Delayed analogue of three-parameter pseudo-Mittag-Leffler functions and their applications to Hilfer pseudo-fractional time retarded differential equations

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In this write-up, we focus on pseudo-Hilfer-type fractional order delayed differential equations with bounded definite integral initial conditions on the time interval $[0,T]$. We begin by establishing relevant lemmas. Then, we derive the solution to the homogeneous Hilfer-type pseudo-fractional order retarded differential equation that satisfies the appropriate initial condition using classical methods. Next, we obtain explicit formulas for solutions to linear inhomogeneous Hilfer-type pseudo-fractional time retarded differential equations with constant coefficients, employing classical ideas. Furthermore, we investigate the existence and uniqueness of the solution of the Hilfer-type pseudo-fractional order delayed differential equation and demonstrate the stability of the given differential equation in the Ulam-Hyers sense on the time interval $[0,T]$.

I. INTRODUCTION.

Differential equations, a fundamental concept in mathematics with ancient roots, saw substantial advancement in the 17th century, primarily due to the contributions of Gottfried Wilhelm Leibniz and Isaac Newton in the field of calculus. This period marked a systematic shift in approaching and solving these equations.

The 18th-century contributions of mathematicians like Leonhard Euler elevated the theory of ordinary differential equations, while the 19th century introduced more sophisticated techniques, including the Laplace transform by Pierre-Simon Laplace, streamlining the resolution of linear differential equations.

The 20th century, with the emergence of computers, brought a transformative shift where numerical methods became indispensable for solving complex differential equations that were previously deemed insurmountable.

Stability theory, a pivotal tool in understanding system behavior, encompasses both linear and nonlinear stability analysis, addressing minor perturbations and intricate non-linear systems, respectively. This facet of differential equations is crucial for ensuring the dependability and practical application of solutions across scientific and engineering domains.

The application of differential equations expanded further to include specialized areas like fractional and delay differential equations. Fractional differential equations, extending the concept of derivatives and integrals to non-integer orders, play a pivotal role in accurately modeling physical and engineering systems, capturing complex dynamics, chaos, and multi-dimensional systems.

Conversely, delay differential equations consider the impact of time delays within systems, enhancing the precision and realism of models in biology, economics, engineering, and beyond.

This broader scope underscores the growing significance and versatility of differential equations in solving diverse problems. The advancements in fractional and delay differential equations underscore the depth of mathematical inquiry and the collaborative, cumulative nature of scientific progress.

In recent times, fractional differential equations (FDEs) have gained prominence for their applications in mechanics, electrical circuits, and time-delay systems stability analysis. By incorporating derivatives of fractional order, FDEs offer a more nuanced modeling approach compared to classical differential equations, capturing behaviors beyond the reach of integer-order derivatives alone. Their applications span various scientific and engineering fields, employing analytical methods like Laplace and Fourier transforms, along with numerical methods. The adaptability of fractional calculus has led to groundbreaking applications in domains such as control theory, signal processing, optimization, image processing, finance, and economics.

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Similarly, pseudo-analysis is a mathematical theory that generalizes classical analysis by using semiconductors defined by pseudo-addition and pseudo-multiplication in the real range, instead of real numbers. This concept has piqued the interest of researchers from different fields such as functionality analysis, functional equations, and variational calculus.

In recent times, many scholars have worked on new formulations of inequalities involving fractional integrals and have investigated the properties of pseudo-fractional operators. For example, J. Vanterler da C. Sousa, Rubens F. Camargo, E. Capelas de Oliveira Gastano S. F. Frederico have studied pseudo-Hilfer-type FDEs([2]).

The existence and uniqueness problems of FDEs with constant delay and the stability of their solutions are crucial topics in the field of fractional differential equations. Many renowned scientists, such as Ahmed H.M., Ahmed A.M.S., Ragusa M.A ([39]), Moniri Z., Moghaddam B.P., Roudbaraki M.Z. ([40]), Vivek D., Kanagarajan K., Elsayed E.M. ([41]), Nazim I. Mahmudov, Ismail T. Huseynov, Arzu Ahmadova,([3],[7]-[12],[17]-[20],[24]-[26]) Khusainov, D.Ya., Ivanov, A.F., Shuklin, G.V.,([15]), Podlubny, I.([21])., J.Vanterler da C.Sousa, Gastao S.F. Frederico, E. Capelas de Oliveira([1],[2]) have made significant contributions to these problems.

In conclusion, fractional differential equations and pseudo-analysis are fascinating areas of research with wide-ranging applications in various fields. The works of renowned scientists in these fields have contributed significantly to the advancement of mathematical theory and its applications in engineering and science.

For instance: J. Vanterler da C. Sousa, Rubens F. Camargo, E. Capelas de Oliveira Gastano S. F. Frederico has looked following pseudo-Hilfer-type FDE([2]).

$$\begin{cases} H^{\alpha,\beta:\psi}_{\oplus,\odot,t_0+}y(t) = Ay(t) \oplus f(t,y(t)), t \in J, \\ I^{1-\gamma}_{\oplus,\odot,t_0+}y(t) = y_0. \end{cases} \tag{I.1}$$

The authors of this study investigate the existence and uniqueness of the global solution for equation (I.1). The equation involves the ψ-Hilfer pseudo-fractional derivative denoted by $H^{\alpha,\beta:\psi}_{\oplus,\odot,t_0+}(\cdot)$, where the order is $0<\alpha\leq 1$ and the type is $0\leq\beta\leq 1$. The parameter γ is defined as $\gamma=\alpha-\beta(1-\alpha)$. The function $f:[t_0,+\infty)\times R^n\times R^n\to R^n$ is continuous. $\mathscr{A}$ is an $n\times n$ matrix.

It is worth mentioning that in a previous study by Sousa et al. in 2020 [6], the existence and uniqueness of the global solution for the initial value problem associated with data (t_0,y_0) was researched. The general form of any solution on the interval $\mathscr{I}:=[a,b]$ is given by the system of equations (I.2), where $\frac{d^{\oplus}}{dt}y(t)$ denotes the pseudo-fractional derivative of $y(t)$ and $\mathscr{F}(t,y(t))=f(t,y(t))$. The initial condition is $y(t_0)=y_0$.

$$\begin{cases} \frac{d^{\oplus}}{dt}y(t) = F(t,y(t)), \\ y(t_0) = y_0. \end{cases} \tag{I.2}$$

with $t_0\in I$. Afterward, in 2020, Sosa et al. ([6]), discussed the reachability of linear and non-linear systems in the sense of the ψ-Hilfer pseudo-fractional derivative in g-calculus by means of the Mittag-Leffler functions with the form

$$\begin{cases} H^{\alpha,\beta;\psi}_{\oplus,\odot,0+}y(t) = Ay(t) \oplus Bu(t), t \in [t_0,t_1], \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t_0) = 0 \end{cases} \tag{I.3}$$

and

$$\begin{cases} H^{\alpha,\beta;\psi}_{\oplus,\odot,0+}y(t) = Ay(t) \oplus Bu(t) \oplus f(t,y(t),u(t)), t \in [t_0,t_1], \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t_0) = 0. \end{cases} \tag{I.4}$$

Where $H^{\alpha,\beta;\psi}_{\oplus,\odot,0+}(\cdot)$ represents the ψ-Hilfer pseudo-fractional derivative with order $0<\alpha\leq 1$ and type $0\leq\beta\leq 1$. The parameter γ is defined as $\gamma=\alpha-\beta(1-\alpha)$, and $I^{1-\gamma}_{\oplus,\odot,0+}(\cdot)$ denotes the Riemann-Liouvile pseudo-fractional integral with respect to another function $1-\gamma$. The state vector is denoted by $y\in R^n$, the control vector by $u\in R^m$, and A and B are constant matrices of dimensions $n\times n$ and $n\times m$, respectively. The non-linear function $f:J\times R^n\times R^m\to R^n$ is continuous in this context.

However, in this research article, we will be considering the following Hilfer-type pseudo-fractional delay differential equation:

$$\begin{cases} H^{\alpha,\beta}_{\oplus,\odot,0+}y(t) = A \odot y(t) \oplus B \odot y(t-\tau) \oplus f(t), t \in (0;T], \tau > 0, \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t) = \phi(t), t \in [-\tau,0]. \end{cases} \tag{I.5}$$

where $m-1<\alpha<m$, $0\leq\beta\leq 1$, $\gamma=(\beta-1)(m-\alpha)+k+1, k=0,...,m-1$.

To achieve our primary objective of obtaining an analytical solution for the Hilfer-type pseudo-fractional time delay differential equation (I.5) with a constant delay using classical methods, we first need to obtain the solution for the homogeneous Hilfer-type pseudo-fractional delay equations (I.6).

$$\begin{cases} H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)=A\odot y(t)\oplus B\odot y(t-\tau), t\in(0;T], \tau>0, \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t)=\phi(t), t\in[-\tau,0]. \end{cases} \tag{I.6}$$

Subsequently, we employ conventional techniques to determine the explicit solution formula for linear inhomogeneous Hilfer-type pseudo-fractional time-retarded differential equations with constant coefficients, as presented in equation (I.5). We utilize well-established methods and refer to equation (I.7) to facilitate the solution.

$$\begin{cases} H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)=A\odot y(t)\oplus B\odot y(t-\tau)\oplus f(t), t\in(0;T], \tau>0, \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t)=0, t\in[-\tau,0]. \end{cases} \tag{I.7}$$

We make use of the solution of equation (I.7) as a particular solution to equation (I.5) to derive the analytic solution, considering the conditions $m-1<\alpha<m$, $0\leq\beta\leq 1$, and $\gamma=(\beta-1)(m-\alpha)+k+1, k=0,...,m-1$. Moreover, we establish the existence and uniqueness of the solution in our study and additionally investigate the stability of the Hilfer-type pseudo-fractional delay differential equation (FDDE) (I.5) in the Ulam-Hyers sense over the time interval $[0,T]$.

II. PRELIMINARIES

In this part, we mention important information that deals with pseudo-analysis, the elements of the fractional analysis, and some necessary lemmas that will use the proof of the theorem. ([16],[21])

- Gamma function:

$$\Gamma(\alpha)=\int_0^\infty \tau^{\alpha-1}e^{-\tau}d\tau, \quad \alpha>0.$$

- Beta function:

$$B(t,s)=\int_0^1 z^{t-1}(1-z)^{s-1}dz, \quad t,s>0.$$

  Let $g:J\to R_+$ be a monotone and continuous function, where $J=[a,b]$ and $R_+=[0,+\infty]$. Then we will define the Mittag-Leffler function as follows.

- The tree parametr Mittag-Leffler function:([22])

$$E^{\delta}_{\alpha,\beta}g(z)=\sum_{s=0}^{\infty}\frac{(\delta)_s}{\Gamma(\alpha s+\beta)}\frac{(g(z))^s}{s!}=\sum_{s=0}^{\infty}\frac{(\delta)_s}{\Gamma(\alpha s+\beta)}\frac{(g(z))^s}{s!}.$$

- Delayed analogue of Mittag-Leffler type function generated by $A,B\in R$ of three parameters:([18])

$$E^{\tau}_{\alpha,\beta,\gamma}(g(A),g(B);t)=\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^n(g(B))^q}{\Gamma(n\alpha+q\beta+\gamma)}(t-n\tau)^{n\alpha+q\beta+\gamma-1}H(t-n\tau).$$

- Exponentially bounded $f:[0,\infty)\to R$ holds an inequality of the form

$$\|f(t)\|\leq Le^{\sigma t}, \quad t>T,$$

  for the real constants $\sigma, L>0$ and $T>0$.

- Laplace transform $\mathfrak{L}\{f(t)\}(s)$:

$$F(s)=\mathfrak{L}\{f(t)\}(s)=\int_0^\infty e^{-st}f(t)dt, \quad s\in C,$$

where $f:[0,\infty)\to R$ is measurable and exponentially bounded on $[0,\infty)$, then the appointed by exists and is an analytic function of s for $Re(s)>0$.

- Timeshift feature of the Laplace transform:

$$\mathfrak{L}\{f(t-a)H(t-a)\}(s)=e^{-as}F(s).$$

- Convolution feature of Laplace transform:

$$\mathfrak{L}\{(f*h)(t)\}=\mathfrak{L}\{f(t)\}(s)\mathfrak{L}\{h(t)\}(s),$$

where $f,h:[0,\infty)\to R$ are exponetially bounded functions.

- Riemann-Liouville fractional integral:

$$I^\alpha_{a+}f(x)=\frac{1}{\Gamma(\alpha)}\int_a^x(x-t)^{\alpha-1}f(t)dt.$$

- Hilfer fractional derivative Let $m-1<\alpha<m$, with $m\in N$. The right-sided Hilfer fractional derivatives, denoted by ${}^HD^{\alpha,\beta}_{a+}(\cdot)$ of a function f of order α and type $0\le\beta\le 1$, are appointed by

$${}^HD^{\alpha,\beta}_{a+}f(x)=I^{\beta(m-\alpha)}_{a+}\frac{d^m}{dx^m}I^{(1-\beta)(m-\alpha)}_{a+}f(x). \tag{II.1}$$

Taking the limit $\beta\to 0$ in Eq.(II.1), we have the Rieman-Liouville derivative, given by:

$${}^{RL}D^\alpha_{a+}f(x)=\frac{d^m}{dx^m}I^{(m-\alpha)}_{a+}f(x).$$

Taking the limit $\beta\to 1$ in Eq.(II.1), we have the Caputo derivative, given by:

$${}^CD^\alpha_{a+}f(x)=I^{(m-\alpha)}_{a+}\frac{d^m}{dx^m}f(x).$$

- For any linear and bounded operator Ω appointed on a Banach space with $\|\Omega\|<1$, the operator $(I-\Omega)^{-1}$ is linear and bounded with property

$$(I-\Omega)^{-1}=\sum_{k=0}^\infty\Omega^k. \tag{II.2}$$

Lemma II.1. *Let $g:J\to R_+$ be a monotone and continious function , where $J=[a,b]$ and $R_+=[0,+\infty]$.Then, for $\alpha>0$,$A\in R$, $n\in N_0=0,1,2,...$, we have*

$$\mathfrak{L}^{-1}\left\{\frac{1}{(s^\alpha-g(A))^{n+1}}\right\}(t)=\sum_{q=0}^\infty\binom{n+q}{q}(g(A))^q\frac{t^{\alpha(n+q+1)-1}}{\Gamma(\alpha(n+q+1))}=t^{(n+1)\alpha-1}E^{n+1}_{\alpha,(n+1)\alpha}(g(A)t^\alpha), \quad Re(s)>0.$$

Proof. Using the expansion

$$\frac{1}{(1-t)^{n+1}}=\sum_{q=0}^\infty\binom{n+q}{q}t^q, \quad |t|<1,$$

for $|t|=\left|\frac{g(A)}{s^\alpha}\right|<1$, we find that

$$\frac{1}{(s^\alpha-g(A))^{n+1}}=\frac{1}{s^{\alpha(n+1)}}\times\frac{1}{\left(1-\frac{g(A)}{s^\alpha}\right)^{n+1}}=\frac{1}{s^{\alpha(n+1)}}\sum_{q=0}^\infty\binom{n+q}{q}\left(\frac{g(A)}{s^\alpha}\right)^q=\sum_{q=0}^\infty\binom{n+q}{q}\frac{(g(A))^q}{s^{q\alpha+\alpha(n+1)}}.$$

Taking the inverse-Laplace transform of the above, we obtain that

$$\mathfrak{L}^{-1}\left\{\frac{1}{(s^{\alpha}-g(A))^{n+1}}\right\}(t)=\mathfrak{L}^{-1}\left\{\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q}}{s^{q\alpha+\alpha(n+1)}}\right\}(t)=\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}$$
$$\times\mathfrak{L}^{-1}\left\{\frac{1}{s^{q\alpha+\alpha(n+1)}}\right\}(t)=\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q}t^{\alpha(n+q+1)-1}}{\Gamma(\alpha(n+q+1))}=t^{(n+1)\alpha-1}E^{n+1}_{\alpha,(n+1)\alpha}(g(A)t^{\alpha}).$$

□

Lemma II.2. *Let $g:J\to R_{+}$ be a monotone and continuous function, where $J=[a,b]$ and $R_{+}=[0,+\infty]$.Then, for $\alpha>0,\alpha>\gamma$, we obtain.*

$$\mathfrak{L}^{-1}\left\{\frac{s^{\gamma}}{s^{\alpha}-g(A)-g(B)e^{-s\tau}}\right\}(t)=E^{\tau}_{\alpha,\alpha,\alpha-\gamma}(g(A),g(B);t).$$

Proof. According to the well-known Neumann series, $\frac{s^{\gamma}}{s^{\alpha}-g(A)-g(B)e^{-s\tau}}$ can be written through a series expansion as below:

$$\frac{s^{\gamma}}{s^{\alpha}-g(A)-g(B)e^{-s\tau}}=\frac{s^{\gamma}}{s^{\alpha}-g(A)}\frac{1}{1-\frac{g(B)e^{-s\tau}}{s^{\alpha}-g(A)}}=\frac{s^{\gamma}}{s^{\alpha}-g(A)}\sum_{n=0}^{\infty}\frac{(g(B))^{n}e^{-sn\tau}}{(s^{\alpha}-g(A))^{n}}=\sum_{n=0}^{\infty}\frac{(g(B))^{n}e^{-sn\tau}s^{\gamma}}{(s^{\alpha}-g(A))^{n+1}}.$$

Then imposing Lemma II.2 to the final consideration we get:

$$\frac{s^{\gamma}}{s^{\alpha}-g(A)-g(B)e^{-s\tau}}=\sum_{n=0}^{\infty}\frac{(g(B))^{n}e^{-sn\tau}s^{\gamma}}{s^{\alpha(n+1)}(1-\frac{g(A)}{s^{\alpha}})^{n+1}}=\sum_{n=0}^{\infty}\frac{(g(B))^{n}e^{-sn\tau}s^{\gamma}}{s^{\alpha(n+1)}}\sum_{q=0}^{\infty}\binom{n+q}{q}\left(\frac{g(A)}{s^{\alpha}}\right)^{q}$$
$$=\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{g(A)^{q}(g(B))^{n}e^{-sn\tau}}{s^{\alpha(n+1)+q\alpha-\gamma}}.$$

From the time delay feature of the Laplace integral transform, we have

$$\mathfrak{L}\{g(t-\tau)\}(s)(H(t-\tau)=e^{-s\tau}\mathfrak{L}\{g(t)\}(s).$$

Then, by taking the Inverse Laplace transform of the aforementioned function, we get

$$\mathfrak{L}^{-1}\left\{\frac{s^{\gamma}}{s^{\alpha}-g(A)-g(B)e^{-s\tau}}\right\}(t)=\mathfrak{L}^{-1}\left\{\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{g(A)^{q}(g(B))^{n}e^{-sn\tau}}{s^{\alpha(n+1)+q\alpha-\gamma}}\right\}(t)$$
$$=\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n}\mathfrak{L}^{-1}\left(\frac{e^{-sn\tau}}{s^{\alpha(n+1)+q\alpha-\gamma}}\right)(t)$$
$$=\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n}\frac{(t-n\tau)^{\alpha(n+1)+q\alpha-\gamma-1}H(t-n\tau)}{\Gamma(\alpha(n+1)+q\alpha-\gamma)}=E^{\tau}_{\alpha,\alpha,\alpha-\gamma}(g(A),g(B);t).$$

We need additional conditions on s, namely: $s^{\alpha}>|A|$ and $|s^{\alpha}-g(A)|>|B|e^{-s\tau}$ for convergence of the series. But, these conditions can be removed at the end of the evaluation with analytical continuation, to obtain the desired conclusion for all $s\in C$ with $Re(s)>0$. □

A. PSEUDO-ANALYSIS

Assume $g:[\alpha,\beta]\to[0,\infty]$ be a monotone and continuous function. We will define pseudo operators as follows. (see, e.g., [1],[2],[4],[27],[28])

- Pseudo operators:

$$\alpha \oplus \beta = g^{-1}(g(\alpha)+g(\beta)) \quad and \quad \alpha \odot \beta = g^{-1}(g(\alpha)g(\beta)),$$

$$\alpha \ominus \beta = g^{-1}(g(\alpha)-g(\beta)) \quad and \quad \alpha \oslash \beta = g^{-1}\left(\frac{g(\alpha)}{g(\beta)}\right).$$

Suppose that $f:[c,d]\to[a,b]$ is a measurable function.

- g-integral:

$$\int_{[c,d]}^{\oplus} f \odot dx = g^{-1}\left(\int_c^d g(f(x))dx\right).$$

- g-Laplace transform:

$$\mathfrak{L}^{\oplus}\{f(x)\}(s) = g^{-1}\left(\mathfrak{L}\{g(f(x))\}(s)\right).$$

Assuming that g is the generator function for the strict pseudo-addition $\oplus$ on the interval $[a,b]$, and g is continuously differentiable on (a,b), the corresponding pseudo-multiplication $\odot$ is defined as $x\odot y = g^{-1}(g(x)g(y))$. If a function f is differentiable on (c,d) and has the same monotonicity as the function g, then the g-derivative of f at the point $x\in(c,d)$ can be defined as follows:

- g-derivative:

$$\frac{d^{\oplus}f(x)}{dx} = g^{-1}\left(\frac{d}{dx}g(f(x))\right).$$

- n^{th}-g-derivative:

$$\frac{d^{(n)\oplus}f(x)}{dx} = g^{-1}\left(\frac{d^n}{dx^n}g(f(x))\right).$$

Now we will give some essential information about the Hilfer operator and Hilfer-type fractional derivative.

- Riemann-Liouville pseudo-fractional integral.

  Assuming that $g:[a,b]\to[0,+\infty]$ is an increasing function that defines pseudo-addition $\oplus$ and pseudo-multiplication $\odot$ operations, the right-sided and left-sided Riemann-Liouville pseudo-fractional integrals of a measurable function $f:[a,b]\to[a,b]$ with a positive order $\alpha>0$ can be defined in the following manner:

$$I^{\alpha}_{\oplus,\odot,a+}f(x) = g^{-1}\left(I^{\alpha}_{a+}g(f(x))\right) = \int_{[a,x]}^{\oplus}\left[g^{-1}\left(\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\right)\odot f(t)\right]\odot dt.$$

  and

$$I^{\alpha}_{\oplus,\odot,b-}f(x) = g^{-1}\left(I^{\alpha}_{b-}g(f(x))\right) = \int_{[x,b]}^{\oplus}\left[g^{-1}\left(\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\right)\odot f(t)\right]\odot dt.$$

- Hilfer pseudo-fractional derivatives.

  Consider a generator function $g:[a,b]\to[0,\infty]$ that is increasing, defining the pseudo-addition $\oplus$ and pseudo-multiplication $\odot$ operations. The right-sided and left-sided Hilfer pseudo-fractional derivatives of a measurable function $f:[a,b]\to[a,b]$, with orders $m-1<\alpha<m$ and type $0\le\beta\le1$, respectively, can be defined as follows:

$$H^{\alpha,\beta}_{\oplus,\odot,a+}f(x) = g^{-1}\left({}^{H}D^{\alpha,\beta}_{a+}\,g(f(x))\right) = I^{\beta(m-\alpha)}_{\alpha,\beta,a+}\,g^{-1}\left(\frac{d^m}{dx^m}\right)\odot I^{1-\gamma}_{\oplus,\odot,a+}f(x),$$

and

$$H^{\alpha,\beta}_{\oplus,\odot,b-}f(x)=g^{-1}\left({}^{H}D^{\alpha,\beta}_{b-}g(f(x))\right)=I^{\beta(m-\alpha)}_{\alpha,\beta,b-}g^{-1}\left(\frac{d^m}{dx^m}\right)\odot I^{1-\gamma}_{\oplus,\odot,b-}f(x).$$

Note that

$$H^{\alpha,\beta}_{\oplus,\odot,a+}f(x)=g^{-1}\left(I^{\gamma-\alpha RL}_{a+}D^{\gamma}_{a+}g(f(x))\right)=I^{\gamma-\alpha}_{\oplus,\odot,a+}{}^{RL}D^{\gamma}_{\oplus,\odot,a+}f(x),$$

and

$$H^{\alpha,\beta}_{\oplus,\odot,b-}f(x)=g^{-1}\left(I^{\gamma-\alpha RL}_{b-}D^{\gamma}_{b-}g(f(x))\right)=I^{\gamma-\alpha}_{\oplus,\odot,b-}{}^{RL}D^{\gamma}_{\oplus,\odot,b-}f(x),$$

where $\gamma=\alpha+\beta(m-\alpha)$. (For extra information about pseudo-analysis, see [27],[28],[37],[38].
In the following, we will first discuss the derivation of the formulas of the pseudo-Mittag-Leffler functions and their definitions based on these calculations.

- The one parameter pseudo-Mittag-Leffler function::

$$E^{\oplus}_{\alpha}(z)=g^{-1}\Big(E_{\alpha}g(z)\Big)=g^{-1}\left(\sum_{s=0}^{\infty}\frac{(g(z))^s}{\Gamma(\alpha s+1)}\right)=\bigoplus_{s=0}^{\infty}g^{-1}\left(\frac{(g(z))^s}{\Gamma(\alpha s+1)}\right)=\bigoplus_{s=0}^{\infty}\left[g^{-1}\Big((g(z))^s\Big)\oslash g^{-1}\Big(\Gamma(\alpha s+1)\Big)\right].$$

Where $(\delta)_s$ is the famous Pochhammer symbol denoting $\frac{\Gamma(\delta+s)}{\Gamma(\delta)}$.

- The two-parameter pseudo-Mittag-Leffler function:

$$E^{\oplus}_{\alpha,\beta}(z)=g^{-1}\Big(E_{\alpha,\beta}g(z)\Big)=g^{-1}\left(\sum_{s=0}^{\infty}\frac{(g(z))^s}{\Gamma(\alpha s+\beta)}\right)=\bigoplus_{s=0}^{\infty}g^{-1}\left(\frac{(g(z))^s}{\Gamma(\alpha s+\beta)}\right)=\bigoplus_{s=0}^{\infty}\left[g^{-1}\Big((g(z))^s\Big)\oslash g^{-1}\Big(\Gamma(\alpha s+\beta)\Big)\right].$$

- The three-parameter pseudo-Mittag-Leffler function:

$$E^{\delta,\oplus}_{\alpha,\beta}(z)=g^{-1}\Big(E^{\delta}_{\alpha,\beta}g(z)\Big)=g^{-1}\left(\sum_{s=0}^{\infty}\frac{(\delta)_s}{\Gamma(\alpha s+\beta)}\frac{(g(z))^s}{s!}\right)=\bigoplus_{s=0}^{\infty}g^{-1}\left(\frac{(\delta)_s}{\Gamma(\alpha s+\beta)}\frac{(g(z))^s}{s!}\right)$$

$$=\bigoplus_{s=0}^{\infty}g^{-1}\left[g\left(g^{-1}\left(\frac{(\delta)_s}{\Gamma(\alpha s+\beta)}\right)\right)g\left(g^{-1}\left(\frac{(g(z))^s}{s!}\right)\right)\right]=\bigoplus_{s=0}^{\infty}\left[g^{-1}\left(\frac{(\delta)_s}{\Gamma(\alpha s+\beta)}\right)\odot g^{-1}\left(\frac{(g(z))^s}{s!}\right)\right]$$

$$=\bigoplus_{s=0}^{\infty}\left[\Big(g^{-1}((\delta)_s)\oslash g^{-1}(\Gamma(\alpha s+\beta))\Big)\odot\Big(g^{-1}((g(z))^s)\oslash g^{-1}(s!)\Big)\right].$$

- The pseudo-bivariate Mittag-Leffler function:

$$E^{\delta,\oplus}_{\alpha,\beta,\gamma}(a,b)=g^{-1}\Big(E^{\delta}_{\alpha,\beta,\gamma}(g(a),g(b))\Big)=g^{-1}\left(\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\delta)_{l+s}}{\Gamma(l\alpha+s\beta+\gamma)}\frac{(g(a))^l(g(b))^s}{l!s!}\right)$$

$$=\bigoplus_{l=0}^{\infty}\bigoplus_{s=0}^{\infty}g^{-1}\left[\frac{(\delta)_{l+s}}{\Gamma(l\alpha+s\beta+\gamma)}\frac{(g(a))^l(g(b))^s}{l!s!}\right]=\bigoplus_{l=0}^{\infty}\bigoplus_{s=0}^{\infty}\left[g^{-1}\left(\frac{(\delta)_{l+s}}{\Gamma(l\alpha+s\beta+\gamma)}\right)\odot g^{-1}\left(\frac{(g(a))^l(g(b))^s}{l!s!}\right)\right]$$

$$=\bigoplus_{l=0}^{\infty}\bigoplus_{s=0}^{\infty}\left[\Big(g^{-1}((\delta)_{l+s})\oslash g^{-1}(\Gamma(l\alpha+s\beta+\gamma))\Big)\odot\Big(\Big(g^{-1}((g(a))^l(g(b))^s)\Big)\oslash\Big(g^{-1}(l!\times s!)\Big)\Big)\right]$$

$$=\bigoplus_{l=0}^{\infty}\bigoplus_{s=0}^{\infty}\left[\Big(g^{-1}((\delta)_{l+s})\oslash g^{-1}(\Gamma(l\alpha+s\beta+\gamma))\Big)\odot\Big(\Big(g^{-1}((g(a))^l)\odot g^{-1}((g(b))^s)\Big)\oslash\Big(g^{-1}(l!)\odot g^{-1}(s!)\Big)\Big)\right].$$

- Delayed analogue of pseudo-Mittag-Leffler type function generated by $A, B \in R$ of three parameters:

$$E^{\tau,\oplus}_{\alpha,\beta,\gamma}(A,B;t) = g^{-1}\left(E^{\tau}_{\alpha,\beta,\gamma}(g(A),g(B);g(t))\right)$$

$$= g^{-1}\left(\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^n(g(B))^q}{\Gamma(n\alpha+q\beta+\gamma)}(g(t-n\tau))^{n\alpha+q\beta+\gamma-1}H(g(t-n\tau))\right)$$

$$= \bigoplus_{n=0}^{\infty}\bigoplus_{q=0}^{\infty} g^{-1}\left(\binom{n+q}{q}(g(A))^n(g(B))^n\frac{(g(t-n\tau))^{n\alpha+q\beta+\gamma-1}H(g(t-n\tau))}{\Gamma(n\alpha+q\beta+\gamma)}\right)$$

$$= \bigoplus_{n=0}^{\infty}\bigoplus_{q=0}^{\infty}\left[g^{-1}\left(\binom{n+q}{q}\right)\odot g^{-1}\left((g(A))^n\right)\odot g^{-1}\left((g(B))^q\right)\right.$$

$$\left.\odot g^{-1}\left((g(t-n\tau))^{n\alpha+q\beta+\gamma-1}\right)\odot g^{-1}\left(H(g(t-n\tau))\right)\oslash g^{-1}\left(\Gamma(n\alpha+q\beta+\gamma)\right)\right],$$

where $H(\cdot): R \to R$ is the Heaviside function appointed as follows

$$H(t) = \begin{cases} 1, & \text{if } t \geq 0, \\ 0, & \text{if } t < 0. \end{cases}$$

Theorem II.1. *([1], p.254, theorem 27.)*
Assume that g is the additive generator of the strict-pseudo-addition $\oplus$ on [a,b], so that g is continuously differentiable on(a,b), $0 < m-1 \leq \alpha < m, \quad 0 \leq \beta \leq 1$ and $s \in R$ Then, the g-Laplace transform of the pseudo-Hilfer pseudo-fractional derivative of order α is given by:

$$\mathscr{L}^{\oplus}\left\{{}^{H}D^{\alpha,\beta}_{\oplus,\odot,0+}f(x)\right\} = [g^{-1}(s^{\alpha})\odot\mathfrak{L}^{\oplus}\{f(x)\}]\ominus\bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1})\odot I^{(1-\beta)(m-\alpha)-k}f(0)\right]. \tag{II.3}$$

III. EXPLICIT SOLUTIONS OF HOMOGENEOUS HILFER-TYPE PSEUDO-FRACTIONAL DIFFERENTIAL EQUATION

This section has demonstrated the explicit solution to the Hilfer-type pseudo-fractional differential equation system (III.1).

$$\begin{cases} H^{\alpha,\beta}_{\oplus,\odot,0+}y(t) = A\odot y(t)\oplus B\odot y(t-\tau), t\in(0;T], \tau>0, \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t) = \phi(t), t\in[-\tau,0]. \end{cases} \tag{III.1}$$

where $m-1<\alpha<m, 0\leq\beta\leq 1, \gamma=(\beta-1)(m-\alpha)+k+1, k=0,...,m-1$.

Theorem III.1. *A unique analytical solution $y\in C^m([-\tau,T],R)$ of the initial problem (III.1) has as shown below:*

$$y(t) = \bigoplus_{k=0}^{m-2}\left(g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\oplus(A\oplus B)\odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;t-\tau)\right)\odot\phi_0^{(k)}$$

$$\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;t)\odot\phi_0^{(m-1)}\oplus B\odot\int^{\oplus}_{[-\tau,\min(t-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;t-\tau-s)\odot\phi(s)\odot ds.$$

Proof. Suppose that $T=\infty$. Assume that (I.5) has a unique m times continuously differentiable solution y and f are continuous and exponentially bounded, and $H^{\alpha,\beta}_{\oplus,\odot,0+}y$ is exponentially bounded on $[0,\infty)$, then Laplace transforms of them exist. We are going to receive an integral representation of the solution to the linear homogeneous Hilfer-type pseudo-fractional differential equation.

First of all, we are imposing the Laplace integral transform to both sides of (III.1) with the help of Theorem II.1.

$$\begin{aligned}
&\mathfrak{L}^{\oplus}\left\{H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)\right\}(s)=g^{-1}\left[\mathfrak{L}\left\{g\left(H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)\right)\right\}(s)\right]=g^{-1}\left[\mathfrak{L}\left\{{}^{H}D^{\alpha,\beta}_{0+}g(y(t)\right\}\right]\\
=&g^{-1}\left[s^{\alpha}\mathfrak{L}\left\{g(y(t))\right\}(s)-\sum_{k=0}^{m-1}s^{m(1-\beta)+\alpha\beta-k-1}\left(I^{(1-\beta)(m-\alpha)-k}_{0+}g(y)\right)(0)\right]\\
=&g^{-1}(s^{\alpha})\odot\mathfrak{L}\left\{y(t)\right\}(s)\ominus\bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1})\odot I^{(1-\beta)(m-\alpha)-k}_{\oplus,\odot,0+}y(0)\right]\\
=&g^{-1}(s^{\alpha})\odot Y(s)\ominus\bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1})\odot\phi_0^k\right],
\end{aligned}$$

$$\mathfrak{L}^{\oplus}[H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)](s)=g^{-1}(s^{\alpha})\odot Y(s)\ominus\bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1})\odot\phi_0^k\right]. \tag{III.2}$$

where, $\mathfrak{L}^{\oplus}\left\{y(t)\right\}(s)=Y(s)$.

$$\begin{aligned}
&\mathfrak{L}^{\oplus}\left\{A\odot y(t)\oplus B\odot y(t-\tau)\right\}(s)=g^{-1}\left(\mathfrak{L}\left\{g(A\odot y(t)\oplus B\odot y(t-\tau)\right\}\right)\\
=&g^{-1}\left(\mathfrak{L}\left\{g(A)g(y(t))+g(B)g\big(y(t-\tau)\big)\right\}\right)=A\odot\mathfrak{L}^{\oplus}(y(t))\oplus B\odot\mathfrak{L}^{\oplus}(y(t-\tau))\\
=&A\odot Y(s)\oplus B\odot\mathfrak{L}^{\oplus}(y(t-\tau)).
\end{aligned}$$

we get

$$\mathfrak{L}^{\oplus}\left\{A\odot y(t)\oplus B\odot y(t-\tau)\right\}(s)=A\odot Y(s)\oplus B\odot\mathfrak{L}^{\oplus}(y(t-\tau)) \tag{III.3}$$

$$\mathfrak{L}^{\oplus}(y(t-\tau))(s)=g^{-1}(\mathfrak{L}(g(t-\tau))(s)).$$

and by using substitution of $t-\tau=\theta$, we receive that

$$\begin{aligned}
&\mathfrak{L}\left\{g(t-\tau)\right\}(s)=\int_0^{\infty}g(t-\tau)e^{-st}dt=\int_{-\tau}^{\infty}g(y(\theta))e^{-s(\tau+\theta)}d\theta=e^{-s\tau}\int_{-\tau}^{\infty}g(y(\theta))e^{-s(\theta)}d\theta\\
=&e^{-s\tau}\left[\int_{-\tau}^{0}g(y(\theta))e^{-s(\theta)}d\theta+\int_0^{\infty}g(y(\theta))e^{-s(\theta)}d\theta\right]=\int_{-\tau}^{0}g(y(\theta))e^{-s(\tau+\theta)}d\theta\\
&+e^{-s\tau}\mathfrak{L}(g(y(\theta))(s))=\int_0^{\tau}g(y(t-\tau))e^{-st}dt+e^{-s\tau}\mathfrak{L}(g(y(\theta))(s)).
\end{aligned}$$

On the other hand, due to the integral property of the pseudo-Riemann-Liouville-fraction, we obtain the following results. Let's also note that the initial condition of the issue we are reviewing is manifested in the following case.

$$I^{0}_{\oplus,\odot,0+}y(t)=y(t)\implies y(t)=\phi(t),t\in[-\tau,0].$$

in there $\tilde{\phi}(\cdot):R\to R$ is the unit-step function, which it has defined as bellow:

$$\tilde{\phi}(t)=\begin{cases}\phi(t) & \text{if } -\tau\le t\le 0,\\ 0 & \text{if } t>0.\end{cases}$$

Therefore we get the following relations:

$$\mathfrak{L}\left\{g(t-\tau)\right\}(s)=\int_0^{\tau}g\big(y(t-\tau)\big)e^{-st}dt+e^{-s\tau}\mathfrak{L}\left\{g(y(\theta))\right\}(s)=\int_0^{\infty}g(\tilde{\phi}(t-\tau))e^{-st}dt+e^{-s\tau}\mathfrak{L}\left\{g(y(\theta))\right\}(s),$$

$$\mathfrak{L}^{\oplus}(y(t-\tau))(s)=g^{-1}(e^{-s\tau})\odot Y(s)\oplus\mathfrak{L}^{\oplus}(\tilde{\phi}(t-\tau))(s). \tag{III.4}$$

By using the formula III.2, III.3, III.4 we get the following results.

$$g^{-1}(s^{\alpha})\odot Y(s)\ominus\bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1})\odot\phi_0^{(k)}\right]=A\odot Y(s)\oplus B\odot\left[g^{-1}(e^{-s\tau})\odot Y(s)\oplus\mathfrak{L}^{\oplus}\left\{\tilde{\phi}(t-\tau)\right\}(s)\right].$$

Afterward, we write the above relation in the following explicit form

$$\left[g^{-1}(s^{\alpha})\ominus A\ominus B\odot g^{-1}(e^{-s\tau})\right]\odot Y(s)=\bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1})\odot\phi_0^{k}\right]\oplus B\odot\mathfrak{L}^{\oplus}\left\{\tilde{\phi}(t-\tau)\right\}(s). \tag{III.5}$$

Then, we solve ((III.5)) with respect to $Y(s)$,

$$\begin{aligned}
Y(s)=&\left[\bigoplus_{k=0}^{m-1}\left(g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1})\odot\phi_0^{(k)}\right)\oplus B\odot\mathfrak{L}^{\oplus}\left\{\tilde{\phi}(t-\tau)\right\}(s)\right]\oslash\left[g^{-1}(s^{\alpha})\ominus A\ominus B\odot g^{-1}(e^{-s\tau})\right]\\
=&g^{-1}\left(\frac{\sum_{k=0}^{m-1}[s^{m(1-\beta)+\alpha\beta-k-1}g(\phi_0^{k})]+g(B)g(\mathfrak{L}^{\oplus}(\tilde{\phi}(t-\tau))(s))}{s^{\alpha}-g(A)-g(B)e^{-s\tau}}\right)\\
=&g^{-1}\left(\sum_{k=0}^{m-2}\frac{s^{m(1-\beta)+\alpha\beta-k-1}g(\phi_0^{(k)})}{s^{\alpha}-g(A)-g(B)e^{-s\tau}}+\frac{s^{-m\beta+\alpha\beta}}{s^{\alpha}-g(A)-g(B)e^{-s\tau}}g(\phi_0^{(m-1)})+\frac{g(B)}{s^{\alpha}-g(A)-g(B)e^{-s\tau}}\mathfrak{L}\left\{g(\tilde{\phi}(t-\tau))\right\}\right)\\
=&g^{-1}\left[\left(1+\frac{g(A)+g(B)e^{-s\tau}}{s^{\alpha}-g(A)-g(B)}\right)\sum_{k=0}^{m-2}\frac{s^{m(1-\beta)+\alpha\beta-k-1}g(\phi_0^{(k)})}{s^{\alpha}-g(A)-g(B)e^{-s\tau}}+\frac{g(B)}{s^{\alpha}-g(A)-g(B)e^{-s\tau}}\mathfrak{L}\left\{g(\tilde{\phi}(t-\tau))\right\}\right].
\end{aligned}$$

By relation (II.2), we have

$$\begin{aligned}
\left[s^{\alpha}-g(A)-g(B)e^{-s\tau}\right]^{-1}&=(s^{\alpha}-g(A))^{-1}\left[1-(s^{\alpha}-g(A))^{-1}g(B)e^{-s\tau}\right]^{-1}\\
=&(s^{\alpha}-g(A))^{-1}\sum_{n=0}^{\infty}\left[(s^{\alpha}-g(A))^{-n}(g(B))^{n}e^{-sn\tau}\right],
\end{aligned}$$

$$\left[s^{\alpha}-g(A)-g(B)e^{-s\tau}\right]^{-1}=\sum_{n=0}^{\infty}\left[(s^{\alpha}-g(A))^{-(n+1)}(g(B))^{n}e^{-sn\tau}\right]. \tag{III.6}$$

If we replace the expression (III.6) in the $Y(s)$ formula obtained above, we get the following results.

$$\begin{aligned}
Y(s)=g^{-1}\Bigg\{&\left(\sum_{k=0}^{m-2}s^{m(1-\beta)+\alpha\beta-k-1}g(\phi_0^{(k)})+(g(A)+g(B)e^{-s\tau})\sum_{k=0}^{m-2}s^{m(1-\beta)+\alpha\beta-k-1}g(\phi_0^{(k)})\right)\\
&\times\sum_{n=0}^{\infty}\left[(s^{\alpha}-g(A))^{-n}(g(B))^{n}e^{-sn\tau}\right]+g(\phi_0^{(m-1)})s^{-m\beta+\alpha\beta}\sum_{n=0}^{\infty}\left[(s^{\alpha}-g(A))^{-(n+1)}(g(B))^{n}e^{-sn\tau}\right]\\
&+g(B)\mathfrak{L}\left\{g(\tilde{\phi}[t-\tau])\right\}(s)\sum_{n=0}^{\infty}\left[(s^{\alpha}-g(A))^{-(n+1)}(g(B))^{n}e^{-sn\tau}\right]\Bigg\}.
\end{aligned}$$

Imposing the inverse g-Laplace transform to the above result, we get:

$$
\begin{aligned}
y(t) = g^{-1}\Bigg(& \mathfrak{L}^{-1}\left[\sum_{k=0}^{m-2} s^{m(1-\beta)+\alpha\beta-k-1} g(\phi_0^{(k)}) + (g(A)+g(B)e^{-s\tau}) \sum_{k=0}^{m-2} s^{m(1-\beta)+\alpha\beta-k-1} g(\phi_0^{(k)}) \right.\\
& \times \sum_{n=0}^{\infty}\left[(s^{\alpha}-g(A))^{-n}(g(B))^{n}e^{-sn\tau}\right] + g(\phi_0^{(m-1)})s^{-m\beta+\alpha\beta}\sum_{n=0}^{\infty}\left[(s^{\alpha}-g(A))^{-(n+1)}(g(B))^{n}e^{-sn\tau}\right]\\
& \left. + g(B)\sum_{n=0}^{\infty}\left[(s^{\alpha}-g(A))^{-(n+1)}(g(B))^{n}e^{-sn\tau}\right]\mathfrak{L}\left\{g(\tilde{\phi}[t-\tau])\right\}(s)\right](t)\Bigg).
\end{aligned}
$$

Taking the inverse Laplace transform of the statement above and by using Lemma II.1, Lemma II.2 and time shift and convolution property of the Laplace transform, we gain an explicit representation of solution for an initial issue (III.1)

$$
\begin{aligned}
y(t) = g^{-1}\Bigg\{ & \mathfrak{L}^{-1}\left[\sum_{k=0}^{m-2} s^{m(1-\beta)+\alpha\beta-k-1} g(\phi_0^{(k)})\right.\\
& + \sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\binom{n+q}{q}(g(A))^{q+1}(g(B))^{n}\frac{s^{(1-\beta)(m-\alpha)-1}}{s^{\alpha(n+q+1)}}e^{-sn\tau}g(\phi_0)\\
& + \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n+1}\frac{s^{(1-\beta)(m-\alpha)-1}e^{-sn\tau}}{s^{\alpha(n+q+1)}}g(\phi_0)+\cdots+\\
& + \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q+1}(g(B))^{n}\frac{s^{(1-\beta)(m-\alpha)-m+1e^{-sn\tau}}}{s^{\alpha(n+q+1)}}g(\phi_0^{(m-2)})\\
& + \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n+1}\frac{s^{(1-\beta)(m-\alpha)-m+1}e^{-sn\tau}}{s^{\alpha(n+q+1)}}g(\phi_0^{(m-2)})\\
& + \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q}(g(B))^{n}e^{-sn\tau}}{s^{\alpha(n+q+1)-\beta(\alpha-m)}}g(\phi_0^{(m-1)})\\
& \left. + \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q}(g(B))^{n+1}e^{-sn\tau}}{s^{\alpha(n+q+1)}}\mathfrak{L}\big(g(\tilde{\phi}[t-\tau])\big)(s)\right](t)\Bigg\},
\end{aligned}
$$

$$
\begin{aligned}
y(t) = g^{-1}\Bigg\{ & \mathfrak{L}^{-1}\left[\sum_{k=0}^{m-2} s^{m(1-\beta)+\alpha\beta-k-1} g(\phi_0^{(k)})\right.\\
& + \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q+1}(g(B))^{n}}{s^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+1}}e^{-sn\tau}g(\phi_0)\\
& + \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q}(g(B))^{n+1}e^{-sn\tau}}{s^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+1}}g(\phi_0)+\cdots+\\
& + \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q+1}(g(B))^{n}e^{-sn\tau}}{s^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+m-1}}g(\phi_0^{(m-2)})\\
& + \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n+1}\frac{e^{-sn\tau}}{s^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+m-1}}g(\phi_0^{(m-2)})\\
& + \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q}(g(B))^{n}e^{-sn\tau}}{s^{\alpha(n+q+1)-\beta(\alpha-m)}}g(\phi_0^{(m-1)})\\
& \left. + \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q}(g(B))^{n+1}e^{-sn\tau}}{s^{\alpha(n+q+1)}}\mathfrak{L}\left\{g(\tilde{\phi}[t-\tau])\right\}(s)\right](t)\Bigg\}.
\end{aligned}
$$

Then we get the following result.

$$\begin{aligned}
y(t) &= g^{-1}\Bigg(\sum_{k=0}^{m-2}\frac{t^{(1-\beta)(m-\alpha)-k}}{\Gamma((1-\beta)(m-\alpha)-k+1)}g(\phi_0^{(k)})\\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q+1}(g(B))^{n}\frac{(t-(n+1)\tau)^{\alpha(n+q+1)-(1-\beta)(m-\alpha)}}{\Gamma(\alpha(n+q+1)-(1-\beta)(m-\alpha)+1)}H(t-(n+1)\tau)(\phi_0)\\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n+1}\frac{(t-(n+1)\tau)^{\alpha(n+q+1)-(1-\beta)(m-\alpha)}}{\Gamma(\alpha(n+q+1)-(1-\beta)(m-\alpha)+1)}H(t-(n+1)\tau)g(\phi_0)+\cdots+\\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q+1}(g(B))^{n}\frac{(t-(n+1)\tau)^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+m-2}}{\Gamma(\alpha(n+q+1)-(1-\beta)(m-\alpha)+m-1)}H(t-(n+1)\tau)g(\phi_0^{(m-2)})\\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n+1}\frac{(t-(n+1)\tau)^{\alpha(n+q+1)-(1-\beta)(m-\alpha)}}{\Gamma(\alpha(n+q+1)-(1-\beta)(m-\alpha)+1)}H(t-(n+1)\tau)g(\phi_0^{(m-2)})\\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n}\frac{(t-n\tau)^{\alpha(n+q+1)-\beta(\alpha-m)-1}}{\Gamma(\alpha(n+q+1)-\beta(\alpha-m))}H(t-n\tau)g(\phi_0^{(m-1)})\\
&+g(B)\int_0^t\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n}\frac{(t-n\tau-s)^{\alpha(n+q+1)-1}}{\Gamma(\alpha(n+q+1))}H(t-n\tau-s)g\big(\tilde{\phi}(s-\tau)\big)ds\Bigg)
\end{aligned}$$

$$\begin{aligned}
y(t) &= g^{-1}\Bigg(\sum_{k=0}^{m-2}\Bigg(\frac{t^{(1-\beta)(m-\alpha)-k}}{\Gamma((1-\beta)(m-\alpha)-k+1)}+(g(A)+g(B))\\
&\times\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n}\frac{(t-(n+1)\tau)^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+k}}{\Gamma(\alpha(n+q+1)-(1-\beta)(m-\alpha)+k+1)}\Bigg)g(\phi_0^{(k)})\\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n}\frac{(t-n\tau)^{\alpha(n+q+1)-\beta(\alpha-m)-1}}{\Gamma(\alpha(n+q+1)-\beta(\alpha-m))}H(t-n\tau)g(\phi_0^{(m-1)})\\
&+g(B)\int_{-\tau}^{t-\tau}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n}\frac{(t-(n+1)\tau-s)^{\alpha(n+q+1)-1}}{\Gamma(\alpha(n+q+1))}H(t-(n+1)\tau-s)g\big(\tilde{\phi}(s)\big)ds\Bigg)\\
&=g^{-1}\Bigg(\sum_{k=0}^{m-2}\Bigg(\frac{t^{(1-\beta)(m-\alpha)-k}}{\Gamma((1-\beta)(m-\alpha)-k+1)}+(g(A)+g(B))E^{\tau}_{\alpha,\alpha,\alpha+(\beta-1)(m-\alpha)+k+1}(g(A),g(B);t-\tau)\phi_0^{(k)}\Bigg)\\
&+E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t)\phi_0^{(m-1)}+g(B)\int_{-\tau}^{min(t-\tau,0)}E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-\tau-s)g\big(\tilde{\phi}(s)\big)ds\Bigg)\\
&=\bigoplus_{k=0}^{m-2}\Bigg(g^{-1}\Bigg(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+j+1)}\Bigg)\oplus(A\oplus B)\odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))\Bigg)\odot\phi_0^{(k)}\\
&\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t))\odot\phi_0^{m-1}\oplus B\odot\int^{\oplus}_{[-\tau,\min(t-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds.
\end{aligned}$$

We get

$$y(t)=\bigoplus_{k=0}^{m-2}\Bigg(g^{-1}\Bigg(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\Bigg)\oplus(A\oplus B)\odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))\Bigg)\odot\phi_0^{(k)}\oplus$$
$$E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t))\odot\phi_0^{(m-1)}\oplus B\odot\int^{\oplus}_{[-\tau,\min(t-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds. \tag{III.7}$$

If we take $t\geq\tau$ then,

$$\int^{\oplus}_{[-\tau,t-\tau]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\tilde{\phi}(s)\odot ds=\int^{\oplus}_{[-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds. \tag{III.8}$$

If we take $t < \tau$ then,

$$\int_{[-\tau,t-\tau]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s)) \odot \tilde{\phi}(s) \odot ds = \int_{[-\tau,t-\tau]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s)) \odot \phi(s) \odot ds. \tag{III.9}$$

By using (III.8) and (III.9) we will get following result.

$$\int_{[-\tau,t-\tau]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s)) \odot \tilde{\phi}(s) \odot ds = \int_{[-\tau,\min(t-\tau,0)]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s)) \odot \phi(s) \odot ds. \tag{III.10}$$

□

IV. INTEGRAL REPRESENTATION OF SOLUTION TO LINEAR INHOMOGENEOUS HILFER-TYPE PSEUDO-FRACTIONAL TIME DELAY DIFFERENTIAL EQUATIONS

In this part, by imposing the classical manners to solve (I.5), we will obtain the explicit formula for the solutions of linear inhomogeneous fractional Hilfer-type pseudo-fractional differential equations with invariable coefficients and time delay.

Let us examine the following two Hilfer-type pseudo-FDDEs with constant coefficients:

$$\begin{cases} H^{\alpha,\beta}_{\oplus,\odot,0+} y(t) = A \odot y(t) \oplus B \odot y(t-\tau) \oplus f(t), t \in (0;T], \tau > 0, \\ I^{1-\gamma}_{\oplus,\odot,0+} y(t) = 0, t \in [-\tau, 0]. \end{cases} \tag{IV.1}$$

and

$$\begin{cases} H^{\alpha,\beta}_{\oplus,\odot,0+} y(t) = A \odot y(t) \oplus B \odot y(t-\tau), t \in (0;T], \tau > 0, \\ I^{1-\gamma}_{\oplus,\odot,0+} y(t) = \phi(t), t \in [-\tau, 0]. \end{cases} \tag{IV.2}$$

where $m-1 < \alpha < m$, $0 \leq \beta \leq 1$, $\gamma = (\beta-1)(m-\alpha)+k+1, k = 0,...,m-1$.

The following lemma plays an important role in the proof of the subsequent theorem, which can be obtained from classical ways about the solution of the system (I.5).

Lemma IV.1. *If y_1 and y_2 are the solutions systems (IV.1) and (IV.2) ,respectively, then $y(t) = y_1 \oplus y_2$ is the general solution of system* (I.5).

Mention that the solution y_2 of (IV.2) is investigated in paragraph III. In other words, to reach our goal, we need to find y_1 which is a particular solution of (I.5).

Lemma IV.2. *Assume $m-1 < \alpha < m$, $0 < \beta \leq 1$ for $m \geq 2$. Then, we have the following relation:*

$$\int_{\eta+l\tau}^{t} (t-s)^{(1-\beta)(m-\alpha)-1} (s-l\tau-\eta)^{l\alpha+p\alpha+\alpha-1} ds = (t-l\tau-\eta)^{m-\beta(m-\alpha)+l\alpha+p\alpha-2} B\Big((1-\beta)(m-\alpha), (l+1)\alpha+p\alpha\Big).$$

Proof. To prove the lemma, we use the definition of the Beta function and substitution of $u = \frac{t-s}{t-l\tau-\eta}$. Consequently, we obtain

$$\begin{aligned} &\int_{\eta+l\tau}^{t} (t-s)^{(1-\beta)(m-\alpha)-1} (s-l\tau-\eta)^{l\alpha+p\alpha+\alpha-1} ds \\ =&(t-l\tau-\eta)^{m-\beta(m-\alpha)+l\alpha+p\alpha-2} \int_0^1 u^{(1-\beta)(m-\alpha)-1} (1-u)^{l\alpha+\alpha-1} du \\ =&(t-l\tau-\eta)^{m-\beta(m-\alpha)+l\alpha+p\alpha-2} B\Big((1-\beta)(m-\alpha), (l+1)\alpha+p\alpha\Big). \end{aligned}$$

□

We denote the following theorem for the particular solution of equation (I.5).

Theorem IV.1. *A solution $\tilde{y} \in C^m([0,T],R)$ of* (I.5) *holding zero initial conditions $\tilde{y}(t) = 0$, $t \in [-\tau, 0)$, $\tilde{y}^{(k)}(0) = 0$, $0 \leq k \leq m-1$ has the following form:*

$$\tilde{y}(t) = \int_{[0,t]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot f(s) \odot ds, \quad t > 0. \tag{IV.3}$$

Proof. Using the method of variation of constants, any solution $\tilde{y}$ of the the inhomogeneous system must be provided in the following shape:

$$\tilde{y}(t) = \int_{[0,t]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot h(s) \odot ds, \quad t > 0, \tag{IV.4}$$

where $h(s)$, $0 \leq s \leq t$ is a sought vector function and $\tilde{y}(0) = 0$.

$$\begin{aligned}
&\tilde{y}(t) = \int_{[0,t]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot h(s) \odot ds = g^{-1}\left(\int_0^t E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)g(h(s))ds\right)\\
&H^{\alpha,\beta}_{\oplus,\odot,0+}\tilde{y}(t) = g^{-1}({}^H D^{\alpha,\beta}_{0+} g(\tilde{y}(t))) = g^{-1}({}^H D^{\alpha,\beta}_{0+}(\int_0^t E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)g(h(s))ds))\\
&{}^H D^{\alpha,\beta}_{0+} g(\tilde{y}(t)) = {}^H D^{\alpha,\beta}_{0+}(\int_0^t E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)g(h(s))ds)\\
=&I^{\beta(m-\alpha)}\frac{d^m}{dt^m} I^{(1-\beta)(m-\alpha)}\left(\int_0^t E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)g(h(s))ds\right)\\
=&I^{\beta(m-\alpha)}\frac{d^m}{dt^m}\left(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\int_0^t (t-s)^{(1-\beta)(m-\alpha)-1}\int_0^s E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);s-\eta)g(h(\eta))d\eta ds\right)\\
=&I^{\beta(m-\alpha)}\left(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\frac{d^m}{dt^m}\int_0^t\int_0^s (t-s)^{(1-\beta)(m-\alpha)-1} E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);s-\eta)g(h(\eta))d\eta ds\right)\\
=&I^{\beta(m-\alpha)}\left(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\frac{d^m}{dt^m}\int_0^t\int_{\eta+l\tau}^t (t-s)^{(1-\beta)(m-\alpha)-1} E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);s-\eta)g(h(\eta))d\eta ds\right)\\
=&I^{\beta(m-\alpha)}\left(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\frac{d^m}{dt^m}\int_0^t g(h(\eta))\left(\int_{\eta+l\tau}^t (t-s)^{(1-\beta)(m-\alpha)-1} E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);s-\eta)ds\right)d\eta\right)\\
=&I^{\beta(m-\alpha)}\left(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\frac{d^m}{dt^m}\int_0^t g(h(\eta))\left(\int_{\eta+n\tau}^t (t-s)^{(1-\beta)(m-\alpha)-1}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q(g(B))^n\right.\right.\\
&\left.\left.\frac{(s-n\tau-\eta)H(s-n\tau-\eta)}{\Gamma(q\alpha+n\alpha+\alpha)}ds\right)d\eta\right) = I^{\beta(m-\alpha)}\left(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q(g(B))^n\right.\\
&\left.\frac{d^m}{dt^m}\int_0^t\frac{(t-n\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+q\alpha-2}H(t-n\tau-\eta)}{\Gamma((n+1)\alpha+q\alpha)}g(h(\eta))d\eta B\Big((1-\beta)(m-\alpha),(n+1)\alpha+q\alpha\Big)\right)\\
=&I^{\beta(m-\alpha)}\left(\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q(g(B))^n\frac{d^m}{dt^m}\int_0^t\frac{(t-n\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+q\alpha-2}H(t-n\tau-\eta)}{\Gamma(m-\beta(m-\alpha)+n\alpha+q\alpha-1)}g(h(\eta))d\eta\right).
\end{aligned}$$

On the other hand, $I^{\beta(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}(f(t)) = {}^C D^{\beta(\alpha+m)}_{0+} f(t)$, and according to the formula between Riemann-Luovile and Caputo fractional derivatives, we have

$$I^{\beta(m-\alpha)}\frac{d^m}{dt^m}f(t) = {}^C D^{\beta(\alpha+m)}_{0+} f(t) = {}^{RL} D^{\beta(\alpha+m)}_{0+} f(t) - \sum_{k=0}^{m-1}\frac{t^{k-\beta(\alpha+m)}}{\Gamma(k-\beta(\alpha+m))}f^{(k)}(0), \quad t > 0.$$

With the help of the following binomial identity.

$$\binom{n+q}{q} = \binom{n+q-1}{q} + \binom{n+q-1}{q-1}, \quad n,q \geq 1,$$

and imposing the Leibniz rule for higher-order derivatives (Ismail T.Huseynov et al ., 2021 [3], see Theorem 3.2), we achieve

$$
\begin{aligned}
&{}^{H}D^{\alpha,\beta}_{0+}g(\tilde{y}(t)) = I^{\beta(m-\alpha)}\frac{d^m}{dt^m}\left(\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q(g(B))^n\right.\\
&\times\left.\int_0^t \frac{(t-n\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+q\alpha-2}H(t-n\tau-\eta)}{\Gamma(m-\beta(m-\alpha)+n\alpha+q\alpha-1)}g(h(\eta))d\eta\right)\\
&= \ {}^{C}D^{\beta(\alpha-m)+m}\left(\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q(g(B))^n\int_0^t \frac{(t-n\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+p\alpha-2}H(t-n\tau-\eta)}{\Gamma(m-\beta(m-\alpha)+n\alpha+p\alpha-1)}g(h(\eta))d\eta\right)\\
&=\frac{d^m}{dt^m}\int_0^t \frac{(t-\eta)^{m-\beta(m-\alpha)-2}H(t-\eta)}{\Gamma(m-\beta(m-\alpha)-1)}g(h(\eta))d\eta\\
&+\sum_{n=1}^{\infty}\sum_{q=0}^{\infty}\binom{n+q-1}{q}(g(A))^q(g(B))^n\frac{d^m}{dt^m}\int_0^t \frac{(t-l\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+q\alpha-2}H(t-n\tau-\eta)}{\Gamma(m-\beta(m-\alpha)+n\alpha+q\alpha-1)}g(h(\eta))d\eta\\
&+\sum_{n=0}^{\infty}\sum_{q=1}^{\infty}\binom{n+q-1}{q-1}(g(A))^q(g(B))^n\frac{d^m}{dt^m}\int_0^t \frac{(t-n\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+q\alpha-2}H(t-n\tau-\eta)}{\Gamma(m-\beta(m-\alpha)+n\alpha+q\alpha-1)}g(h(\eta))d\eta\\
&=g(h(t))+\sum_{n=1}^{\infty}\sum_{q=0}^{\infty}\binom{n+q-1}{q}(g(A))^q(g(B))^n\int_0^t \frac{(t-n\tau-\eta)^{\alpha\beta+n\alpha+q\alpha-2}H(t-n\tau-\eta)}{\Gamma(\alpha\beta+n\alpha+q\alpha-1)}g(h(\eta))d\eta
\end{aligned}
$$

$$
\begin{aligned}
&+\sum_{n=0}^{\infty}\sum_{q=1}^{\infty}\binom{n+q-1}{q-1}(g(A))^q(g(B))^n\int_0^t \frac{(t-n\tau-\eta)^{\alpha\beta+n\alpha+q\alpha-2}H(t-n\tau-\eta)}{\Gamma(\alpha\beta+n\alpha+q\alpha-1)}g(h(\eta))d\eta\\
&=g(h(t))+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q(g(B))^{n+1}\int_0^t \frac{(t-n\tau-\eta)^{\alpha\beta+n\alpha+q\alpha-2}H(t-(n+1)\tau-\eta)}{\Gamma(\alpha\beta+(n+1)\alpha+q\alpha-1)}g(h(\eta))d\eta\\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q-1}(g(A))^{q+1}(g(B))^n\int_0^t \frac{(t-n\tau-\eta)^{\alpha\beta+n\alpha+(q+1)\alpha-2}H(t-n\tau-\eta)}{\Gamma(\alpha\beta+n\alpha+(q+1)\alpha-1)}g(h(\eta))d\eta\\
&=g(h(t))+g(A)\int_0^t E^{\tau}_{\alpha,\alpha,\alpha\beta+\alpha-1}(g(A),g(B);t-\eta)g(h(\eta))d\eta+g(B)\int_0^t E^{\tau}_{\alpha,\alpha,\alpha\beta+\alpha-1}(g(A),g(B);t-\tau-\eta)g(h(\eta))d\eta
\end{aligned}
$$

$$
\begin{aligned}
&H^{\alpha,\beta}_{\oplus,\odot,0+}\tilde{y}(t) = g^{-1}({}^{H}D^{\alpha,\beta}_{0+}g(\tilde{y}(t))) = g^{-1}\left(g(h(t))+g(A)\int_0^t E^{\tau}_{\alpha,\alpha,\alpha\beta+\alpha-1}(g(A),g(B);t-\eta)g(h(\eta))d\eta\right.\\
&\left.+g(B)\int_0^t E^{\tau}_{\alpha,\alpha,\alpha\beta+\alpha-1}(g(A),g(B);t-\tau-\eta)g(h(\eta))d\eta\right) = A\oplus\tilde{y}(t)\oplus B\tilde{y}(t-\tau)\oplus h(t) = A\oplus\tilde{y}(t)\oplus B\tilde{y}(t-\tau)\oplus f(t).
\end{aligned}
$$

Therefore, we obtain that $h(t) = f(t)$ for $t\in[0,T]$. □

Eventually, we obtain the next theorem for the unique analytical solution of the Cauchy problem (I.5).

Theorem IV.2. *A unique analytical solution $y\in C^m([-\tau,T],R)$ of the initial issue* (I.1) *has the following form:*

$$
\begin{aligned}
y(t) = \bigoplus_{k=0}^{m-2}&\left(g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\oplus(A\oplus B)\odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))\right)\odot\phi_0^{(k)}\\
&\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t))\odot\phi_0^{(m-1)}\oplus B\odot\int^{\oplus}_{[-\tau,\min(t-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds\\
&\oplus\int^{\oplus}_{[0,t]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))\odot f(s)\odot ds,\quad t>0.
\end{aligned}
$$

Proof. The proof of the theorem is immediate. Therefore, we pass above it. □

V. EXISTENCE AND UNIQUENESS PROBLEM FOR NONLINEAR TIME RETARDED HILFER-TYPE PSEUDO-FRACTIONAL DIFFERENTIAL EQUATIONS

In the following section, we will look at the initial issue of a nonlinear Hilfer-type pseudo-fractional differential equation with constant delay.

$$\begin{cases} H^{\alpha,\beta}_{\oplus,\odot,0+}y(t) = A\odot y(t)\oplus B\odot y(t-\tau)\oplus f(t,y(t)), t\in(0;T], \tau>0, \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t) = \phi(t), t\in[-\tau,0]. \end{cases} \tag{V.1}$$

Where $m-1<\alpha\leq m$, $0<\beta\leq 1$, $y(\cdot)\in R$, $f(\cdot,y(\cdot)):[0,\infty)\times R\to R$ is a nonlinear perturbation and also a continuous function. And we will also suppose that $(t\to f(t,0))\in C([0,\infty),R)$. Then, according to Theorem IV.2, we obtain the solution of the nonlinear Hilfer-type pseudo-FDE (V.1) as follows:

$$y(t) = \bigoplus_{k=0}^{m-2}\left(g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\oplus(A\oplus B)\odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))\right)\odot\phi_0^{(k)}$$

$$\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t))\odot\phi_0^{(m-1)}\oplus B\odot\int^{\oplus}_{[-\tau,\min(t-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds$$

$$\oplus\int^{\oplus}_{[0,t]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))\odot f(s,y(s))\odot ds,\quad t>0.$$

First of all, we denote the following lemmas and notes: For $x(\cdot):[a,b]\to R_+$, we will define the norm of the function as a follow:

$$\|x(t)\|_g = g^{-1}(|g(x(t)))|).$$

Lemma V.1. *([3], page 12, lemma 5.1) The following estimation satisfies true:*

$$|E^{\tau}_{\alpha,\alpha-\beta,\alpha+k}(A,B;t)|\leq t^{\alpha+k-1}\exp(|A|t^{\alpha}+|B|t^{\alpha-\beta}). \tag{V.2}$$

for $k=0,1,\dots,m-1$

Corollary V.1. *([3], page 12, corollary 5.1)*
For $m\geq 2$, the following conclusion satisfies:

$$|E^{\tau}_{\alpha,\alpha-\beta,m}(A,B;t)|\leq t^{m-1}\exp(|A|t^{\alpha}+|B|t^{\alpha-\beta}). \tag{V.3}$$

Analogously, we will get the following results for pseudo-Mittag-Leffler functions.

Lemma V.2. *Assume a generator $g:[a,b]\to[0,\infty]$ and $A,B\in R$. For following delayed pseudo-Mittag-Leffler function estimation holds true:*

$$|E^{\tau,\oplus}_{\alpha,\alpha-\beta,\alpha+k}(A,B;g^{-1}(t))|_g\leq g^{-1}(t^{\alpha+k-1})\odot g^{-1}(\exp(|A|t^{\alpha}+|B|t^{\alpha-\beta})). \tag{V.4}$$

for $k=0,1,\dots,m-1$

Proof.

$$\begin{aligned} &|E^{\tau,\oplus}_{\alpha,\alpha-\beta,\alpha+k}(A,B;g^{-1}(t))|_g = g^{-1}\left(g\left(|E^{\tau,\oplus}_{\alpha,\alpha-\beta,\alpha+k}(A,B;g^{-1}(t))|\right)\right) \\ =&g^{-1}\left(|E^{\tau}_{\alpha,\alpha-\beta,\alpha+k}(g(A),g(B);t)|\right)\leq g^{-1}\left(t^{\alpha+k-1}\exp(|g(A)|t^{\alpha}+|g(B)|t^{\alpha-\beta})\right) \\ \leq&g^{-1}(t^{\alpha+k-1})\odot g^{-1}(\exp(|A|t^{\alpha}+|B|t^{\alpha-\beta})). \end{aligned}$$

$\square$

Then, we can denote analogously following the corollary.

Corollary V.2. *Let a generator $g : [a,b] \to [0,\infty]$ and $A, B \in R$. For $m \geq 2$, the following inequality holds:*

$$|E^{\tau,\oplus}_{\alpha,\alpha-\beta,m}(A,B;g^{-1}(t))|_g \leq g^{-1}\left(t^{m-1}\right) \odot g^{-1}\left(\exp(|A|t^{\alpha} + |B|t^{\alpha-\beta})\right). \tag{V.5}$$

Theorem V.1. *Assume that the following hypothesizes are true:*

$$(H_1) f : [0,T] \times R \to R \quad be \quad a \quad continuous \quad function:$$
$$(H_2) there \quad exist \quad C > 0 \quad such \quad that \quad f \quad holds \quad the \quad Lipschitz \quad condition:$$

$$|f(t,y) \ominus f(t,\sigma)|_g \leq C \odot |y \ominus \sigma|_g, \quad \forall (t,y),(t,\sigma) \in [0,T] \times R; \tag{V.6}$$

Then, the problem (V.1) *has a unique global continuous solution on* $[0,T]$.

Proof. Assume that a ball be appointed as $B_R := y \in C([0,T],R) : \|y\|_{\omega} \leq R, \omega > 0$ where $R > 0$ with

$$R \geq \left[W \odot |\phi_0^{(k)}|_g \oplus g^{-1}(T^{\alpha-1}) \odot S \odot |\phi_0^{(m-1)}|_g \oplus S \odot g^{-1}(\Gamma(\alpha)) \odot |B| \odot \|\phi\|_{\omega} \oplus D\right] \oslash (g^{-1}(\omega^{\alpha} \ominus S \odot g^{-1}(\Gamma(\alpha)) \odot C)), \tag{V.7}$$

where

$$W = \bigoplus_{k=0}^{m-2} \left(g^{-1}\left(\frac{T^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right) \oplus (|A| \oplus |B|) \odot g^{-1}\left(T^{(\beta-1)(m-\alpha)+\alpha+k}\right) \odot S\right),$$
$$D = \max_{t \in [0,T]} \left\{|f(t,0)_g \oslash \exp(\omega t)\right\}; S = \exp\left((|g(A)| + |g(B)|)T^{\alpha}\right).$$

Now, we set an integral operator F on B_R as below:

$$F : C([0,T],R) \supset B_R \ni y \to F(y) := (t \to (Fy)(t)) \in C([0,T],R),$$

through the following formula

$$(Fy)(t) = \bigoplus_{k=0}^{m-2} \left(g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right) \oplus (A \oplus B) \odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))\right) \odot \phi_0^{(k)}$$
$$\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t)) \odot \phi_0^{(m-1)} \oplus B \odot \int^{\oplus}_{[-\tau,\min(t-\tau,0)]} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s)) \odot \phi(s) \odot ds$$
$$\oplus \int^{\oplus}_{[0,t]} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot f(s,y(s)) \odot ds, \quad t \in [0,T].$$

We can establish that operator F is well-defined based on condition (H_1), and thus, the existence of a solution to the initial issue (V.1) is equivalent to the existence of a fixed point for the integral operator F on the set B_R. To prove the uniqueness of the fixed point, we will apply the contraction mapping principle. However, instead of using the maximum norm $C([0,T],R)$, which only yields a local solution within the subinterval $[0,T]$, we will consider equipping $C([0,T],R)$ with the weighted maximum norm $\|\cdot\|_{\omega}$ with respect to the exponential function, defined as:

$$|y|_{\omega} := \max_{t \in [0,T]} \left\{|y(t)|_g \oslash \exp(\omega t)\right\}, \forall y \in C([0,T],R).$$

Since two norms $\|\cdot\|_{\infty}$ and $\|\cdot\|_{\omega}$ are equivalent, $C([0,T],R,\|\cdot\|_{\omega})$ is also a Banach space. The proof is separated into two parts.

Step 1: We prove that $F(B_R) \subset B_R$. In this part, we look at the following estimation.

$$|(Fy)(t)|_g \oslash \exp(\omega t) = g^{-1}\left(\frac{g|(Py)(t)|_g}{g(\exp(\omega t))}\right) = g^{-1}\left(\frac{|(Pg(y))(t)|}{g(\exp(\omega t))}\right). \tag{V.8}$$

First of all, we denote the following notes for use in the process of proof.

$$(Fg(y))(t)=\sum_{k=0}^{m-2}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}+(g(A)+g(B))E^{\tau}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(g(A),g(B);t-\tau)\right)g(\phi_0^{(k)})$$
$$+E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t)g(\phi_0^{(m-1)})+g(B)\int_{-\tau}^{min(t-\tau,0)}E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-\tau-s)g(\phi(s))ds$$
$$+\int_0^t E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)g(f(s,y(s)))ds, t\in[0,T]$$

Then, we will get.

$$\frac{|F(g(y))(t)|}{g(\exp(\omega t))}\leq\frac{1}{g(\exp(\omega t))}\sum_{k=0}^{m-2}\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}|g(\phi_0^{(k)})|+\frac{|g(A)|+|g(B)|}{g(\exp(\omega t))}$$
$$+\sum_{k=0}^{m-2}|E^{\tau}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(g(A),g(B);t-\tau)||g(\phi_0^{(k)})|+\frac{1}{g(\exp(\omega t))}|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t)||g(\phi_0^{(m-1)})|$$
$$+\frac{|g(B)|}{g(\exp(\omega t))}\int_{-\tau}^{0}|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-\tau-s)||g(\phi(s)|$$
$$+\frac{1}{g(\exp(\omega t))}\int_0^t|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)||g(f(s,y(s)))-g(f(s.0))|+|g(f(s,0))|ds$$
$$\leq\sum_{k=0}^{m-2}\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}|g(\phi_0^{(k)})|+(|g(A)|+|g(B)|)\sum_{k=0}^{m-2}|E^{\tau}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(g(A),g(B);t-\tau)||g(\phi_0^{(k)})|$$
$$+\frac{1}{g(\exp(\omega t))}|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t)||g(\phi_0^{(m-1)})|+\frac{|g(B)|}{g(\exp(\omega t))}\int_{-\tau}^{0}|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-\tau-s)||g(\phi(s)|ds$$
$$+\frac{1}{g(\exp(\omega t))}\int_0^t|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)||g(f(s,y(s)))-g(f(s.0))|+|g(f(s,0))|ds$$
$$\leq\sum_{k=0}^{m-2}\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}|g(\phi_0^{(k)})|+(|g(A)|+|g(B)|)\sum_{k=0}^{m-2}|E^{\tau}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(g(A),g(B);t-\tau)||g(\phi_0^{(k)})|$$
$$+\frac{1}{g(\exp(\omega t))}|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t)||g(\phi_0^{(m-1)})|+\frac{|g(B)|}{g(\exp(\omega t))}\int_{-\tau}^{0}|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-\tau-s)|\frac{g(\exp(\omega s))}{g(\exp(\omega s))}|g(\phi(s)|ds$$
$$+\frac{1}{g(\exp(\omega t))}\int_0^t|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)||g(f(s,y(s)))-g(f(s.0))|\frac{g(\exp(\omega s))}{g(\exp(\omega s))}ds$$
$$+\int_0^t|E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)||g(f(s,0))|\frac{g(\exp(\omega s))}{g(\exp(\omega s))}ds.$$

By using from this formula and (V.8) we obtain

$$|(Fy)(t)|_g\oslash\exp(\omega t)\leq\bigoplus_{k=0}^{m-2}g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\odot|\phi_0^{(k)}|_g$$
$$\oplus(|A|\oplus|B|)\odot\bigoplus_{k=0}^{m-2}|E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))|_g\odot|\phi_0^{(k)}|_g$$
$$\oplus|E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t))|_g\odot|\phi_0^{(m-1)}|_g\oslash\exp(\omega t)$$
$$\oplus|B|\oslash\exp(\omega t)\odot\int_{[-\tau,0]}^{\oplus}|E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))|_g\odot\exp(\omega s)\odot|\phi(s)|_g\oslash\exp(\omega s)\odot ds$$
$$\oplus\int_{[0,t]}^{\oplus}|E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))|_g\odot|f(s,y(s))\ominus f(s.0)|_g\odot\exp(\omega s)\oslash\exp(\omega s)\odot ds\oslash\exp(\omega t)$$
$$\oplus\int_{[0,t]}^{\oplus}|E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))|_g\odot|f(s,0)|_g\oslash\exp(\omega s)\odot\exp(\omega s)\odot ds.$$

Now take $\forall t\in[0,T]$ and $\forall y\in B_R$. By using (H_2) by means of Lemma V.2, we receive:

$$
\begin{aligned}
&|(Fy)(t)|_g \oslash \exp(\omega t) \leq \bigoplus_{k=0}^{m-2} g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\odot|\phi_0^{(k)}|_g\\
&\oplus(|A|\oplus|B|)\odot\bigoplus_{k=0}^{m-2} g^{-1}\left((t-\tau)^{(\beta-1)(m-\alpha)+\alpha+k}\right)\odot g^{-1}((\exp(|A|+|B|)(t-\tau)^{\alpha})\odot|\phi_0^{(k)}|_g\\
&\oplus g^{-1}(t^{\alpha-1})\odot g^{-1}((\exp(|A|+|B|)t^{\alpha})\odot|\phi_0^{(m-1)}|_g\oslash\exp(\omega t)\\
&\oplus|B|\oslash\exp(\omega t)\odot\int_{[-\tau,0]}^{\oplus} g^{-1}\left((t-\tau-s)^{\alpha-1}\right)\odot g^{-1}((\exp(|A|+|B|)(t-\tau-s)^{\alpha})\odot\exp(\omega s)\odot|\phi(s)|_g\oslash\exp(\omega s)\odot ds\\
&\oplus\int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right)\odot g^{-1}((\exp(|A|+|B|)(t-s)^{\alpha})\odot C\odot|y(s)|_g\odot\exp(\omega s)\oslash\exp(\omega s)\odot ds\oslash\exp(\omega t)\\
&\oplus\int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right)\odot g^{-1}((\exp(|A|+|B|)(t-s)^{\alpha})\odot|f(s,0)|_g\oslash\exp(\omega s)\odot\exp(\omega s)\odot ds.
\end{aligned}
$$

Using the substitution $r-s=u$ and Lipschitz condition (H_2), we get

$$
\begin{aligned}
&|(Fy)(t)|_g \oslash \exp(\omega t) \leq \bigoplus_{k=0}^{m-2} g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\odot|\phi_0^{(k)}|_g\\
&\oplus(|A|\oplus|B|)\odot\bigoplus_{k=0}^{m-2} g^{-1}\left(t^{(\beta-1)(m-\alpha)+\alpha+k}\right)\odot g^{-1}((\exp(|A|+|B|)t^{\alpha})\odot|\phi_0^{(k)}|_g\\
&\oplus g^{-1}(t^{\alpha-1})\odot g^{-1}((\exp(|A|+|B|)t^{\alpha})\odot|\phi_0^{(m-1)}|_g\oslash\exp(\omega t)\\
&\oplus|B|\oslash\exp(\omega t)\odot\int_{[-\tau,0]}^{\oplus} g^{-1}\left((t-\tau-s)^{\alpha-1}\right)\odot|\phi(s)|_g\oslash\exp(\omega s)\odot\exp(\omega s)\odot ds\odot g^{-1}((\exp(|A|+|B|)(t)^{\alpha})\\
&\oplus C\oslash\exp(\omega t)\odot\int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right)\odot|y(s)|_g\odot\exp(\omega s)\oslash\exp(\omega s)\odot ds\odot g^{-1}((\exp(|A|+|B|)(t)^{\alpha})\\
&\oplus\int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right)\odot|f(s,0)|_g\oslash\exp(\omega s)\odot\exp(\omega s)\odot ds\odot g^{-1}((\exp(|A|+|B|)(t)^{\alpha})\oslash\exp(\omega t)\\
&\leq\bigoplus_{k=0}^{m-2} g^{-1}\left(\frac{T^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\odot|\phi_0^{(k)}|_g\oplus(|A|\oplus|B|)\odot\bigoplus_{k=0}^{m-2} g^{-1}\left(T^{(\beta-1)(m-\alpha)+\alpha+k}\right)\\
&\odot g^{-1}((\exp(|A|+|B|)T^{\alpha})\odot|\phi_0^{(k)}|_g\oplus g^{-1}(T^{\alpha-1})\odot g^{-1}((\exp(|A|+|B|)T^{\alpha})\odot|\phi_0^{(m-1)}|_g\\
&\oplus|B|\oslash\exp(\omega t)\odot\int_{[0,\tau]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right)\odot\exp(\omega(s-\tau))\odot ds\odot\max_{t\in[0,T]}\left\{|\phi(t)|_g\oslash\exp(\omega t)\right\}\odot g^{-1}((\exp(|A|+|B|)(T)^{\alpha})\\
&\oplus C\oslash\exp(\omega t)\odot\int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right)\odot\exp(\omega s)\odot ds\odot\max_{t\in[0,T]}\left\{|y(t)|_g\oslash\exp(\omega t)\right\}\odot g^{-1}((\exp(|A|+|B|)(T)^{\alpha})\\
&\oplus\int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right)\odot\exp(\omega s)\odot ds\odot\max_{t\in[0,T]}\left\{|f(s,0)|_g\oslash\exp(\omega t)\right\}\odot g^{-1}((\exp(|A|+|B|)(T)^{\alpha})\oslash\exp(\omega t)\\
&|(Fy)(t)|_g \oslash \exp(\omega t) \leq \bigoplus_{k=0}^{m-2} g^{-1}\left(\frac{T^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\odot|\phi_0^{(k)}|_g\oplus(|A|\oplus|B|)\odot\bigoplus_{k=0}^{m-2} g^{-1}\left(T^{(\beta-1)(m-\alpha)+\alpha+k}\right)\\
&\odot S\odot|\phi_0^{(k)}|_g\oplus g^{-1}(T^{\alpha-1})\odot S\odot|\phi_0^{(m-1)}|_g\oplus|B|\odot S\oslash\exp(\omega t)\odot\int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right)\odot\exp(\omega s)\odot ds\odot\|\phi\|_{\omega}\\
&\oplus C\odot S\oslash\exp(\omega t)\odot\int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right)\odot\exp(\omega s)\odot ds\odot\|y\|_{\omega}\oplus D\odot S\oslash\exp(\omega t)\int_{[0,t]}^{\oplus} g^{-1}\left((t-s)^{\alpha-1}\right)\odot\exp(\omega s)\odot ds
\end{aligned}
$$

$$
\begin{aligned}
&= \bigoplus_{k=0}^{m-2}\left(g^{-1}\left(\frac{T^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\oplus(|A|\oplus|B|)\odot g^{-1}\left(T^{(\beta-1)(m-\alpha)+\alpha+k}\right)\odot S\right)\odot|\phi_0^{(k)}|_g\\
&\oplus g^{-1}(T^{\alpha-1})\odot S\odot|\phi_0^{(m-1)}|_g\oplus|B|\odot S\oslash\exp(\omega t)\odot\int_{[0,t]}^{\oplus}g^{-1}\left(u^{\alpha-1}\right)\odot\exp(\omega t)\odot\exp(-\omega u)\odot du\odot\|\phi\|_\omega\\
&\oplus C\odot S\oslash\exp(\omega t)\odot\int_{[0,t]}^{\oplus}g^{-1}\left(u^{\alpha-1}\right)\odot\exp(\omega t)\odot\exp(-\omega u)\odot du\odot\|y\|_\omega\\
&\oplus D\odot S\oslash\exp(\omega t)\int_{[0,t]}^{\oplus}g^{-1}\left(u^{\alpha-1}\right)\odot\exp(\omega t)\odot\exp(-\omega u)\odot du\\
&= W\odot|\phi_0^{(k)}|_g\oplus g^{-1}(T^{\alpha-1})\odot S\odot|\phi_0^{(m-1)}|_g\oplus|B|\odot S\odot\int_{[0,t]}^{\oplus}g^{-1}\left(u^{\alpha-1}\right)\odot\exp(-\omega u)\odot du\odot\|\phi\|_\omega\\
&\oplus C\odot S\odot\int_{[0,t]}^{\oplus}g^{-1}\left(u^{\alpha-1}\right)\odot\exp(-\omega u)\odot du\odot\|y\|_\omega\oplus D\odot S\int_{[0,t]}^{\oplus}g^{-1}\left(u^{\alpha-1}\right)\odot\exp(-\omega u)\odot du\\
&= W\odot|\phi_0^{(k)}|_g\oplus g^{-1}(T^{\alpha-1})\odot S\odot|\phi_0^{(m-1)}|_g\oplus|B|\odot S\oslash g^{-1}(\omega^\alpha)\odot\int_{[0,\omega t]}^{\oplus}g^{-1}\left(v^{\alpha-1}\right)\odot\exp(-v)\odot dv\odot\|\phi\|_\omega\\
&\oplus C\odot S\oslash g^{-1}(\omega^\alpha)\odot\int_{[0,\omega t]}^{\oplus}g^{-1}\left(v^{\alpha-1}\right)\odot\exp(-v)\odot dv\odot\|y\|_\omega\oplus D\odot S\oslash g^{-1}(\omega^\alpha)\int_{[0,\omega t]}^{\oplus}g^{-1}\left(v^{\alpha-1}\right)\odot\exp(-v)\odot dv\\
&= W\odot|\phi_0^{(k)}|_g\oplus g^{-1}(T^{\alpha-1})\odot S\odot|\phi_0^{(m-1)}|_g\oplus S\oslash g^{-1}(\omega^\alpha)\int_{[0,\omega t]}^{\oplus}g^{-1}\left(v^{\alpha-1}\right)\odot\exp(-v)\odot dv\\
&\odot\left(|B|\odot\|\phi\|_\omega\oplus C\odot\|y\|_\omega\oplus D\right)\\
&\leq W\odot|\phi_0^{(k)}|_g\oplus g^{-1}(T^{\alpha-1})\odot S\odot|\phi_0^{(m-1)}|_g\oplus S\oslash g^{-1}(\omega^\alpha)\int_{[0,\infty]}^{\oplus}g^{-1}\left(v^{\alpha-1}\right)\odot\exp(-v)\odot dv\\
&\odot\left(|B|\odot\|\phi\|_\omega\oplus C\odot\|y\|_\omega\oplus D\right)=W\odot|\phi_0^{(k)}|_g\oplus g^{-1}(T^{\alpha-1})\odot S\odot|\phi_0^{(m-1)}|_g\\
&\oplus S\dot{g}^{-1}(\Gamma(\alpha))\oslash g^{-1}(\omega^\alpha)\odot\left(|B|\odot\|\phi\|_\omega\oplus C\odot\|y\|_\omega\oplus D\right)\\
&\leq W\odot|\phi_0^{(k)}|_g\oplus g^{-1}(T^{\alpha-1})\odot S\odot|\phi_0^{(m-1)}|_g\oplus S\dot{g}^{-1}(\Gamma(\alpha))\oslash g^{-1}(\omega^\alpha)\odot\left(|B|\odot\|\phi\|_\omega\oplus C\odot R\oplus D\right)
\end{aligned}
$$

Taking the maximum over $[0,T]$ and using inequality (V.6), we obtain the following relation:

$$\|Fy\|_\omega\leq R$$

For this reason, $F: B_R\rightarrow B_R$. In other words, F is well-defined on B_R.

Step 2.In this step, we will represent that F is a contractive mapping. We should demonstrate that F is a contraction over B_R. To see this, let $\forall y,\sigma\in B_R$. Mention that

$$(Fy)(t)\ominus(F\sigma)(t)=\int_{[0,t]}^{\oplus}E_{\alpha,\alpha,\alpha}^{\tau;\oplus}(A,B;g^{-1}(t-s))\odot(f(s,y(s))\ominus f(s,\sigma(s))\odot ds,\quad t>0. \tag{V.9}$$

Thus, for any $t \in [0,T]$, from Lemma V.2 and (H_2)-Lipschitz condition, it follows that

$$|(Fy)(t) \ominus (F\sigma)(t)|_g \oslash \exp(\omega t) = g^{-1}\left(\frac{|(Fg(y))(t) - (Fg(\sigma))(t)|}{g(\exp(\omega t))}\right)$$

$$\leq g^{-1}\left(\frac{1}{g(\exp(\omega t))}\int_0^t |E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)||g(f(s,y(s))) - g(f(s,\sigma(s)))|ds\right)$$

$$= g^{-1}\left(\frac{1}{g(\exp(\omega t)}\right) \odot \int^{\oplus}_{[0,t]} |E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))|_g \odot |f(s,y(s)) \ominus f(s,\sigma(s))|_g \odot ds$$

$$\leq (C \odot \exp((|A|+|B|)t^{\alpha})) \oslash \exp(\omega t) \odot \int^{\oplus}_{[0,t]} g^{-1}\left((t-s)^{\alpha-1}\right) \odot |y(s) \ominus \sigma(s)|_g \oslash \exp(\omega s) \odot \exp(\omega s) \odot ds$$

$$\leq (C \odot \exp((|A|+|B|)t^{\alpha})) \oslash \exp(\omega t) \odot \int^{\oplus}_{[0,t]} g^{-1}\left((t-s)^{\alpha-1}\right) \odot \exp(\omega s) \odot ds \odot \max_{t\in[0,T]} \left\{|y(t) - \sigma(t)|_g \oslash \exp(\omega t)\right\}$$

$$= (C \odot \exp((|A|+|B|)t^{\alpha})) \oslash \exp(\omega t) \odot \int^{\oplus}_{[0,t]} g^{-1}\left((t-s)^{\alpha-1}\right) \odot \exp(\omega s) \odot ds \odot \|y - \sigma\|_{\omega}$$

$$= (C \odot \exp((|A|+|B|)t^{\alpha})) \oslash \exp(\omega t) \odot \int^{\oplus}_{[0,t]} g^{-1}\left(u^{\alpha-1}\right) \odot \exp(\omega t) \odot \exp(-\omega u) \odot du \odot \|y - \sigma\|_{\omega}$$

$$= C \odot \exp((|A|+|B|)t^{\alpha}) \odot \int^{\oplus}_{[0,t]} g^{-1}\left(u^{\alpha-1}\right) \odot \exp(-\omega u) \odot du \odot \|y - \sigma\|_{\omega}$$

$$= (C \odot \exp((|A|+|B|)t^{\alpha})) \oslash g^{-1}(\omega^{\alpha}) \odot \int^{\oplus}_{[0,\omega t]} g^{-1}\left(v^{\alpha-1}\right) \odot \exp(-v) \odot dv \odot \|y - \sigma\|_{\omega}$$

$$\leq (C \odot \exp((|A|+|B|)t^{\alpha})) \oslash g^{-1}(\omega^{\alpha}) \odot \int^{\oplus}_{[0,\infty]} g^{-1}\left(v^{\alpha-1}\right) \odot \exp(-v) \odot dv \odot \|y - \sigma\|_{\omega}$$

$$= \exp((|A|+|B|)t^{\alpha}) \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \|y - \sigma\|_{\omega}$$

$$\leq \exp((|A|+|B|)T^{\alpha}) \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \|y - \sigma\|_{\omega} := S \odot C \odot g^{-1}(\Gamma(\alpha)) \odot g^{-1}(\omega^{\alpha}) \odot \|y - \sigma\|_{\omega}$$

Then, we get.

$$|(Fy)(t) \ominus (F\sigma)(t)|_g \oslash \exp(\omega t) \leq S \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \|y - \sigma\|_{\omega}$$

Taking maximum on $[0,T]$, we will get the following conclusion:

$$\|F(y) \ominus F(\sigma)\|_{\omega} \leq S \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \|y - \sigma\|_{\omega} \tag{V.10}$$

If we choose $\omega > (S \odot C \odot g^{-1}(\Gamma(\alpha)) \odot g^{-1}(\omega^{\alpha}))^{\frac{1}{\alpha}}$, then F is a contraction. Thus, by Banach's fixed point theorem, there exists a unique fixed point of F which is just the unique global continuous solution of (V.1). □

Remark V.1. *If the assumptions (H_1) and (H_2) are satisfied for all $t \in [0,\infty)$, then the claim of this theorem holds on the half-real line R, i.e. for any $(m-1)$-times continuously differentiable initial data $\phi : [-\tau, 0] \to R$, the non-linear Hilfer-type pseudo-fractional order differential equation with a constant delay (V.1) has a unique global continuous solution on* $[0,\infty)$.

VI. ULAM-HYERS STABILITY ANALYSIS ON HILFER-TYPE PSEUDO-FRACTIONAL DIFFERENTIAL EQUATION WITH A CONSTANT DELAY

In the following part, we debate the stability of the Hilfer-type pseudo-fractional DDE (V.1) in the Ulam-Hyers sense on $[0,T]$.

Suppose that $\varepsilon > 0$. Let us imagine the Hilfer-type pseudo-fractional delay differential equation (V.1) and the Initial issue for the following inequality

$$|H^{\alpha,\beta}_{\oplus,\odot,0+}\sigma(t) \ominus A \odot \sigma(t) \ominus B \odot \sigma(t-\tau) \ominus f(t,\sigma(t))|_g \leq \varepsilon, \quad for \quad t \in [0,T] \tag{VI.1}$$

Definition VI.1. *Equation* (VI.1) *is Ulam-Hyers stable if there is $\theta > 0$ such that for every $\varepsilon > 0$ and for every solution $\sigma \in C([0,T],R)$ of inequality* (VI.1)*, there is a solution $y \in C([0,T],R)$ of equation* (V.1) *that holds the inequality due to a weighted norm:*

$$\|y \ominus \sigma\|_{\omega} \leq \varepsilon \odot \theta, \quad t \in [0,T] \tag{VI.2}$$

Remark VI.1. *A function $\sigma \in C([0,T],R)$ is a solution of the inequality* (VI.1) *if and only if there is a function $f \in C([0,T],R)$ which fulfills the following conditions:*
1) $|f(t)|_g \leq \varepsilon$;
2) $H^{\alpha,\beta}_{\oplus,\odot,0+}\sigma(t) \ominus A \odot \sigma(t) \ominus B \odot \sigma(t-\tau) \ominus f(t,\sigma(t)) := f(t), t \in [0,T]$.

Due to the Remark VI.1, the solution of following equation:

$$H^{\alpha,\beta}_{\oplus,\odot,0+}\sigma(t) \ominus A \odot \sigma(t) \ominus B \odot \sigma(t-\tau) = f(t,\sigma(t)) \oplus f(t), t \in [0,T]. \tag{VI.3}$$

can be demonstrated by

$$\begin{aligned}
\sigma(t) =& \bigoplus_{k=0}^{m-2}\left(g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right) \oplus (A \oplus B) \odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))\right) \odot \phi_0^{(k)} \\
&\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t)) \odot \phi_0^{(m-1)} \oplus B \odot \int^{\oplus}_{[-\tau,\min(t-\tau,0)]} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s)) \odot \phi(s) \odot ds \\
&\oplus \int^{\oplus}_{[0,t]} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot f(s,\sigma(s)) \odot ds \oplus \int^{\oplus}_{[0,t]} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot f(s) \odot ds \\
:=& (F(\sigma))(t) \oplus \int^{\oplus}_{[0,t]} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot f(s) \odot ds, \quad t \in [0,T].
\end{aligned}$$

To use Lemma V.2, the difference $\sigma(t) \ominus (F(z))(t)$ can be evaluated as follows:

$$\begin{aligned}
|\sigma(t) \ominus (F(\sigma))(t)|_g =& \left|\int^{\oplus}_{[0,t]} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot f(s) \odot ds\right|_g \leq \int^{\oplus}_{[0,t]} |E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))|_g \odot |f(s)|_g \odot ds \\
\leq& \varepsilon \odot g^{-1}(t^{\alpha-1}) \odot g^{-1}(\exp((|A|+|B|)t^{\alpha})) \odot \int^{\oplus}_{[0,t]} ds \leq \varepsilon \odot g^{-1}(T^{\alpha}) \odot g^{-1}(\exp((|A|+|B|)T^{\alpha})) := \varepsilon \odot g^{-1}(T^{\alpha}) \odot S.
\end{aligned} \tag{VI.4}$$

Finally, with constant delay, we are ready to assert and prove the Ulam-Hyers stability result for Hilfer-type pseudo-FDE.

Theorem VI.1. *Suppose that (H_1 and H_2) are satisfied. Then the equation* (V.1) *is Ulam-Hyers stable on $[0,T]$.*

Proof. Assume that $\sigma \in C[0,T],R$ is a solution of the inequality (VI.1). Let y be a unique solution of the Cauchy problem for Hilfer-type pseudo-fractional DDE(V.1), that is

$$\begin{aligned}
y(t) =& \bigoplus_{k=0}^{m-2}\left(g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right) \oplus (A \oplus B) \odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))\right) \odot \phi_0^{(k)} \\
&\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t)) \odot \phi_0^{(m-1)} \oplus B \odot \int^{\oplus}_{[-\tau,\min(t-\tau,0)]} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s)) \odot \phi(s) \odot ds \\
&\oplus \int^{\oplus}_{[0,t]} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot f(s,\sigma(s)) \odot ds := (Fy)(t), \quad t \in [0,T]
\end{aligned} \tag{VI.5}$$

By using estimation (V.9) and (VI.5), we obtain

$$\begin{aligned}
|y(t) \ominus \sigma(t)|_g \oslash \exp(\omega t) =& \left|(Fy)(t) \ominus (F\sigma)(t) \ominus \int^{\oplus}_{[0,t]} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot f(s) \odot ds\right|_g \oslash \exp(\omega t) \\
\leq& |(Fy)(t) \ominus (F\sigma)(t)|_g \oslash \exp(\omega t) \oplus \int^{\oplus}_{[0,t]} |E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))|_g \odot |f(s)|_g \odot ds
\end{aligned}$$

$$\leq C \odot g^{-1}(\Gamma(\alpha)) \odot \exp((|A|+|B|)T^{\alpha}) \oslash g^{-1}(\omega^{\alpha}) \odot \|y-\sigma\|_{\omega} \oplus \varepsilon \odot g^{-1}(T^{\alpha}) \odot g^{-1}(\exp((|A|+|B|)T^{\alpha}))$$
$$:= S \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \|y-\sigma\|_{\omega} \oplus \varepsilon \odot g^{-1}(T^{\alpha}) \odot S$$

We take maximum on $[0,T]$, then we obtain

$$\|y-\sigma\|_{\omega} \leq S \odot L \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \|y-\sigma\|_{\omega} \oplus \varepsilon \odot g^{-1}(T^{\alpha}) \odot S$$

that gives that

$$\|y-\sigma\|_{\omega} \leq \varepsilon \odot (g^{-1}(T^{\alpha}) \odot S) \oslash (1 \ominus S \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}))$$

By choosing $\omega > \Big(g(S \odot C \odot g^{-1}(\Gamma(\alpha)))\Big)^{\frac{1}{\alpha}}$ which implies that

$$\|y-\sigma\|_{\omega} \leq \varepsilon \odot \theta \tag{VI.6}$$

where

$$\theta := (g^{-1}(T^{\alpha}) \odot S) \oslash (1 \ominus S \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}))$$

□

VII. AN EXAMPLE

In this section, we present an example validating the major theoretical results stated in Sections V and VI. The existence, uniqueness, and stability analysis of solutions in this example rely on the application of Theorem VI.1.

Consider the Hilfer pseudo-fractional differential equation with a constant delay, given by:

$$\begin{cases} H^{1.4,0.5}_{\oplus,\odot,0+} y(t) = 3 \odot y(t) \oplus 7 \odot y(t-2) \oplus \frac{cos(y(t))}{t^2+1}, t \in (0;2], \\ I^{1-\gamma}_{\oplus,\odot,0+} y(t) = t+5, \quad t \in [-2,0]. \end{cases} \tag{VII.1}$$

where $\alpha = 1.4, \beta = 0.5, m = 2, \tau = 2, T = 2$, $A = 3$, $B = 7$, and $\phi(t) = t+5$. The function $\phi(t)$ is continuously differentiable for $t \in [-2,0]$ and the nonlinear perturbation $f(t) = \frac{\cos(t)}{t^2+1}$ is continuous on $[-2,0] \times \mathbb{R}$. Let $g(t) = 2t+1$ for all $t \in \mathbb{R}$ be a monotone and continuous function, with its inverse $g^{-1}(t) = \frac{t-1}{2}$. We have $\phi_0 = 5$ and $\phi_0' = 1$.

The parameter γ is defined as $(\beta-1)(m-\alpha)+k+1$, where $k = 0,\ldots,m-1$. Substituting $\alpha = 1.4$, $\beta = 0.5$, and $m = 2$ into the expression for γ, we obtain $\gamma = k+0.7$, where k takes the values 0 and 1. Since $y(0) = 5$ and $y'(0) = 1$, the exact analytical representation of the solution of (VII.1) can be represented as follows:

$$y(t) = \left(g^{-1}\left(\frac{t^{-0.3}}{\Gamma((0.7)}\right) \oplus (3 \oplus 7) \odot E^{2,\oplus}_{1.4,1.4,2.1}(3,7;g^{-1}(t-2))\right) \odot 5$$

$$\oplus E^{2,\oplus}_{1.4,1.4,1.4}(3,7;g^{-1}(t)) \odot \phi_0' \oplus 7 \odot \int^{\oplus}_{[-2,\min(t-2,0)]} E^{2,\oplus}_{1.4,1.4,1.4}(3,7;g^{-1}(t-2-s)) \odot \phi(s) \odot ds$$

$$\oplus \int^{\oplus}_{[0,t]} E^{2,\oplus}_{1.4,1.4,1.4}(3,7;g^{-1}(t-s)) \odot \frac{\cos(y(s))}{s^2+1} \odot ds$$

It is clear that, by using the above basic pseudo-operations and conditions, we can simplify the exact solution of (VII.1). Such that we will obtain the following result for the solution of the Hilfe pseudo-fractional delay differential equation, which is equivalent to the exact solution, so that it can express pseudo-operations.

$$\begin{aligned} y(t) =& \frac{t^{-0.3}}{2\Gamma(0.7)} - 2 + \frac{11}{2} E^{2}_{1.4,1.4,2.1}\Big(7,15;t-2\Big) + \frac{3}{2} \cdot E^{2}_{1.4,1.4,1.4}(7,15;t) \\ &+ \int_{-2}^{\min(t-2,0)} E^{2}_{1.4,1.4,1.4}\Big(7,15;(t-2-s)\Big)\Big(15s + \frac{165}{2}\Big) ds \\ &+ \int_0^t E^{2}_{1.4,1.4,1.4}(7,15;(t-s))\Big(\frac{\cos(y(s))}{s^2+1} + \frac{1}{2}\Big) ds \end{aligned}$$

It is not difficult to see that condition $H(2)$ holds. By mean value theorem, for any $y,z \in R$, there exists $\xi \in (y,z)$such that

$$|f(t,y) \ominus f(t,z)|_g \leq |y \ominus z|_g$$

The statement $H(2)$ is valid with C being equivalent to 1, as per Theorem (VI.1) and Equation (V.1). This implies that the Hilfer pseudo-fractional differential equation with a constant delay, as given in equation (VII.1), has a single solution that is stable in the Ulam-Hyers sense over the interval $[0,2]$. Finally, we will describe the graphical representation of the solution set for equation (VII.1) within the interval [0, 2].

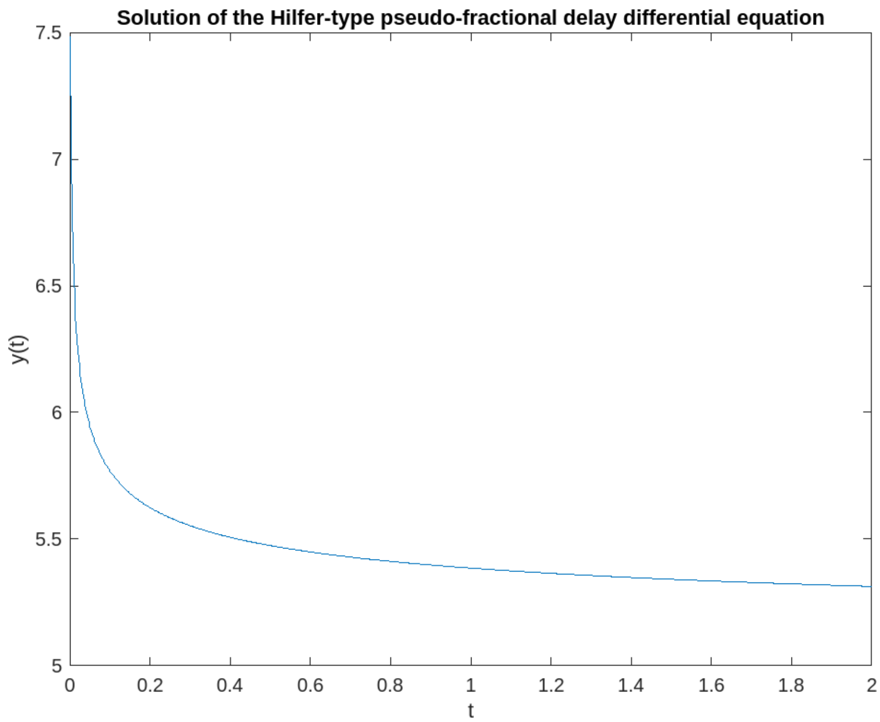

FIG. 1.

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