alpha(n+q+1)}}g(\phi_0^{(m-2)})$$
$$+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q}(g(B))^{n}e^{-sn\tau}}{s^{\alpha(n+q+1)-\beta(\alpha-m)}}g(\phi_0^{(m-1)})$$
$$+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q}(g(B))^{n+1}e^{-sn\tau}}{s^{\alpha(n+q+1)}}\mathfrak{L}\big(g(\tilde{\phi}[t-\tau])\big)(s)\Bigg](t)\Bigg)$$

$$y(t) = g^{-1}\Bigg(\mathfrak{L}^{-1}\Bigg[\sum_{k=0}^{m-2} s^{m(1-\beta)+\alpha\beta-k-1} g(\phi_0^{(k)})$$
$$+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q+1}(g(B))^n}{s^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+1}}e^{-sn\tau}g(\phi_0)$$
$$+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q}(g(B))^{n+1}e^{-sn\tau}}{s^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+1}}g(\phi_0)+\cdots+$$
$$+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q+1}(g(B))^{n}e^{-sn\tau}}{s^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+m-1}}g(\phi_0^{(m-2)})$$
$$+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n+1}\frac{e^{-sn\tau}}{s^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+m-1}}g(\phi_0^{(m-2)})$$
$$+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q}(g(B))^{n}e^{-sn\tau}}{s^{\alpha(n+q+1)-\beta(\alpha-m)}}g(\phi_0^{(m-1)})$$
$$+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q}(g(B))^{n+1}e^{-sn\tau}}{s^{\alpha(n+q+1)}}\mathfrak{L}\left\{g(\tilde{\phi}[t-\tau])\right\}(s)\Bigg](t)\Bigg)$$

Then we get following result.

$$
\begin{aligned}
&y(t) = g^{-1}\Bigg(\sum_{k=0}^{m-2} \frac{t^{(1-\beta)(m-\alpha)-k}}{\Gamma((1-\beta)(m-\alpha)-k+1)} g(\phi_0^{(k)}) \\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q+1}(g(B))^{n}\frac{(t-(n+1)\tau)^{\alpha(n+q+1)-(1-\beta)(m-\alpha)}}{\Gamma(\alpha(n+q+1)-(1-\beta)(m-\alpha)+1)}H(t-(n+1)\tau)g(\phi_0) \\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n+1}\frac{(t-(n+1)\tau)^{\alpha(n+q+1)-(1-\beta)(m-\alpha)}}{\Gamma(\alpha(n+q+1)-(1-\beta)(m-\alpha)+1)}H(t-(n+1)\tau)g(\phi_0)+\cdots+ \\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q+1}(g(B))^{n}\frac{(t-(n+1)\tau)^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+m-2}}{\Gamma(\alpha(n+q+1)-(1-\beta)(m-\alpha)+m-1)}H(t-(n+1)\tau)g(\phi_0^{(m-2)}) \\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n+1}\frac{(t-(n+1)\tau)^{\alpha(n+q+1)-(1-\beta)(m-\alpha)}}{\Gamma(\alpha(n+q+1)-(1-\beta)(m-\alpha)+1)}H(t-(n+1)\tau)g(\phi_0^{(m-2)}) \\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n}\frac{(t-n\tau)^{\alpha(n+q+1)-\beta(\alpha-m)-1}}{\Gamma(\alpha(n+q+1)-\beta(\alpha-m))}H(t-n\tau)g(\phi_0^{(m-1)}) \\
&+g(B)\int_0^t \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n}\frac{(t-n\tau-s)^{\alpha(n+q+1)-1}}{\Gamma(\alpha(n+q+1))}H(t-n\tau-s)g\big(\tilde{\phi}(s-\tau)\big)ds\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
&y(t) = g^{-1}\Bigg(\sum_{k=0}^{m-2} \Bigg(\frac{t^{(1-\beta)(m-\alpha)-k}}{\Gamma((1-\beta)(m-\alpha)-k+1)} + (g(A)+g(B)) \\
&\times\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n}\frac{(t-(n+1)\tau)^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+k}}{\Gamma(\alpha(n+q+1)-(1-\beta)(m-\alpha)+k+1)}\Bigg)g(\phi_0^{(k)}) \\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n}\frac{(t-n\tau)^{\alpha(n+q+1)-\beta(\alpha-m)-1}}{\Gamma(\alpha(n+q+1)-\beta(\alpha-m))}H(t-n\tau)g(\phi_0^{(m-1)}) \\
&+g(B)\int_{-\tau}^{t-\tau} \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n}\frac{(t-(n+1)\tau-s)^{\alpha(n+q+1)-1}}{\Gamma(\alpha(n+q+1))}H(t-(n+1)\tau-s)g\big(\tilde{\phi}(s)\big)ds\Bigg) \\
&=g^{-1}\Bigg(\sum_{k=0}^{m-2}\Bigg(\frac{t^{(1-\beta)(m-\alpha)-k}}{\Gamma((1-\beta)(m-\alpha)-k+1)} + (g(A)+g(B))E^{\tau}_{\alpha,\alpha,\alpha+(\beta-1)(m-\alpha)+k+1}(g(A),g(B);t-\tau)\phi_0^{(k)}\Bigg) \\
&+E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t)\phi_0^{(m-1)} + g(B)\int_{-\tau}^{min(t-\tau,0)} E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-\tau-s)g\big(\tilde{\phi}(s)\big)ds\Bigg) \\
&=\bigoplus_{k=0}^{m-2}\Bigg(g^{-1}\Bigg(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+j+1)}\Bigg) \oplus (A\oplus B)\odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))\Bigg)\odot\phi_0^{(k)} \\
&\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t))\odot\phi_0^{m-1}\oplus B\odot \int_{[-\tau,\min(t-\tau,0)]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds
\end{aligned}
$$

We get

$$y(t)=\bigoplus_{k=0}^{m-2}\left(g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\oplus(A\oplus B)\odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))\right)\odot\phi_0^{(k)}\tag{17}$$

$$\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t))\odot\phi_0^{(m-1)}\oplus B\odot\int^{\oplus}_{[-\tau,\min(t-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds.\tag{18}$$

If we take $t\geq\tau$ then,

$$\int^{\oplus}_{[-\tau,t-\tau]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\tilde{\phi}(s)\odot ds=\int^{\oplus}_{[-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds\tag{19}$$

If we take $t<\tau$ then,

$$\int^{\oplus}_{[-\tau,t-\tau]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\tilde{\phi}(s)\odot ds=\int^{\oplus}_{[-\tau,t-\tau]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds\tag{20}$$

By using (3.8) and (3.9) we will get following result.

$$\int^{\oplus}_{[-\tau,t-\tau]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\tilde{\phi}(s)\odot ds=\int^{\oplus}_{[-\tau,\min(t-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds\tag{21}$$

□

4 | INTEGRAL REPRESENTATION OF SOLLUTION TO LINEAR INHOMOGENEOUS PSEUDO-HILFER-TYPE FRACTIONAL TIME DELAY DIFFERENTIAL EQUATIONS

In this part, by imposing the classical manners to solve (1.5), we will obtain the explicit formula for the solutions of linear inhomogeneous fractional pseudo-Hilfer-type differential equations with invariable coefficients and time delay.

Let us examine the following two pseudo-Hilfer-type FDDEs with constant coefficients:

$$\begin{cases}H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)=A\odot y(t)\oplus B\odot y(t-\tau)\oplus f(t),t\in(0;T],\tau>0,\\ I^{1-\gamma}_{\oplus,\odot,0+}y(t)=0,t\in[-\tau,0].\end{cases}\tag{22}$$

and

$$\begin{cases}H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)=A\odot y(t)\oplus B\odot y(t-\tau),t\in(0;T],\tau>0,\\ I^{1-\gamma}_{\oplus,\odot,0+}y(t)=\phi(t),t\in[-\tau,0].\end{cases}\tag{23}$$

where $m-1<\alpha<m$, $0\leq\beta\leq1$, $\gamma=(\beta-1)(m-\alpha)+k+1, k=0,...,m-1$.

The following lemma plays an important role in the proof of the subsequent theorem, which can be obtained from classical ways about the solution of the system (1.5).

Lemma 3. If y_1 and y_2 are the solutions systems (4.1) and (4.2),respectively, then $y(t)=y_1\oplus y_2$ is the general solution of system (1.5).

Mention that the solution y_2 of (4.2) is investigated in paragraph 3. In other words, to reach our goal, we need to find y_1 which is a particular solution of (1.5).

Lemma 4. Assume $m-1<\alpha<m$, $0<\beta\leq1$ for $m\geq2$. Then, we have the following relation:

$$\int_{\eta+l\tau}^{t}(t-s)^{(1-\beta)(m-\alpha)-1}(s-l\tau-\eta)^{l\alpha+p\alpha+\alpha-1}ds=(t-l\tau-\eta)^{m-\beta(m-\alpha)+l\alpha+p\alpha-2}B\Big((1-\beta)(m-\alpha),(l+1)\alpha+p\alpha\Big)$$

Proof. To prove the lemma, we use the definition of Beta function and substitution of $u=\frac{t-s}{t-l\tau-\eta}$. Consequently, we obtain

$$\int_{\eta+l\tau}^{t} (t-s)^{(1-\beta)(m-\alpha)-1}(s-l\tau-\eta)^{l\alpha+p\alpha+\alpha-1}ds$$
$$=(t-l\tau-\eta)^{m-\beta(m-\alpha)+l\alpha+p\alpha-2}\int_0^1 u^{(1-\beta)(m-\alpha)-1}(1-u)^{l\alpha+\alpha-1}du$$
$$=(t-l\tau-\eta)^{m-\beta(m-\alpha)+l\alpha+p\alpha-2}B\Big((1-\beta)(m-\alpha),(l+1)\alpha+p\alpha\Big)$$

□

We denote the following theorem for the particular solution of equation (1.5).

Theorem 3. A solution $\tilde{y}\in C^m([0,T],R)$ of (1.5) holding zero initial conditions $\tilde{y}(t)=0, t\in[-\tau,0)$, $\tilde{y}^{(k)}(0)=0, 0\leq k\leq m-1$ has the following form:

$$\tilde{y}(t)=\int_{[0,t]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))\odot f(s)\odot ds,\quad t>0 \tag{24}$$

Proof. Using the method of variation of constants, any solution $\tilde{y}$ of the inhomogeneous system must be provided in the following shape:

$$\tilde{y}(t)=\int_{[0,t]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))\odot h(s)\odot ds,\quad t>0 \tag{25}$$

where $h(s)$, $0\leq s\leq t$ is a sought vector function and $\tilde{y}(0)=0$.

$$\tilde{y}(t)=\int_{[0,t]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))\odot h(s)\odot ds=g^{-1}\Bigg(\int_0^t E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)g(h(s))ds\Bigg)$$

$$H^{\alpha,\beta}_{\oplus,\odot,0+}\tilde{y}(t)=g^{-1}({}^H D^{\alpha,\beta}_{0+}g(\tilde{y}(t)))=g^{-1}({}^H D^{\alpha,\beta}_{0+}(\int_0^t E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)g(h(s))ds))$$

$${}^H D^{\alpha,\beta}_{0+}g(\tilde{y}(t))={}^H D^{\alpha,\beta}_{0+}(\int_0^t E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)g(h(s))ds)$$
$$=I^{\beta(m-\alpha)}\frac{d^m}{dt^m}I^{(1-\beta)(m-\alpha)}\Bigg(\int_0^t E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)g(h(s))ds\Bigg)$$
$$=I^{\beta(m-\alpha)}\frac{d^m}{dt^m}\Bigg(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\int_0^t (t-s)^{(1-\beta)(m-\alpha)-1}\int_0^s E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);s-\eta)g(h(\eta))d\eta ds\Bigg)$$
$$=I^{\beta(m-\alpha)}\Bigg(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\frac{d^m}{dt^m}\int_0^t\int_0^s (t-s)^{(1-\beta)(m-\alpha)-1}E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);s-\eta)g(h(\eta))d\eta ds\Bigg)$$
$$=I^{\beta(m-\alpha)}\Bigg(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\frac{d^m}{dt^m}\int_0^t\int_{\eta+l\tau}^t (t-s)^{(1-\beta)(m-\alpha)-1}E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);s-\eta)g(h(\eta))d\eta ds\Bigg)$$
$$=I^{\beta(m-\alpha)}\Bigg(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\frac{d^m}{dt^m}\int_0^t g(h(\eta))\Bigg(\int_{\eta+l\tau}^t (t-s)^{(1-\beta)(m-\alpha)-1}E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);s-\eta)ds\Bigg)d\eta\Bigg)$$

$$=I^{\beta(m-\alpha)}\Bigg(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\frac{d^m}{dt^m}\int_0^t g(h(\eta))\Bigg(\int_{\eta+n\tau}^t (t-s)^{(1-\beta)(m-\alpha)-1}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q(g(B))^n$$

$$\frac{(s-n\tau-\eta)H(s-n\tau-\eta)}{\Gamma(q\alpha+n\alpha+\alpha)}ds\Bigg)d\eta\Bigg)=I^{\beta(m-\alpha)}\Bigg(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q(g(B))^n$$

$$\frac{d^m}{dt^m}\int_0^t \frac{(t-n\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+q\alpha-2}H(t-n\tau-\eta)}{\Gamma((n+1)\alpha+q\alpha)}g(h(\eta))d\eta B\Big((1-\beta)(m-\alpha),(n+1)\alpha+q\alpha\Big)\Bigg)$$

$$=I^{\beta(m-\alpha)}\Bigg(\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q(g(B))^n\frac{d^m}{dt^m}\int_0^t \frac{(t-n\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+q\alpha-2}H(t-n\tau-\eta)}{\Gamma(m-\beta(m-\alpha)+n\alpha+q\alpha-1)}g(h(\eta))d\eta\Bigg)$$

On the other hand, $I^{\beta(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}(f(t)) = {}^{C}D_{0+}^{\beta(\alpha+m)}f(t)$ and according to formula between Riemann-Luovile and Caputo fractional derivative, we have

$$I^{\beta(m-\alpha)}\frac{d^m}{dt^m}f(t)={}^{C}D_{0+}^{\beta(\alpha+m)}f(t)={}^{RL}D_{0+}^{\beta(\alpha+m)}f(t)-\sum_{k=0}^{m-1}\frac{t^{k-\beta(\alpha+m)}}{\Gamma(k-\beta(\alpha+m))}f^{(k)}(0),\quad t>0$$

With the help of following binomial identity.

$$\binom{n+q}{q}=\binom{n+q-1}{q}+\binom{n+q-1}{q-1},\quad n,q\geq 1,$$

and imposing Leibniz rule for higher-order derivatives (Ismail T.Huseynov et al ., 2021)(see Theorem 3.2), we achieve

$${}^{H}D_{0+}^{\alpha,\beta}g(\tilde{y}(t))=I^{\beta(m-\alpha)}\frac{d^m}{dt^m}\Bigg(\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q(g(B))^n$$

$$\times\int_0^t \frac{(t-n\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+q\alpha-2}H(t-n\tau-\eta)}{\Gamma(m-\beta(m-\alpha)+n\alpha+q\alpha-1)}g(h(\eta))d\eta\Bigg)$$

$$=\ {}^{C}D^{\beta(\alpha-m)+m}\Bigg(\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q(g(B))^n\int_0^t \frac{(t-n\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+p\alpha-2}H(t-n\tau-\eta)}{\Gamma(m-\beta(m-\alpha)+n\alpha+p\alpha-1)}g(h(\eta))d\eta\Bigg)$$

$$=\frac{d^m}{dt^m}\int_0^t \frac{(t-\eta)^{m-\beta(m-\alpha)-2}H(t-\eta)}{\Gamma(m-\beta(m-\alpha)-1)}g(h(\eta))d\eta$$

$$+\sum_{n=1}^{\infty}\sum_{q=0}^{\infty}\binom{n+q-1}{q}(g(A))^q(g(B))^n\frac{d^m}{dt^m}\int_0^t \frac{(t-l\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+q\alpha-2}H(t-n\tau-\eta)}{\Gamma(m-\beta(m-\alpha)+n\alpha+q\alpha-1)}g(h(\eta))d\eta$$

$$+\sum_{n=0}^{\infty}\sum_{q=1}^{\infty}\binom{n+q-1}{q-1}(g(A))^q(g(B))^n\frac{d^m}{dt^m}\int_0^t \frac{(t-n\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+q\alpha-2}H(t-n\tau-\eta)}{\Gamma(m-\beta(m-\alpha)+n\alpha+q\alpha-1)}g(h(\eta))d\eta$$

$$=g(h(t))+\sum_{n=1}^{\infty}\sum_{q=0}^{\infty}\binom{n+q-1}{q}(g(A))^q(g(B))^n\int_0^t \frac{(t-n\tau-\eta)^{\alpha\beta+n\alpha+q\alpha-2}H(t-n\tau-\eta)}{\Gamma(\alpha\beta+n\alpha+q\alpha-1)}g(h(\eta))d\eta$$

$$+\sum_{n=0}^{\infty}\sum_{q=1}^{\infty}\binom{n+q-1}{q-1}(g(A))^{q}(g(B))^{n}\int_{0}^{t}\frac{(t-n\tau-\eta)^{\alpha\beta+n\alpha+q\alpha-2}H(t-n\tau-\eta)}{\Gamma(\alpha\beta+n\alpha+q\alpha-1)}g(h(\eta))d\eta$$

$$=g(h(t))+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n+1}\int_{0}^{t}\frac{(t-n\tau-\eta)^{\alpha\beta+n\alpha+q\alpha-2}H(t-(n+1)\tau-\eta)}{\Gamma(\alpha\beta+(n+1)\alpha+q\alpha-1)}g(h(\eta))d\eta$$

$$+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q-1}(g(A))^{q+1}(g(B))^{n}\int_{0}^{t}\frac{(t-n\tau-\eta)^{\alpha\beta+n\alpha+(q+1)\alpha-2}H(t-n\tau-\eta)}{\Gamma(\alpha\beta+n\alpha+(q+1)\alpha-1)}g(h(\eta))d\eta$$

$$=g(h(t))+g(A)\int_{0}^{t}E^{\tau}_{\alpha,\alpha,\alpha\beta+\alpha-1}(g(A),g(B);t-\eta)g(h(\eta))d\eta+g(B)\int_{0}^{t}E^{\tau}_{\alpha,\alpha,\alpha\beta+\alpha-1}(g(A),g(B);t-\tau-\eta)g(h(\eta))d\eta$$

$$H^{\alpha,\beta}_{\oplus,\odot,0+}\tilde{y}(t)=g^{-1}({}^{H}D^{\alpha,\beta}_{0+}g(\tilde{y}(t)))=g^{-1}\Bigg(g(h(t))+g(A)\int_{0}^{t}E^{\tau}_{\alpha,\alpha,\alpha\beta+\alpha-1}(g(A),g(B);t-\eta)g(h(\eta))d\eta$$

$$+g(B)\int_{0}^{t}E^{\tau}_{\alpha,\alpha,\alpha\beta+\alpha-1}(g(A),g(B);t-\tau-\eta)g(h(\eta))d\eta\Bigg)=A\oplus\tilde{y}(t)\oplus B\tilde{y}(t-\tau)\oplus h(t)=A\oplus\tilde{y}(t)\oplus B\tilde{y}(t-\tau)\oplus f(t)$$

Therefore, we obtain that $h(t)=f(t)$ for $t\in[0,T]$. □

Eventually, we obtain the next theorem for the unique analytical solution of the Cauchy problem (1.5).

Theorem 4. A unique analytical solution $y\in C^{m}([-\tau,T],R)$ of the initial issue (1.1) has the following form:

$$y(t)=\bigoplus_{k=0}^{m-2}\left(g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\oplus(A\oplus B)\odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))\right)\odot\phi_{0}^{(k)}$$
$$\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t))\odot\phi_{0}^{(m-1)}\oplus B\odot\int^{\oplus}_{[-\tau,\min(t-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds$$
$$\oplus\int^{\oplus}_{[0,t]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))\odot f(s)\odot ds,\quad t>0.$$

Proof. The proof of the theorem is immediate. Therefore, we pass above it. □

5 | EXISTENCE AND UNIQUNESS PROBLEM FOR NONLINIEAR TIME RETARDED PSEUDO-HILFER-TYPE FRACTIONAL DIFFERENTIONAL EQUATIONS

In the following section, we will look the initial issue for a nonlinear pseudo-Hilfer-type fractional differential equation with constant delay.

$$\begin{cases}H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)=A\odot y(t)\oplus B\odot y(t-\tau)\oplus f(t,y(t)),t\in(0;T],\tau>0,\\ I^{1-\gamma}_{\oplus,\odot,0+}y(t)=\phi(t),t\in[-\tau,0].\end{cases}\tag{26}$$

Where $m-1<\alpha\leq m$, $0<\beta\leq 1$, $y(\cdot)\in R$, $f(\cdot,y(\cdot)):[0,\infty)\times R\to R$ is a nonlinear perturbation and also a continuous function. And we will also suppose that $(t\to f(t,0))\in C([0,\infty),R)$. Then, according to Theorem 4.2, we obtain the solution of the nonlinear Hilfer-type FDE (5.1) as follows:

$$y(t)=\bigoplus_{k=0}^{m-2}\left(g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\oplus(A\oplus B)\odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))\right)\odot\phi_{0}^{(k)}$$

$$\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t))\odot\phi_0^{(m-1)}\oplus B\odot\int^{\oplus}_{[-\tau,\min(t-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds$$

$$\oplus\int^{\oplus}_{[0,t]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))\odot f(s,y(s))\odot ds,\quad t>0.$$

First of all, we denote following lemmas and notes: For $x(\cdot):[a,b]\to R_+$, we will define the norm of the function as a follow:

$$||x(t)||_g=g^{-1}(|g(x(t))|)$$

Lemma 5. ([3], page 12, lemma 5.1) The following estimation satisfies true:

$$|E^{\tau}_{\alpha,\alpha-\beta,\alpha+k}(A,B;t)|\le t^{\alpha+k-1}\exp(|A|t^{\alpha}+|B|t^{\alpha-\beta}) \tag{27}$$

for $k=0,1,\dots,m-1$

Corollary 1. ([3], page 12, corollary 5.1)

For $m\ge 2$, the following conclusion satisfies:

$$|E^{\tau}_{\alpha,\alpha-\beta,m}(A,B;t)|\le t^{m-1}\exp(|A|t^{\alpha}+|B|t^{\alpha-\beta}). \tag{28}$$

Analogously, we will get the following results for pseudo-Mittag-Leffler functions.

Lemma 6. Assume a generator $g:[a,b]\to[0,\infty]$ and $A,B\in R$. For following delayed pseudo-Mittag-Leffler function estimation holds true:

$$|E^{\tau,\oplus}_{\alpha,\alpha-\beta,\alpha+k}(A,B;g^{-1}(t))|_g\le g^{-1}(t^{\alpha+k-1})\odot g^{-1}(\exp(|A|t^{\alpha}+|B|t^{\alpha-\beta})) \tag{29}$$

for $k=0,1,\dots,m-1$

Proof.

$$|E^{\tau,\oplus}_{\alpha,\alpha-\beta,\alpha+k}(A,B;g^{-1}(t))|_g=g^{-1}\Bigg(g\Bigg(|E^{\tau,\oplus}_{\alpha,\alpha-\beta,\alpha+k}(A,B;g^{-1}(t))|\Bigg)\Bigg)$$
$$=g^{-1}\Bigg(|E^{\tau}_{\alpha,\alpha-\beta,\alpha+k}(g(A),g(B);t)|\Bigg)\le g^{-1}\Bigg(t^{\alpha+k-1}\exp(|g(A)|t^{\alpha}+|g(B)|t^{\alpha-\beta})\Bigg)$$
$$\le g^{-1}(t^{\alpha+k-1})\odot g^{-1}(\exp(|A|t^{\alpha}+|B|t^{\alpha-\beta}))$$

□

Then, we can denote analogously following corollary.

Corollary 2. Let a generator $g:[a,b]\to[0,\infty]$ and $A,B\in R$. For $m\ge 2$, the following inequality holds:

$$|E^{\tau,\oplus}_{\alpha,\alpha-\beta,m}(A,B;g^{-1}(t))|_g\le g^{-1}\Bigg(t^{m-1}\Bigg)\odot g^{-1}\Bigg(\exp(|A|t^{\alpha}+|B|t^{\alpha-\beta})\Bigg). \tag{30}$$

Theorem 5. Assume that the following hypothesises are true:

$$(H_1)f:[0,T]\times R\to R\quad be\quad a\quad continious\quad function:$$
$$(H_2)there\quad exisit\quad C>0\quad such\quad that\quad f\quad holds\quad the\quad Lipschitz\quad condition:$$

$$|f(t,y)\ominus f(t,\sigma)|_g\le C\odot|y\ominus\sigma|_g,\quad\forall(t,y),(t,\sigma)\in[0,T]\times R; \tag{31}$$

Then, the problem (5.1) has a unique global continuous solution on $[0,T]$.

Proof. Assume that a ball be appointed as $B_R:=y\in C([0,T],R):||y||_{\omega}\le R,\omega>0$ where $R>0$ with

$$R\ge\Bigg[W\odot|\phi_0^{(k)}|_g\oplus g^{-1}(T^{\alpha-1})\odot S\odot|\phi_0^{(m-1)}|_g\oplus S\odot g^{-1}(\Gamma(\alpha))\odot|B|\odot||\phi||_{\omega}\oplus D\Bigg]\oslash(g^{-1}(\omega^{\alpha}\ominus S\odot g^{-1}(\Gamma(\alpha))\odot C)) \tag{32}$$

where

$$W = \bigoplus_{k=0}^{m-2}\left(g^{-1}\left(\frac{T^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\oplus(|A|\oplus|B|)\odot g^{-1}\left(T^{(\beta-1)(m-\alpha)+\alpha+k}\right)\odot S\right)$$

$$D = \max_{t\in[0,T]}\left\{|f(t,0)_g\oslash \exp(\omega t)\right\}; S = \exp\left((|g(A)|+|g(B)|)T^{\alpha}\right)$$

Now, we set an integral operator F on B_R as below:

$$F : C([0,T],R)\supset B_R \ni y \to F(y) := (t\to (Fy)(t))\in C([0,T],R),$$

through the following formula

$$(Fy)(t) = \bigoplus_{k=0}^{m-2}\left(g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\oplus(A\oplus B)\odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))\right)\odot\phi_0^{(k)}$$

$$\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t))\odot\phi_0^{(m-1)}\oplus B\odot\int^{\oplus}_{[-\tau,\min(t-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds$$

$$\oplus\int^{\oplus}_{[0,t]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))\odot f(s,y(s))\odot ds,\quad t\in[0,T].$$

It is obvious that F is well-defined according to (H_1).Thus, the availability of a solution to the Initial issue (5.1) is equivalent to the fact that the integral operator F has a fixed point on B_R.To prove that F has a unique fixed point, we will impose the contraction mapping principle. On the other hand, we will not use the maximum norm $C([0,T],R)$. Because choosing the maximum norm only brings us to the local solution appointed in the subinterval [0,T]. Let C([0,T],R) be fitted with the weighted maximum norm $||\cdot||_{\omega}$ with respect to the exponential function, where it is appointed as:

$$||y|_{\omega} := \max_{t\in[0,T]}\left\{|y(t)|_g\oslash\exp(\omega t)\right\}, \forall y\in C([0,T],R).$$

Since two norms $||\cdot||_{\infty}$ and $||\cdot||_{\omega}$ are equivalent, $C([0,T],R,||\cdot||_{\omega})$ is also a Banach space. The proof is separated into two parts.

step 1: We prove that $F(B_R)\subset B_R$. in this part, we look following estimation.

$$|(Fy)(t)|_g\oslash\exp(\omega t) = g^{-1}\left(\frac{g|(Py)(t)|_g}{g(\exp(\omega t))}\right) = g^{-1}\left(\frac{|(Pg(y))(t)|}{g(\exp(\omega t))}\right) \tag{33}$$

First of all, we denote the following notes for use in process of proof.

$$(Fg(y))(t) = \sum_{k=0}^{m-2}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}+(g(A)+g(B))E^{\tau}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(g(A),g(B);t-\tau)\right)g(\phi_0^{(k)})$$

$$+E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t)g(\phi_0^{(m-1)})+g(B)\int_{-\tau}^{min(t-\tau,0)}E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-\tau-s)g(\phi(s))ds$$

$$+\int_0^t E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)g(f(s,y(s)))ds, t\in[0,T]$$

Then, we will get.

$$\frac{|F(g(y))(t)|}{g(\exp(\omega t))}\le\frac{1}{g(\exp(\omega t))}\sum_{k=0}^{m-2}\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}|g(\phi_0^{(k)})|+\frac{|g(A)|+|g(B)|}{g(\exp(\omega t))}$$

$$+\sum_{k=0}^{m-2}|E^{\tau}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(g(A),g(B);t-\tau)||g(\phi_0^{(k)})|+\frac{1}{g(\exp(\omega t))}|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t)||g(\phi_0^{(m-1)})|$$

$$+\frac{|g(B)|}{g(\exp(\omega t))}\int_{-\tau}^{0}|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-\tau-s)||g(\phi(s)|$$

$$+\frac{1}{g(\exp(\omega t))}\int_{0}^{t}|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)||g(f(s,y(s)))-g(f(s.0))|+|g(f(s,0))|ds$$

$$\leq\sum_{k=0}^{m-2}\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}|g(\phi_0^{(k)})|+(|g(A)|+|g(B)|)\sum_{k=0}^{m-2}|E^{\tau}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(g(A),g(B);t-\tau)||g(\phi_0^{(k)})|$$

$$+\frac{1}{g(\exp(\omega t))}|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t)||g(\phi_0^{(m-1)})|+\frac{|g(B)|}{g(\exp(\omega t))}\int_{-\tau}^{0}|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-\tau-s)||g(\phi(s)|ds$$

$$+\frac{1}{g(\exp(\omega t))}\int_{0}^{t}|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)||g(f(s,y(s)))-g(f(s.0))|+|g(f(s,0))|ds$$

$$\leq\sum_{k=0}^{m-2}\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}|g(\phi_0^{(k)})|+(|g(A)|+|g(B)|)\sum_{k=0}^{m-2}|E^{\tau}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(g(A),g(B);t-\tau)||g(\phi_0^{(k)})|$$

$$+\frac{1}{g(\exp(\omega t))}|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t)||g(\phi_0^{(m-1)})|+\frac{|g(B)|}{g(\exp(\omega t))}\int_{-\tau}^{0}|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-\tau-s)|\frac{g(\exp(\omega s))}{g(\exp(\omega s))}|g(\phi(s)|ds$$

$$+\frac{1}{g(\exp(\omega t))}\int_{0}^{t}|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)||g(f(s,y(s)))-g(f(s.0))|\frac{g(\exp(\omega s))}{g(\exp(\omega s))}ds$$

$$+\int_{0}^{t}|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)||g(f(s,0))|\frac{g(\exp(\omega s))}{g(\exp(\omega s))}ds$$

By using from this formula and (5.8) we obtain

$$|(Fy)(t)|_g\oslash\exp(\omega t)\leq\bigoplus_{k=0}^{m-2}g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\odot|\phi_0^{(k)}|_g$$

$$\oplus(|A|\oplus|B|)\odot\bigoplus_{k=0}^{m-2}|E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))|_g\odot|\phi_0^{(k)}|_g$$

$$\oplus|E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t))|_g\odot|\phi_0^{(m-1)}|_g\oslash\exp(\omega t)$$

$$\oplus|B|\oslash\exp(\omega t)\odot\int_{[-\tau,0]}^{\oplus}|E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))|_g\odot\exp(\omega s)\odot|\phi(s)|_g\oslash\exp(\omega s)\odot ds$$

$$\oplus\int_{[0,t]}^{\oplus}|E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))|_g\odot|f(s,y(s))\ominus f(s.0)|_g\odot\exp(\omega s)\oslash\exp(\omega s)\odot ds\oslash\exp(\omega t)$$

$$\oplus \int_{[0,t]}^{\oplus} |E_{\alpha,\alpha,\alpha}^{\tau,\oplus}(A,B;g^{-1}(t-s))|_g \odot |f(s,0)|_g \oslash \exp(\omega s) \odot \exp(\omega s) \odot ds$$

Now take $\forall t \in [0,T]$ and $\forall y \in B_R$. By using (H_2) by means of Lemma 5.2, we receive:

$$\begin{aligned}
&|(Fy)(t)|_g \oslash \exp(\omega t) \leq \bigoplus_{k=0}^{m-2} g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right) \odot |\phi_0^{(k)}|_g \\
&\oplus(|A| \oplus |B|) \odot \bigoplus_{k=0}^{m-2} g^{-1}\left((t-\tau)^{(\beta-1)(m-\alpha)+\alpha+k}\right) \odot g^{-1}((\exp(|A|+|B|)(t-\tau)^{\alpha}) \odot |\phi_0^{(k)}|_g \\
&\oplus g^{-1}(t^{\alpha-1}) \odot g^{-1}((\exp(|A|+|B|)t^{\alpha}) \odot |\phi_0^{(m-1)}|_g \oslash \exp(\omega t) \\
&\oplus |B| \oslash \exp(\omega t) \odot \int_{[-\tau,0]}^{\oplus} g^{-1}\left((t-\tau-s)^{\alpha-1}\right) \odot g^{-1}((\exp(|A|+|B|)(t-\tau-s)^{\alpha}) \odot \exp(\omega s) \odot |\phi(s)|_g \oslash \exp(\omega s) \odot ds \\
&\oplus \int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right) \odot g^{-1}((\exp(|A|+|B|)(t-s)^{\alpha}) \odot C \odot |y(s)|_g \odot \exp(\omega s) \oslash \exp(\omega s) \odot ds \oslash \exp(\omega t) \\
&\oplus \int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right) \odot g^{-1}((\exp(|A|+|B|)(t-s)^{\alpha}) \odot |f(s,0)|_g \oslash \exp(\omega s) \odot \exp(\omega s) \odot ds
\end{aligned}$$

Using the substitution $r - s = u$ and Lipschitz condition (H_2), we get

$$\begin{aligned}
&|(Fy)(t)|_g \oslash \exp(\omega t) \leq \bigoplus_{k=0}^{m-2} g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right) \odot |\phi_0^{(k)}|_g \\
&\oplus(|A| \oplus |B|) \odot \bigoplus_{k=0}^{m-2} g^{-1}\left(t^{(\beta-1)(m-\alpha)+\alpha+k}\right) \odot g^{-1}((\exp(|A|+|B|)t^{\alpha}) \odot |\phi_0^{(k)}|_g \\
&\oplus g^{-1}(t^{\alpha-1}) \odot g^{-1}((\exp(|A|+|B|)t^{\alpha}) \odot |\phi_0^{(m-1)}|_g \oslash \exp(\omega t) \\
&\oplus |B| \oslash \exp(\omega t) \odot \int_{[-\tau,0]}^{\oplus} g^{-1}\left((t-\tau-s)^{\alpha-1}\right) \odot |\phi(s)|_g \oslash \exp(\omega s) \odot \exp(\omega s) \odot ds \odot g^{-1}((\exp(|A|+|B|)(t)^{\alpha}) \\
&\oplus C \oslash \exp(\omega t) \odot \int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right) \odot |y(s)|_g \odot \exp(\omega s) \oslash \exp(\omega s) \odot ds \odot g^{-1}((\exp(|A|+|B|)(t)^{\alpha}) \\
&\oplus \int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right) \odot |f(s,0)|_g \oslash \exp(\omega s) \odot \exp(\omega s) \odot ds \odot g^{-1}((\exp(|A|+|B|)(t)^{\alpha}) \oslash \exp(\omega t) \\
&\leq \bigoplus_{k=0}^{m-2} g^{-1}\left(\frac{T^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right) \odot |\phi_0^{(k)}|_g \oplus (|A| \oplus |B|) \odot \bigoplus_{k=0}^{m-2} g^{-1}\left(T^{(\beta-1)(m-\alpha)+\alpha+k}\right) \\
&\odot g^{-1}((\exp(|A|+|B|)T^{\alpha}) \odot |\phi_0^{(k)}|_g \oplus g^{-1}(T^{\alpha-1}) \odot g^{-1}((\exp(|A|+|B|)T^{\alpha}) \odot |\phi_0^{(m-1)}|_g
\end{aligned}$$

$$\oplus|B| \oslash \exp(\omega t) \odot \int_{[0,\tau]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right) \odot \exp(\omega(s-\tau)) \odot ds \odot \max_{t\in[0,T]}\left\{|\phi(t)|_g \oslash \exp(\omega t)\right\} \odot g^{-1}((\exp(|A|+|B|)(T)^{\alpha})$$

$$\oplus C \oslash \exp(\omega t) \odot \int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right) \odot \exp(\omega s) \odot ds \odot \max_{t\in[0,T]}\left\{|y(t)|_g \oslash \exp(\omega t)\right\} \odot g^{-1}((\exp(|A|+|B|)(T)^{\alpha})$$

$$\oplus \int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right) \odot \exp(\omega s) \odot ds \odot \max_{t\in[0,T]}\left\{|f(s,0)|_g \oslash \exp(\omega t)\right\} \odot g^{-1}((\exp(|A|+|B|)(T)^{\alpha}) \oslash \exp(\omega t)$$

$$|(Fy)(t)|_g \oslash \exp(\omega t) \leq \bigoplus_{k=0}^{m-2} g^{-1}\left(\frac{T^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right) \odot |\phi_0^{(k)}|_g \oplus (|A| \oplus |B|) \odot \bigoplus_{k=0}^{m-2} g^{-1}\left(T^{(\beta-1)(m-\alpha)+\alpha+k}\right)$$

$$\odot S \odot |\phi_0^{(k)}|_g \oplus g^{-1}(T^{\alpha-1}) \odot S \odot |\phi_0^{(m-1)}|_g \oplus |B| \odot S \oslash \exp(\omega t) \odot \int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right) \odot \exp(\omega s) \odot ds \odot \|\phi\|_{\omega}$$

$$\oplus C \odot S \oslash \exp(\omega t) \odot \int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right) \odot \exp(\omega s) \odot ds \odot \|y\|_{\omega}$$

$$\oplus D \odot S \oslash \exp(\omega t) \int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right) \odot \exp(\omega s) \odot ds$$

$$= \bigoplus_{k=0}^{m-2}\left(g^{-1}\left(\frac{T^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right) \oplus (|A| \oplus |B|) \odot g^{-1}\left(T^{(\beta-1)(m-\alpha)+\alpha+k}\right) \odot S\right) \odot |\phi_0^{(k)}|_g$$

$$\oplus g^{-1}(T^{\alpha-1}) \odot S \odot |\phi_0^{(m-1)}|_g \oplus |B| \odot S \oslash \exp(\omega t) \odot \int_{[0,t]}^{\oplus} g^{-1}\left(u^{\alpha-1}\right) \odot \exp(\omega t) \odot \exp(-\omega u) \odot du \odot ||\phi||_{\omega}$$

$$\oplus C \odot S \oslash \exp(\omega t) \odot \int_{[0,t]}^{\oplus} g^{-1}\left(u^{\alpha-1}\right) \odot \exp(\omega t) \odot \exp(-\omega u) \odot du \odot \|y\|_{\omega}$$

$$\oplus D \odot S \oslash \exp(\omega t) \int_{[0,t]}^{\oplus} g^{-1}\left(u^{\alpha-1}\right) \odot \exp(\omega t) \odot \exp(-\omega u) \odot du$$

$$= W \odot |\phi_0^{(k)}|_g \oplus g^{-1}(T^{\alpha-1}) \odot S \odot |\phi_0^{(m-1)}|_g \oplus |B| \odot S \odot \int_{[0,t]}^{\oplus} g^{-1}\left(u^{\alpha-1}\right) \odot \exp(-\omega u) \odot du \odot \|\phi\|_{\omega}$$

$$\oplus C \odot S \odot \int_{[0,t]}^{\oplus} g^{-1}\left(u^{\alpha-1}\right) \odot \exp(-\omega u) \odot du \odot \|y\|_{\omega} \oplus D \odot S \int_{[0,t]}^{\oplus} g^{-1}\left(u^{\alpha-1}\right) \odot \exp(-\omega u) \odot du$$

$$= W \odot |\phi_0^{(k)}|_g \oplus g^{-1}(T^{\alpha-1}) \odot S \odot |\phi_0^{(m-1)}|_g \oplus |B| \odot S \oslash g^{-1}(\omega^{\alpha}) \odot \int_{[0,\omega t]}^{\oplus} g^{-1}\left(v^{\alpha-1}\right) \odot \exp(-v) \odot dv \odot \|\phi\|_{\omega}$$

$$\oplus C \odot S \oslash g^{-1}(\omega^{\alpha}) \odot \int_{[0,\omega t]}^{\oplus} g^{-1}\left(v^{\alpha-1}\right) \odot \exp(-v) \odot dv \odot \|y\|_{\omega} \oplus D \odot S \oslash g^{-1}(\omega^{\alpha}) \int_{[0,\omega t]}^{\oplus} g^{-1}\left(v^{\alpha-1}\right) \odot \exp(-v) \odot dv$$

$$= W \odot |\phi_0^{(k)}|_g \oplus g^{-1}(T^{\alpha-1}) \odot S \odot |\phi_0^{(m-1)}|_g \oplus S \oslash g^{-1}(\omega^{\alpha}) \int_{[0,\omega t]}^{\oplus} g^{-1}\left(v^{\alpha-1}\right) \odot \exp(-v) \odot dv$$

$$\odot\left(|B| \odot \|\phi\|_{\omega} \oplus C \odot \|y\|_{\omega} \oplus D\right)$$

$$\leq W \odot |\phi_0^{(k)}|_g \oplus g^{-1}(T^{\alpha-1}) \odot S \odot |\phi_0^{(m-1)}|_g \oplus S \oslash g^{-1}(\omega^{\alpha}) \int_{[0,\infty]}^{\oplus} g^{-1}\left(v^{\alpha-1}\right) \odot \exp(-v) \odot dv$$

$$\odot\left(|B| \odot \|\phi\|_{\omega} \oplus C \odot ||y||_{\omega} \oplus D\right) = W \odot |\phi_0^{(k)}|_g \oplus g^{-1}(T^{\alpha-1}) \odot S \odot |\phi_0^{(m-1)}|_g$$

$$\oplus S \dot{g}^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \left(|B| \odot \|\phi\|_{\omega} \oplus C \odot \|y\|_{\omega} \oplus D\right)$$

$$\leq W \odot |\phi_0^{(k)}|_g \oplus g^{-1}(T^{\alpha-1}) \odot S \odot |\phi_0^{(m-1)}|_g \oplus S \dot{g}^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \left(|B| \odot \|\phi\|_{\omega} \oplus C \odot R \oplus D\right)$$

Taking the maximum over $[0, T]$ and using inequality (5.6), we obtain the following relation:

$$\|Fy\|_{\omega} \leq R$$

For this reason, $F : B_R \to B_R$. In other words, F is well-defined on B_R.

Step 2.In this step, we will represent that F is a contractive mapping. We should demonstrate that F is a contraction over B_R. To see this, let $\forall y, \sigma \in B_R$. Mention that

$$(Fy)(t) \ominus (F\sigma)(t) = \int_{[0,t]}^{\oplus} E_{\alpha,\alpha,\alpha}^{\tau,\oplus}(A, B; g^{-1}(t-s)) \odot (f(s, y(s)) \ominus f(s, \sigma(s)) \odot ds, \quad t > 0. \tag{34}$$

Thus, for any $t \in [0, T]$, from lemma 5.2 and (H_2)-Lipschitz condition, it follows that

$$|(Fy)(t) \ominus (F\sigma)(t)|_g \oslash \exp(\omega t) = g^{-1}\left(\frac{|(Fg(y))(t) - (Fg(\sigma))(t)|}{g(\exp(\omega t))}\right)$$

$$\leq g^{-1}\left(\frac{1}{g(\exp(\omega t))} \int_0^t |E_{\alpha,\alpha,\alpha}^{\tau}(g(A), g(B); t-s)||g(f(s, y(s))) - g(f(s, \sigma(s)))|ds\right)$$

$$= g^{-1}\left(\frac{1}{g(\exp(\omega t)}\right) \odot \int_{[0,t]}^{\oplus} |E_{\alpha,\alpha,\alpha}^{\tau,\oplus}(A, B; g^{-1}(t-s))|_g \odot |f(s, y(s)) \ominus f(s, \sigma(s))|_g \odot ds$$

$$\leq (C \odot \exp((|A| + |B|)t^{\alpha})) \oslash \exp(\omega t) \odot \int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right) \odot |y(s) \ominus \sigma(s)|_g \oslash \exp(\omega s) \odot \exp(\omega s) \odot ds$$

$$\leq (C \odot \exp((|A| + |B|)t^{\alpha})) \oslash \exp(\omega t) \odot \int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right) \odot \exp(\omega s) \odot ds \odot \max_{t\in[0,T]} \left\{|y(t) - \sigma(t)|_g \oslash \exp(\omega t)\right\}$$

$$= (C \odot \exp((|A| + |B|)t^{\alpha})) \oslash \exp(\omega t) \odot \int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right) \odot \exp(\omega s) \odot ds \odot \|y - \sigma\|_{\omega}$$

$$
\begin{aligned}
&=(C \odot \exp((|A|+|B|)t^{\alpha})) \oslash \exp(\omega t) \odot \int_{[0,t]}^{\oplus} g^{-1}\left(u^{\alpha-1}\right) \odot \exp(\omega t) \odot \exp(-\omega u) \odot du \odot \|y-\sigma\|_{\omega} \\
&=C \odot \exp((|A|+|B|)t^{\alpha}) \odot \int_{[0,t]}^{\oplus} g^{-1}\left(u^{\alpha-1}\right) \odot \exp(-\omega u) \odot du \odot \|y-\sigma\|_{\omega} \\
&=(C \odot \exp((|A|+|B|)t^{\alpha})) \oslash g^{-1}(\omega^{\alpha}) \odot \int_{[0,\omega t]}^{\oplus} g^{-1}\left(v^{\alpha-1}\right) \odot \exp(-v) \odot dv \odot \|y-\sigma\|_{\omega} \\
&\leq (C \odot \exp((|A|+|B|)t^{\alpha})) \oslash g^{-1}(\omega^{\alpha}) \odot \int_{[0,\infty]}^{\oplus} g^{-1}\left(v^{\alpha-1}\right) \odot \exp(-v) \odot dv \odot \|y-\sigma\|_{\omega} \\
&= \exp((|A|+|B|)t^{\alpha}) \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \|y-\sigma\|_{\omega} \\
&\leq \exp((|A|+|B|)T^{\alpha}) \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \|y-\sigma\|_{\omega} := S \odot C \odot g^{-1}(\Gamma(\alpha)) \odot g^{-1}(\omega^{\alpha}) \odot \|y-\sigma\|_{\omega}
\end{aligned}
$$

Then, we get.

$$|(Fy)(t) \ominus (F\sigma)(t)|_g \oslash \exp(\omega t) \leq S \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \|y-\sigma\|_{\omega}$$

Taking maximum on $[0,T]$, we will get the following conclusion:

$$\|F(y) \ominus F(\sigma)\|_{\omega} \leq S \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \|y-\sigma\|_{\omega} \tag{35}$$

If we choose $\omega > (S \odot C \odot g^{-1}(\Gamma(\alpha)) \odot g^{-1}(\omega^{\alpha}))^{\frac{1}{\alpha}}$, then F is a contraction. Thus, by Banach's fixed point theorem, there exists a unique fixed point of F which is just the unique global continuous solution of (5.1). □

Remark 1. If the assumptions (H_1) and (H_2) are satisfied for all $t \in [0,\infty)$, then the claim of this theorem holds on the half-real line R, i.e. for any $(m-1)$-times continuously differentiable initial data $\phi : [-\tau, 0] \to R$, the non-linear pseudo-Hilfer equation type equation of fractional order with a constant delay (5.1) has a unique global continuous solution on $[0,\infty)$.

6 | ULAM-HYERS STABILITY ANALYSIS ON PSEUDO-HILFER TYPE FRACTIONAL DIFFERENTIONAL EQUATION WITH A CONSTANT DELAY

In the following part, we debate the stability of the pseudo-Hilfer-type DDE (5.1) in the Ulam-Hyers sense on $[0,T]$.

Suppose that $\epsilon > 0$. Let us imagine the pseudo-Hilfer type fractional delay differential equation (5.1) and the Initial issue for the following inequality:

$$|H^{\alpha,\beta}_{\oplus,\odot,0+}\sigma(t) \ominus A \odot \sigma(t) \ominus B \odot \sigma(t-\tau) \ominus f(t,\sigma(t))|_g \leq \epsilon, \quad for \quad t \in [0,T] \tag{36}$$

Definition 1. Equation (6.1) is Ulam-Hyers stable if there is $\theta > 0$ such that for every $\epsilon > 0$ and for every solution $\sigma \in C([0,T],R)$ of inequality (6.1), there is a solution $y \in C([0,T],R)$ of equation (5.1) that holds the inequality due to a weighted norm:

$$\|y \ominus \sigma\|_{\omega} \leq \epsilon \odot \theta, \quad t \in [0,T] \tag{37}$$

Remark 2. A function $\sigma \in C([0,T],R)$ is a solution of the inequality (6.1) if and only if there is a function $f \in C([0,T],R)$ which fulfills the following conditions:
1) $|f(t)|_g \leq \epsilon$;
2) $H^{\alpha,\beta}_{\oplus,\odot,0+}\sigma(t) \ominus A \odot \sigma(t) \ominus B \odot \sigma(t-\tau) \ominus f(t,\sigma(t)) := f(t), t \in [0,T]$.

Due to the Remark 6.1, the solution of following equation:

$$H^{\alpha,\beta}_{\oplus,\odot,0+}\sigma(t)\ominus A\odot\sigma(t)\ominus B\odot\sigma(t-\tau)=f(t,\sigma(t))\oplus f(t),t\in[0,T]. \tag{38}$$

can be demonstrate by

$$\sigma(t)=\bigoplus_{k=0}^{m-2}\left(g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\oplus(A\oplus B)\odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))\right)\odot\phi_0^{(k)}$$

$$\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t))\odot\phi_0^{(m-1)}\oplus B\odot\int^{\oplus}_{[-\tau,\min(t-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds$$

$$\oplus\int^{\oplus}_{[0,t]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))\odot f(s,\sigma(s))\odot ds\oplus\int^{\oplus}_{[0,t]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))\odot f(s)\odot ds$$

$$:=(F(\sigma))(t)\oplus\int^{\oplus}_{[0,t]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))\odot f(s)\odot ds,\quad t\in[0,T].$$

To use Lemma 5.2, the difference $\sigma(t)\ominus(F(z))(t)$ can be evaluated as follows:

$$|\sigma(t)\ominus(F(\sigma))(t)|_g=|\int^{\oplus}_{[0,t]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))\odot f(s)\odot ds|_g\leq\int^{\oplus}_{[0,t]}|E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))|_g\odot|f(s)|_g\odot ds$$

$$\leq\epsilon\odot g^{-1}(t^{\alpha-1})\odot g^{-1}(\exp((|A|+|B|)t^{\alpha}))\odot\int^{\oplus}_{[0,t]}ds\leq\epsilon\odot g^{-1}(T^{\alpha})\odot g^{-1}(\exp((|A|+|B|)T^{\alpha})):=\epsilon\odot g^{-1}(T^{\alpha})\odot S. \tag{39}$$

Finally, with constant delay, we are ready to assert and prove the Ulam-Hyers stability result for pseudo-Hilfer FDE.

Theorem 6. Suppose that (H_1 and H_2) are satisfied. Then the equation (5.1) is Ulam-Hyers stable on $[0,T]$.

Proof. Assume that $\sigma\in C[0,T]$, R is a solution of the inequality (6.1). Let y be a unique solution of the Cauchy problem for pseudo-Hilfer type fractional-order DDE(5.1), that is

$$y(t)=\bigoplus_{k=0}^{m-2}\left(g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\oplus(A\oplus B)\odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))\right)\odot\phi_0^{(k)}$$

$$\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t))\odot\phi_0^{(m-1)}\oplus B\odot\int^{\oplus}_{[-\tau,\min(t-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds$$

$$\oplus\int^{\oplus}_{[0,t]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))\odot f(s,\sigma(s))\odot ds:=(Fy)(t),\quad t\in[0,T] \tag{40}$$

By using estimation (5.9) and (6.5), we obtain

$$|y(t)\ominus\sigma(t)|_g\oslash\exp(\omega t)=|(Fy)(t)\ominus(F\sigma)(t)\ominus\int^{\oplus}_{[0,t]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))\odot f(s)\odot ds|_g\oslash\exp(\omega t)$$

$$\leq|(Fy)(t)\ominus(F\sigma)(t)|_g\oslash\exp(\omega t)\oplus\int^{\oplus}_{[0,t]}|E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))|_g\odot|f(s)|_g\odot ds$$

$$\leq C \odot g^{-1}(\Gamma(\alpha)) \odot \exp((|A|+|B|)T^{\alpha}) \oslash g^{-1}(\omega^{\alpha}) \odot \|y-\sigma\|_{\omega} \oplus \epsilon \odot g^{-1}(T^{\alpha}) \odot g^{-1}(\exp((|A|+|B|)T^{\alpha}))$$
$$:= S \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \|y-\sigma\|_{\omega} \oplus \epsilon \odot g^{-1}(T^{\alpha}) \odot S$$

We take maximum on $[0, T]$, then we obtain

$$\|y-\sigma\|_{\omega} \leq S \odot L \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot ||y-\sigma||_{\omega} \oplus \epsilon \odot g^{-1}(T^{\alpha}) \odot S$$

that gives that

$$\|y-\sigma\|_{\omega} \leq \epsilon \odot (g^{-1}(T^{\alpha}) \odot S) \oslash (1 \ominus S \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}))$$

By choosing $\omega > \left(g(S \odot C \odot g^{-1}(\Gamma(\alpha)))\right)^{\frac{1}{\alpha}}$ which implies that

$$\|y-\sigma\|_{\omega} \leq \epsilon \odot \theta \tag{41}$$

where

$$\theta := (g^{-1}(T^{\alpha}) \odot S) \oslash (1 \ominus S \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}))$$

□

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