

Dissipative models of swell propagation across the Pacific

Camille R. Zaug¹ and John D. Carter¹

¹Mathematics Department; Seattle University; 901 12th Avenue, Seattle, WA; USA

Key Points:

- Dissipation is a physically important effect in the evolution of ocean swell
- Dissipative generalizations of the nonlinear Schrödinger equation are reasonably effective models of swell propagation
- Linearizations of nonlinear models performed a bit better than nonlinear models, suggesting linear models are sufficient

Abstract

Ocean swell plays an important role in the transport of energy across the ocean, yet its evolution is still not well understood. In the late 1960s, the nonlinear Schrödinger (NLS) equation was derived as a model for the propagation of ocean swell over large distances. More recently, a number of dissipative generalizations of the NLS equation based on a simple dissipation assumption have been proposed. These models have been shown to accurately model wave evolution in the laboratory setting, but their validity in modeling ocean swell has not previously been examined. We study the efficacy of the NLS equation and four of its generalizations in modeling the evolution of swell in the ocean. The dissipative generalizations perform significantly better than conservative models and are overall reasonable models for swell amplitudes, indicating dissipation is an important physical effect in ocean swell evolution. The nonlinear models did not out-perform their linearizations, indicating linear models may be sufficient in modeling ocean swell evolution.

1 Introduction

Swell in the ocean is composed of slowly modulated surface wave trains with relatively long periods. It is typically formed after waves created by distant storms have had a chance to disperse. Swell can travel thousands of kilometers, see for example, Snodgrass et al. (1966) and Collard et al. (2009). This coherence over long distances might suggest that there is a simple underlying model that governs the evolution of swell. However, Rogers (2002) and Rasche et al. (2008) show that swell amplitudes are relatively poorly predicted. Although Snodgrass et al. (1966) neglected dissipative effects, more recent work suggests that dissipation may play an important role in swell evolution, see for example, Collard et al. (2009), Ardhuin et al. (2009), Henderson & Segur (2013), and Young et al. (2013).

1.1 Model Equations

The dimensionless cubic nonlinear Schrödinger (NLS) equation,

$$iu_\chi + u_{\xi\xi} + 4|u|^2u = 0, \quad (1)$$

is an approximate model for the slow evolution of a nearly monochromatic wave train of gravity waves on deep water (i.e. swell propagating over large distances). Here $u = u(\xi, \chi)$ is a dimensionless complex-valued function that describes the evolution of the envelope of the oscillations of a carrier wave, χ represents dimensionless distance across the ocean, and ξ represents dimensionless time. The leading-order approximation to the dimensional surface displacement, $\eta(x, t)$, can be obtained from an NLS solution, $u(\xi, \chi)$, via the relation

$$\eta(x, t) = \frac{\epsilon}{k_0} (u(\xi, \chi)e^{i\omega_0 t - ik_0 x} + u^*(\xi, \chi)e^{-i\omega_0 t + ik_0 x}) + \mathcal{O}(\epsilon^2), \quad (2)$$

where η , x , and t are dimensional variables and u^* represents complex conjugate of u . Here ω_0 , k_0 , and a_0 are parameters that represent the dimensional frequency, wavenumber, and amplitude of the carrier wave respectively, and $\epsilon = 2a_0k_0$ is a measure of wave steepness/nonlinearity. The dimensionless and dimensional independent variables are related by

$$\xi = \epsilon\omega_0 t - 2\epsilon k_0 x, \quad \chi = \epsilon^2 k_0 x. \quad (3)$$

Zakharov (1968) derived the NLS equation as a model for the propagation of ocean swell over large distances. See Johnson (1997) for a more detailed and modern derivation, but note that both of these derivations rely on an ansatz that is slightly different than the one given in equation (2). In deriving the NLS equation (and the generalizations presented below), one assumes that the surface displacement is small, that the spectrum is narrow banded, and that the spectrum is centered about the carrier wave. The

NLS equation has been studied extensively from a mathematical perspective, see for example, Sulem & Sulem (1999), as well as from a physical perspective. The NLS equation has also been shown to favorably predict measurements from laboratory experiments when the waves have small amplitude and steepness (i.e. $\epsilon < 0.1$), see for example, Lo & Mei (1985).

In order to weaken the NLS equation's narrow-bandedness restriction, Dysthe (1979) extended the NLS derivation asymptotics one additional order and derived the equation that now bears his name

$$iu_\chi + u_{\xi\xi} + 4|u|^2u + \epsilon(-8iu^2u_\xi^* - 32i|u|^2u_\xi - 8iu(\mathcal{H}(|u|^2))_\xi) = 0. \quad (4)$$

Here \mathcal{H} represents the Hilbert transform, which is defined by

$$\mathcal{H}(f(\xi)) = \sum_{k=-\infty}^{\infty} -i \operatorname{sgn}(k) \hat{f}(k) e^{2\pi i k \xi / L}, \quad (5)$$

where $\hat{f}(k)$ is the Fourier transform of the function $f(\xi)$ and is defined by

$$\hat{f}(k) = \frac{1}{L} \int_0^L f(\xi) e^{-2\pi i k \xi / L} d\xi, \quad (6)$$

and L is the ξ -period of the measurements. Lo & Mei (1985) showed that the Dysthe equation accurately predicts laboratory experimental measurements for a wider range of wave amplitude and steepness values than does the NLS equation.

Neither the NLS equation nor the Dysthe equation include terms that account for dissipative effects. In other words, both are conservative partial differential equations (PDEs). In order to address this limitation, a number of dissipative generalizations of the NLS equation have been proposed and studied. In this work, we focus on three dissipative generalizations of the NLS equation. Segur et al. (2005) and Wu et al. (2006) showed that predictions obtained from the dissipative nonlinear Schrödinger (dNLS) equation

$$iu_\chi + u_{\xi\xi} + 4|u|^2u + i\delta u = 0, \quad (7)$$

where δ is a nonnegative constant representing dissipative effects, compared favorably with a range of laboratory experiments. In this dissipative model and those included below, dissipative effects from all sources are accounted for by the single, constant parameter δ . This is simplest dissipative generalization of the NLS equation as the dissipation is constant and frequency independent. While Young et al. (2013) showed that the ocean swell decay rate is proportional to the wavenumber squared, since the ocean data we examine is narrow banded and the models rely on a narrow-bandedness assumption, it is a reasonable first-order assumption that the dissipation rate is wave-number independent. The dNLS equation is the most common dissipative generalization of the NLS equation. Henderson & Segur (2013) use the dNLS equation as a basis for a comparison of dissipation rates, frequency downshift, and evolution of swell in laboratory experiments and in the ocean using the Snodgrass et al. (1966) data.

Recently, following the work of Dysthe (1979) and Dias et al. (2008), Carter & Govan (2016) derived the viscous Dysthe (vDysthe) equation

$$iu_\chi + u_{\xi\xi} + 4|u|^2u + i\delta u + \epsilon(-8iu^2u_\xi^* - 32i|u|^2u_\xi - 8iu(\mathcal{H}(|u|^2))_\xi + 5\delta u_\xi) = 0, \quad (8)$$

from the dissipative generalization of the water-wave problem presented by Wu et al. (2006). Additionally, they showed that the vDysthe equation accurately predicts the evolution of slowly-modulated wave trains from two series of experiments. In this model, the dissipation rate depends linearly on the wavenumber. A flaw arises in the vDysthe equation because of this linear dissipation rate: Any lower sideband with frequency further

than $1/(5\epsilon)$ from the carrier wave will grow exponentially. This is obviously not physical. The flaw results from limitations associated with the narrow-bandwidth assumption used in the derivation of the vDysthe equation. See Section 2.2 of Carter et al. (2019) for more details.

Motivated by the work of Gramstad & Trulsen (2011), Carter et al. (2019) showed that the ad-hoc dissipative Gramstad-Trulsen (dGT) equation,

$$iu_\chi + u_{\xi\xi} + 4|u|^2u + i\delta u + \epsilon(-32i|u|^2u_\xi - 8u(\mathcal{H}(|u|^2))_\xi + 5\delta u_\xi) - 10i\epsilon^2\delta u_{\xi\xi} = 0, \quad (9)$$

accurately predicts the evolution of slowly-modulated wave trains from four series of laboratory experiments. In this model, the dissipation rate depends quadratically on the wavenumber, which is at least qualitatively similar with the observations of Young et al. (2013). Although the accuracy of the vDysthe and dGT equations were similar for the experiments examined, it is important to note that the dGT equation does not have the same non-physical growth flaw as the vDysthe equation because of the addition of the $\epsilon^2\delta u_{\xi\xi}$ term. *The main goal of this paper is to test the accuracy of these equations as models for swell traveling across the Pacific Ocean.*

1.2 Frequency Downshift

Frequency downshift (FD) is said to occur when the carrier wave loses a significant amount of energy to its lower sidebands. FD was first observed in wave tank experiments conducted by Lake et al. (1977) and Lake & Yuen (1977). Using a wave maker located at one end of the tank, they created a wave train with a particular frequency. As the waves traveled down the tank, they experienced the growth of the Benjamin & Feir (1967) instability and disintegrated. Further down the tank, the waves regained coherence and coalesced into a wave train with a lower frequency than the one created by the wave maker.

There are two common metrics used to quantify FD: a monotonic decrease in the wave's spectral peak or a monotonic decrease in the wave's spectral mean. FD is said to be temporary if an initial decrease in either the spectral peak or mean is followed by an increase. The spectral peak, ω_p , is defined to be the frequency with maximal amplitude. The spectral mean, ω_m , is defined by

$$\omega_m = \frac{\mathcal{P}}{\mathcal{M}}, \quad (10)$$

where \mathcal{P} describes the “linear momentum” of the wave and is given by

$$\mathcal{P} = \frac{i}{2L} \int_0^L (uu_\xi^* - u_\xi u^*) d\xi, \quad (11)$$

and \mathcal{M} describes the “mass” of the wave and is given by

$$\mathcal{M} = \frac{1}{L} \int_0^L |u|^2 d\xi, \quad (12)$$

where L is the period of the ξ measurement. Since ω_p is a “local” frequency measurement and ω_m is a “global” frequency measurement, it is possible for a particular wave train to exhibit FD in neither, either, or both senses. The experiments of Lake et al. (1977) and Lake & Yuen (1977) provide a clear demonstration of FD in the spectral peak sense. Their experiments also likely exhibited FD in the spectral mean sense, but neither \mathcal{P} nor ω_m was measured, so a definitive statement regarding FD in the spectral mean sense cannot be made. Most physical explanations for FD rely on wind and wave breaking, see for example Trulsen & Dysthe (1990), Hara & Mei (1991), and Brunetti et al. (2014). However, the laboratory experiments examined by Segur et al. (2005) exhibited FD in both senses without wind or wave breaking. Thus, there must be a mechanism for this phenomenon which does not rely on these effects. The dGT and vDysthe equations, which

predict FD in the spectral mean sense without relying on wind or wave breaking, were proposed as models for this phenomenon.

The ocean swell data collected on the Pacific Ocean by Snodgrass et al. (1966) displays evidence of FD in both senses. We examine this data in detail below. Understanding the mechanisms for FD will contribute to knowledge about how energy propagates across the ocean, with the potential to improve predictive abilities as swell nears the shore, impacting fields from shipping to surfing. *A secondary goal of this paper is to examine frequency downshift in ocean swell data and in these models.*

1.3 Model Properties

Unfortunately, there is no mathematical theory that governs the evolution of the spectral peak over large distances for any of the equations under consideration. However, the theory for the evolution of the spectral mean for these equations is well known. The NLS equation preserves both \mathcal{M} and \mathcal{P} and therefore the NLS equation cannot predict FD in the spectral mean sense. The Dysthe equation preserves \mathcal{M} , but does not necessarily preserve \mathcal{P} . Since the sign in the change of \mathcal{P} depends on the solution under consideration, the Dysthe equation predicts FD for some waves, frequency upshift for other waves, and constant spectral mean for other waves. The dNLS equation does not preserve \mathcal{M} , nor does it preserve \mathcal{P} . However, the dNLS equation preserves ω_m , so it cannot exhibit FD in the spectral mean sense. This result is related to the fact that dissipation in the dNLS equation is frequency independent. The vDysthe equation does not preserve \mathcal{M} or \mathcal{P} . The sign in the change of \mathcal{P} is indefinite for the vDysthe equation just as it is for the Dysthe equation. Therefore, the vDysthe equation can exhibit frequency downshift or upshift depending on the solution under consideration. Finally, the dGT equation predicts FD in the spectral mean sense for all nontrivial wave trains.

In the remainder of this paper, we compare the efficacy of these generalizations of the NLS equation at modeling the evolution of ocean swell as it travels across the Pacific Ocean. In order to test the accuracy of these models, we focus on two questions: (i) how important are dissipative effects? and (ii) how important are nonlinear effects? These questions have been addressed using laboratory data, but to our knowledge, have not been addressed using ocean data.

The remainder of the paper is outlined as follows. Section 2 contains a description of the Snodgrass et al. (1966) ocean data and how we processed it. Section 3 contains the results of comparisons between the model predictions and the ocean data. Finally, Section 4 summarizes our observations and results.

2 Ocean Data

2.1 Description of Data

During the southern hemisphere winter of 1963, a team of researchers led by Frank E. Snodgrass and Walter Munk from the University of California Institute of Geophysics and Planetary Physics set out to measure the evolution of swell across the Pacific Ocean. In order to track waves originating from storms in the southern hemisphere propagating northwards, the team manned six stations along a great circle. The locations included: Cape Palliser in New Zealand, Tutuila in American Samoa, Palmyra Atoll, Honolulu in Hawaii, the vessel *FLIP* in the north Pacific, and Yakutat in Alaska. However, the Cape Palliser and *FLIP* locations did not produce data included in this work. Due to dispersion, the ocean swell was recorded at any particular station for up to one week. Three hours of time series pressure data was collected twice daily and converted to surface wave spectra. Taking a ridge cut of these narrow-banded spectra at each station resulted in a composite spectrum removing the effects of dispersion. Snodgrass et al. (1966) presented

this data in the form of a power density spectrum, $C(f)$, with units of energy density (dB above 1 cm²/mHz) per frequency (mHz) on a logarithmic scale. They corrected the data to account for geometric spreading and island shadowing. In addition, they determined the impact of effects such as instrument placement, refraction, oblateness of the earth, and wave-wave interactions, including scattering and wave breaking.

Overall, swells from twelve storms were observed, but detailed ridge spectra were only provided for five swells. In this study, we focus on the swells named August 1.9, August 13.7, and July 23.2. The spectra corresponding to these swells are included in Figure 1. The swells of August 13.7 and July 23.2 exhibit FD in the spectral mean sense, while only the swell of August 13.7 definitively exhibits FD in the spectral peak sense. The swell of August 1.9 exhibits only temporary FD in the spectral mean sense, and its momentum, \mathcal{P} , increases as the waves propagate.

Additionally, a narrow-bandedness assumption is reasonable for these three swell. The second-to-last column of Table 1 contains a measure of each swell’s narrow-bandedness, $\Delta\omega/\omega_0$ at the first gauge using half-width-half-max to determine $\Delta\omega$. The values for the August 13.7 and July 23.2 are both quite small, while value for August 1.9 is reasonably small. We do not consider the other two swells presented by Snodgrass et al. (1966). because their spectra have are not narrow banded or have multiple peaks, rendering them outside of the range of validity of the mathematical models considered herein.

There are aspects of the swell data that limited our work. First, for the swells considered in this study, data is only provided at four gauges. As we used the data at the first gauge to determine the initial conditions for our simulations, there were only three gauges to compare the simulation results against. This limited our ability to make strong conclusions. Additionally, in the July 23.2 spectra, the energy at the third gauge is higher than the energy at the second gauge, which is evidence of uncertainty in the data. Furthermore, the data collection and processing techniques used by Snodgrass et al. (1966) resulted in a loss of phase data, which is necessary to produce the physical surface displacement time series realizations required by the PDE models presented above. Finally, the domain of frequencies present in each spectra varies across gauges. We accounted for this in our measurements of model accuracy (see below).

2.2 Data Processing

Given the power density spectra presented in Snodgrass et al. (1966) for a swell, to create initial conditions for our models, we created realizations of surface displacement time series at each gauge. To do this, the data was digitized and interpolated to create a continuous spectrum. (We did not have access to the original data.) We then converted from decibels to units of energy density, cm²/mHz, by taking $\Phi(f) = 10^{C(f)/10}$. Next, we discretized the continuous data into bands of width Δf , where $\Delta f = 1/L$, with L representing the collection period of three hours, and computed the Fourier amplitudes, $a = 0.01\sqrt{\Phi(f)\Delta f}$, which have units of meters. We created a discretization grid around the spectral peak at the first gauge and ensured that each subsequent gauge maintained the same grid. To compensate for the lack of phase data, each Fourier mode was assigned a random phase, preserving the magnitude of each amplitude. Assigning random phases is appropriate when the phase data is missing, see for example, Holthuijsen (2007). We then generated a two-sided Hermitian spectrum and took an inverse discrete Fourier transform to find a realization of the surface displacement time series at each gauge.

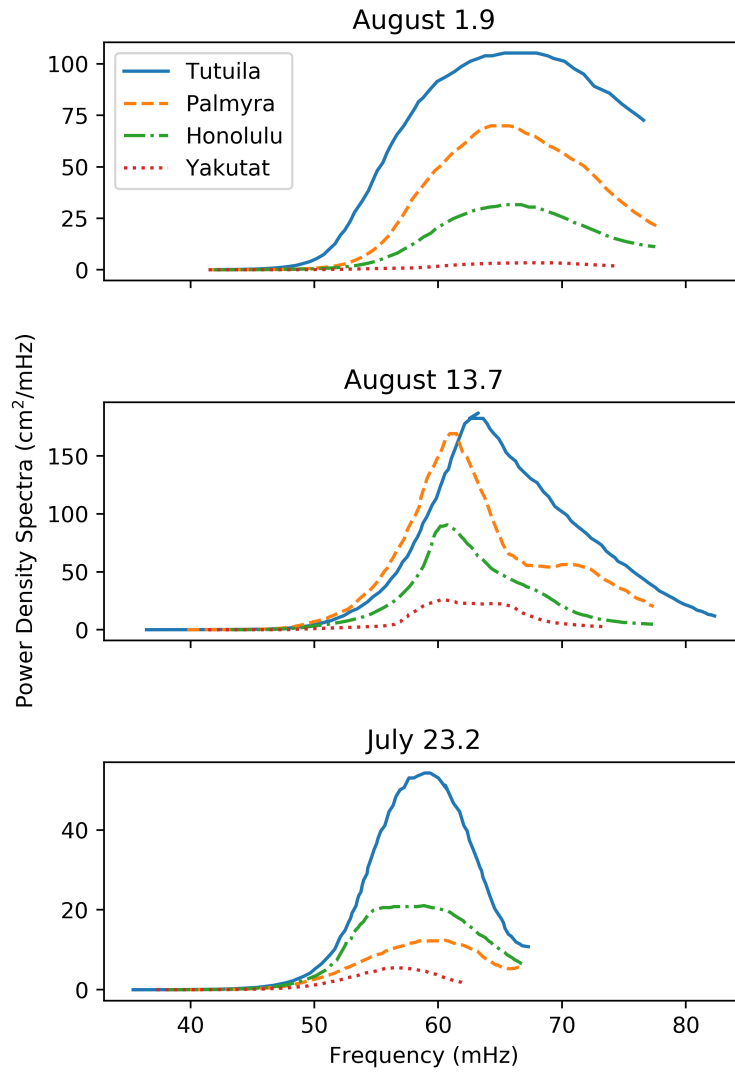


Figure 1. The power density spectra for the swells of August 1.9, August 13.7, and July 23.2.

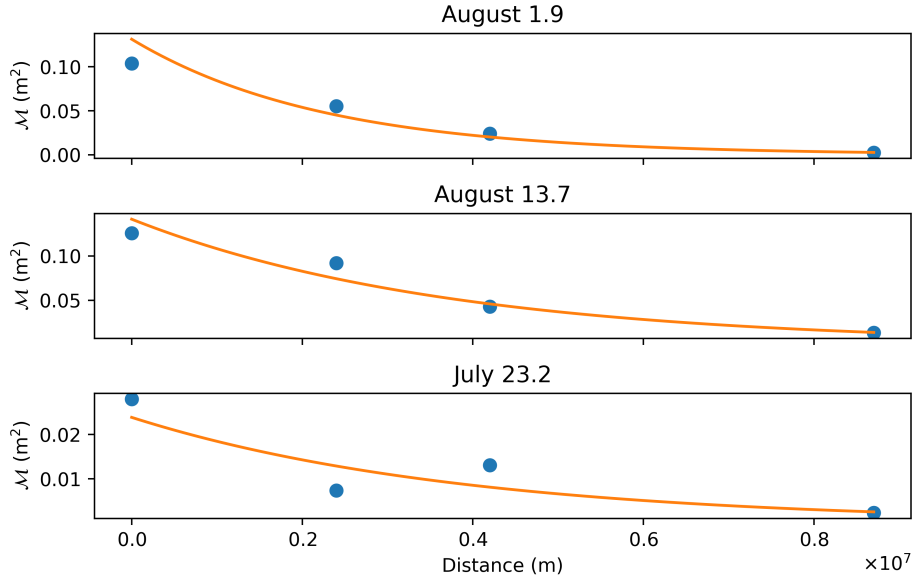


Figure 2. Plots of the dimensional mass, \mathcal{M} , versus the distance along the great circle. The dots represent the physical measurements and the curve represents the best exponential fit. From left to right, the dots refer to the stations in Tutuila, Palmyra, Honolulu, and Yakutat.

3 Model Comparisons

3.1 Computation of Parameters

There are two dimensionless parameters that appear in the models examined in this work. The wave-steepness/nonlinearity parameter, ϵ , is defined by $\epsilon = 2a_0k_0$ where a_0 and k_0 are the amplitude and wavenumber of the carrier wave. The values of ω_0 , the frequency of the carrier wave, and a_0 were obtained directly from the spectrum at the first gauge. The value of k_0 was determined using the deep-water linear dispersion relation, $\omega_0^2 = gk_0$. Table 1 contains the values of these parameters. We note that our ϵ values are different than those in Henderson & Segur (2013) because we used a different definition for ϵ . However, this is irrelevant because the results we present below are independent of the value of ϵ due to an invariance of the PDEs. The dissipation parameter, δ , was determined empirically by best-fitting an exponential through the decay of \mathcal{M} with respect to the dimensionless distance χ , along the great circle. Figure 2 contains plots of dimensional \mathcal{M} and the best exponential fit for each of the three swells. The values of δ for each swell are included in Table 1.

3.2 Simulation Methods

All model PDEs were solved numerically in dimensionless form by assuming periodic boundary conditions in ξ and using the sixth-order operator splitting algorithm developed by Yoshida (1990) in χ in Python. The linear parts of the PDEs were solved exactly in Fourier space using the fast Fourier transform (FFT). The nonlinear parts of the PDEs were either solved exactly (NLS, dNLS) or using fourth-order Runge-Kutta (Dysthe, vDysthe, dGT) in physical space. The evolution of the quantities \mathcal{M} and \mathcal{P} was compared against model predictions and was found to be consistent, indicating that the implemented numerical methods correctly solved each PDE.

| Swell | ω_0 (Hz) | k_0 (m ⁻¹) | a_0 (m) | ϵ | \mathcal{M}_0 (m ²) | δ | $\Delta\omega/\omega_0$ | Downshift |
|-------------|-----------------|--------------------------|-----------|------------|-----------------------------------|----------|-------------------------|----------------------|
| August 1.9 | 4.129e-1 | 1.738e-2 | 2.209e-2 | 7.676e-4 | 1.037e-1 | 2.173e1 | 1.497e-1 | NA |
| August 13.7 | 3.973e-1 | 1.609e-2 | 2.942e-2 | 9.470e-4 | 1.257e-1 | 9.294e0 | 9.421e-2 | ω_m, ω_p |
| July 23.2 | 3.719e-1 | 1.410e-2 | 1.586e-2 | 4.472e-4 | 2.797e-2 | 4.547e1 | 8.321e-2 | ω_m |

Table 1. Empirically determined parameters for each swell. The parameter ω_0 represents the carrier wave frequency, k_0 represents the carrier wavenumber, a_0 represents the amplitude of the carrier wave, ϵ is the dimensionless nonlinearity parameter, \mathcal{M}_0 is the value of \mathcal{M} at the first gauge (Tutula), and δ is the dimensionless dissipation parameter. The second-to-last column shows a measure of the narrow bandedness of the spectra. The final column shows whether FD in the spectral mean, ω_m , or spectral peak, ω_p , sense occurred in each swell as it propagated northwards.

The (dimensionless) initial conditions were generated by factoring the carrier wave out of the nondimensionalized one-sided processed spectrum at the first gauge (Tutuila), choosing random phases for each mode, and taking an inverse DFT. Similarly, the time series of the modulating envelope was computed at each of the three remaining gauges. These results were re-dimensionalized and compared with the ocean swell measurements using the error norm

$$\mathcal{E} = \sum_{n=2}^4 \sum_{j=-J_n}^{J_n} \frac{1}{3\mathcal{M}_n} \left| |\hat{B}_n^{\text{sim}}(j)| - |\hat{B}_n^{\text{data}}(j)| \right|^2, \quad (13)$$

n represents the gauge number, where J_n is the number of nonzero Fourier modes at gauge n , \mathcal{M}_n is the value of \mathcal{M} at the n^{th} gauge, and $\hat{B}_n(j)$ is the j^{th} nonzero Fourier amplitude at the n^{th} gauge from the numerical simulation (sim) or the ocean swell data (data). This process was repeated 100 times with different random phases for each swell. The mean of the results is reported to account for random effects. Additionally, to compare nonlinear and linear theories, solutions to both the full (nonlinear) PDEs and their linearizations were computed. Note that phase does not affect the linear results, so only one random phase simulation was computed for each linearized PDE for each swell.

3.3 Results

Figure 3 shows plots comparing the ocean data with the numerical predictions for the carrier wave and six sidebands for the August 13.7 swell. The plots for the August 1.9 and July 23.2 swells are similar. The sidebands shown represent a broad range of the swell's spectrum and demonstrate the nonphysical exponential growth predicted by the vDysthe equation in the (far) lower sidebands. Quantitative comparisons between the ocean data and simulations of the full PDEs using the error norm given in equation (13) are reported in Table 2. Quantitative comparisons between the ocean data and simulations of the linearized PDEs are reported in Table 3. Note that the linearizations of the NLS and Dysthe equations result in the same linear PDE. For all three swells, the dissipative models (dNLS, vDysthe, dGT) performed between one to two orders of magnitude better than the conservative models (NLS, Dysthe). This result is predictable because the spectra shown in Figure 1 show that the swells generally lose energy as they traveled northwards. These results demonstrate that including dissipation is necessary to accurately model the evolution of swell as it travels across the Pacific Ocean.

Considering only the nonlinear PDEs, dNLS produced the smallest error for the August 1.9 swell; vDysthe produced the smallest error for the August 13.7 swell; and dGT produced the smallest error for the July 23.2 swell, though vDysthe produced a very similar result. Although the differences between the linear and nonlinear results were small, the linearized dNLS equation performed best for the swell of August 1.9 and the linearized vDysthe equation performed best for the swells of August 13.7 and July 23.2. These results suggest that including nonlinear effects is not necessary to accurately model the evolution of swell across the Pacific. However, because nonlinear effects occur over short distances, we hypothesize that the linear models sometimes appear more effective than the nonlinear models due to the low spatial resolution in the data. Ideally, we would compare our models against data with more resolution to resolve the nonlinear behavior.

Other observations:

- The swell of August 13.7 had the most energy, while the swell of July 23.2 had the least. The fact that the linear models provided the best predictions in both of these cases suggests either that a swell needs to have even more energy than the August 13.7 swell for nonlinearity to be important or that there is not a simple relationship between energy and the importance of nonlinearity.

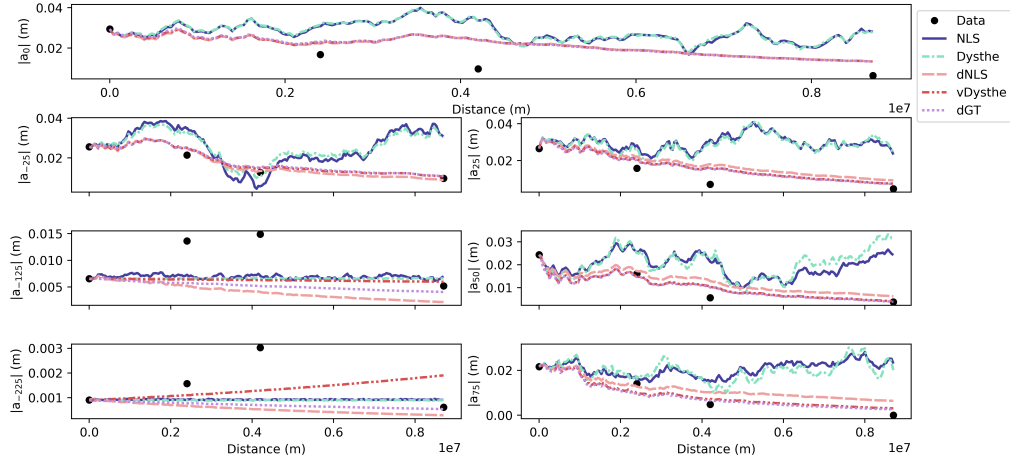


Figure 3. Plots of seven Fourier amplitudes versus distance traveled comparing the PDE predictions (curves) with the August 13.7 swell data (dots). The top plot is of the carrier wave amplitude. The left column contains plots of three lower sideband amplitudes and the right column contains plots of three upper sideband amplitudes.

- All of the swells had comparable carrier wave frequencies, and the slight variations do not appear to strongly affect the simulation predictions.
- All of the swells were relatively narrow banded and the degree of narrow-bandedness does not appear to strongly affect the accuracy of the simulation predictions.
- Swell exhibiting FD in the spectral mean sense are best predicted by the vDysthe or dGT equations, which can both predict this phenomena, though these models were not significantly better than the dNLS equation. There is no clear pattern regarding the effect of spectral peak FD on the simulation results. However, none of the models accurately qualitatively model the evolution of the spectral mean or peak, either underestimating the amount of the spectral mean decreases or predicting too much variation in the spectral peak.
- The vDysthe equation predicts nonphysical exponential growth in lower sidebands that are further than $5/(\epsilon k_0)$ away from the carrier wave, see the lower left plot in Figure 3. The amplitudes of these modes was small enough that their exponential growth did not greatly increase the value of the error, \mathcal{E} .
- According to Snodgrass et al. (1966), the swell of July 23.2 had more energy in Honolulu than in Palmyra. This means that energy did not decay monotonically as the swell propagated northwards. Switching the order of the data from these two gauges (so that the energy decays monotonically) does not have a large impact on the qualitative results. However, switching the order causes the accuracy of the vDysthe and dGT equations to increase significantly.
- We attempted to test the accuracy of the Islas & Schober (2011) model. However, we found that the optimal value of their free parameter β was negative, violating the model's assumptions. Thus, this is not a good model for this ocean swell data.

| Model | August 1.9 | August 13.7 | July 23.2 |
|---------|----------------------------|---------------------------|---------------------------|
| NLS | $0.0910 \pm 4\text{e-}4$ | $0.0142 \pm 2\text{e-}4$ | $0.0173 \pm 1\text{e-}4$ |
| Dysthe | $0.0908 \pm 5\text{e-}4$ | $0.0140 \pm 2\text{e-}4$ | $0.0172 \pm 1\text{e-}4$ |
| dNLS | $0.000731 \pm 1\text{e-}4$ | $0.00314 \pm 9\text{e-}5$ | $0.00229 \pm 2\text{e-}5$ |
| vDysthe | $0.0117 \pm 9\text{e-}5$ | $0.00173 \pm 9\text{e-}5$ | $0.00173 \pm 2\text{e-}5$ |
| dGT | $0.00401 \pm 9\text{e-}5$ | $0.00198 \pm 9\text{e-}5$ | $0.00172 \pm 2\text{e-}5$ |

Table 2. Averaged error results for ensembles of 100 simulations of the full PDEs using the error norm defined in equation (13).

| Model | August 1.9 | August 13.7 | July 23.2 |
|-----------------------|------------|-------------|-----------|
| Linearized NLS/Dysthe | 0.08743 | 0.01288 | 0.01711 |
| Linearized dNLS | 0.00040 | 0.00278 | 0.00225 |
| Linearized vDysthe | 0.01143 | 0.00139 | 0.00169 |
| Linearized dGT | 0.00379 | 0.00166 | 0.00170 |

Table 3. Error results for simulations of the linearized PDEs using the error norm defined in equation (13). Note that the linearized versions of the NLS and Dysthe equations are the same.

4 Conclusions

We compared the ocean swell data collected by Snodgrass et al. (1966) with predictions from the nonlinear Schrödinger, Dysthe, dissipative nonlinear Schrödinger, viscous Dysthe, and dissipative Gramstad-Trulsen equations. As only amplitude data was provided, we made the random phase assumption, ran 100 simulations with different random phases, and averaged the error between the ocean measurements and PDE predictions. We found that the dissipative models (dNLS, vDysthe, dGT) performed orders of magnitude better than the conservative models (NLS, Dysthe), suggesting that dissipation is a physically important effect for swell propagating across the Pacific Ocean. Additionally, for swells exhibiting frequency downshift in the spectral mean sense, models that can predict this behavior (vDysthe, dGT) performed the best. The dissipative models, which are based upon a simple dissipation ansatz, provided good predictions for the swells as they propagated across the ocean. We also found that the linear models performed slightly better than the nonlinear models. This suggests that (dissipative) linear models may be sufficient for modeling the evolution of swell across the Pacific.

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