Delayed analogue of three-parameter pseudo-Mittag-Leffler functions and their applications to Hilfer pseudo-fractional time retarded differential equations

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Abstract

In this write-up, we focus on pseudo-Hilfer-type fractional order delayed differential equations with bounded definite integral initial conditions on the time interval $[0, T]$. We begin by establishing relevant lemmas. Then, we derive the solution to the homogeneous pseudo-Hilfer-type fractional order retarded differential equation that satisfies the appropriate initial condition using classical methods. Next, we obtain explicit formulas for solutions to linear inhomogeneous pseudo-Hilfer-type fractional time retarded differential equations with constant coefficients, employing classical ideas. Furthermore, we investigate the existence and uniqueness of the solution of the pseudo-Hilfer-type fractional order delayed differential equation, and demonstrate the stability of the given differential equation in the Ulam-Hyers sense on the time interval $[0, T]$.

Keywords: Pseudo-fractional operator. Existence and uniqueness. Delayed analouge pseudo-Mittag-Leffler type function. Fractional differential equations.

1. Introduction

Fractional differential equations (FDEs) are a generalized form of classical differential equations that involve derivatives of fractional order. Fractional calculus is a mathematical field that deals with derivatives and integrals of fractional order, and it includes important concepts such as fractional derivatives and integrals. Fractional derivatives are defined using operators like Caputo, Riemann-Liouville, or Grünwald-Letnikov, and exhibit non-local behavior. Fractional integrals, on the other hand, extend classical integrals and can describe memory effects and long-range dependencies. FDEs find widespread applications in various scientific and engineering fields, such as physics, biology, finance, signal processing, control theory, and image processing. Analytical methods like Laplace and Fourier transforms, as well as numerical methods like fractional-order numerical schemes and finite difference methods, are commonly used to solve FDEs. Fractional calculus is also employed in control theory, signal processing, and optimization, with diverse applications in domains such as image processing, audio processing, communication systems, finance, economics, and engineering.

Fractional differential equations (FDEs) have gained increasing attention in recent times due to their wide-ranging applications in various fields such as mechanics, electrical circuits, and time-delay systems stability analysis. FDEs are a generalization of classical differential equations, as they involve derivatives of arbitrary (fractional) order. The use of fractional-order derivatives allows for modeling diverse behaviors that cannot be captured by integer-order derivatives alone, making FDEs a powerful tool in engineering and science.

Similarly, pseudo-analysis is a mathematical theory that generalizes classical analysis by using semiconductors defined by pseudo-addition and pseudo-multiplication in the real range, instead of real numbers. This

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concept has piqued the interest of researchers from different fields such as functionality analysis, functional equations, and variational calculus.

In recent times, many scholars have worked on new formulations of inequalities involving fractional integrals and have investigated the properties of pseudo-fractional operators. For example, J. Vanterler da C. Sousa, Rubens F. Camargo, E. Capelas de Oliveira Gastano S. F. Frederico have studied pseudo-Hilfer-type FDEs([2]).

The existence and uniqueness problems of FDEs with constant delay and the stability of their solutions are crucial topics in the field of fractional differential equations. Many renowned scientists, such as Ahmed H.M., Ahmed A.M.S., Ragusa M.A ([39]), Moniri Z., Moghaddam B.P., Roudbaraki M.Z. ([40]), Vivek D., Kanagarajan K., Elsayed E.M. ([41]), Nazim I. Mahmudov, Ismail T. Huseynov, Arzu Ahmadova,([3],[7]-[12],[17]-[20],[24]-[26]) Khusainov, D.Ya., Ivanov, A.F., Shuklin, G.V.,([15]), Podlubny, I.([21])., J.Vanterler da C.Sousa, Gastao S.F. Frederico, E. Capelas de Oliveira([1],[2]) have made significant contributions to these problems.

In conclusion, fractional differential equations and pseudo-analysis are fascinating areas of research with wide-ranging applications in various fields. The works of renowned scientists in these fields have contributed significantly to the advancement of mathematical theory and its applications in engineering and science.

For instance: J. Vanterler da C. Sousa, Rubens F. Camargo, E. Capelas de Oliveira Gastano S. F. Frederico have looked following pseudo-Hilfer-type FDE([2]).

$$\begin{cases} H^{\alpha,\beta:\psi}_{\oplus,\odot,t_0+}y(t) = Ay(t) \oplus f(t, y(t)), t \in J, \\ I^{1-\gamma}_{\oplus,\odot,t_0+}y(t) = y_0. \end{cases} \tag{1.1}$$

The authors of this study investigate the existence and uniqueness of the global solution for equation (1.1). The equation involves the ψ-Hilfer pseudo-fractional derivative denoted by $H^{\alpha,\beta:\psi}_{\oplus,\odot,t_0+}(\cdot)$, where the order is $0 < \alpha \leq 1$ and the type is $0 \leq \beta \leq 1$. The parameter γ is defined as $\gamma = \alpha - \beta(1 - \alpha)$. The function $f : [t_0, +\infty) \times R^n \times R^n \to R^n$ is continuous. $\mathcal{A}$ is an $n \times n$ matrix.

It is worth mentioning that in a previous study by Sousa et al. in 2020 [6], the existence and uniqueness of the global solution for the initial value problem associated with data (t_0, y_0) was researched. The general form of any solution on the interval $\mathcal{I} := [a, b]$ is given by the system of equations (1.2), where $\frac{d^{\oplus}}{dt}y(t)$ denotes the pseudo-fractional derivative of $y(t)$ and $\mathcal{F}(t, y(t)) = f(t, y(t))$. The initial condition is $y(t_0) = y_0$.

$$\begin{cases} \frac{d^{\oplus}}{dt}y(t) = F(t, y(t)), \\ y(t_0) = y_0. \end{cases} \tag{1.2}$$

with $t_0 \in I$. Afterwards, in 2020, Sosa et al. ([6]), discussed the reachability of linear and non-linear systems in the sense of the ψ-Hilfer pseudo-fractional derivative in g-calculus by means of the Mittag-Leffler functions with the form

$$\begin{cases} H^{\alpha,\beta;\psi}_{\oplus,\odot,0+}y(t) = Ay(t) \oplus Bu(t), t \in [t_0, t_1], \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t_0) = 0 \end{cases} \tag{1.3}$$

and

$$\begin{cases} H^{\alpha,\beta;\psi}_{\oplus,\odot,0+}y(t) = Ay(t) \oplus Bu(t) \oplus f(t, y(t), u(t)), t \in [t_0, t_1], \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t_0) = 0. \end{cases} \tag{1.4}$$

In these equations, $H^{\alpha,\beta;\psi}_{\oplus,\odot,0+}(\cdot)$ represents the ψ-Hilfer pseudo-fractional derivative with order $0 < \alpha \leq 1$ and type $0 \leq \beta \leq 1$. The parameter γ is defined as $\gamma = \alpha - \beta(1 - \alpha)$, and $I^{1-\gamma}_{\oplus,\odot,0+}(\cdot)$ denotes the Riemann-Liouvile pseudo-fractional integral with respect to another function $1 - \gamma$. The state vector is denoted by $y \in R^n$, the control vector by $u \in R^m$, and A and B are constant matrices of dimensions $n \times n$ and $n \times m$, respectively. The non-linear function $f : J \times R^n \times R^m \to R^n$ is continuous in this context.

However, in this research article, we will be considering the following pseudo-Hilfer-type fractional delay differential equation:

$$\begin{cases} H^{\alpha,\beta}_{\oplus,\odot,0+}y(t) = A \odot y(t) \oplus B \odot y(t - \tau) \oplus f(t), t \in (0; T], \tau > 0, \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t) = \phi(t), t \in [-\tau, 0]. \end{cases} \tag{1.5}$$

where $m-1<\alpha<m$, $0\leq\beta\leq 1$, $\gamma=(\beta-1)(m-\alpha)+k+1, k=0,...,m-1$.

To achieve our primary objective of obtaining an analytical solution for the pseudo-Hilfer-type fractional time delay differential equation (1.5) with a constant delay using classical methods, we first need to obtain the solution for the homogeneous pseudo-Hilfer-type fractional delay equations (1.6).

$$\begin{cases} H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)=A\odot y(t)\oplus B\odot y(t-\tau), t\in(0;T], \tau>0, \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t)=\phi(t), t\in[-\tau,0]. \end{cases} \tag{1.6}$$

Subsequently, we employ conventional techniques to determine the explicit solution formula for linear inhomogeneous pseudo-Hilfer-type fractional time-retarded differential equations with constant coefficients, as presented in equation (1.5). We utilize well-established methods and refer to equation (1.7) to facilitate the solution.

$$\begin{cases} H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)=A\odot y(t)\oplus B\odot y(t-\tau)\oplus f(t), t\in(0;T], \tau>0, \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t)=0, t\in[-\tau,0]. \end{cases} \tag{1.7}$$

We make use of the solution of equation (1.7) as a particular solution to equation (1.5) in order to derive the analytic solution, considering the conditions $m-1<\alpha<m$, $0\leq\beta\leq 1$, and $\gamma=(\beta-1)(m-\alpha)+k+1, k=0,...,m-1$. Moreover, we establish the existence and uniqueness of the solution in our study, and additionally investigate the stability of the pseudo-Hilfer-type delay differential equation (DDE) (1.5) in the Ulam-Hyers sense over the time interval $[0,T]$.

2. PRELIMINARIES

In this part, we mention that important information which it deals with pseudo-analysis, the elements of the fractional analysis and some necessary lemmas which will use the proof of the theorem. ([16],[21])

- Gamma function:

$$\Gamma(\alpha)=\int_0^\infty \tau^{\alpha-1}e^{-\tau}d\tau, \quad \alpha>0.$$

- Beta function:

$$B(t,s)=\int_0^1 z^{t-1}(1-z)^{s-1}dz, \quad t,s>0.$$

  Let $g:J\to R_+$ be a monotone and continuous function, where $J=[a,b]$ and $R_+=[0,+\infty]$. Then we will defined Mittag-Leffler function as follow.

- The tree parametr Mittag-Leffler function:([22])

$$E^{\delta}_{\alpha,\beta}g(z)=\sum_{s=0}^{\infty}\frac{(\delta)_s}{\Gamma(\alpha s+\beta)}\frac{(g(z))^s}{s!}=\sum_{s=0}^{\infty}\frac{(\delta)_s}{\Gamma(\alpha s+\beta)}\frac{(g(z))^s}{s!}$$

- Delayed analogue of Mittag-Leffler type function generated by $A,B\in R$ of three parameters:([18])

$$E^{\tau}_{\alpha,\beta,\gamma}(g(A),g(B);t)=\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^n(g(B))^q}{\Gamma(n\alpha+q\beta+\gamma)}(t-n\tau)^{n\alpha+q\beta+\gamma-1}H(t-n\tau)$$

- Exponentially bounded $f:[0,\infty)\to R$ holds an inequality of the form

$$||f(t)||\leq Le^{\sigma t}, \quad t>T,$$

  for the real constants $\sigma, L>0$ and $T>0$.

- Laplace transform $\mathfrak{L}\{f(t)\}(s):$

$$F(s)=\mathfrak{L}\{f(t)\}(s)=\int_0^\infty e^{-st}f(t)dt, \quad s\in C,$$

  where $f:[0,\infty)\to R$ is measurable and exponentially bounded on $[0,\infty)$, then the appointed by exists and is an analytic function of s for $Re(s)>0$.

- Time shift feature of the Laplace transform:

$$\mathfrak{L}\left\{f(t-a)H(t-a)\right\}(s) = e^{-as}F(s).$$

- Convolution feature of Laplace transform:

$$\mathfrak{L}\left\{(f*h)(t)\right\} = \mathfrak{L}\left\{f(t)\right\}(s)\mathfrak{L}\left\{h(t)\right\}(s),$$

where $f, h : [0,\infty) \to R$ are exponetially bounded functions.

- Riemann-Liouville fractional integral:

$$I^{\alpha}_{a+}f(x) = \frac{1}{\Gamma(\alpha)}\int_a^x (x-t)^{\alpha-1}f(t)dt.$$

- Hilfer fractional derivative Let $m-1 < \alpha < m$, with $m \in N$. The right-sided Hilfer fractional derivatives, denoted by ${}^{H}D^{\alpha,\beta}_{a+}(\cdot)$ of a function f of order α and type $0 \le \beta \le 1$, are appointed by

$${}^{H}D^{\alpha,\beta}_{a+}f(x) = I^{\beta(m-\alpha)}_{a+}\frac{d^m}{dx^m}I^{(1-\beta)(m-\alpha)}_{a+}f(x). \tag{2.1}$$

Taking the limit $\beta \to 0$ in Eq.(2.1), we have the Rieman-Liouville derivative, given by:

$${}^{RL}D^{\alpha}_{a+}f(x) = \frac{d^m}{dx^m}I^{(m-\alpha)}_{a+}f(x).$$

Taking the limit $\beta \to 1$ in Eq.(2.1), we have the Caputo derivative, given by:

$${}^{C}D^{\alpha}_{a+}f(x) = I^{(m-\alpha)}_{a+}\frac{d^m}{dx^m}f(x).$$

- For any linear and bounded operator Ω appointed on a Banach space with $||\Omega|| < 1$, the operator $(I-\Omega)^{-1}$ is linear and bounded with property

$$(I-\Omega)^{-1} = \sum_{k=0}^{\infty}\Omega^k. \tag{2.2}$$

Lemma 2.1. *Let $g : J \to R_+$ be a monotone and continious function , where $J = [a,b]$ and $R_+ = [0,+\infty]$. Then, for $\alpha > 0$, $A \in R$, $n \in N_0 = 0,1,2,...,$ we have*

$$\mathfrak{L}^{-1}\left\{\frac{1}{(s^{\alpha}-g(A))^{n+1}}\right\}(t) = \sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q\frac{t^{\alpha(n+q+1)-1}}{\Gamma(\alpha(n+q+1)} = t^{(n+1)\alpha-1}E^{n+1}_{\alpha,(n+1)\alpha}(g(A)t^{\alpha}), \quad Re(s) > 0.$$

Proof. Using the expansion

$$\frac{1}{(1-t)^{n+1}} = \sum_{q=0}^{\infty}\binom{n+q}{q}t^q, \quad |t| < 1,$$

for $|t| = \left|\frac{g(A)}{s^{\alpha}}\right| < 1$, we find that

$$\frac{1}{(s^{\alpha}-g(A))^{n+1}} = \frac{1}{s^{\alpha(n+1)}} \times \frac{1}{\left(1-\frac{g(A)}{s^{\alpha}}\right)^{n+1}} = \frac{1}{s^{\alpha(n+1)}}\sum_{q=0}^{\infty}\binom{n+q}{q}\left(\frac{g(A)}{s^{\alpha}}\right)^q = \sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^q}{s^{q\alpha+\alpha(n+1)}}$$

Taking inverse-Laplace transform of the above, we obtain that

$$\mathfrak{L}^{-1}\left\{\frac{1}{(s^{\alpha}-g(A))^{n+1}}\right\}(t) = \mathfrak{L}^{-1}\left\{\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^q}{s^{q\alpha+\alpha(n+1)}}\right\}(t) = \sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q$$

$$\times\mathfrak{L}^{-1}\left\{\frac{1}{s^{q\alpha+\alpha(n+1)}}\right\}(t) = \sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^q t^{\alpha(n+q+1)-1}}{\Gamma(\alpha(n+q+1))} = t^{(n+1)\alpha-1}E^{n+1}_{\alpha,(n+1)\alpha}(g(A)t^{\alpha}).$$

□

Lemma 2.2. *Let $g : J \to R_+$ be a monotone and continuous function, where $J = [a,b]$ and $R_+ = [0,+\infty]$. Then, for $\alpha > 0, \alpha > \gamma$, we obtain.*

$$\mathfrak{L}^{-1}\left\{\frac{s^\gamma}{s^\alpha - g(A) - g(B)e^{-s\tau}}\right\}(t) = E^{\tau}_{\alpha,\alpha,\alpha-\gamma}(g(A), g(B); t).$$

Proof. According to the well-known Neumann series, $\frac{s^\gamma}{s^\alpha - g(A) - g(B)e^{-s\tau}}$ can be written through a series expansion as below:

$$\frac{s^\gamma}{s^\alpha - g(A) - g(B)e^{-s\tau}} = \frac{s^\gamma}{s^\alpha - g(A)}\frac{1}{1-\frac{g(B)e^{-s\tau}}{s^\alpha - g(A)}} = \frac{s^\gamma}{s^\alpha - g(A)}\sum_{n=0}^{\infty}\frac{(g(B))^n e^{-sn\tau}}{(s^\alpha - g(A))^n} = \sum_{n=0}^{\infty}\frac{(g(B))^n e^{-sn\tau}s^\gamma}{(s^\alpha - g(A))^{n+1}}.$$

Then imposing Lemma 2.2 to the final consideration we get:

$$\frac{s^\gamma}{s^\alpha - g(A) - g(B)e^{-s\tau}} = \sum_{n=0}^{\infty}\frac{(g(B))^n e^{-sn\tau}s^\gamma}{s^{\alpha(n+1)}(1-\frac{g(A)}{s^\alpha})^{n+1}} = \sum_{n=0}^{\infty}\frac{(g(B))^n e^{-sn\tau}s^\gamma}{s^{\alpha(n+1)}}\sum_{q=0}^{\infty}\binom{n+q}{q}\left(\frac{g(A)}{s^\alpha}\right)^q$$
$$= \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{g(A)^q(g(B))^n e^{-sn\tau}}{s^{\alpha(n+1)+q\alpha-\gamma}}.$$

From the time delay feature of the Laplace integral transform, we have

$$\mathfrak{L}\left\{g(t-\tau)\right\}(s)(H(t-\tau) = e^{-s\tau}\mathfrak{L}\left\{g(t)\right\}(s).$$

Then, by taking the Inverse Laplace transform of the aforementioned function, we get

$$\mathfrak{L}^{-1}\left\{\frac{s^\gamma}{s^\alpha - g(A) - g(B)e^{-s\tau}}\right\}(t) = \mathfrak{L}^{-1}\left\{\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{g(A)^q(g(B))^n e^{-sn\tau}}{s^{\alpha(n+1)+q\alpha-\gamma}}\right\}(t)$$
$$= \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q(g(B))^n\mathfrak{L}^{-1}\left(\frac{e^{-sn\tau}}{s^{\alpha(n+1)+q\alpha-\gamma}}\right)(t)$$
$$= \sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q(g(B))^n\frac{(t-n\tau)^{\alpha(n+1)+q\alpha-\gamma-1}H(t-n\tau)}{\Gamma(\alpha(n+1)+q\alpha-\gamma)} = E^{\tau}_{\alpha,\alpha,\alpha-\gamma}(g(A), g(B); t).$$

We need additional conditions on s, namely: $s^\alpha > |A|$ and $|s^\alpha - g(A)| > |B|e^{-s\tau}$ for convergence of the series. But, these conditions can be removed at the end of the evaluation with analytical continuation, to obtain the desired conclusion for all $s \in C$ with $Re(s) > 0$. □

2.1. ***PSEUDO-ANALYSIS***

Assume $g : [\alpha, \beta] \to [0, \infty]$ be monotone and continuous function. We will define pseudo operators as follow. (see, e.g., [1],[2],[4],[27],[28])

- Pseudo operators:

$$\alpha \oplus \beta = g^{-1}(g(\alpha) + g(\beta)) \quad \textit{and} \quad \alpha \odot \beta = g^{-1}(g(\alpha)g(\beta)),$$
$$\alpha \ominus \beta = g^{-1}(g(\alpha) - g(\beta)) \quad \textit{and} \quad \alpha \oslash \beta = g^{-1}\left(\frac{g(\alpha)}{g(\beta)}\right).$$

Suppose that $f : [c,d] \to [a,b]$ is measurable function.

- g-integral:

$$\int_{[c,d]}^{\oplus} f \odot dx = g^{-1}\left(\int_c^d g(f(x))dx\right).$$

- g-Laplace transform:
$$\mathfrak{L}^{\oplus}\left\{f(x)\right\}(s) = g^{-1}\big(\mathfrak{L}\left\{g(f(x))\right\}(s)\big).$$

Assuming that g is the generator function for the strict pseudo-addition $\oplus$ on the interval $[a,b]$, and g is continuously differentiable on (a,b), the corresponding pseudo-multiplication $\odot$ is defined as $x \odot y = g^{-1}(g(x)g(y))$. If a function f is differentiable on (c,d) and has the same monotonicity as the function g, then the g-derivative of f at the point $x \in (c,d)$ can be defined as follows:

- g-derivative:
$$\frac{d^{\oplus} f(x)}{dx} = g^{-1}\left(\frac{d}{dx} g(f(x))\right).$$

- n^{th}-g-derivative:
$$\frac{d^{(n)\oplus} f(x)}{dx} = g^{-1}\left(\frac{d^n}{dx^n} g(f(x))\right).$$

Now we will give some essential information about Hilfer operator and Hilfer-type fractional derivative .

- Riemann-Liouville pseudo-fractional integral.

  Assuming that $g : [a,b] \to [0,+\infty]$ is an increasing function that defines pseudo-addition $\oplus$ and pseudo-multiplication $\odot$ operations, the right-sided and left-sided Riemann-Liouville pseudo-fractional integrals of a measurable function $f : [a,b] \to [a,b]$ with a positive order $\alpha > 0$ can be defined in the following manner:
$$I^{\alpha}_{\oplus,\odot,a+} f(x) = g^{-1}\left(I^{\alpha}_{a+} g(f(x))\right) = \int^{\oplus}_{[a,x]} \left[g^{-1}\left(\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\right) \odot f(t)\right] \odot dt$$
  and
$$I^{\alpha}_{\oplus,\odot,b-} f(x) = g^{-1}\left(I^{\alpha}_{b-} g(f(x))\right) = \int^{\oplus}_{[x,b]} \left[g^{-1}\left(\frac{(x-t)^{\alpha-1}}{\Gamma(\alpha)}\right) \odot f(t)\right] \odot dt$$

- Hilfer pseudo-fractional derivatives.

  Consider a generator function $g : [a,b] \to [0,\infty]$ that is increasing, defining the pseudo-addition $\oplus$ and pseudo-multiplication $\odot$ operations. The right-sided and left-sided Hilfer pseudo-fractional derivatives of a measurable function $f : [a,b] \to [a,b]$, with orders $m-1 < \alpha < m$ and type $0 \leq \beta \leq 1$, respectively, can be defined as follows:
$$H^{\alpha,\beta}_{\oplus,\odot,a+} f(x) = g^{-1}\left({}^{H}D^{\alpha,\beta}_{a+} g(f(x))\right) = I^{\beta(m-\alpha)}_{\alpha,\beta,a+} g^{-1}\left(\frac{d^m}{dx^m}\right) \odot I^{1-\gamma}_{\oplus,\odot,a+} f(x)$$
  and
$$H^{\alpha,\beta}_{\oplus,\odot,b-} f(x) = g^{-1}\left({}^{H}D^{\alpha,\beta}_{b-} g(f(x))\right) = I^{\beta(m-\alpha)}_{\alpha,\beta,b-} g^{-1}\left(\frac{d^m}{dx^m}\right) \odot I^{1-\gamma}_{\oplus,\odot,b-} f(x)$$
  Note that
$$H^{\alpha,\beta}_{\oplus,\odot,a+} f(x) = g^{-1}\left(I^{\gamma-\alpha}_{a+}\, {}^{RL}D^{\gamma}_{a+} g(f(x))\right) = I^{\gamma-\alpha}_{\oplus,\odot,a+}\, {}^{RL}D^{\gamma}_{\oplus,\odot,a+} f(x)$$
  and
$$H^{\alpha,\beta}_{\oplus,\odot,b-} f(x) = g^{-1}\left(I^{\gamma-\alpha}_{b-}\, {}^{RL}D^{\gamma}_{b-} g(f(x))\right) = I^{\gamma-\alpha}_{\oplus,\odot,b-}\, {}^{RL}D^{\gamma}_{\oplus,\odot,b-} f(x)$$
  where $\gamma = \alpha + \beta(m-\alpha)$.

  For extra information about pseudo-analysis, see [29, 39, 40, 41].

  In the following, we will first discuss the derivation of the formulas of the pseudo-Mittag-Leffler functions and their definitions based on these calculations.

- The one parameter pseudo-Mittag-Leffler function::

$$E_{\alpha}^{\oplus}(z) = g^{-1}\Big(E_{\alpha}g(z)\Big) = g^{-1}\Bigg(\sum_{s=0}^{\infty}\frac{(g(z))^s}{\Gamma(\alpha s+1)}\Bigg) = \bigoplus_{s=0}^{\infty} g^{-1}\Bigg(\frac{(g(z))^s}{\Gamma(\alpha s+1)}\Bigg) = \bigoplus_{s=0}^{\infty}\Big[g^{-1}\Big((g(z))^s\Big)\oslash g^{-1}\Big(\Gamma(\alpha s+1)\Big)\Big]$$

Where $(\delta)_s$ is the famous Pochhammer symbol denoting $\frac{\Gamma(\delta+s)}{\Gamma(\delta)}$.

- The two parameter pseudo-Mittag-Leffler function:

$$E_{\alpha,\beta}^{\oplus}(z) = g^{-1}\Big(E_{\alpha,\beta}g(z)\Big) = g^{-1}\Bigg(\sum_{s=0}^{\infty}\frac{(g(z))^s}{\Gamma(\alpha s+\beta)}\Bigg) = \bigoplus_{s=0}^{\infty} g^{-1}\Bigg(\frac{(g(z))^s}{\Gamma(\alpha s+\beta)}\Bigg) = \bigoplus_{s=0}^{\infty}\Big[g^{-1}\Big((g(z))^s\Big)\oslash g^{-1}\Big(\Gamma(\alpha s+\beta)\Big)\Big]$$

- The three parameter pseudo-Mittag-Leffler function:

$$E_{\alpha,\beta}^{\delta,\oplus}(z) = g^{-1}\Big(E_{\alpha,\beta}^{\delta}g(z)\Big) = g^{-1}\Bigg(\sum_{s=0}^{\infty}\frac{(\delta)_s}{\Gamma(\alpha s+\beta)}\frac{(g(z))^s}{s!}\Bigg) = \bigoplus_{s=0}^{\infty} g^{-1}\Bigg(\frac{(\delta)_s}{\Gamma(\alpha s+\beta)}\frac{(g(z))^s}{s!}\Bigg)$$

$$= \bigoplus_{s=0}^{\infty} g^{-1}\Bigg[g\Bigg(g^{-1}\Bigg(\frac{(\delta)_s}{\Gamma(\alpha s+\beta)}\Bigg)\Bigg)g\Bigg(g^{-1}\Bigg(\frac{(g(z))^s}{s!}\Bigg)\Bigg)\Bigg] = \bigoplus_{s=0}^{\infty}\Bigg[g^{-1}\Bigg(\frac{(\delta)_s}{\Gamma(\alpha s+\beta)}\Bigg)\odot g^{-1}\Bigg(\frac{(g(z))^s}{s!}\Bigg)\Bigg]$$

$$= \bigoplus_{s=0}^{\infty}\Big[\Big(g^{-1}((\delta)_s)\oslash g^{-1}(\Gamma(\alpha s+\beta))\Big)\odot\Big(g^{-1}((g(z))^s)\oslash g^{-1}(s!)\Big)\Big]$$

- The pseudo-bivariate Mittag-Leffler function:

$$E_{\alpha,\beta,\gamma}^{\delta,\oplus}(a,b) = g^{-1}\Big(E_{\alpha,\beta,\gamma}^{\delta}(g(a),g(b))\Big) = g^{-1}\Bigg(\sum_{l=0}^{\infty}\sum_{s=0}^{\infty}\frac{(\delta)_{l+s}}{\Gamma(l\alpha+s\beta+\gamma)}\frac{(g(a))^l(g(b))^s}{l!s!}\Bigg)$$

$$= \bigoplus_{l=0}^{\infty}\bigoplus_{s=0}^{\infty} g^{-1}\Bigg[\frac{(\delta)_{l+s}}{\Gamma(l\alpha+s\beta+\gamma)}\frac{(g(a))^l(g(b))^s}{l!s!}\Bigg] = \bigoplus_{l=0}^{\infty}\bigoplus_{s=0}^{\infty}\Bigg[g^{-1}\Bigg(\frac{(\delta)_{l+s}}{\Gamma(l\alpha+s\beta+\gamma)}\Bigg)\odot g^{-1}\Bigg(\frac{(g(a))^l(g(b))^s}{l!s!}\Bigg)\Bigg]$$

$$= \bigoplus_{l=0}^{\infty}\bigoplus_{s=0}^{\infty}\Bigg[\Bigg(g^{-1}((\delta)_{l+s})\oslash g^{-1}(\Gamma(l\alpha+s\beta+\gamma))\Bigg)\odot\Bigg(\Big(g^{-1}((g(a))^l(g(b))^s)\Big)\oslash\Big(g^{-1}(l!\times s!)\Big)\Bigg)\Bigg]$$

$$= \bigoplus_{l=0}^{\infty}\bigoplus_{s=0}^{\infty}\Bigg[\Bigg(g^{-1}((\delta)_{l+s})\oslash g^{-1}(\Gamma(l\alpha+s\beta+\gamma))\Bigg)\odot\Bigg(\Big(g^{-1}((g(a))^l)\odot g^{-1}((g(b))^s)\Big)\oslash\Big(g^{-1}(l!)\odot g^{-1}(s!)\Big)\Bigg)\Bigg]$$

- Delayed analogue of pseudo-Mittag-Leffler type function generated by $A, B \in R$ of three parameters:

$$E_{\alpha,\beta,\gamma}^{\tau,\oplus}(A,B;t) = g^{-1}\Big(E_{\alpha,\beta,\gamma}^{\tau}(g(A),g(B);g(t))\Big)$$

$$= g^{-1}\Bigg(\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^n(g(B))^q}{\Gamma(n\alpha+q\beta+\gamma)}(g(t-n\tau))^{n\alpha+q\beta+\gamma-1}H(g(t-n\tau))\Bigg)$$

$$= \bigoplus_{n=0}^{\infty}\bigoplus_{q=0}^{\infty} g^{-1}\Bigg(\binom{n+q}{q}(g(A))^n(g(B))^n\frac{(g(t-n\tau))^{n\alpha+q\beta+\gamma-1}H(g(t-n\tau))}{\Gamma(n\alpha+q\beta+\gamma)}\Bigg)$$

$$= \bigoplus_{n=0}^{\infty}\bigoplus_{q=0}^{\infty}\Bigg[g^{-1}\Bigg(\binom{n+q}{q}\Bigg)\odot g^{-1}\Big((g(A))^n\Big)\odot g^{-1}\Big((g(B))^q\Big)$$

$$\odot g^{-1}\Big((g(t-n\tau))^{n\alpha+q\beta+\gamma-1}\Big)\odot g^{-1}\Big(H(g(t-n\tau))\Big)\oslash g^{-1}\Big(\Gamma(n\alpha+q\beta+\gamma)\Big)\Bigg]$$

where $H(\cdot): R\to R$ is the Heaviside function appointed as follows

$$H(t) = \begin{cases} 1, & \text{if } t \geq 0 \\ 0, & \text{if } t < 0. \end{cases}$$

Theorem 2.1. *([1], p.254, theorem 27.)*

Assume that g is the additive generator of the strict-pseudo-addition $\oplus$ on [a,b], so that g is continiously differentiable on(a,b), $0 < m-1 \le \alpha < m$, $\quad 0 \le \beta \le 1$ and $s \in R$ Then, the g-Laplace transform of the pseudo-Hilfer pseudo-fractional derivative of order α is given by:

$$\mathscr{L}^{\oplus}\left\{{}^{H}D^{\alpha,\beta}_{\oplus,\odot,0+}f(x)\right\} = \left[g^{-1}(s^{\alpha}) \odot \mathfrak{L}^{\oplus}\left\{f(x)\right\}\right] \ominus \bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1}) \odot I^{(1-\beta)(m-\alpha)-k}f(0)\right] \quad (2.3)$$

3. EXPLICIT SOLUTIONS OF HOMOGENEOUS PSEUDO-HILFER-TYPE FRACTIONAL DIFERENTIONAL EQUATION

In this part, we have proved the explicit solution given by following (3.1) pseudo-Hilfer-type fractional differential equation system.

$$\begin{cases} H^{\alpha,\beta}_{\oplus,\odot,0+}y(t) = A \odot y(t) \oplus B \odot y(t-\tau), t \in (0;T], \tau > 0, \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t) = \phi(t), t \in [-\tau, 0]. \end{cases} \quad (3.1)$$

where $m-1 < \alpha < m$, $0 \le \beta \le 1$, $\gamma = (\beta-1)(m-\alpha)+k+1, k = 0, ..., m-1$.

Theorem 3.1. *A unique analytical solution $y \in C^m([-\tau, T], R)$ of the initial problem (3.1) has as shown below:*

$$y(t) = \bigoplus_{k=0}^{m-2}\left(g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right) \oplus (A \oplus B) \odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A, B; t-\tau)\right) \odot \phi_0^{(k)}$$

$$\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A, B; t) \odot \phi_0^{(m-1)} \oplus B \odot \int^{\oplus}_{[-\tau,\min(t-\tau,0)]} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A, B; t-\tau-s) \odot \phi(s) \odot ds$$

Proof. Suppose that $T = \infty$. Assume that (1.5) has a unique m times continuously differentiable solution y and f are continuous and exponentially bounded, and $H^{\alpha,\beta}_{\oplus,\odot,0+}y$ is exponentially bounded on $[0, \infty)$, then Laplace transforms of them exist. And we are going to receive an integral representation of the solution to the linear homogeneous pseudo-Hilfer-type fractional differential equation.

First of all, we are imposing Laplace integral transform to both sides of (3.1) with the help of Theorem 2.1.

$$\begin{aligned} \mathfrak{L}^{\oplus}\left\{H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)\right\}(s) &= g^{-1}\left[\mathfrak{L}\left\{g\left(H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)\right)\right\}(s)\right] = g^{-1}\left[\mathfrak{L}\left\{{}^{H}D^{\alpha,\beta}_{0+}g(y(t)\right\}\right] \\ &= g^{-1}\left[s^{\alpha}\mathfrak{L}\left\{g(y(t))\right\}(s) - \sum_{k=0}^{m-1}s^{m(1-\beta)+\alpha\beta-k-1}\left(I^{(1-\beta)(m-\alpha)-k}_{0+}g(y)\right)(0)\right] \\ &= g^{-1}(s^{\alpha}) \odot \mathfrak{L}\left\{y(t)\right\}(s) \ominus \bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1}) \odot I^{(1-\beta)(m-\alpha)-k}_{\oplus,\odot,0+}y(0)\right] \\ &= g^{-1}(s^{\alpha}) \odot Y(s) \ominus \bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1}) \odot \phi_0^{k}\right] \end{aligned}$$

$$\mathfrak{L}^{\oplus}[H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)](s) = g^{-1}(s^{\alpha}) \odot Y(s) \ominus \bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1}) \odot \phi_0^{k}\right]. \quad (3.2)$$

where, $\mathfrak{L}^{\oplus}\left\{y(t)\right\}(s) = Y(s)$

$$\mathfrak{L}^{\oplus}\left\{A \odot y(t) \oplus B \odot y(t-\tau)\right\}(s) = g^{-1}\Bigg(\mathfrak{L}\left\{g(A \odot y(t) \oplus B \odot y(t-\tau)\right\}\Bigg)$$

$$= g^{-1}\Bigg(\mathfrak{L}\left\{g(A)g(y(t)) + g(B)g\big(y(t-\tau)\big)\right\}\Bigg) = A \odot \mathfrak{L}^{\oplus}(y(t)) \oplus B \odot \mathfrak{L}^{\oplus}(y(t-\tau))$$

$$= A \odot Y(s) \oplus B \odot \mathfrak{L}^{\oplus}(y(t-\tau))$$

we get

$$\mathfrak{L}^{\oplus}\left\{A \odot y(t) \oplus B \odot y(t-\tau)\right\}(s) = A \odot Y(s) \oplus B \odot \mathfrak{L}^{\oplus}(y(t-\tau)) \tag{3.3}$$

$$\mathfrak{L}^{\oplus}(y(t-\tau))(s) = g^{-1}(\mathfrak{L}(g(t-\tau))(s))$$

and by using substitution of $t-\tau=\theta$, we receive that

$$\mathfrak{L}\left\{g(t-\tau)\right\}(s) = \int_0^{\infty} g(t-\tau)e^{-st}dt = \int_{-\tau}^{\infty} g(y(\theta))e^{-s(\tau+\theta)}d\theta = e^{-s\tau}\int_{-\tau}^{\infty} g(y(\theta))e^{-s(\theta)}d\theta$$

$$= e^{-s\tau}\left[\int_{-\tau}^{0} g(y(\theta))e^{-s(\theta)}d\theta + \int_0^{\infty} g(y(\theta))e^{-s(\theta)}d\theta\right] = \int_{-\tau}^{0} g(y(\theta))e^{-s(\tau+\theta)}d\theta$$

$$+ e^{-s\tau}\mathfrak{L}(g(y(\theta))(s)) = \int_0^{\tau} g(y(t-\tau))e^{-st}dt + e^{-s\tau}\mathfrak{L}(g(y(\theta))(s))$$

On the other hand, due to the integral property of the pseudo-Riemann-Liouville-fraction, we obtain the following results. Let's also note that the initial condition of the issue we are reviewing is manifested in the following case.

$$I^{0}_{\oplus,\odot,0+}y(t) = y(t) \implies y(t) = \phi(t), t \in [-\tau, 0]$$

in there $\tilde{\phi}(\cdot) : R \to R$ is the unit-step function,which it has defined as bellow:

$$\tilde{\phi}(t) = \begin{cases} \phi(t) & \text{if } -\tau \le t \le 0 \\ 0 & \text{if } t > 0 \end{cases}$$

Therefore we get following relations:

$$\mathfrak{L}\left\{g(t-\tau)\right\}(s) = \int_0^{\tau} g\big(y(t-\tau)\big)e^{-st}dt + e^{-s\tau}\mathfrak{L}\left\{g(y(\theta))\right\}(s) = \int_0^{\infty} g(\tilde{\phi}(t-\tau))e^{-st}dt + e^{-s\tau}\mathfrak{L}\left\{g(y(\theta))\right\}(s)$$

$$\mathfrak{L}^{\oplus}(y(t-\tau))(s) = g^{-1}(e^{-s\tau}) \odot Y(s) \oplus \mathfrak{L}^{\oplus}(\tilde{\phi}(t-\tau))(s) \tag{3.4}$$

By using formula (3.2), (3.3), (3.4) we get the following results.

$$g^{-1}(s^{\alpha}) \odot Y(s) \ominus \bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1}) \odot \phi_0^{(k)}\right] = A \odot Y(s) \oplus B \odot \left[g^{-1}(e^{-s\tau}) \odot Y(s) \oplus \mathfrak{L}^{\oplus}\left\{\tilde{\phi}(t-\tau)\right\}(s)\right]$$

Afterward, we write the above relation in the following explicit form

$$\left[g^{-1}(s^{\alpha}) \ominus A \ominus B \odot g^{-1}(e^{-s\tau})\right] \odot Y(s) = \bigoplus_{k=0}^{m-1}\left[g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1}) \odot \phi_0^{k}\right] \oplus B \odot \mathfrak{L}^{\oplus}\left\{\tilde{\phi}(t-\tau)\right\}(s) \tag{3.5}$$

Then, we solve (3.5) with respect to Y(s),

$$Y(s) = \left[\bigoplus_{k=0}^{m-1}\left(g^{-1}(s^{m(1-\beta)+\alpha\beta-k-1}) \odot \phi_0^{(k)}\right) \oplus B \odot \mathfrak{L}^{\oplus}\left\{\tilde{\phi}(t-\tau)\right\}(s)\right] \oslash \left[g^{-1}(s^{\alpha}) \ominus A \ominus B \odot g^{-1}(e^{-s\tau})\right]$$

$$=g^{-1}\left(\frac{\sum_{k=0}^{m-1}[s^{m(1-\beta)+\alpha\beta-k-1}g(\phi_0^{k})] + g(B)g(\mathfrak{L}^{\oplus}(\tilde{\phi}(t-\tau))(s))}{s^{\alpha} - g(A) - g(B)e^{-s\tau}}\right)$$

$$=g^{-1}\left(\sum_{k=0}^{m-2}\frac{s^{m(1-\beta)+\alpha\beta-k-1}g(\phi_0^{(k)})}{s^{\alpha}-g(A)-g(B)e^{-s\tau}} + \frac{s^{-m\beta+\alpha\beta}}{s^{\alpha}-g(A)-g(B)e^{-s\tau}}g(\phi_0^{(m-1)}) + \frac{g(B)}{s^{\alpha}-g(A)-g(B)e^{-s\tau}}\mathfrak{L}\left\{g(\tilde{\phi}(t-\tau))\right\}\right)$$

$$=g^{-1}\left[\left(1+\frac{g(A)+g(B)e^{-s\tau}}{s^{\alpha}-g(A)-g(B)}\right)\sum_{k=0}^{m-2}\frac{s^{m(1-\beta)+\alpha\beta-k-1}g(\phi_0^{(k)})}{s^{\alpha}-g(A)-g(B)e^{-s\tau}} + \frac{g(B)}{s^{\alpha}-g(A)-g(B)e^{-s\tau}}\mathfrak{L}\left\{g(\tilde{\phi}(t-\tau))\right\}\right]$$

In accordance with relation (2.2), we have

$$\left[s^{\alpha}-g(A)-g(B)e^{-s\tau}\right]^{-1} = (s^{\alpha}-g(A))^{-1}\left[1-(s^{\alpha}-g(A))^{-1}g(B)e^{-s\tau}\right]^{-1}$$

$$=(s^{\alpha}-g(A))^{-1}\sum_{n=0}^{\infty}\left[(s^{\alpha}-g(A))^{-n}(g(B))^{n}e^{-sn\tau}\right]$$

$$\left[s^{\alpha}-g(A)-g(B)e^{-s\tau}\right]^{-1} = \sum_{n=0}^{\infty}\left[(s^{\alpha}-g(A))^{-(n+1)}(g(B))^{n}e^{-sn\tau}\right] \tag{3.6}$$

If we replace the expression (3.6) in the Y(s) formula obtained above, we get the following results.

$$Y(s) = g^{-1}\Bigg(\left(\sum_{k=0}^{m-2}s^{m(1-\beta)+\alpha\beta-k-1}g(\phi_0^{(k)}) + (g(A)+g(B)e^{-s\tau})\sum_{k=0}^{m-2}s^{m(1-\beta)+\alpha\beta-k-1}g(\phi_0^{(k)})\right)$$
$$\times\sum_{n=0}^{\infty}\left[(s^{\alpha}-g(A))^{-n}(g(B))^{n}e^{-sn\tau}\right] + g(\phi_0^{(m-1)})s^{-m\beta+\alpha\beta}\sum_{n=0}^{\infty}\left[(s^{\alpha}-g(A))^{-(n+1)}(g(B))^{n}e^{-sn\tau}\right]$$
$$+g(B)\mathfrak{L}\left\{g(\tilde{\phi}[t-\tau])\right\}(s)\sum_{n=0}^{\infty}\left[(s^{\alpha}-g(A))^{-(n+1)}(g(B))^{n}e^{-sn\tau}\right]\Bigg)$$

Imposing the inverse g-Laplace transform to the above result, we get:

$$y(t) = g^{-1}\Bigg(\mathfrak{L}^{-1}\Bigg[\sum_{k=0}^{m-2}s^{m(1-\beta)+\alpha\beta-k-1}g(\phi_0^{(k)}) + (g(A)+g(B)e^{-s\tau})\sum_{k=0}^{m-2}s^{m(1-\beta)+\alpha\beta-k-1}g(\phi_0^{(k)})$$
$$\times\sum_{n=0}^{\infty}\left[(s^{\alpha}-g(A))^{-n}(g(B))^{n}e^{-sn\tau}\right] + g(\phi_0^{(m-1)})s^{-m\beta+\alpha\beta}\sum_{n=0}^{\infty}\left[(s^{\alpha}-g(A))^{-(n+1)}(g(B))^{n}e^{-sn\tau}\right]$$
$$+g(B)\sum_{n=0}^{\infty}\left[(s^{\alpha}-g(A))^{-(n+1)}(g(B))^{n}e^{-sn\tau}\right]\mathfrak{L}\left\{g(\tilde{\phi}[t-\tau])\right\}(s)\Bigg](t)\Bigg)$$

Taking inverse Laplace transform of the statement above and by using Lemma 2.1, Lemma 2.2 and time shift and convolution property of the Laplace transform, we gain an explict representation of solution for a initial issue (3.1)

$$
\begin{aligned}
y(t) &= g^{-1}\Bigg(\mathfrak{L}^{-1}\Bigg[\sum_{k=0}^{m-2} s^{m(1-\beta)+\alpha\beta-k-1} g(\phi_0^{(k)}) \\
&+\sum_{n=0}^{\infty}\sum_{p=0}^{\infty}\binom{n+q}{q}(g(A))^{q+1}(g(B))^{n}\frac{s^{(1-\beta)(m-\alpha)-1}}{s^{\alpha(n+q+1)}}e^{-sn\tau}g(\phi_0) \\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n+1}\frac{s^{(1-\beta)(m-\alpha)-1}e^{-sn\tau}}{s^{\alpha(n+q+1)}}g(\phi_0)+\cdots+ \\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q+1}(g(B))^{n}\frac{s^{(1-\beta)(m-\alpha)-m+1e^{-sn\tau}}}{s^{\alpha(n+q+1)}}g(\phi_0^{(m-2)}) \\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n+1}\frac{s^{(1-\beta)(m-\alpha)-m+1}e^{-sn\tau}}{s^{\alpha(n+q+1)}}g(\phi_0^{(m-2)}) \\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q}(g(B))^{n}e^{-sn\tau}}{s^{\alpha(n+q+1)-\beta(\alpha-m)}}g(\phi_0^{(m-1)}) \\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q}(g(B))^{n+1}e^{-sn\tau}}{s^{\alpha(n+q+1)}}\mathfrak{L}\big(g(\tilde{\phi}[t-\tau])\big)(s)\Bigg](t)\Bigg)
\end{aligned}
$$

$$
\begin{aligned}
y(t) &= g^{-1}\Bigg(\mathfrak{L}^{-1}\Bigg[\sum_{k=0}^{m-2} s^{m(1-\beta)+\alpha\beta-k-1} g(\phi_0^{(k)}) \\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q+1}(g(B))^{n}}{s^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+1}}e^{-sn\tau}g(\phi_0) \\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q}(g(B))^{n+1}e^{-sn\tau}}{s^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+1}}g(\phi_0)+\cdots+ \\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q+1}(g(B))^{n}e^{-sn\tau}}{s^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+m-1}}g(\phi_0^{(m-2)}) \\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^{q}(g(B))^{n+1}\frac{e^{-sn\tau}}{s^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+m-1}}g(\phi_0^{(m-2)}) \\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q}(g(B))^{n}e^{-sn\tau}}{s^{\alpha(n+q+1)-\beta(\alpha-m)}}g(\phi_0^{(m-1)}) \\
&+\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}\frac{(g(A))^{q}(g(B))^{n+1}e^{-sn\tau}}{s^{\alpha(n+q+1)}}\mathfrak{L}\left\{g(\tilde{\phi}[t-\tau])\right\}(s)\Bigg](t)\Bigg)
\end{aligned}
$$

Then we get following result.

$$
\begin{aligned}
y(t) = g^{-1}\Big(&\sum_{k=0}^{m-2} \frac{t^{(1-\beta)(m-\alpha)-k}}{\Gamma((1-\beta)(m-\alpha)-k+1)} g(\phi_0^{(k)}) \\
&+ \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} \binom{n+q}{q} (g(A))^{q+1}(g(B))^{n} \frac{(t-(n+1)\tau)^{\alpha(n+q+1)-(1-\beta)(m-\alpha)}}{\Gamma(\alpha(n+q+1)-(1-\beta)(m-\alpha)+1)} H(t-(n+1)\tau) g(\phi_0) \\
&+ \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} \binom{n+q}{q} (g(A))^{q}(g(B))^{n+1} \frac{(t-(n+1)\tau)^{\alpha(n+q+1)-(1-\beta)(m-\alpha)}}{\Gamma(\alpha(n+q+1)-(1-\beta)(m-\alpha)+1)} H(t-(n+1)\tau) g(\phi_0) + \cdots + \\
&+ \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} \binom{n+q}{q} (g(A))^{q+1}(g(B))^{n} \frac{(t-(n+1)\tau)^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+m-2}}{\Gamma(\alpha(n+q+1)-(1-\beta)(m-\alpha)+m-1)} H(t-(n+1)\tau) g(\phi_0^{(m-2)}) \\
&+ \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} \binom{n+q}{q} (g(A))^{q}(g(B))^{n+1} \frac{(t-(n+1)\tau)^{\alpha(n+q+1)-(1-\beta)(m-\alpha)}}{\Gamma(\alpha(n+q+1)-(1-\beta)(m-\alpha)+1)} H(t-(n+1)\tau) g(\phi_0^{(m-2)}) \\
&+ \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} \binom{n+q}{q} (g(A))^{q}(g(B))^{n} \frac{(t-n\tau)^{\alpha(n+q+1)-\beta(\alpha-m)-1}}{\Gamma(\alpha(n+q+1)-\beta(\alpha-m))} H(t-n\tau) g(\phi_0^{(m-1)}) \\
&+ g(B)\int_0^t \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} \binom{n+q}{q} (g(A))^{q}(g(B))^{n} \frac{(t-n\tau-s)^{\alpha(n+q+1)-1}}{\Gamma(\alpha(n+q+1))} H(t-n\tau-s) g\big(\tilde{\phi}(s-\tau)\big) ds\Big)
\end{aligned}
$$

$$
\begin{aligned}
y(t) = g^{-1}\Big(&\sum_{k=0}^{m-2} \bigg(\frac{t^{(1-\beta)(m-\alpha)-k}}{\Gamma((1-\beta)(m-\alpha)-k+1)} + (g(A)+g(B)) \\
&\times \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} \binom{n+q}{q} (g(A))^{q}(g(B))^{n} \frac{(t-(n+1)\tau)^{\alpha(n+q+1)-(1-\beta)(m-\alpha)+k}}{\Gamma(\alpha(n+q+1)-(1-\beta)(m-\alpha)+k+1)} \bigg) g(\phi_0^{(k)}) \\
&+ \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} \binom{n+q}{q} (g(A))^{q}(g(B))^{n} \frac{(t-n\tau)^{\alpha(n+q+1)-\beta(\alpha-m)-1}}{\Gamma(\alpha(n+q+1)-\beta(\alpha-m))} H(t-n\tau) g(\phi_0^{(m-1)}) \\
&+ g(B)\int_{-\tau}^{t-\tau} \sum_{n=0}^{\infty}\sum_{q=0}^{\infty} \binom{n+q}{q} (g(A))^{q}(g(B))^{n} \frac{(t-(n+1)\tau-s)^{\alpha(n+q+1)-1}}{\Gamma(\alpha(n+q+1))} H(t-(n+1)\tau-s) g\big(\tilde{\phi}(s)\big) ds\Big) \\
=& g^{-1}\Big(\sum_{k=0}^{m-2} \bigg(\frac{t^{(1-\beta)(m-\alpha)-k}}{\Gamma((1-\beta)(m-\alpha)-k+1)} + (g(A)+g(B)) E^{\tau}_{\alpha,\alpha,\alpha+(\beta-1)(m-\alpha)+k+1}(g(A),g(B);t-\tau)\phi_0^{(k)} \bigg) \\
&+ E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t)\phi_0^{(m-1)} + g(B)\int_{-\tau}^{min(t-\tau,0)} E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-\tau-s) g\big(\tilde{\phi}(s)\big) ds\Big) \\
=& \bigoplus_{k=0}^{m-2} \bigg(g^{-1}\bigg(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+j+1)} \bigg) \oplus (A\oplus B)\odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau)) \bigg) \odot \phi_0^{(k)} \\
&\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t))\odot \phi_0^{m-1} \oplus B \odot \int^{\oplus}_{[-\tau,\min(t-\tau,0)]} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot \phi(s) \odot ds
\end{aligned}
$$

We get

$$
y(t) = \bigoplus_{k=0}^{m-2} \bigg(g^{-1}\bigg(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)} \bigg) \oplus (A\oplus B)\odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau)) \bigg) \odot \phi_0^{(k)} \tag{3.7}
$$

$$\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t))\odot\phi_0^{(m-1)}\oplus B\odot\int^{\oplus}_{[-\tau,\min(t-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds. \tag{3.8}$$

If we take $t\geq\tau$ then,

$$\int^{\oplus}_{[-\tau,t-\tau]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\tilde{\phi}(s)\odot ds=\int^{\oplus}_{[-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds \tag{3.9}$$

If we take $t<\tau$ then,

$$\int^{\oplus}_{[-\tau,t-\tau]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\tilde{\phi}(s)\odot ds=\int^{\oplus}_{[-\tau,t-\tau]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds \tag{3.10}$$

By using (3.8) and (3.9) we will get following result.

$$\int^{\oplus}_{[-\tau,t-\tau]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\tilde{\phi}(s)\odot ds=\int^{\oplus}_{[-\tau,\min(t-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds \tag{3.11}$$

□

4. INTEGRAL REPRESENTATION OF SOLLUTION TO LINEAR INHOMOGENEOUS PSEUDO-HILFER-TYPE FRACTIONAL TIME DELAY DIFFERENTIAL EQUATIONS

In this part, by imposing the classical manners to solve (1.5), we will obtain the explicit formula for the solutions of linear inhomogeneous fractional pseudo-Hilfer-type differential equations with invariable coefficients and time delay.

Let us examine the following two pseudo-Hilfer-type FDDEs with constant coefficients:

$$\begin{cases}H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)=A\odot y(t)\oplus B\odot y(t-\tau)\oplus f(t), t\in(0;T],\tau>0,\\ I^{1-\gamma}_{\oplus,\odot,0+}y(t)=0, t\in[-\tau,0].\end{cases} \tag{4.1}$$

and

$$\begin{cases}H^{\alpha,\beta}_{\oplus,\odot,0+}y(t)=A\odot y(t)\oplus B\odot y(t-\tau), t\in(0;T],\tau>0,\\ I^{1-\gamma}_{\oplus,\odot,0+}y(t)=\phi(t), t\in[-\tau,0].\end{cases} \tag{4.2}$$

where $m-1<\alpha<m$, $0\leq\beta\leq1$, $\gamma=(\beta-1)(m-\alpha)+k+1, k=0,...,m-1$.

The following lemma plays an important role in the proof of the subsequent theorem, which can be obtained from classical ways about the solution of the system (1.5).

Lemma 4.1. *If y_1 and y_2 are the solutions systems* (4.1) *and* (4.2)*,respectively, then $y(t)=y_1\oplus y_2$ is the general solution of system* (1.5).

Mention that the solution y_2 of (4.2) is investigated in paragraph 3. In other words, to reach our goal, we need to find y_1 which is a particular solution of (1.5).

Lemma 4.2. *Assume $m-1<\alpha<m$, $0<\beta\leq1$ for $m\geq2$. Then, we have the following relation:*

$$\int_{\eta+l\tau}^{t}(t-s)^{(1-\beta)(m-\alpha)-1}(s-l\tau-\eta)^{l\alpha+p\alpha+\alpha-1}ds=(t-l\tau-\eta)^{m-\beta(m-\alpha)+l\alpha+p\alpha-2}B\Big((1-\beta)(m-\alpha),(l+1)\alpha+p\alpha\Big)$$

Proof. To prove the lemma, we use the definition of Beta function and substitution of $u=\frac{t-s}{t-l\tau-\eta}$. Consequently, we obtain

$$\begin{aligned}&\int_{\eta+l\tau}^{t}(t-s)^{(1-\beta)(m-\alpha)-1}(s-l\tau-\eta)^{l\alpha+p\alpha+\alpha-1}ds\\ =&(t-l\tau-\eta)^{m-\beta(m-\alpha)+l\alpha+p\alpha-2}\int_0^1u^{(1-\beta)(m-\alpha)-1}(1-u)^{l\alpha+\alpha-1}du\\ =&(t-l\tau-\eta)^{m-\beta(m-\alpha)+l\alpha+p\alpha-2}B\Big((1-\beta)(m-\alpha),(l+1)\alpha+p\alpha\Big)\end{aligned}$$

□

We denote the following theorem for the particular solution of equation (1.5).

Theorem 4.1. *A solution $\tilde{y} \in C^m([0,T],R)$ of (1.5) holding zero initial conditions $\tilde{y}(t) = 0$, $t \in [-\tau, 0)$, $\tilde{y}^{(k)}(0) = 0$, $0 \le k \le m-1$ has the following form:*

$$\tilde{y}(t) = \int_{[0,t]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot f(s) \odot ds, \quad t > 0 \tag{4.3}$$

Proof. Using the method of variation of constants, any solution $\tilde{y}$ of the inhomogeneous system must be provided in the following shape:

$$\tilde{y}(t) = \int_{[0,t]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot h(s) \odot ds, \quad t > 0 \tag{4.4}$$

where $h(s)$, $0 \le s \le t$ is a sought vector function and $\tilde{y}(0) = 0$.

$$\tilde{y}(t) = \int_{[0,t]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot h(s) \odot ds = g^{-1}\bigg(\int_0^t E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)g(h(s))ds\bigg)$$

$$H^{\alpha,\beta}_{\oplus,\odot,0+}\tilde{y}(t) = g^{-1}({}^{H}D^{\alpha,\beta}_{0+}g(\tilde{y}(t))) = g^{-1}({}^{H}D^{\alpha,\beta}_{0+}(\int_0^t E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)g(h(s))ds))$$

$${}^{H}D^{\alpha,\beta}_{0+}g(\tilde{y}(t)) = {}^{H}D^{\alpha,\beta}_{0+}(\int_0^t E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)g(h(s))ds)$$

$$= I^{\beta(m-\alpha)}\frac{d^m}{dt^m}I^{(1-\beta)(m-\alpha)}\bigg(\int_0^t E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)g(h(s))ds\bigg)$$

$$= I^{\beta(m-\alpha)}\frac{d^m}{dt^m}\bigg(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\int_0^t (t-s)^{(1-\beta)(m-\alpha)-1}\int_0^s E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);s-\eta)g(h(\eta))d\eta ds\bigg)$$

$$= I^{\beta(m-\alpha)}\bigg(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\frac{d^m}{dt^m}\int_0^t\int_0^s (t-s)^{(1-\beta)(m-\alpha)-1}E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);s-\eta)g(h(\eta))d\eta ds\bigg)$$

$$= I^{\beta(m-\alpha)}\bigg(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\frac{d^m}{dt^m}\int_0^t\int_{\eta+l\tau}^t (t-s)^{(1-\beta)(m-\alpha)-1}E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);s-\eta)g(h(\eta))d\eta ds\bigg)$$

$$= I^{\beta(m-\alpha)}\bigg(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\frac{d^m}{dt^m}\int_0^t g(h(\eta))\bigg(\int_{\eta+l\tau}^t (t-s)^{(1-\beta)(m-\alpha)-1}E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);s-\eta)ds\bigg)d\eta\bigg)$$

$$= I^{\beta(m-\alpha)}\bigg(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\frac{d^m}{dt^m}\int_0^t g(h(\eta))\bigg(\int_{\eta+n\tau}^t (t-s)^{(1-\beta)(m-\alpha)-1}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q(g(B))^n$$

$$\frac{(s-n\tau-\eta)H(s-n\tau-\eta)}{\Gamma(q\alpha+n\alpha+\alpha)}ds\bigg)d\eta\bigg) = I^{\beta(m-\alpha)}\bigg(\frac{1}{\Gamma((1-\beta)(m-\alpha))}\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q(g(B))^n$$

$$\frac{d^m}{dt^m}\int_0^t \frac{(t-n\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+q\alpha-2}H(t-n\tau-\eta)}{\Gamma((n+1)\alpha+q\alpha)}g(h(\eta))d\eta B\Big((1-\beta)(m-\alpha),(n+1)\alpha+q\alpha\Big)\bigg)$$

$$= I^{\beta(m-\alpha)}\bigg(\sum_{n=0}^{\infty}\sum_{q=0}^{\infty}\binom{n+q}{q}(g(A))^q(g(B))^n\frac{d^m}{dt^m}\int_0^t \frac{(t-n\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+q\alpha-2}H(t-n\tau-\eta)}{\Gamma(m-\beta(m-\alpha)+n\alpha+q\alpha-1)}g(h(\eta))d\eta\bigg)$$

On the other hand, $I^{\beta(m-\alpha)}\frac{\mathrm{d}^m}{\mathrm{d}x^m}(f(t)) = {}^{C}D^{\beta(\alpha+m)}_{0+}f(t)$ and according to formula between Riemann-Luovile and Caputo fractional derivative, we have

$$I^{\beta(m-\alpha)}\frac{d^m}{dt^m}f(t) = {}^{C}D^{\beta(\alpha+m)}_{0+}f(t) = {}^{RL}D^{\beta(\alpha+m)}_{0+}f(t) - \sum_{k=0}^{m-1}\frac{t^{k-\beta(\alpha+m)}}{\Gamma(k-\beta(\alpha+m))}f^{(k)}(0), \quad t > 0$$

With the help of following binomial identity.

$$\binom{n+q}{q} = \binom{n+q-1}{q} + \binom{n+q-1}{q-1}, \quad n,q \geq 1,$$

and imposing Leibniz rule for higher-order derivatives (Ismail T.Huseynov et al ., 2021)(see Theorem 3.2), we achieve

$$
\begin{aligned}
&{}^{H}D^{\alpha,\beta}_{0+} g(\tilde{y}(t)) = I^{\beta(m-\alpha)} \frac{d^m}{dt^m} \Bigg(\sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \binom{n+q}{q} (g(A))^q (g(B))^n \\
&\times \int_0^t \frac{(t-n\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+q\alpha-2} H(t-n\tau-\eta)}{\Gamma(m-\beta(m-\alpha)+n\alpha+q\alpha-1)} g(h(\eta)) d\eta \Bigg) \\
&= \ {}^{C}D^{\beta(\alpha-m)+m} \Bigg(\sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \binom{n+q}{q} (g(A))^q (g(B))^n \int_0^t \frac{(t-n\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+p\alpha-2} H(t-n\tau-\eta)}{\Gamma(m-\beta(m-\alpha)+n\alpha+p\alpha-1)} g(h(\eta)) d\eta \Bigg) \\
&= \frac{d^m}{dt^m} \int_0^t \frac{(t-\eta)^{m-\beta(m-\alpha)-2} H(t-\eta)}{\Gamma(m-\beta(m-\alpha)-1)} g(h(\eta)) d\eta \\
&+ \sum_{n=1}^{\infty} \sum_{q=0}^{\infty} \binom{n+q-1}{q} (g(A))^q (g(B))^n \frac{d^m}{dt^m} \int_0^t \frac{(t-l\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+q\alpha-2} H(t-n\tau-\eta)}{\Gamma(m-\beta(m-\alpha)+n\alpha+q\alpha-1)} g(h(\eta)) d\eta \\
&+ \sum_{n=0}^{\infty} \sum_{q=1}^{\infty} \binom{n+q-1}{q-1} (g(A))^q (g(B))^n \frac{d^m}{dt^m} \int_0^t \frac{(t-n\tau-\eta)^{m-\beta(m-\alpha)+n\alpha+q\alpha-2} H(t-n\tau-\eta)}{\Gamma(m-\beta(m-\alpha)+n\alpha+q\alpha-1)} g(h(\eta)) d\eta \\
&= g(h(t)) + \sum_{n=1}^{\infty} \sum_{q=0}^{\infty} \binom{n+q-1}{q} (g(A))^q (g(B))^n \int_0^t \frac{(t-n\tau-\eta)^{\alpha\beta+n\alpha+q\alpha-2} H(t-n\tau-\eta)}{\Gamma(\alpha\beta+n\alpha+q\alpha-1)} g(h(\eta)) d\eta
\end{aligned}
$$

$$
\begin{aligned}
&+ \sum_{n=0}^{\infty} \sum_{q=1}^{\infty} \binom{n+q-1}{q-1} (g(A))^q (g(B))^n \int_0^t \frac{(t-n\tau-\eta)^{\alpha\beta+n\alpha+q\alpha-2} H(t-n\tau-\eta)}{\Gamma(\alpha\beta+n\alpha+q\alpha-1)} g(h(\eta)) d\eta \\
&= g(h(t)) + \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \binom{n+q}{q} (g(A))^q (g(B))^{n+1} \int_0^t \frac{(t-n\tau-\eta)^{\alpha\beta+n\alpha+q\alpha-2} H(t-(n+1)\tau-\eta)}{\Gamma(\alpha\beta+(n+1)\alpha+q\alpha-1)} g(h(\eta)) d\eta \\
&+ \sum_{n=0}^{\infty} \sum_{q=0}^{\infty} \binom{n+q}{q-1} (g(A))^{q+1} (g(B))^n \int_0^t \frac{(t-n\tau-\eta)^{\alpha\beta+n\alpha+(q+1)\alpha-2} H(t-n\tau-\eta)}{\Gamma(\alpha\beta+n\alpha+(q+1)\alpha-1)} g(h(\eta)) d\eta \\
&= g(h(t)) + g(A) \int_0^t E^{\tau}_{\alpha,\alpha,\alpha\beta+\alpha-1}(g(A), g(B); t-\eta) g(h(\eta)) d\eta + g(B) \int_0^t E^{\tau}_{\alpha,\alpha,\alpha\beta+\alpha-1}(g(A), g(B); t-\tau-\eta) g(h(\eta)) d\eta
\end{aligned}
$$

$$
\begin{aligned}
&H^{\alpha,\beta}_{\oplus,\odot,0+} \tilde{y}(t) = g^{-1}({}^{H}D^{\alpha,\beta}_{0+} g(\tilde{y}(t))) = g^{-1} \Bigg(g(h(t)) + g(A) \int_0^t E^{\tau}_{\alpha,\alpha,\alpha\beta+\alpha-1}(g(A), g(B); t-\eta) g(h(\eta)) d\eta \\
&+ g(B) \int_0^t E^{\tau}_{\alpha,\alpha,\alpha\beta+\alpha-1}(g(A), g(B); t-\tau-\eta) g(h(\eta)) d\eta \Bigg) = A \oplus \tilde{y}(t) \oplus B\tilde{y}(t-\tau) \oplus h(t) = A \oplus \tilde{y}(t) \oplus B\tilde{y}(t-\tau) \oplus f(t)
\end{aligned}
$$

Therefore, we obtain that $h(t) = f(t)$ for $t \in [0, T]$. □

Eventually, we obtain the next theorem for the unique analytical solution of the Cauchy problem (1.5).

Theorem 4.2. *A unique analytical solution $y \in C^m([-\tau, T], R)$ of the initial issue* (1.1) *has the following form:*

$$y(t) = \bigoplus_{k=0}^{m-2} \left(g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)} \right) \oplus (A \oplus B) \odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A, B; g^{-1}(t-\tau)) \right) \odot \phi_0^{(k)}$$

$$\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A, B; g^{-1}(t)) \odot \phi_0^{(m-1)} \oplus B \odot \int^{\oplus}_{[-\tau,\min(t-\tau,0)]} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A, B; g^{-1}(t-\tau-s)) \odot \phi(s) \odot ds$$

$$\oplus \int^{\oplus}_{[0,t]} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A, B; g^{-1}(t-s)) \odot f(s) \odot ds, \quad t > 0.$$

Proof. The proof of the theorem is immediate. Therefore, we pass above it. □

5. EXISTENCE AND UNIQUNESS PROBLEM FOR NONLINIEAR TIME RETARDED PSEUDO-HILFER-TYPE FRACTIONAL DIFFERENTIONAL EQUATIONS

In the following section, we will look the initial issue for a nonlinear pseudo-Hilfer-type fractional differential equation with constant delay.

$$\begin{cases} H^{\alpha,\beta}_{\oplus,\odot,0+} y(t) = A \odot y(t) \oplus B \odot y(t-\tau) \oplus f(t, y(t)), t \in (0; T], \tau > 0, \\ I^{1-\gamma}_{\oplus,\odot,0+} y(t) = \phi(t), t \in [-\tau, 0]. \end{cases} \tag{5.1}$$

Where $m-1 < \alpha \le m$, $0 < \beta \le 1$, $y(\cdot) \in R$, $f(\cdot, y(\cdot)) : [0, \infty) \times R \to R$ is a nonlinear perturbation and also a continuous function. And we will also suppose that $(t \to f(t, 0)) \in C([0, \infty), R)$. Then, according to Theorem 4.2, we obtain the solution of the nonlinear Hilfer-type FDE (5.1) as follows:

$$y(t) = \bigoplus_{k=0}^{m-2} \left(g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)} \right) \oplus (A \oplus B) \odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A, B; g^{-1}(t-\tau)) \right) \odot \phi_0^{(k)}$$

$$\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A, B; g^{-1}(t)) \odot \phi_0^{(m-1)} \oplus B \odot \int^{\oplus}_{[-\tau,\min(t-\tau,0)]} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A, B; g^{-1}(t-\tau-s)) \odot \phi(s) \odot ds$$

$$\oplus \int^{\oplus}_{[0,t]} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A, B; g^{-1}(t-s)) \odot f(s, y(s)) \odot ds, \quad t > 0.$$

First of all, we denote following lemmas and notes: For $x(\cdot) : [a, b] \to R_+$, we will define the norm of the function as a follow:

$$||x(t)||_g = g^{-1}(|g(x(t)))|)$$

Lemma 5.1. *([3], page 12, lemma 5.1) The following estimation satisfies true:*

$$|E^{\tau}_{\alpha,\alpha-\beta,\alpha+k}(A, B; t)| \le t^{\alpha+k-1} \exp(|A|t^{\alpha} + |B|t^{\alpha-\beta}) \tag{5.2}$$

for $k = 0, 1, \ldots, m-1$

Corollary 5.1. *([3], page 12, corollary 5.1)*

For $m \ge 2$, the following conclusion satisfies:

$$|E^{\tau}_{\alpha,\alpha-\beta,m}(A, B; t)| \le t^{m-1} \exp(|A|t^{\alpha} + |B|t^{\alpha-\beta}). \tag{5.3}$$

Analogously, we will get the following results for pseudo-Mittag-Leffler functions.

Lemma 5.2. *Assume a generator $g : [a, b] \to [0, \infty]$ and $A, B \in R$. For following delayed pseudo-Mittag-Leffler function estimation holds true:*

$$|E^{\tau,\oplus}_{\alpha,\alpha-\beta,\alpha+k}(A, B; g^{-1}(t))|_g \le g^{-1}(t^{\alpha+k-1}) \odot g^{-1}(\exp(|A|t^{\alpha} + |B|t^{\alpha-\beta})) \tag{5.4}$$

for $k = 0, 1, \ldots, m-1$

Proof.

$$\begin{aligned}
&|E^{\tau,\oplus}_{\alpha,\alpha-\beta,\alpha+k}(A,B;g^{-1}(t))|_g = g^{-1}\bigg(g\Big(|E^{\tau,\oplus}_{\alpha,\alpha-\beta,\alpha+k}(A,B;g^{-1}(t))|\Big)\bigg)\\
=&g^{-1}\Big(|E^{\tau}_{\alpha,\alpha-\beta,\alpha+k}(g(A),g(B);t)|\Big) \leq g^{-1}\Big(t^{\alpha+k-1}\exp(|g(A)|t^{\alpha}+|g(B)|t^{\alpha-\beta})\Big)\\
\leq& g^{-1}(t^{\alpha+k-1})\odot g^{-1}(\exp(|A|t^{\alpha}+|B|t^{\alpha-\beta}))
\end{aligned}$$

□

Then, we can denote analogously following corollary.

Corollary 5.2. *Let a generator $g:[a,b]\to[0,\infty]$ and $A,B\in R$. For $m\geq 2$, the following inequality holds:*

$$|E^{\tau,\oplus}_{\alpha,\alpha-\beta,m}(A,B;g^{-1}(t))|_g \leq g^{-1}\Big(t^{m-1}\Big)\odot g^{-1}\Big(\exp(|A|t^{\alpha}+|B|t^{\alpha-\beta})\Big). \tag{5.5}$$

Theorem 5.1. *Assume that the following hypothesises are true:*

$$\begin{aligned}
&(H_1) f:[0,T]\times R\to R \quad be \quad a \quad continious \quad function:\\
&(H_2) there \quad exisit \quad C>0 \quad such \quad that \quad f \quad holds \quad the \quad Lipschitz \quad condition:
\end{aligned}$$

$$|f(t,y)\ominus f(t,\sigma)|_g \leq C\odot|y\ominus\sigma|_g, \quad \forall(t,y),(t,\sigma)\in[0,T]\times R; \tag{5.6}$$

Then, the problem (5.1) *has a unique global continuous solution on* $[0,T]$.

Proof. Assume that a ball be appointed as $B_R := y\in C([0,T],R) : ||y||_\omega \leq R, \omega>0$ where $R>0$ with

$$R\geq\Big[W\odot|\phi_0^{(k)}|_g\oplus g^{-1}(T^{\alpha-1})\odot S\odot|\phi_0^{(m-1)}|_g\oplus S\odot g^{-1}(\Gamma(\alpha))\odot|B|\odot||\phi||_\omega\oplus D\Big]\oslash(g^{-1}(\omega^{\alpha}\ominus S\odot g^{-1}(\Gamma(\alpha))\odot C)) \tag{5.7}$$

where

$$\begin{aligned}
W&=\bigoplus_{k=0}^{m-2}\bigg(g^{-1}\Big(\frac{T^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\Big)\oplus(|A|\oplus|B|)\odot g^{-1}\Big(T^{(\beta-1)(m-\alpha)+\alpha+k}\Big)\odot S\bigg)\\
D&=\max_{t\in[0,T]}\{|f(t,0)_g\oslash\exp(\omega t)\};S=\exp\Big((|g(A)|+|g(B)|)T^{\alpha}\Big)
\end{aligned}$$

Now, we set an integral operator F on B_R as below:

$$F:C([0,T],R)\supset B_R\ni y\to F(y):=(t\to(Fy)(t))\in C([0,T],R),$$

through the following formula

$$\begin{aligned}
(Fy)(t)&=\bigoplus_{k=0}^{m-2}\bigg(g^{-1}\bigg(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\bigg)\oplus(A\oplus B)\odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))\bigg)\odot\phi_0^{(k)}\\
&\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t))\odot\phi_0^{(m-1)}\oplus B\odot\int^{\oplus}_{[-\tau,\min(t-\tau,0)]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))\odot\phi(s)\odot ds\\
&\oplus\int^{\oplus}_{[0,t]}E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))\odot f(s,y(s))\odot ds, \quad t\in[0,T].
\end{aligned}$$

We can establish that the operator F is well-defined based on condition (H_1), and thus, the existence of a solution to the initial issue (5.1) is equivalent to the existence of a fixed point for the integral operator F

on the set B_R. To prove the uniqueness of the fixed point, we will apply the contraction mapping principle. However, instead of using the maximum norm $C([0,T],R)$, which only yields a local solution within the subinterval $[0,T]$, we will consider equipping $C([0,T],R)$ with the weighted maximum norm $||\cdot||_\omega$ with respect to the exponential function, defined as:

$$||y|_\omega := \max_{t\in[0,T]} \{|y(t)|_g \oslash \exp(\omega t)\}, \forall y \in C([0,T],R).$$

Since two norms $||\cdot||_\infty$ and $||\cdot||_\omega$ are equivalent, $C([0,T],R,||\cdot||_\omega)$ is also a Banach space. The proof is separated into two parts.

Step 1: We prove that $F(B_R)\subset B_R$. In this part, we look following estimation.

$$|(Fy)(t)|_g \oslash \exp(\omega t) = g^{-1}\left(\frac{g|(Py)(t)|_g}{g(\exp(\omega t))}\right) = g^{-1}\left(\frac{|(Pg(y))(t)|}{g(\exp(\omega t))}\right) \tag{5.8}$$

First of all, we denote the following notes for use in process of proof.

$$(Fg(y))(t) = \sum_{k=0}^{m-2}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)} + (g(A)+g(B))E^\tau_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(g(A),g(B);t-\tau)\right)g(\phi_0^{(k)})$$
$$+E^\tau_{\alpha,\alpha,\alpha}(g(A),g(B);t)g(\phi_0^{(m-1)}) + g(B)\int_{-\tau}^{min(t-\tau,0)} E^\tau_{\alpha,\alpha,\alpha}(g(A),g(B);t-\tau-s)g(\phi(s))ds$$
$$+\int_0^t E^\tau_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)g(f(s,y(s)))ds, t\in[0,T]$$

Then, we will get.

$$\frac{|F(g(y))(t)|}{g(\exp(\omega t))} \le \frac{1}{g(\exp(\omega t))}\sum_{k=0}^{m-2}\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}|g(\phi_0^{(k)})| + \frac{|g(A)|+|g(B)|}{g(\exp(\omega t))}$$

$$+\sum_{k=0}^{m-2}|E^\tau_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(g(A),g(B);t-\tau)||g(\phi_0^{(k)})| + \frac{1}{g(\exp(\omega t))}|E^\tau_{\alpha,\alpha,\alpha}(g(A),g(B);t)||g(\phi_0^{(m-1)})|$$
$$+\frac{|g(B)|}{g(\exp(\omega t))}\int_{-\tau}^0 |E^\tau_{\alpha,\alpha,\alpha}(g(A),g(B);t-\tau-s)||g(\phi(s)|$$
$$+\frac{1}{g(\exp(\omega t))}\int_0^t |E^\tau_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)||g(f(s,y(s)))-g(f(s.0))|+|g(f(s,0))|ds$$
$$\le \sum_{k=0}^{m-2}\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}|g(\phi_0^{(k)})| + (|g(A)|+|g(B)|)\sum_{k=0}^{m-2}|E^\tau_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(g(A),g(B);t-\tau)||g(\phi_0^{(k)})|$$
$$+\frac{1}{g(\exp(\omega t))}|E^\tau_{\alpha,\alpha,\alpha}(g(A),g(B);t)||g(\phi_0^{(m-1)})| + \frac{|g(B)|}{g(\exp(\omega t))}\int_{-\tau}^0 |E^\tau_{\alpha,\alpha,\alpha}(g(A),g(B);t-\tau-s)||g(\phi(s)|ds$$
$$+\frac{1}{g(\exp(\omega t))}\int_0^t |E^\tau_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)||g(f(s,y(s)))-g(f(s.0))|+|g(f(s,0))|ds$$
$$\le \sum_{k=0}^{m-2}\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}|g(\phi_0^{(k)})| + (|g(A)|+|g(B)|)\sum_{k=0}^{m-2}|E^\tau_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(g(A),g(B);t-\tau)||g(\phi_0^{(k)})|$$
$$+\frac{1}{g(\exp(\omega t))}|E^\tau_{\alpha,\alpha,\alpha}(g(A),g(B);t)||g(\phi_0^{(m-1)})| + \frac{|g(B)|}{g(\exp(\omega t))}\int_{-\tau}^0 |E^\tau_{\alpha,\alpha,\alpha}(g(A),g(B);t-\tau-s)|\frac{g(\exp(\omega s))}{g(\exp(\omega s))}|g(\phi(s)|ds$$

$$+\frac{1}{g(\exp(\omega t))}\int_0^t |E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)||g(f(s,y(s)))-g(f(s.0))|\frac{g(\exp(\omega s))}{g(\exp(\omega s))}ds$$
$$+\int_0^t |E^{\tau}_{\alpha,\alpha,\alpha}(g(A),g(B);t-s)||g(f(s,0))|\frac{g(\exp(\omega s))}{g(\exp(\omega s))}ds$$

By using from this formula and (5.8) we obtain

$$|(Fy)(t)|_g \oslash \exp(\omega t) \leq \bigoplus_{k=0}^{m-2} g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\odot|\phi_0^{(k)}|_g$$
$$\oplus(|A|\oplus|B|)\odot\bigoplus_{k=0}^{m-2}|E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))|_g\odot|\phi_0^{(k)}|_g$$
$$\oplus|E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t))|_g\odot|\phi_0^{(m-1)}|_g\oslash\exp(\omega t)$$
$$\oplus|B|\oslash\exp(\omega t)\odot\int_{[-\tau,0]}^{\oplus}|E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s))|_g\odot\exp(\omega s)\odot|\phi(s)|_g\oslash\exp(\omega s)\odot ds$$
$$\oplus\int_{[0,t]}^{\oplus}|E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))|_g\odot|f(s,y(s))\ominus f(s.0)|_g\odot\exp(\omega s)\oslash\exp(\omega s)\odot ds\oslash\exp(\omega t)$$

$$\oplus\int_{[0,t]}^{\oplus}|E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))|_g\odot|f(s,0)|_g\oslash\exp(\omega s)\odot\exp(\omega s)\odot ds$$

Now take $\forall t\in[0,T]$ and $\forall y\in B_R$. By using (H_2) by means of Lemma 5.2, we receive:

$$|(Fy)(t)|_g \oslash \exp(\omega t) \leq \bigoplus_{k=0}^{m-2} g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\odot|\phi_0^{(k)}|_g$$
$$\oplus(|A|\oplus|B|)\odot\bigoplus_{k=0}^{m-2} g^{-1}\Big((t-\tau)^{(\beta-1)(m-\alpha)+\alpha+k}\Big)\odot g^{-1}((\exp(|A|+|B|)(t-\tau)^{\alpha})\odot|\phi_0^{(k)}|_g$$
$$\oplus g^{-1}(t^{\alpha-1})\odot g^{-1}((\exp(|A|+|B|)t^{\alpha})\odot|\phi_0^{(m-1)}|_g\oslash\exp(\omega t)$$
$$\oplus|B|\oslash\exp(\omega t)\odot\int_{[-\tau,0]}^{\oplus}g^{-1}\Big((t-\tau-s)^{\alpha-1}\Big)\odot g^{-1}((\exp(|A|+|B|)(t-\tau-s)^{\alpha})\odot\exp(\omega s)\odot|\phi(s)|_g\oslash\exp(\omega s)\odot ds$$
$$\oplus\int_{[0,t]}^{\oplus}g^{-1}\Big((t-s)^{\alpha-1}\Big)\odot g^{-1}((\exp(|A|+|B|)(t-s)^{\alpha})\odot C\odot|y(s)|_g\odot\exp(\omega s)\oslash\exp(\omega s)\odot ds\oslash\exp(\omega t)$$
$$\oplus\int_{[0,t]}^{\oplus}g^{-1}\Big((t-s)^{\alpha-1}\Big)\odot g^{-1}((\exp(|A|+|B|)(t-s)^{\alpha})\odot|f(s,0)|_g\oslash\exp(\omega s)\odot\exp(\omega s)\odot ds$$

Using the substitution $r - s = u$ and Lipschitz condition (H_2), we get

$$|(Fy)(t)|_g \oslash \exp(\omega t) \leq \bigoplus_{k=0}^{m-2} g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right) \odot |\phi_0^{(k)}|_g$$

$$\oplus(|A| \oplus |B|) \odot \bigoplus_{k=0}^{m-2} g^{-1}\left(t^{(\beta-1)(m-\alpha)+\alpha+k}\right) \odot g^{-1}((\exp(|A|+|B|)t^{\alpha}) \odot |\phi_0^{(k)}|_g$$

$$\oplus g^{-1}(t^{\alpha-1}) \odot g^{-1}((\exp(|A|+|B|)t^{\alpha}) \odot |\phi_0^{(m-1)}|_g \oslash \exp(\omega t)$$

$$\oplus|B| \oslash \exp(\omega t) \odot \int_{[-\tau,0]}^{\oplus} g^{-1}\Big((t-\tau-s)^{\alpha-1}\Big) \odot |\phi(s)|_g \oslash \exp(\omega s) \odot \exp(\omega s) \odot ds \odot g^{-1}((\exp(|A|+|B|)(t)^{\alpha})$$

$$\oplus C \oslash \exp(\omega t) \odot \int_{[0,t]}^{\oplus} g^{-1}\Big((t-s)^{\alpha-1}\Big) \odot |y(s)|_g \odot \exp(\omega s) \oslash \exp(\omega s) \odot ds \odot g^{-1}((\exp(|A|+|B|)(t)^{\alpha})$$

$$\oplus \int_{[0,t]}^{\oplus} g^{-1}\Big((t-s)^{\alpha-1}\Big) \odot |f(s,0)|_g \oslash \exp(\omega s) \odot \exp(\omega s) \odot ds \odot g^{-1}((\exp(|A|+|B|)(t)^{\alpha}) \oslash \exp(\omega t)$$

$$\leq \bigoplus_{k=0}^{m-2} g^{-1}\left(\frac{T^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right) \odot |\phi_0^{(k)}|_g \oplus (|A| \oplus |B|) \odot \bigoplus_{k=0}^{m-2} g^{-1}\left(T^{(\beta-1)(m-\alpha)+\alpha+k}\right)$$

$$\odot g^{-1}((\exp(|A|+|B|)T^{\alpha}) \odot |\phi_0^{(k)}|_g \oplus g^{-1}(T^{\alpha-1}) \odot g^{-1}((\exp(|A|+|B|)T^{\alpha}) \odot |\phi_0^{(m-1)}|_g$$

$$\oplus|B|\oslash\exp(\omega t)\odot\int_{[0,\tau]}^{\oplus}g^{-1}\Big((t-s)^{\alpha-1}\Big)\odot\exp(\omega(s-\tau))\odot ds\odot\max_{t\in[0,T]}\{|\phi(t)|_g\oslash\exp(\omega t)\}\odot g^{-1}((\exp(|A|+|B|)(T)^{\alpha})$$

$$\oplus C\oslash\exp(\omega t)\odot\int_{[0,t]}^{\oplus}g^{-1}\Big((t-s)^{\alpha-1}\Big)\odot\exp(\omega s)\odot ds\odot\max_{t\in[0,T]}\{|y(t)|_g\oslash\exp(\omega t)\}\odot g^{-1}((\exp(|A|+|B|)(T)^{\alpha})$$

$$\oplus\int_{[0,t]}^{\oplus}g^{-1}\Big((t-s)^{\alpha-1}\Big)\odot\exp(\omega s)\odot ds\odot\max_{t\in[0,T]}\{|f(s,0)|_g\oslash\exp(\omega t)\}\odot g^{-1}((\exp(|A|+|B|)(T)^{\alpha})\oslash\exp(\omega t)$$

$$|(Fy)(t)|_g\oslash\exp(\omega t)\leq\bigoplus_{k=0}^{m-2}g^{-1}\left(\frac{T^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\odot|\phi_0^{(k)}|_g\oplus(|A|\oplus|B|)\odot\bigoplus_{k=0}^{m-2}g^{-1}\Big(T^{(\beta-1)(m-\alpha)+\alpha+k}\Big)$$

$$\odot S\odot|\phi_0^{(k)}|_g\oplus g^{-1}(T^{\alpha-1})\odot S\odot|\phi_0^{(m-1)}|_g\oplus|B|\odot S\oslash\exp(\omega t)\odot\int_{[0,t]}^{\oplus}g^{-1}\Big((t-s)^{\alpha-1}\Big)\odot\exp(\omega s)\odot ds\odot\|\phi\|_{\omega}$$

$$\oplus C\odot S\oslash\exp(\omega t)\odot\int_{[0,t]}^{\oplus}g^{-1}\Big((t-s)^{\alpha-1}\Big)\odot\exp(\omega s)\odot ds\odot\|y\|_{\omega}$$

$$\oplus D\odot S\oslash\exp(\omega t)\int_{[0,t]}^{\oplus}g^{-1}\Big((t-s)^{\alpha-1}\Big)\odot\exp(\omega s)\odot ds$$

$$=\bigoplus_{k=0}^{m-2}\left(g^{-1}\left(\frac{T^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right)\oplus(|A|\oplus|B|)\odot g^{-1}\Big(T^{(\beta-1)(m-\alpha)+\alpha+k}\Big)\odot S\right)\odot|\phi_0^{(k)}|_g$$

$$\oplus g^{-1}(T^{\alpha-1})\odot S\odot|\phi_0^{(m-1)}|_g\oplus|B|\odot S\oslash\exp(\omega t)\odot\int_{[0,t]}^{\oplus}g^{-1}\Big(u^{\alpha-1}\Big)\odot\exp(\omega t)\odot\exp(-\omega u)\odot du\odot||\phi||_{\omega}$$

$$\oplus C\odot S\oslash\exp(\omega t)\odot\int_{[0,t]}^{\oplus}g^{-1}\Big(u^{\alpha-1}\Big)\odot\exp(\omega t)\odot\exp(-\omega u)\odot du\odot\|y\|_{\omega}$$

$$\oplus D\odot S\oslash\exp(\omega t)\int_{[0,t]}^{\oplus}g^{-1}\Big(u^{\alpha-1}\Big)\odot\exp(\omega t)\odot\exp(-\omega u)\odot du$$

$$=W\odot|\phi_0^{(k)}|_g\oplus g^{-1}(T^{\alpha-1})\odot S\odot|\phi_0^{(m-1)}|_g\oplus|B|\odot S\odot\int_{[0,t]}^{\oplus}g^{-1}\Big(u^{\alpha-1}\Big)\odot\exp(-\omega u)\odot du\odot\|\phi\|_{\omega}$$

$$\oplus C\odot S\odot\int_{[0,t]}^{\oplus}g^{-1}\Big(u^{\alpha-1}\Big)\odot\exp(-\omega u)\odot du\odot\|y\|_{\omega}\oplus D\odot S\int_{[0,t]}^{\oplus}g^{-1}\Big(u^{\alpha-1}\Big)\odot\exp(-\omega u)\odot du$$

$$=W\odot|\phi_0^{(k)}|_g\oplus g^{-1}(T^{\alpha-1})\odot S\odot|\phi_0^{(m-1)}|_g\oplus|B|\odot S\oslash g^{-1}(\omega^{\alpha})\odot\int_{[0,\omega t]}^{\oplus}g^{-1}\Big(v^{\alpha-1}\Big)\odot\exp(-v)\odot dv\odot\|\phi\|_{\omega}$$

$$\oplus C \odot S \oslash g^{-1}(\omega^{\alpha}) \odot \int_{[0,\omega t]}^{\oplus} g^{-1}\Big(v^{\alpha-1}\Big) \odot \exp(-v) \odot dv \odot \|y\|_{\omega} \oplus D \odot S \oslash g^{-1}(\omega^{\alpha}) \int_{[0,\omega t]}^{\oplus} g^{-1}\Big(v^{\alpha-1}\Big) \odot \exp(-v) \odot dv$$

$$= W \odot |\phi_0^{(k)}|_g \oplus g^{-1}(T^{\alpha-1}) \odot S \odot |\phi_0^{(m-1)}|_g \oplus S \oslash g^{-1}(\omega^{\alpha}) \int_{[0,\omega t]}^{\oplus} g^{-1}\Big(v^{\alpha-1}\Big) \odot \exp(-v) \odot dv$$

$$\odot \Big(|B| \odot \|\phi\|_{\omega} \oplus C \odot \|y\|_{\omega} \oplus D\Big)$$

$$\leq W \odot |\phi_0^{(k)}|_g \oplus g^{-1}(T^{\alpha-1}) \odot S \odot |\phi_0^{(m-1)}|_g \oplus S \oslash g^{-1}(\omega^{\alpha}) \int_{[0,\infty]}^{\oplus} g^{-1}\Big(v^{\alpha-1}\Big) \odot \exp(-v) \odot dv$$

$$\odot \Big(|B| \odot \|\phi\|_{\omega} \oplus C \odot ||y||_{\omega} \oplus D\Big) = W \odot |\phi_0^{(k)}|_g \oplus g^{-1}(T^{\alpha-1}) \odot S \odot |\phi_0^{(m-1)}|_g$$

$$\oplus S \dot{g}^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \Big(|B| \odot \|\phi\|_{\omega} \oplus C \odot \|y\|_{\omega} \oplus D\Big)$$

$$\leq W \odot |\phi_0^{(k)}|_g \oplus g^{-1}(T^{\alpha-1}) \odot S \odot |\phi_0^{(m-1)}|_g \oplus S \dot{g}^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \Big(|B| \odot \|\phi\|_{\omega} \oplus C \odot R \oplus D\Big)$$

Taking the maximum over $[0, T]$ and using inequality (5.6), we obtain the following relation:

$$\|Fy\|_{\omega} \leq R$$

For this reason, $F : B_R \to B_R$. In other words, F is well-defined on B_R.

Step 2.In this step, we will represent that F is a contractive mapping. We should demonstrate that F is a contraction over B_R. To see this, let $\forall y, \sigma \in B_R$. Mention that

$$(Fy)(t) \ominus (F\sigma)(t) = \int_{[0,t]}^{\oplus} E_{\alpha,\alpha,\alpha}^{\tau,\oplus}(A, B; g^{-1}(t-s)) \odot (f(s, y(s)) \ominus f(s, \sigma(s)) \odot ds, \quad t > 0. \tag{5.9}$$

Thus, for any $t \in [0, T]$, from Lemma 5.2 and (H_2)-Lipschitz condition, it follows that

$$|(Fy)(t) \ominus (F\sigma)(t)|_g \oslash \exp(\omega t) = g^{-1}\Big(\frac{|(Fg(y))(t) - (Fg(\sigma))(t)|}{g(\exp(\omega t))}\Big)$$

$$\leq g^{-1}\Big(\frac{1}{g(\exp(\omega t))} \int_0^t |E_{\alpha,\alpha,\alpha}^{\tau}(g(A), g(B); t-s)||g(f(s, y(s))) - g(f(s, \sigma(s)))|ds\Big)$$

$$= g^{-1}\Big(\frac{1}{g(\exp(\omega t)}\Big) \odot \int_{[0,t]}^{\oplus} |E_{\alpha,\alpha,\alpha}^{\tau,\oplus}(A, B; g^{-1}(t-s))|_g \odot |f(s, y(s)) \ominus f(s, \sigma(s))|_g \odot ds$$

$$\leq (C \odot \exp((|A| + |B|)t^{\alpha})) \oslash \exp(\omega t) \odot \int_{[0,t]}^{\oplus} g^{-1}\Big((t-s)^{\alpha-1}\Big) \odot |y(s) \ominus \sigma(s)|_g \oslash \exp(\omega s) \odot \exp(\omega s) \odot ds$$

$$\leq (C \odot \exp((|A| + |B|)t^{\alpha})) \oslash \exp(\omega t) \odot \int_{[0,t]}^{\oplus} g^{-1}\Big((t-s)^{\alpha-1}\Big) \odot \exp(\omega s) \odot ds \odot \max_{t \in [0,T]} \{|y(t) - \sigma(t)|_g \oslash \exp(\omega t)\}$$

$$= (C \odot \exp((|A| + |B|)t^{\alpha})) \oslash \exp(\omega t) \odot \int_{[0,t]}^{\oplus} g^{-1}\Big((t-s)^{\alpha-1}\Big) \odot \exp(\omega s) \odot ds \odot \|y - \sigma\|_{\omega}$$

$$=(C \odot \exp((|A|+|B|)t^{\alpha})) \oslash \exp(\omega t) \odot \int_{[0,t]}^{\oplus} g^{-1}\Big(u^{\alpha-1}\Big) \odot \exp(\omega t) \odot \exp(-\omega u) \odot du \odot \|y-\sigma\|_{\omega}$$

$$=C \odot \exp((|A|+|B|)t^{\alpha}) \odot \int_{[0,t]}^{\oplus} g^{-1}\Big(u^{\alpha-1}\Big) \odot \exp(-\omega u) \odot du \odot \|y-\sigma\|_{\omega}$$

$$=(C \odot \exp((|A|+|B|)t^{\alpha})) \oslash g^{-1}(\omega^{\alpha}) \odot \int_{[0,\omega t]}^{\oplus} g^{-1}\Big(v^{\alpha-1}\Big) \odot \exp(-v) \odot dv \odot \|y-\sigma\|_{\omega}$$

$$\leq(C \odot \exp((|A|+|B|)t^{\alpha})) \oslash g^{-1}(\omega^{\alpha}) \odot \int_{[0,\infty]}^{\oplus} g^{-1}\Big(v^{\alpha-1}\Big) \odot \exp(-v) \odot dv \odot \|y-\sigma\|_{\omega}$$

$$=\exp((|A|+|B|)t^{\alpha}) \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \|y-\sigma\|_{\omega}$$

$$\leq \exp((|A|+|B|)T^{\alpha}) \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \|y-\sigma\|_{\omega} := S \odot C \odot g^{-1}(\Gamma(\alpha)) \odot g^{-1}(\omega^{\alpha}) \odot \|y-\sigma\|_{\omega}$$

Then, we get.

$$|(Fy)(t) \ominus (F\sigma)(t)|_g \oslash \exp(\omega t) \leq S \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \|y-\sigma\|_{\omega}$$

Taking maximum on $[0,T]$, we will get the following conclusion:

$$\|F(y) \ominus F(\sigma)\|_{\omega} \leq S \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \|y-\sigma\|_{\omega} \tag{5.10}$$

If we choose $\omega > (S \odot C \odot g^{-1}(\Gamma(\alpha)) \odot g^{-1}(\omega^{\alpha}))^{\frac{1}{\alpha}}$, then F is a contraction. Thus, by Banach's fixed point theorem, there exists a unique fixed point of F which is just the unique global continuous solution of (5.1). □

Remark 5.1. If the assumptions (H_1) and (H_2) are satisfied for all $t \in [0,\infty)$, then the claim of this theorem holds on the half-real line R, i.e. for any $(m-1)$-times continuously differentiable initial data $\phi : [-\tau, 0] \to R$, the non-linear pseudo-Hilfer equation type equation of fractional order with a constant delay (5.1) has a unique global continuous solution on $[0,\infty)$.

6. ULAM-HYERS STABILITY ANALYSIS ON PSEUDO-HILFER TYPE FRACTIONAL DIFFERENTIONAL EQUATION WITH A CONSTANT DELAY

In the following part, we debate the stability of the pseudo-Hilfer-type DDE (5.1) in the Ulam-Hyers sense on $[0,T]$.

Suppose that $\epsilon > 0$. Let us imagine the pseudo-Hilfer type fractional delay differential equation (5.1) and the Initial issue for the following inequality:

$$|H^{\alpha,\beta}_{\oplus,\odot,0+}\sigma(t) \ominus A \odot \sigma(t) \ominus B \odot \sigma(t-\tau) \ominus f(t,\sigma(t))|_g \leq \epsilon, \quad for \quad t \in [0,T] \tag{6.1}$$

Definition 6.1. Equation (6.1) is Ulam-Hyers stable if there is $\theta > 0$ such that for every $\epsilon > 0$ and for every solution $\sigma \in C([0,T],R)$ of inequality (6.1), there is a solution $y \in C([0,T],R)$ of equation (5.1) that holds the inequality due to a weighted norm:

$$\|y \ominus \sigma\|_{\omega} \leq \epsilon \odot \theta, \quad t \in [0,T] \tag{6.2}$$

Remark 6.1. A function $\sigma \in C([0,T],R)$ is a solution of the inequality (6.1) if and only if there is a function $f \in C([0,T],R)$ which fulfills the following conditions:
1) $|f(t)|_g \leq \epsilon$;
2) $H^{\alpha,\beta}_{\oplus,\odot,0+}\sigma(t) \ominus A \odot \sigma(t) \ominus B \odot \sigma(t-\tau) \ominus f(t,\sigma(t)) := f(t), t \in [0,T]$.

Due to the Remark 6.1, the solution of following equation:

$$H^{\alpha,\beta}_{\oplus,\odot,0+}\sigma(t) \ominus A \odot \sigma(t) \ominus B \odot \sigma(t-\tau) = f(t,\sigma(t)) \oplus f(t), t \in [0,T]. \tag{6.3}$$

can be demonstrate by

$$\begin{aligned}
\sigma(t) =& \bigoplus_{k=0}^{m-2}\left(g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right) \oplus (A \oplus B) \odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))\right) \odot \phi_0^{(k)} \\
&\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t)) \odot \phi_0^{(m-1)} \oplus B \odot \int_{[-\tau,\min(t-\tau,0)]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s)) \odot \phi(s) \odot ds \\
&\oplus \int_{[0,t]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot f(s,\sigma(s)) \odot ds \oplus \int_{[0,t]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot f(s) \odot ds \\
:=& (F(\sigma))(t) \oplus \int_{[0,t]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot f(s) \odot ds, \quad t \in [0,T].
\end{aligned}$$

To use Lemma 5.2, the difference $\sigma(t) \ominus (F(z))(t)$ can be evaluated as follows:

$$\begin{aligned}
&|\sigma(t) \ominus (F(\sigma))(t)|_g = |\int_{[0,t]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot f(s) \odot ds|_g \le \int_{[0,t]}^{\oplus} |E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))|_g \odot |f(s)|_g \odot ds \\
\le& \epsilon \odot g^{-1}(t^{\alpha-1}) \odot g^{-1}(\exp((|A|+|B|)t^{\alpha})) \odot \int_{[0,t]}^{\oplus} ds \le \epsilon \odot g^{-1}(T^{\alpha}) \odot g^{-1}(\exp((|A|+|B|)T^{\alpha})) := \epsilon \odot g^{-1}(T^{\alpha}) \odot S.
\end{aligned} \tag{6.4}$$

Finally, with constant delay, we are ready to assert and prove the Ulam-Hyers stability result for pseudo-Hilfer FDE.

Theorem 6.1. *Suppose that (H_1 and H_2) are satisfied. Then the equation* (5.1) *is Ulam-Hyers stable on* $[0,T]$.

Proof. Assume that $\sigma \in C[0,T], R$ is a solution of the inequality (6.1). Let y be a unique solution of the Cauchy problem for pseudo-Hilfer type fractional-order DDE(5.1), that is

$$\begin{aligned}
y(t) =& \bigoplus_{k=0}^{m-2}\left(g^{-1}\left(\frac{t^{(\beta-1)(m-\alpha)+k}}{\Gamma((\beta-1)(m-\alpha)+k+1)}\right) \oplus (A \oplus B) \odot E^{\tau,\oplus}_{\alpha,\alpha,(\beta-1)(m-\alpha)+\alpha+k+1}(A,B;g^{-1}(t-\tau))\right) \odot \phi_0^{(k)} \\
&\oplus E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t)) \odot \phi_0^{(m-1)} \oplus B \odot \int_{[-\tau,\min(t-\tau,0)]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-\tau-s)) \odot \phi(s) \odot ds \\
&\oplus \int_{[0,t]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot f(s,\sigma(s)) \odot ds := (Fy)(t), \quad t \in [0,T]
\end{aligned} \tag{6.5}$$

By using estimation (5.9) and (6.5), we obtain

$$\begin{aligned}
&|y(t) \ominus \sigma(t)|_g \oslash \exp(\omega t) = |(Fy)(t) \ominus (F\sigma)(t) \ominus \int_{[0,t]}^{\oplus} E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s)) \odot f(s) \odot ds|_g \oslash \exp(\omega t) \\
\le& |(Fy)(t) \ominus (F\sigma)(t)|_g \oslash \exp(\omega t) \oplus \int_{[0,t]}^{\oplus} |E^{\tau,\oplus}_{\alpha,\alpha,\alpha}(A,B;g^{-1}(t-s))|_g \odot |f(s)|_g \odot ds
\end{aligned}$$

$$\begin{aligned}
\le& C \odot g^{-1}(\Gamma(\alpha)) \odot \exp((|A|+|B|)T^{\alpha}) \oslash g^{-1}(\omega^{\alpha}) \odot \|y-\sigma\|_{\omega} \oplus \epsilon \odot g^{-1}(T^{\alpha}) \odot g^{-1}(\exp((|A|+|B|)T^{\alpha})) \\
:=& S \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^{\alpha}) \odot \|y-\sigma\|_{\omega} \oplus \epsilon \odot g^{-1}(T^{\alpha}) \odot S
\end{aligned}$$

We take maximum on $[0,T]$, then we obtain

$$\|y-\sigma\|_\omega \leq S \odot L \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^\alpha) \odot ||y-\sigma||_\omega \oplus \epsilon \odot g^{-1}(T^\alpha) \odot S$$

that gives that

$$\|y-\sigma\|_\omega \leq \epsilon \odot (g^{-1}(T^\alpha) \odot S) \oslash (1 \ominus S \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^\alpha))$$

By choosing $\omega > \left(g(S \odot C \odot g^{-1}(\Gamma(\alpha)))\right)^{\frac{1}{\alpha}}$ which implies that

$$\|y-\sigma\|_\omega \leq \epsilon \odot \theta \tag{6.6}$$

where

$$\theta := (g^{-1}(T^\alpha) \odot S) \oslash (1 \ominus S \odot C \odot g^{-1}(\Gamma(\alpha)) \oslash g^{-1}(\omega^\alpha))$$

□

7. An Example

In this section, we present an example that serves as a validation of the major theoretical results stated in Sections 5 and 6. The demonstration of the existence, uniqueness, and stability analysis of solutions in the following example relies on the application of Theorem 6.1.

Let $\alpha = 1.4, \beta = 0.5, m = 2, \tau = 2$, and $T = 2$. Consider the following pseudo-Hilfer delay diferential equation with a constant delay:

$$\begin{cases} H^{1.4,0.5}_{\oplus,\odot,0+}y(t) = 3 \odot y(t) \oplus 7 \odot y(t-2) \oplus \frac{cos(y(t))}{t^2+1}, t \in (0;2], \\ I^{1-\gamma}_{\oplus,\odot,0+}y(t) = t+5, \quad t \in [-2,0]. \end{cases} \tag{7.1}$$

with constants $A = 3, B = 7$ and $\phi(t) = t + 5$ is continuously differentiable function for $t \in [-2,0]$ and nonlinear perturbation$f(t) = \frac{cos(t)}{t^2+1}$is continuous on a Cartesian product $[-2,0] \times R$. Let $g(t) = 2t+1, \quad \forall t \in R$ be be monotone and continuous function. Such that the inverse of $g(t)$ be $g^{-1}(t) = \frac{t-1}{2}$. And we will denote $\phi(t) = t+5$,and we have $\phi_0 = 5, \quad \phi_0' = 1$.

Let's clarify the notation: γ is defined as $(\beta-1)(m-\alpha)+k+1$, where $k = 0, ..., m-1$. Now, if we substitute $\alpha = 1.4$, $\beta = 0.5$, and $m = 2$ into the expression for γ, we obtain $\gamma = k + 0.7$, where k takes the values 0 and 1.

Since $y(0) = 5$, and $y'(0) = 1$, the exact analytical representation of solution of (7.1) can be represented as follows:

$$y(t) = \left(g^{-1}\left(\frac{t^{-0.3}}{\Gamma((0.7)}\right) \oplus (3 \oplus 7) \odot E^{2,\oplus}_{1.4,1.4,2.1}(3,7;g^{-1}(t-2))\right) \odot 5$$

$$\oplus E^{2,\oplus}_{1.4,1.4,1.4}(3,7;g^{-1}(t)) \odot \phi_0' \oplus 7 \odot \int^{\oplus}_{[-2,\min(t-2,0)]} E^{2,\oplus}_{1.4,1.4,1.4}(3,7;g^{-1}(t-2-s)) \odot \phi(s) \odot ds$$

$$\oplus \int^{\oplus}_{[0,t]} E^{2,\oplus}_{1.4,1.4,1.4}(3,7;g^{-1}(t-s)) \odot \frac{\cos(y(s))}{s^2+1} \odot ds$$

By using some basic pseudo-operations and above conditions, we can simplify the exact solution of (7.1).

$$3\oplus 7=g^{-1}(g(3)+g(7))=g^{-1}(22)=10.5,\quad g^{-1}\left(\frac{t^{-0.3}}{\Gamma((0.7)}\right)=\frac{t^{-0.3}}{2\Gamma(0.7)}-\frac{1}{2}$$

$$\begin{aligned}
&10.5\odot E^{2,\oplus}_{1.4,1.4,2.1}(3,7;g^{-1}(t-2))=g^{-1}\Big[g(10.5)\times g\Big(E^{2,\oplus}_{1.4,1.4,2.1}(3,7;g^{-1}(t-2))\Big)\Big]\\
=&g^{-1}\Big[22\times E^{2}_{1.4,1.4,2.1}\Big(g(3),g(7);t-2\Big)\Big]=g^{-1}\Big[22\times E^{2}_{1.4,1.4,2.1}\Big(7,15;t-2\Big)\Big]\\
=&11E^{2}_{1.4,1.4,2.1}\Big(7,15;t-2\Big)-\frac{1}{2}\\
&\frac{t^{-0.3}}{2\Gamma(0.7)}-\frac{1}{2}\oplus\Big[11E^{2}_{1.4,1.4,2.1}\Big(7,15;t-2\Big)-\frac{1}{2}\Big]=g^{-1}\Big[g\Big(\frac{t^{-0.3}}{2\Gamma(0.7)}-\frac{1}{2}\Big)+g\Big(11E^{2}_{1.4,1.4,2.1}\Big(7,15;t-2\Big)-\frac{1}{2}\Big)\Big]\\
=&g^{-1}\Big[\frac{t^{-0.3}}{\Gamma(0.7)}+22E^{2}_{1.4,1.4,2.1}\Big(7,15;t-2\Big)\Big]=\frac{t^{-0.3}}{2\Gamma(0.7)}+11E^{2}_{1.4,1.4,2.1}\Big(7,15;t-2\Big)-\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
&\Big[\frac{t^{-0.3}}{2\Gamma(0.7)}+11E^{2}_{1.4,1.4,2.1}\Big(7,15;t-2\Big)-\frac{1}{2}\Big]\odot 5=g^{-1}\Big[g\Big(\frac{t^{-0.3}}{2\Gamma(0.7)}+11E^{2}_{1.4,1.4,2.1}\Big(7,15;t-2\Big)-\frac{1}{2}\Big)\cdot g(5)\Big]\\
=&g^{-1}\Big[\Big(\frac{t^{-0.3}}{\Gamma(0.7)}+22E^{2}_{1.4,1.4,2.1}\Big(7,15;t-2\Big)\Big)\cdot 11\Big]=g^{-1}\Big[\frac{11t^{-0.3}}{\Gamma(0.7)}+242E^{2}_{1.4,1.4,2.1}\Big(7,15;t-2\Big)\Big]\\
=&\frac{11t^{-0.3}}{2\Gamma(0.7)}+121E^{2}_{1.4,1.4,2.1}\Big(7,15;t-2\Big)-\frac{1}{2}\\
&E^{2,\oplus}_{1.4,1.4,1.4}(3,7;g^{-1}(t))\odot\phi_0'=g^{-1}\Big[E^{2}_{1.4,1.4,1.4}(g(3),g(7);t)\cdot g(1)\Big]=g^{-1}\Big[3\cdot E^{2}_{1.4,1.4,1.4}(7,15;t)\Big]\\
=&\frac{3}{2}\cdot E^{2}_{1.4,1.4,1.4}(7,15;t)-\frac{1}{2}
\end{aligned}$$

$$\begin{aligned}
&7\odot\int^{\oplus}_{[-2,\min(t-2,0)]}E^{2,\oplus}_{1.4,1.4,1.4}(3,7;g^{-1}(t-2-s))\odot\phi(s)\odot ds\\
=&g^{-1}\Big[g(7)\cdot\int_{-2}^{\min(t-2,0)}E^{2}_{1.4,1.4,1.4}(g(3),g(7);(t-2-s))g(\phi(s))ds\Big]\\
=&g^{-1}\Big[15\cdot\int_{-2}^{\min(t-2,0)}E^{2}_{1.4,1.4,1.4}(7,15;(t-2-s))(2s+11)ds\Big]\\
=&15\cdot\int_{-2}^{\min(t-2,0)}E^{2}_{1.4,1.4,1.4}(7,15;(t-2-s))(s+5.5)ds-\frac{1}{2}\\
=&15\cdot\int_{-2}^{\min(t-2,0)}E^{2}_{1.4,1.4,1.4}\Big(7,15;(t-2-s)\Big)sds\\
&+\frac{165}{2}\cdot\int_{-2}^{\min(t-2,0)}E^{2}_{1.4,1.4,1.4}\Big(7,15;(t-2-s)\Big)ds-\frac{1}{2}
\end{aligned}$$

$$
\begin{aligned}
&\int_{[0,t]}^{\oplus} E^{2,\oplus}_{1.4,1.4,1.4}(3,7;g^{-1}(t-s)) \odot \frac{\cos(y(s))}{s^2+1} \odot ds \\
=&g^{-1}\Big[\int_0^t E^{2}_{1.4,1.4,1.4}(7,15;(t-s))\Big(2\cdot\frac{\cos(y(s))}{s^2+1}+1\Big)ds\Big] \\
=&\int_0^t E^{2}_{1.4,1.4,1.4}(7,15;(t-s))\Big(\frac{\cos(y(s))}{s^2+1}+\frac{1}{2}\Big)ds-\frac{1}{2} \\
=&\int_0^t E^{2}_{1.4,1.4,1.4}(7,15;(t-s))\frac{\cos(y(s))}{s^2+1}ds \\
+&\frac{1}{2}\int_0^t E^{2}_{1.4,1.4,1.4}(7,15;(t-s))ds-\frac{1}{2}
\end{aligned}
$$

$$
\begin{aligned}
&\Big[\frac{t^{-0.3}}{2\Gamma(0.7)}+11E^{2}_{1.4,1.4,2.1}\Big(7,15;t-2\Big)-\frac{1}{2}\Big]\oplus\Big[\frac{3}{2}\cdot E^{2}_{1.4,1.4,1.4}(7,15;t)-\frac{1}{2}\Big] \\
&\oplus\Big[15\cdot\int_{-2}^{\min(t-2,0)} E^{2}_{1.4,1.4,1.4}\Big(7,15;(t-2-s)\Big)sds+\frac{165}{2}\cdot\int_{-2}^{\min(t-2,0)} E^{2}_{1.4,1.4,1.4}\Big(7,15;(t-2-s)\Big)ds-\frac{1}{2}\Big] \\
&\oplus\Big[\int_0^t E^{2}_{1.4,1.4,1.4}(7,15;(t-s))\frac{\cos(y(s))}{s^2+1}ds+\frac{1}{2}\int_0^t E^{2}_{1.4,1.4,1.4}(7,15;(t-s))ds-\frac{1}{2}\Big] \\
&=g^{-1}\Bigg[g\Big(\frac{t^{-0.3}}{2\Gamma(0.7)}+11E^{2}_{1.4,1.4,2.1}\Big(7,15;t-2\Big)-\frac{1}{2}\Big)+g\Big(\frac{3}{2}\cdot E^{2}_{1.4,1.4,1.4}(7,15;t)-\frac{1}{2}\Big) \\
&+g\Big(15\cdot\int_{-2}^{\min(t-2,0)} E^{2}_{1.4,1.4,1.4}\Big(7,15;(t-2-s)\Big)sds+\frac{165}{2}\cdot\int_{-2}^{\min(t-2,0)} E^{2}_{1.4,1.4,1.4}\Big(7,15;(t-2-s)\Big)ds-\frac{1}{2}\Big) \\
&+\Big(\int_0^t E^{2}_{1.4,1.4,1.4}(7,15;(t-s))\frac{\cos(y(s))}{s^2+1}ds+\frac{1}{2}\int_0^t E^{2}_{1.4,1.4,1.4}(7,15;(t-s))ds-\frac{1}{2}\Big)\Bigg] \\
&=\frac{t^{-0.3}}{2\Gamma(0.7)}+\frac{11}{2}E^{2}_{1.4,1.4,2.1}\Big(7,15;t-2\Big)-\frac{1}{2}+\frac{3}{2}\cdot E^{2}_{1.4,1.4,1.4}(7,15;t)-\frac{1}{2} \\
&+15\cdot\int_{-2}^{\min(t-2,0)} E^{2}_{1.4,1.4,1.4}\Big(7,15;(t-2-s)\Big)sds+\frac{165}{2}\cdot\int_{-2}^{\min(t-2,0)} E^{2}_{1.4,1.4,1.4}\Big(7,15;(t-2-s)\Big)ds-\frac{1}{2} \\
&+\int_0^t E^{2}_{1.4,1.4,1.4}(7,15;(t-s))\frac{\cos(y(s))}{s^2+1}ds+\frac{1}{2}\int_0^t E^{2}_{1.4,1.4,1.4}(7,15;(t-s))ds-\frac{1}{2} \\
&=\frac{t^{-0.3}}{2\Gamma(0.7)}-2+\frac{11}{2}E^{2}_{1.4,1.4,2.1}\Big(7,15;t-2\Big)+\frac{3}{2}\cdot E^{2}_{1.4,1.4,1.4}(7,15;t) \\
&+15\cdot\int_{-2}^{\min(t-2,0)} E^{2}_{1.4,1.4,1.4}\Big(7,15;(t-2-s)\Big)sds+\frac{165}{2}\cdot\int_{-2}^{\min(t-2,0)} E^{2}_{1.4,1.4,1.4}\Big(7,15;(t-2-s)\Big)ds \\
&+\int_0^t E^{2}_{1.4,1.4,1.4}(7,15;(t-s))\Big(\frac{\cos(y(s))}{s^2+1}+\frac{1}{2}\Big)ds
\end{aligned}
$$

Finally, we will obtain the following result for the solution of pseudo-Hilfer delay diferensial equation, which it is equivalent with exact solution, so that it can express pseudo-operations.

$$
\begin{aligned}
y(t)=&\frac{t^{-0.3}}{2\Gamma(0.7)}-2+\frac{11}{2}E^{2}_{1.4,1.4,2.1}\Big(7,15;t-2\Big)+\frac{3}{2}\cdot E^{2}_{1.4,1.4,1.4}(7,15;t) \\
&+\int_{-2}^{\min(t-2,0)} E^{2}_{1.4,1.4,1.4}\Big(7,15;(t-2-s)\Big)\Big(15s+\frac{165}{2}\Big)ds \\
&+\int_0^t E^{2}_{1.4,1.4,1.4}(7,15;(t-s))\Big(\frac{\cos(y(s))}{s^2+1}+\frac{1}{2}\Big)ds
\end{aligned}
$$

It is not difficult to see that condition $H(2)$ holds. By mean value theorem, for any $y, z \in R$, there exists $\xi \in (y, z)$such that

$$|f(t, y) \ominus f(t, z)|_g \leq |y \ominus z|_g$$

The statement $H(2)$ is valid with C being equivalent to 1, as per Theorem 6.1 and equation (5.1). This implies that the pseudo-Hilfer differential equation with a constant delay, as given in equation (7.1), has a single solution that is stable in the Ulam-Hyers sense over the interval $[0, 2]$.

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