Supplementary information on the reduced physics model: Evolution of alongshore velocity.

The along-isobath flow is initially balanced by a cross-shelf surface pressure gradient in a stratified ocean which starts with horizontal isopycnals. Given an alongshore periodic domain and the assumption that the relative change in water depth with time is small, the depth and alongshore integrated cross-shelf velocity u must be:

$$\int_{alongshore} dy \int_{-H}^{0} u \, dz = 0 \quad . \tag{$S1$}$$

Using this, the alongshelf momentum equation

$$\frac{\partial v}{\partial t} + \nabla \cdot (\vec{u} \cdot v) + fu = \frac{1}{\rho_0} \frac{\partial P}{\partial y} + \frac{\partial}{\partial z} A \frac{\partial v}{\partial z} \tag{$S2$}$$

averaged from the surface to the bottom and over the alongshore domain, under the assumption that there is no along-shore wind-stress, results in

$$\frac{\partial < V >}{\partial t} + \frac{\partial}{\partial x} \ll u'v' \gg = -\frac{1}{H} < \frac{\tau_y}{\rho_0} > \tag{$S3$}$$

Where <*V*> is the depth and alongshore averaged alongshore velocity, <> a depth and alongshore average, and *u'* and *v'* are the deviations of the cross- and along-shelf velocity from the depth and alongshore averaged velocity and τ_{bot}^{y} is the alongshore bottom stress.

To use ($S3$) to estimate <*V*>, the divergence of the cross-shelf advection of alongshelf momentum in that equation – $\frac{\partial}{\partial x} \ll u'v' \gg$ – must be neglected (this is the term analyzed in Lentz & Chapman, 2004 in a coastal upwelling system). In part for this reason, the modeling described above has been focused on relatively deep slope flows to reduce the magnitude of this advection term, and to ensure weak cross-shelf gradients as described above. In this deeper water there are smaller fractional changes in water depth, reduced cross-shelf gradients in alongshore velocity, and thus reduced cross-shelf divergence in the cross-shelf momentum flux. The evolution of the alongshore velocity above the low-PV layer is then

$$\frac{\partial < V >}{\partial t} = -\frac{1}{H} \frac{\tau_{bot}^{y}}{\rho_0} \; . \tag{$S4$}$$

To find the velocity in the region above the low-PV layer, this averaged velocity needs to be adjusted for the velocity deficit in the low-PV layer caused by the thermal wind shear (9) over the depth of the low-PV layer δ_{LPV} to get the velocity above the low-PV layer V:

$$V = < V > + \frac{\delta_{LPV}}{2f^2 H} (f N_0^2 - q_{LPV}) \frac{\partial H}{\partial x} \tag{$S5$}$$

Supplementary information on the reduced physics model: Bottom PV flux boundary condition.

The boundary condition at the slope is the flux $\boldsymbol{J}$ of potential vorticity out of the bottom boundary. $\boldsymbol{J}$ is composed of an advective component $q\boldsymbol{u}$ and frictional and diabatic components (Benthuysen & Thomas, 2012):

$$\boldsymbol{J} = q\boldsymbol{u} + \nabla b \times \boldsymbol{F} - D\omega_a \ , \tag{S6}$$

where $\boldsymbol{F}$ are the frictional terms of the momentum equation

$$\frac{D\vec{u}}{Dt} + f\hat{\boldsymbol{k}} \times \boldsymbol{u} = -\rho_0^{-1}\nabla P + \boldsymbol{F} \tag{S7}$$

and D is the diabatic term of the buoyancy equation

$$\frac{Db}{Dt} = D \ . \tag{S8}$$

In the interior of the ocean and this model, diabatic and frictional processes are weak and the evolution of PV is dominated by advection (e.g. Hallberg & Rhines, 2000; Haynes & McIntyre, 1987); here the advective fluxes are carried by the eddies, and are parameterized by the diffusive model (6). However, in the turbulent BBL diabatic and frictional processes are sinks of PV, and where the isopycnals intersect the bottom these processes must dominate, for there can be no advective fluxes out of the bottom. The flux of PV normal to the bottom can be quantified following the existing literature (Benthuysen & Thomas, 2012; Taylor & Ferrari, 2010; Wenegrat & Thomas, 2020); the nomenclature here is most similar to that of Benthuysen and Thomas (2012), but with *x* offshore and *y* alongshore. It is useful to use a coordinate system $\hat{x}, \hat{y}, \hat{z}$ where $\hat{z}$ is perpendicular to the bottom, $\hat{y}$ is parallel to the bottom and the coast, and $\hat{x}$ is parallel to the bottom and perpendicular to the coast.The PV flux of interest is the flux parallel to the isopycnals, and it is assumed that the alongshore-averaged isopycnals intersect the bottom perpendicularly (in a fully two-dimensional system, this would be exact because there can be no diffusive or advective flux into the bottom). The component of the PV flux due to diabatic processes, $-D\omega_a$ normal to the bottom is then (where $\hat{\boldsymbol{k}}$ is the unit normal perpendicular to the bottom and K is a vertical eddy diffusivity)

$$-D\omega_a \cdot \hat{\boldsymbol{k}} \approx \left(f + \left(\frac{\partial v}{\partial \hat{x}} - \frac{\partial u}{\partial \hat{y}}\right)\right)\frac{\partial}{\partial \hat{z}}\left(K\frac{\partial b}{\partial \hat{z}}\right) \approx 0 \tag{S9}$$

because $\frac{\partial b}{\partial \hat{z}}$, is zero. The PV flux along the isopycnal due to viscosity is:

$$\hat{k} \cdot \nabla b \times \boldsymbol{F} = \frac{\partial b}{\partial \hat{x}}F^y - \frac{\partial b}{\partial \hat{y}}F^x \tag{S10}$$

Since there is no net alongshore variation in the full model, $\frac{\partial b}{\partial \hat{y}}$ is zero in the alongshore average. In the limit of small bottom slope, the cross-shelf bottom-parallel buoyancy derivative $\frac{\partial b}{\partial \hat{x}} = N_{LPV}^2\theta$, where θ is the bottom angle $\tan^{-1}\frac{\partial H}{\partial x}$, $\frac{dH}{dx}$ is the bottom slope and N_{LPV}^2 is the buoyancy frequency $\frac{\partial b}{\partial z}$ near the bottom. The alongshore frictional term F^y is then assumed to be dominated by the divergence of the vertical turbulent mixing of momentum, so $F^y = \frac{\partial}{\partial z}(\rho_0\tau^y)$. To estimate the magnitude of F^y, the magnitude of τ^y at the bottom, τ_{bot}^y is estimated as the bottom-stress computed from the alongshore average bottom velocity v_{bot}, and the length scale of the vertical derivative is called δ, so that the PV flux normal to the bottom is

$$J^n = \hat{k}\cdot\nabla b \times \boldsymbol{F} \approx N^2\theta F^y = N_{LPV}^2\theta\frac{\tau_{bot}^y}{\delta\rho_0} = N_{LPV}^2\theta\frac{\tau_{bot}^y}{\delta_{LPV}\rho_0}\,. \qquad (S11)$$

The vertical length scale δ has in other work (e.g. Wenegrat & Thomas, 2020) been set to the vertical scale of the low-PV layer, which is called δ_{LPV}. The justification for this is unclear, both in the prior literature and to this author. This assumption would appear inconsistent with the assumption that the PV dynamics are primarily along-isopycnal, unless there is divergence of the flux of momentum carried by the eddies along isopycnals across the low-PV region. The justification for using δ_{LPV} remains an open question; however, it is seen below that ($S11$) correctly predicts the flux of PV from the bottom.

The PV flux in ($S11$) is normal to the bottom, but the model in (6) is written in the cross-shelf coordinate, for the isopycnals that leave normal to the bottom quickly become quasi-horizontal (meters of vertical change over kilometers of across-shelf extent). To ensure that the total flux out of the bottom (the integral along the bottom of the bottom-normal flux) is consistent with the total flux into all the possible models along each isopycnal (the vertical integral of the flux in the cross-shelf direction for all isopycnals), the flux into (6) must be divided by the bottom slope. At small slope angles, the slope angle θ is approximately equal to the bottom slope, and the boundary condition on (6) at the slope is

$$A_h\frac{\partial q}{\partial x} = \frac{J^n}{\frac{\partial H}{\partial x}} \approx N^2\frac{\tau_{bot}^y}{\delta\rho_0} \qquad (S12)$$

Supplementary information: Animations

Figures 6 and 7 are provided as animations over time from 1 to 60 days in the files figure6_animated.mp4 and figure7_animated.mp4.

Supplementary information: Reduced physics model

The reduced physics model code, and the code that generates Figure 8, are included in ReducedPhysicsModelCode.zip