Rate and State Friction as a Spatially Regularized Transient Viscous Flow Law

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Key Points:

- We reformulate the empirical rate and state friction law as a bulk viscous flow law in terms of anelastic shear strain rate.
- We show how mesh independence is achieved by including a gradient-like non-local anelastic shear strain rate equivalent.
- We show analytically and numerically that the proposed continuum model closely reproduces existing results of rate and state friction.

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Abstract

The theory of rate and state friction unifies field, laboratory, and theoretical analysis of the evolution of slip on natural faults. While the observational study of earthquakes and aseismic fault slip is hampered by its strong multi-scale character in space and time, numerical simulations are well-positioned to link the laboratory study of grain-scale processes to the scale at which rock masses move. However, challenges remain in accurately representing the complex and permanently evolving sub-surface fault networks that exist in nature. Additionally, the common representation of faults as interfaces may miss important physical aspects governing volumetric fault system behavior. In response, we propose a transient viscous rheology that produces shear bands that closely mimic the rate- and state-dependent sliding behavior of equivalent fault interfaces. Critically, we show that the expected tendency of the continuum rheology for runaway localization and mesh dependence can be halted by including an artificial diffusion-type regularization of anelastic strain rate in the softening law. We demonstrate analytically and numerically using a simplified fault transect that important aspects of the frictional behavior are not significantly affected by the introduced regularization. Any discrepancies with respect to the interfacial description of fault behavior are critically evaluated using 1D numerical velocity stepping and spring-slider experiments. Since no new physical parameters are introduced, our model may be straightforwardly used to extend the existing modeling techniques. The model predicts the emergence of complex patterns of shear localization and delocalization that may inform the interpretation of complex damage distributions observed around faults in nature.

Plain Language Summary

How, where, and when earthquakes nucleate is one of the great questions in science and society that, despite steady progress, has hardly been answered to any practical degree. Based on field observations, laboratory experiments, and theoretical work it is believed that a cocktail of escalating mechanical, chemical, and thermal grain-scale processes cause the sudden and rapid onset of earthquakes. The net effect of these processes are characterized by an immediate strengthening and a gradual weakening response to deformation and are unified in simplified form in the theory of 'rate and state friction'. This theory is commonly used in computer simulations of earthquake sequences. We point out that rate and state friction, unlike some physical theories of earthquake

rupture, does not incorporate a diffusion process such as for example heat conduction. We show the introduction of an artificial diffusion process can prevent the mathematical reduction of a fault zone to a two-dimensional interface while retaining the properties of the original friction law. This in turn enables simulation techniques that rely on an interface-free description of the earth and promise to provide new insights into the spontaneous organization of seismic and aseismic phenomena in developing fault zones.

1 Introduction

Over the second half of the last century the study of the sliding behavior of frictional surfaces, such as those believed to occur in the earth's crust, has led to a general understanding that it is governed by competition of stabilizing viscous-like effects and potentially destabilizing processes that affect the texture of the interface in a time-dependent manner and self-organize into periods during which the interface is arbitrarily close to elastic stick punctuated with periods of anelastic slip (Rabinowicz, 1958; Brace & Byerlee, 1966; Bowden & Tabor, 1966; Scholz et al., 1972; Ida, 1972; Dieterich, 1978; Ruina, 1980, 1983; Dieterich & Kilgore, 1994).

1.1 Rate- and State-dependent friction

This has led to the proposition of the phenomenological *rate- and state-dependent friction law* by Dieterich (1978, 1979a); Ruina (1980, 1983), which reads in its general form as

$$f = f_0 + a \log V + b \log \Theta. \tag{1}$$

In this law, the friction coefficient f is given by the linear combination of a reference friction coefficient f_0, a logarithmic contribution from dimensionless slip rate V multiplied by a coefficient a, and another logarithmic contribution from a dimensionless *state* variable Θ multiplied by a coefficient b. All quantities are taken to be positive.

The rate-effect or *direct effect* is to a variable degree strengthening with increasing slip rate and thus exerts a stabilizing influence on the frictional interface for any strictly positive a. The *evolution effect* is to be governed by an evolution law that has the evolving steady state $\Theta_{\mathrm{ss}} V(t) = 1$. Then, if $b > a$, this allows an externally loaded system with a frictional interface to move to a configuration with lower elastic strain energy and thus a frictional-mechanical instability may occur.

One of the most widely used state evolution laws is the *aging law* of Ruina (1980, 1983), given by

$$\dot{\Theta} = r_0 \left(1 - V\Theta\right), \tag{2}$$

with r_0 a rate constant that is commonly expressed by dividing the reference velocity v_0 by a critical slip distance d_c. It should be noted that several other evolution laws that govern the state variable have been proposed. The most commonly cited contender is the *slip law* (Dieterich, 1979a; Ruina, 1980, 1983). In this work we do not consider it because its functional form ($\dot{\Theta} \propto \Theta V \log \Theta V$) is unfortunately not amenable to the type of analytical treatment that is performed. We note that rate and state friction captures only the first-order behavior of the stick-slip cycle and that laboratory experiments have brought to light many secondary effects (e.g. Ruina, 1983; Marone, 1998; Mair & Marone, 1999; Chester, 1994; Passelègue et al., 2020).

Following Amonton's law the friction coefficient is expressed as the ratio of shear stress τ to normal stress σ, implying a cohesionless fault. It is further assumed that the fault is always critically loaded. This assumption is necessary because (1) degenerates at $V = 0$, but may also be realistic for faults in nature (Bak & Tang, 1989).

The rate and state friction law is frequently applied in numerical studies of fault slip, whether that be a study over the course of a single earthquake or a complex sequence of slip transients (e.g. Dieterich, 1979b; P. G. Okubo, 1989; Rice, 1993; Rice & Ben-Zion, 1996; Ben-Zion & Rice, 1997; Lapusta et al., 2000; Lapusta & Liu, 2009; Rubin & Ampuero, 2005; Gabriel et al., 2012; Jiang & Lapusta, 2016; D. Li & Liu, 2017; Erickson et al., 2020). The use of interfacial friction laws in these models necessitates the treatment of faults as mesh features on which internal boundary conditions can be applied. It can be laborious to construct such meshes for non-trivial fault geometries, and the procedure does not scale well with increasing fault network complexity. There is currently no established method to construct meshes with time-dependent geometry and topology reflecting an evolving fault system, but we refer to K. Okubo et al. (2019) for work in this direction. With some exceptions (e.g. D. Li & Liu, 2017; Sathiakumar et al., 2020; Perez-Silva et al., 2021) many studies focus on the already complex behavior of long-term sequences of slip transients on linear or planar faults in simple domains.

1.2 Continuum fault rheology

There is strong field and experimental evidence that faults in nature are not infinitely thin planes but consist of complex evolving networks of strongly localized shear zones within a wider region of damaged host rock (Tchalenko, 1970; Granier, 1985; Katz et al., 2004; Savage & Brodsky, 2011; Faulkner et al., 2011; Barth et al., 2013; Passelègue et al., 2016; Perrin et al., 2016; Locatelli et al., 2018, 2019; Pozzi et al., 2018, 2019, 2021; Ritter, Rosenau, & Oncken, 2018; Ritter, Santimano, et al., 2018; Ross et al., 2019). This knowledge has spurred a search for intermediate-scale continuum homogenizations of distributed micro-scale processes that obey the rate and state framework in a somewhat generalized sense. Currently proposed models focus on rheological feedback mechanisms involving temperature (Braeck & Podladchikov, 2007; John et al., 2009; Roubíček, 2014; Thielmann et al., 2015; Thielmann, 2018; Pozzi et al., 2021), grain size (Rozel et al., 2011; Thielmann et al., 2015; Thielmann, 2018; Barbot, 2019; Pozzi et al., 2021), porosity (Sleep, 1997; Niemeijer & Spiers, 2007; Chen & Spiers, 2016; Van den Ende et al., 2018), fluid pressure (Poulet et al., 2014; Veveakis et al., 2014; Alevizos et al., 2014; Rattez, Stefanou, & Sulem, 2018; Rattez, Stefanou, Sulem, Veveakis, & Poulet, 2018), damage (Lyakhovsky et al., 2011; Lyakhovsky & Ben-Zion, 2014b, 2014a; Lyakhovsky et al., 2016; Kurzon et al., 2019, 2020), or combinations thereof. Continuum models employing a rate and state formulation with the slip rate and tractions replaced by strain rate and stress invariants have been developed by Herrendörfer et al. (2018) and applied to self-organizing fault systems in Preuss et al. (2019, 2020). Since the state of the subsurface is typically difficult to accurately characterize on a regional basis, these models are important to develop a generic understanding of the long-term evolution of seismogenic fault systems.

A problem commonly associated with strain softening rheologies without internal length scale is unconstrained localization (Hobbs et al., 1990). As a shear zone localizes to the scale of a mesh element or grid cell they become poorly resolved and cause mesh-dependence of the simulation. Models that possess an internal length scale perpendicular to the direction of shear may not suffer from this issue, provided that this length scale is resolved by the discretization. For example, as thermo-rheological feedback mechanism causes localization the heat diffusion term may gain prominence until localizing and delocalizing (diffusing) influences find a balance. Unfortunately some of the proposed fault rheologies don't possess an internal length scale, and some possess one that has physical meaning and can't be changed without changing the behavior of interest. If such a

physical length scale is very small in nature, then that may place insurmountable constraints on the mesh resolution in two- or three-dimensional regional simulations. A few models *do* contain a controllable internal length scale; these are the unified rate and state friction theory of Sleep (1997) and the damage-breakage rheology of Lyakhovsky and Ben-Zion (2014b, 2014a); Lyakhovsky et al. (2016); Kurzon et al. (2019, 2020).

1.3 Objective

In summary, we state the need for a rigorous treatment of a continuum framework in which Dieterich–Ruina-type rate and state friction can be embedded based on the following observations:

1. Rate and state friction is a useful and powerful homogenization of the complex micro-scale processes that give rise to slip transients in nature.
2. Numerical modeling exploration of the complex and evolving relation that fault networks have with their tectonic environment is hampered by the long-standing challenges with mesh adaptive generation as well as lack of detailed knowledge of the structure of the subsurface.
3. The description of fault friction as a continuum process can be an elegant and practical way to avoid these problems provided they possess a controllable internal length scale in order to produce sensible results given reasonable computational resources.
4. Much effort has been dedicated to understanding rate and state friction as an ingredient in numerical models – not all proposed continuum rheologies can equally benefit from this.

In response, we develop a Dieterich–Ruina-type continuum reformulation of rate and state friction that

1. possesses a controllable internal length scale λ that enforces a constraint on localization,
2. yields mesh-independent results once λ is sufficiently well resolved, and
3. retains the parameters to the interfacial friction (1) and evolution (2) laws and yields similar trajectories of V and f for the same choice of parameter values.

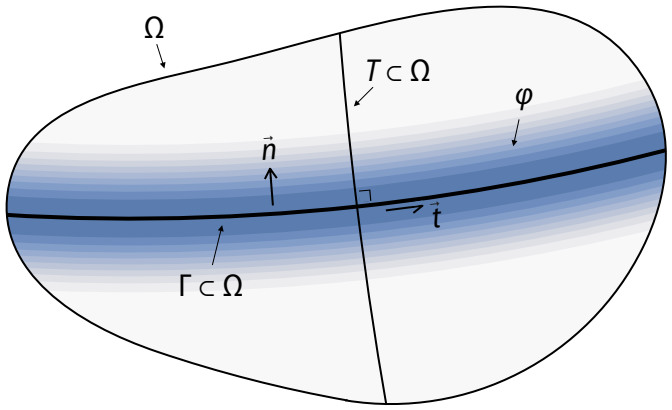

Figure 1. A region Ω in the enclosing a portion of a fault interface Γ, with normal and tangent vectors $\vec{n}$ and $\vec{t}$ indicated. A transect T is drawn perpendicular to the fault. The fault interface may be substituted by a shear band on which anelastic shear strain is distributed according to $\varphi(\vec{x})$, as indicated by the colored contours.

The model we propose here bears close relation to the earlier works of Sleep (1997) and Herrendörfer et al. (2018). Contrary to Sleep (1997) we refrain from in-depth discussion of fault physics but concentrate on the introduced spatial regularization, which is also a necessary extension of Herrendörfer et al. (2018).

In Section 2 we will motivate our choice of constitutive model, focusing on the formal links between interfacial and continuum models. In Section 3 we will present our equivalent continuum rate and state rheology. In Section 4 we build our rheology into a numerical model of a simplified one-dimensional fault transect to demonstrate both its localization and delocalization behavior as well as its similarities and differences with respect to the original laws of Dieterich and Ruina. We close with a discussion and summary in Sections 5 and 6.

2 Material model and continuum mechanics

We will first describe how the existing interfacial description of rate and state friction can be embedded in a continuum mechanical framework with internal discontinuities, and then generalize this to internal friction and distributed anelastic deformation in an interface-free model.

2.1 A fault reference frame

We consider the immediate neighborhood $\Omega \subset \mathbb{R}^3$ of a section of fault $\Gamma \subset \Omega$ (Figure 1) that is described as the set of points belonging to a differentiable 2D manifold. We invoke the existence of a surjective function $\vec{x}_f(\vec{x})$ that maps a coordinate $\vec{x} \in \Omega$ to its nearest Euclidian neighbor $\vec{x}_f \in \Gamma$. Next, we define the fault transect $T(\vec{x}_f) \subset \Omega$ as the 1D curve that includes all the points $\vec{x} \in \Omega$ that map to a particular fault coordinate $\vec{x}_f \in \Gamma$, in short $T = \vec{x}_f^{-1}(\{\vec{x}\}) \subset \Omega$. Finally, we let $\hat{n}(\vec{x}) \in \mathbb{R}^3,\ \vec{x} \in \Omega$ be a unit vector field locally tangent to the transecting curve $T(\vec{x}_f(\vec{x}))$ and therefore normal to the fault.

The *state* of the body Ω is described by a symmetric Cauchy stress tensor field $\boldsymbol{\sigma}(t, \vec{x}) \in \mathbb{R}^3 \otimes \mathbb{R}^3$ and a velocity vector field $v_0\, \vec{v}(t, \vec{x}) \in \mathbb{R}^3$. Their governing equations are stated after the introduction of some further scaffolding. Let t denote time.

We define the normal traction $\sigma = -\hat{n} \cdot \boldsymbol{\sigma} \cdot \hat{n}$ (positive in compression), the shear traction vector $\vec{\tau} = \boldsymbol{\sigma} \cdot \hat{n} + \sigma\, \hat{n}$, and the shear traction magnitude $\tau = \sqrt{\vec{\tau} \cdot \vec{\tau}}$. The rupture process is most efficient when the slip rate vector $\vec{V} \in \mathbb{R}^3$ is parallel to the in-plane shear traction vector $\vec{\tau}$, and so alongside the full velocity field $\vec{v}$ we define a purely slip-induced velocity double couple field $\vec{v}_{\rightleftharpoons}$ around the coordinate $\vec{x}$ as

$$\vec{v}_{\rightleftharpoons}(t, \vec{x}) := \tfrac{1}{2} V(t, \vec{x}_f)\, \hat{t}(\vec{x}_f)\, \mathrm{sgn}(\hat{n}(\vec{x}_f) \cdot [\vec{x} - \vec{x}_f]), \tag{3}$$

with $\hat{t} \in \mathbb{R}^3 := \vec{\tau}/\tau$ the unit tangent vector and sgn the sign function, which has $\mathrm{sgn}(0) = 0$.

2.2 Assumptions

In the following, we will assume that continuum processes that occur around the fault are predominantly confined to the small neighborhood Ω, which itself includes only a small section of a whole fault. The neighborhood Ω is assumed to be sufficiently small that in-plane variations of fault properties such as state, slip rate, and curvature are negligible. This assumption comes at a loss of generality, especially around fault kinks, branches, and tips, but does not neccesarily limit the applicability of our formulation. We will revisit this assumption in the Section 5.5 of the Discussion.

2.3 Generalization of a fault to a shear band

We now generalize the fault interface to a distributed shear band by substituting a smoothed step function $\Phi = \Phi(t, x; \lambda_0) \in [-1, +1]$, $\lambda_0 > 0$ (e.g. $\Phi(t, x; \lambda_0) := \tanh(x/\lambda_0)$) for the sign function in (3). We require that the chosen function converges in a pointwise manner to the sign function as $\lambda_0 \to 0^+$, such that (3) can be seen as the result of taking the limit

$$\begin{aligned} \vec{v}_{\rightleftharpoons}(t, \vec{x}; \lambda_0) &:= \tfrac{1}{2} V(t, \vec{x}_f)\, \hat{t}(\vec{x})\, \Phi\left(t, \hat{n} \cdot \left[\frac{\vec{x} - \vec{x}_f}{\lambda_0}\right]\right) \\ \vec{v}_{\rightleftharpoons}(t, \vec{x}) &:= \lim_{\lambda_0 \to 0^+} \vec{v}_{\rightleftharpoons}(t, \vec{x}; \lambda_0)\,. \end{aligned} \tag{4}$$

Recognizing that fault slip is an anelastic process, the anelastic strain rate tensor $\dot{\boldsymbol{\varepsilon}}_{\neg \mathrm{e}}$ is defined to be the symmetric gradient ∇^s of the (continuous) slip-induced velocity field $\vec{v}_{\rightleftharpoons}(t, \vec{x}; \lambda_0)$:

$$\begin{aligned} \dot{\boldsymbol{\varepsilon}}_{\neg \mathrm{e}}(t, \vec{x}) &:= v_0 \nabla^s \vec{v}_{\rightleftharpoons}(t, \vec{x}; \lambda_0) := \tfrac{1}{2} v_0 \left[(\nabla \vec{v}_{\rightleftharpoons})^{\mathrm{T}} + (\nabla \vec{v}_{\rightleftharpoons})\right](t, \vec{x}; \lambda_0) \\ &\approx \frac{1}{2} \frac{v_0}{\lambda_0} V(t, \vec{x}_f) \varphi\left(t, \hat{n} \cdot \left[\frac{\vec{x} - \vec{x}_f}{\lambda_0}\right]\right) \left[\hat{t} \otimes \hat{n} + \hat{n} \otimes \hat{t}\right](\vec{x}) =: \gamma_0 \gamma(t, \vec{x}) \hat{\boldsymbol{s}}(\vec{x}). \end{aligned} \tag{5}$$

Here the *equivalent anelastic shear strain rate* (a scalar value) has been introduced as $\gamma_0 \gamma(t, x) = v_0 V(t) \varphi(t, x)$, with reference value $\gamma_0 = v_0/\lambda_0$ and *strain rate distribution* $\varphi(t, x; \lambda_0) = \partial_x \Phi(t, x; \lambda_0)$ (illustrated in blue shading in Figure 1). The symbol $\hat{\boldsymbol{s}}$ denotes the unit shear sense tensor $\hat{\boldsymbol{s}} = \frac{1}{2}\left[\hat{t} \otimes \hat{n} + \hat{n} \otimes \hat{t}\right]$, and its *Frobenius norm* $\|\hat{\boldsymbol{s}}\| = \sqrt{\mathrm{tr}\, \hat{\boldsymbol{s}}^{\mathrm{T}} \hat{\boldsymbol{s}}} = 1/\sqrt{2}$, which is an important property to maintain when generalizing $\hat{\boldsymbol{s}}$ later on because it is compatible with the interpretation of $\gamma_0 \gamma$ as the fault-perpendicular derivative of fault-parallel velocity, i.e. as an accurate measure of simple shear. In the derivation of (5) the gradients of V and the orthonormal bases $\hat{t}$ and $\hat{n}$ have been dropped under the assumption set out in Section 2.2 that they are small compared to the gradient of Φ. For this assumption to be met, the across-fault length scale λ_0 must be small compared to the along-fault length scales that exists in the interfacial rate and state formulation.

Since the derivative of half a smoothed step function $\frac{1}{2}\Phi(x) \in [-\frac{1}{2}, +\frac{1}{2}]$ has the properties of a distribution – is a non-negative and importantly integrates to unity over its domain – we may express the integral relation

$$\mathcal{V}(\gamma) := \gamma_0 \int_T \gamma(t, \vec{x}) \mathrm{d}x = v_0 V(t, \vec{x}_f) \tag{6}$$

over the fault transect T, which is again compatible with the notion that $\gamma_0\gamma$ is a measure of simple shear in the fault reference frame. We note that the above integral strictly requires that all anelastic strain occurs over the length of the fault transect T, implying that the anelastic strain rate distribution has compact support in space. In practice we can relax that requirement if said distribution is narrow and thin-tailed.

Subtracting the anelastic shear strain rate tensor $\dot{\boldsymbol{\varepsilon}}_{\neg\mathrm{e}}$ from the full strain rate tensor $\dot{\boldsymbol{\varepsilon}} = \nabla^s \vec{v}$ results in the elastic strain rate tensor. This additive decomposition of elastic and anelastic strains is known as the Maxwell model. Application of Hooke's law of linear elasticity to the elastic strain rate tensor leads to

$$\dot{\boldsymbol{\sigma}} = \boldsymbol{S}\ [\dot{\boldsymbol{\varepsilon}} - \gamma_0\gamma\hat{\boldsymbol{s}}] \tag{7}$$

with fourth-order elastic stiffness tensor $\boldsymbol{S}$ and the dot over a symbol denoting the time derivative. The relation above is standard in the context of elasto-plasticity (e.g. Mühlhaus & Aifantis, 1991), in which $\gamma_0\gamma$ is known as the plastic multiplier and $\hat{\boldsymbol{s}}$ coincides with the derivative of the magnitude of shear traction τ with respect to the full stress tensor $\boldsymbol{\sigma}$. In this case τ is to be interpreted as the *non-associated* plastic potential of a material that does not undergo anelastic volume change.

For completeness we list ordinary differential equation that governs $\vec{v}$, the *momentum balance equation*:

$$v_0\dot{\vec{v}} - \vec{g} = \rho^{-1}\nabla\cdot\boldsymbol{\sigma}. \tag{8}$$

Here $\vec{g}$ is the acceleration due to gravity and ρ the mass density. Gravity may be ignored in the remainder of this work at no loss of generality and the inertial term $\dot{\vec{v}}$ may be ignored at *some* loss of generality under the assumption of static momentum balance.

2.4 Plasticity and coordinate invariance

For a spontaneously developing fault zone the shear and normal stress can not be defined in relation to a known plane. Plasticity models avoid this problem through the use of stress tensor invariants and scalar parameters. The Mohr-Coulomb plasticity model generates anelastic shear strain parallel to the unit shear sense tensor $\hat{\boldsymbol{s}} = \frac{1}{2}(\hat{n}\otimes\hat{t}+\hat{t}\otimes\hat{n})$, in which $\hat{t}$ makes an angle $\phi = \tan^{-1} f$ with respect to the largest principal stress axis $\vec{\sigma}_1$, and $\hat{t}$ and $\hat{n}$ form an orthogonal triad of vectors together with the intermediate principal stress axis $\vec{\sigma}_2$. In other words, anelastic shear strain occurs in a plane perpen-

dicular to $\vec{\sigma}_2$. The corresponding cohesionless yield criterion is written as

$$\sigma_1 - \sigma_3 = -\sin(\phi)\,(\sigma_1 + \sigma_3), \tag{9}$$

with σ_1 and σ_3 the magnitudes of the largest and smallest principal stresses $\vec{\sigma}_1$ and $\vec{\sigma}_3$. We note that the angle of a shear band that is generated does not need to be parallel to the direction of shear strain (Vardoulakis, 1980; Vermeer, 1990; Kaus, 2010; Le Pourhiet, 2013), and moreover that a local change in friction is not expected to immediately alter the macroscopic fault angle (e.g. Preuss et al., 2019).

In the Mohr-Coulomb model, the friction coefficient f of a yielding material may be expressed as a function of the principal stresses σ_1 and σ_3 as

$$f(\boldsymbol{\sigma}) = \frac{1}{2}\frac{\sigma_1(\boldsymbol{\sigma}) - \sigma_3(\boldsymbol{\sigma})}{\sqrt{\sigma_1(\boldsymbol{\sigma})\sigma_3(\boldsymbol{\sigma})}} \tag{10}$$

Herrendörfer et al. (2018) used the Drucker-Prager model (Drucker & Prager, 1952) as a simple and smooth approximation to Mohr-Coulomb plasticity. The model is defined in terms of the straightforwardly computable invariants

1. pressure $p = -\frac{1}{3}\operatorname{tr}\boldsymbol{\sigma}$, and
2. effective shear stress τ_e proportional to the *Frobenius norm* of the deviatoric stress tensor $\boldsymbol{\tau} := \boldsymbol{\sigma} + p\,\boldsymbol{\delta}$, with $\boldsymbol{\delta}$ the Kronecker delta:

$$\tau_\mathrm{e} := \frac{\|\boldsymbol{\tau}\|}{\sqrt{2}} = \sqrt{\tfrac{1}{2}\operatorname{tr}\left(\boldsymbol{\tau}^\mathrm{T}\cdot\boldsymbol{\tau}\right)}. \tag{11}$$

In this model the existing definition of the unit shear sense tensor $\hat{\boldsymbol{s}}$ is no longer applicable because the unit vectors $\hat{t}$ and $\hat{n}$ are not known *a priori*, and are not even uniquely constrained *a posteriori*. Keeping in line with plasticity theory, $\hat{\boldsymbol{s}}$ is redefined as the stress derivative of the non-associated plastic potential τ_e:

$$\hat{\boldsymbol{s}} := \frac{\partial \tau_\mathrm{e}}{\partial \boldsymbol{\sigma}} = \frac{\boldsymbol{\tau}}{\tau_\mathrm{e}}.$$

This definition still satisfies the criterion set out in the preceding section that $\|\hat{\boldsymbol{s}}\| = 1/\sqrt{2}$.

We calibrate the Drucker-Prager model to the Mohr-Coulomb model around a reference stress state $\boldsymbol{\sigma}_0$ of simple shear combined with isotropic compression, which encompasses all stress states possible in 2D plane strain and may be considered the most relevant stress state even in 3D tectonic settings. Let $\vec{\sigma}_0$ be given in ordered principal

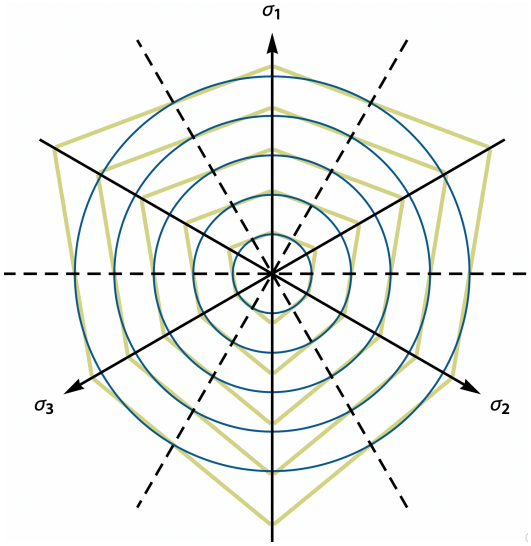

Figure 2. Comparison of Mohr-Coulomb (thick yellow lines) and Drucker-Prager yield envelopes (thin blue lines) at increments of pressure. The field of view is the octahedral plane of principal stress space. Projections of principal stress axes are indicated with arrows, and projections of planes of simple shear are indicated with dashed lines. A friction coefficient of 0.3 was used to generate this figure.

stress space by

$$\vec{\sigma}_0 := -p_0 \begin{bmatrix} 1 \\ 1 \\ 1 \end{bmatrix} + \tau_{\mathrm{e},0} \begin{bmatrix} +1 \\ 0 \\ -1 \end{bmatrix}.$$

At $\vec{\sigma} = \vec{\sigma}_0$, Mohr-Coulomb and Drucker-Prager measures of effective shear and normal stress coincide (up to a sign) and thus the Drucker-Prager model may be written as

$$\tau_{\mathrm{e}}(\boldsymbol{\sigma}) = \sin(\phi)\, p(\boldsymbol{\sigma}). \tag{12}$$

Away from $\vec{\sigma}_0$ (and towards a uniaxial stress state) the Drucker-Prager yield surface becomes an increasingly worse approximation to the Mohr-Coulomb yield surface (Figure 2). Higher-order approximations like the Willam-Warnke yield envelope (Ulm et al., 1999) may be considered too. The yield surfaces discussed in this section serve as part of a general model proposition – in the one-dimensional numerical experiments considered in this work only the two effective stresses of Amonton's friction law are defined.

For a material that is known to be in a state of plastic yielding, the friction coefficient $f = \tan\phi$ may also be expressed from (12) as a function of the stress tensor $\boldsymbol{\sigma}$

as

$$f(\boldsymbol{\sigma}) = \tau_{\mathrm{e}}(\boldsymbol{\sigma}) \left[p(\boldsymbol{\sigma})^2 - \tau_{\mathrm{e}}(\boldsymbol{\sigma})^2 \right]^{-\frac{1}{2}} . \tag{13}$$

The friction coefficient $f(\boldsymbol{\sigma})$ equals one when the pressure $p(\boldsymbol{\sigma}) = 0$ and equals zero when the effective stress $\tau_{\mathrm{e}}(\boldsymbol{\sigma}) = 0$. The friction coefficient falls in the interval $[0, 1]$ at any state of stress $\boldsymbol{\sigma}$, but has discontinuous derivatives at aforementioned zeroes of p and $f(\boldsymbol{\sigma})$.

Despite adopting a plasticity framework to enforce the frictional yield constraint we emphasise that the resulting model is still best regarded as 'frictional viscous' rather than frictional plastic because it lacks a distinct boundary between yielding and non-yielding states in space and time – the yield constraint (an equality, not an inequality) is enforced everywhere and anytime, and at any stress.

3 Continuum evolution law

Our goal is to find a continuum state evolution law that produces nearly the same history of loading and sliding as the original formulation of Dieterich and Ruina on a discrete fault. Like $V(t)$ is a 'global' measure of $\gamma(t, x)$ over a fictitious fault transect T, we introduce a local variable $\theta(t, x)$ of which $\Theta(t)$ is a global measure. We target continuum friction and evolution laws expressed in terms of γ and θ of the form

$$f = f_0 + a \log \gamma + b \log \theta, \tag{14}$$

$$\dot{\theta} = r_0 \left(c_2 - c_1 \mathcal{M}(\gamma)\, \theta \right), \tag{15}$$

with $\mathcal{M}$ a smoothing operator that will be further elaborated in Section 3.3 and c_1 and c_2 additional coefficients that are required to calibrate (14) and (15) to their respective interfacial equivalent. The local friction and evolution laws (14) and (15) retain the structure of their interfacial counterparts (1) and (2) and generalize the continuum formulations of Sleep (1997) and Herrendörfer et al. (2018). We will revisit this connection in the Section 5.4 of the Discussion. The targeted form of the continuum equations does not come out of the blue. In coming sections we hope to make clear how it arises.

3.1 Analytical framework

By

1. reorganizing the rate and state friction law (1) into a definition of $\Theta(V, f)$,
2. differentiating (1) with respect to time and reorganizing it into a definition of $\dot{V}(t)$, and
3. substituting the definition of $\Theta(V, f)$ and the definition of $\dot{\Theta}$ that follows from the aging law (2) into the definition of $\dot{V}(t)$,

we arrive at the ordinary differential equation

$$a\,\dot{V} = b\,r_0\left[V^2 - V\,\Theta(V,f)^{-1}\right] + V\dot{f}, \tag{16a}$$

$$\Theta(V,f)^{-1} := V^{\frac{a}{b}}\exp\left(-[f-f_0]/b\right) \tag{16b}$$

in which the friction coefficient $f = f(\boldsymbol{\sigma}; \hat{t}, \hat{n})$ is treated as a property of – rather than a constraint on – the stress tensor $\boldsymbol{\sigma}$.

By applying the same procedure to the proposed continuum friction and aging laws (14) and (15) we obtain the analogous ODE for γ:

$$a\,\dot{\gamma} = b\,r_0\left[c_1\gamma\mathcal{M}(\gamma) - c_2\gamma\,\theta(\gamma,f)^{-1}\right] + \gamma\dot{f}, \tag{17a}$$

$$\theta(\gamma,f)^{-1} := \gamma^{\frac{a}{b}}\exp\left(-[f-f_0]/b\right), \tag{17b}$$

in which $f = f(\boldsymbol{\sigma})$ is given by the yield criterion (13) and $\dot{f} = \dot{f}(\boldsymbol{\sigma}, \dot{\boldsymbol{\sigma}})$ is expressible using the same yield criterion and the elastic constitutive equation (7).

Equations (16a,b) and (17a,b) are useful in mathematical and numerical analysis of the problem because they eliminate a variable and an algebraic constraint, and at the same time provide an ideal reference frame for establishing, and if necessary influencing, the ability of (17a,b) to reproduce the predictions made by (16a,b) in the sense of the integral relation (6).

In the following derivation we will assume that the field of effective friction $f(\vec{x}), \vec{x} \in \Omega$ in the neighborhood of a point $\vec{x}_f$ on the fault is the same whether it is generated by a hard or a soft discontinuity. This assumption ties into the principal assumption set out in Section 2.2, which is revisited in Discussion section 5.5. Combining (16a,b), (17a,b), and (6) leads to the following useful relation that describes the evolution of the anelastic shear strain rate distribution over time:

$$\dot{\varphi} \propto V\left[\lambda_0\,c_1\,\varphi\,\mathcal{M}(\varphi) - \varphi\right] - \Theta(V,f)^{-1}\left[\lambda_0^{\frac{a}{b}}\,c_2\,\varphi^{1+\frac{a}{b}} - \varphi\right]. \tag{18}$$

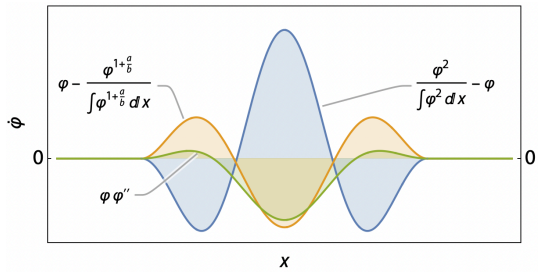

Figure 3. Influence of the different terms in Eq. (18) on the time derivative $\dot{\varphi}$. The first term (blue) can be seen to promote localization, while the second term (orange) favors the opposite. Finally the delocalizing influence of the nonlinear Laplacian term $\varphi(x)\varphi''(x)$ is shown in green. The input function φ follows a cosine-squared distribution.

This differential equation tells us several things. Most importantly, for γ to remain in accordance with V through the integral relation (6) over the transect T, the same integral of the left hand side of (18) must be zero. This requirement can only be met at abitrary (V, f) if both the term multiplied by V and the term multiplied by $\Theta(V, f)^{-1}$ integrate to zero over T. This in turn can not in general be guaranteed unless

$$c_1 := \mathcal{C}_1(\varphi) := \lambda_0^{-1} \left[\int \varphi(x) \mathcal{M}(\varphi)(x) \, \mathrm{d}x \right]^{-1}, \tag{19a}$$

$$c_2 := \mathcal{C}_2(\varphi) := \lambda_0^{-\frac{a}{b}} \left[\int \varphi(x)^{1+\frac{a}{b}} \, \mathrm{d}x \right]^{-1}. \tag{19b}$$

We consider it undesirable for coefficients to depend in a time-dependent way on integrals of the modeled quantities and therefore will be restricting our attention to specific regimes of interest during which the values of the coefficients $c_{1,2}$ can be predicted analytically. We then employ those predicted values as model-specific constants in time and space. We accept that this incurs a potential error whenever the state of the model is outside the selected regime, and will critically evaluate this error using numerical models in Section 4.

Assuming the coefficients $c_{1,2}$ have been chosen appropriately and for simplicity that $\mathcal{M}(\varphi) = \varphi$, two additional observations can be made on the basis of (18):

1. The term that is multiplied by V promotes localization if $\mathcal{M}(\varphi) = \varphi$ because φ^2 is a narrower distribution than φ (Figure 3).

2. The opposite (delocalization; Figure 3) holds true for the term that is multiplied by $\Theta(V,f)^{-1}$ on account of the corresponding minus sign and the requirement that $a, b > 0$.

This localizating and delocalizing behavior is of great interest and will be treated in detail in the following sections.

In the sections that follow we will assume to be operating on the fault-perpendicular transect T and will denote with the scalar coordinate x the distance with respect to the fault core, i.e. $x = \hat{n}(\vec{x}) \cdot [\vec{x} - \vec{x}_f(\vec{x})]$. This setting is in line with our assumption set out in Section 2.2 that accross-fault variations in anelastic strain rate are more compact than its along-fault variations.

3.2 Runaway sliding and localization

Earthquakes occur as superexponential solutions to (16a,b). Prior to inertial damping the seismic slip rate V behaves as

$$\dot{V} \propto V^2.$$

This ODE has the solution

$$V(t) \propto \left(1 - \frac{t}{t_*}\right)^{-1}, \tag{20}$$

in which t_* is the time of the (hypothetical) singularity.

The same behavior occurs in (17a,b) in the limit

$$\dot{\gamma} \propto c_1 \gamma \, \mathcal{M}(\gamma), \tag{21}$$

and in (18) in the limit

$$\dot{\varphi} \propto V\big[\lambda_0 \, c_1 \, \varphi \, \mathcal{M}(\varphi) - \varphi\big].$$

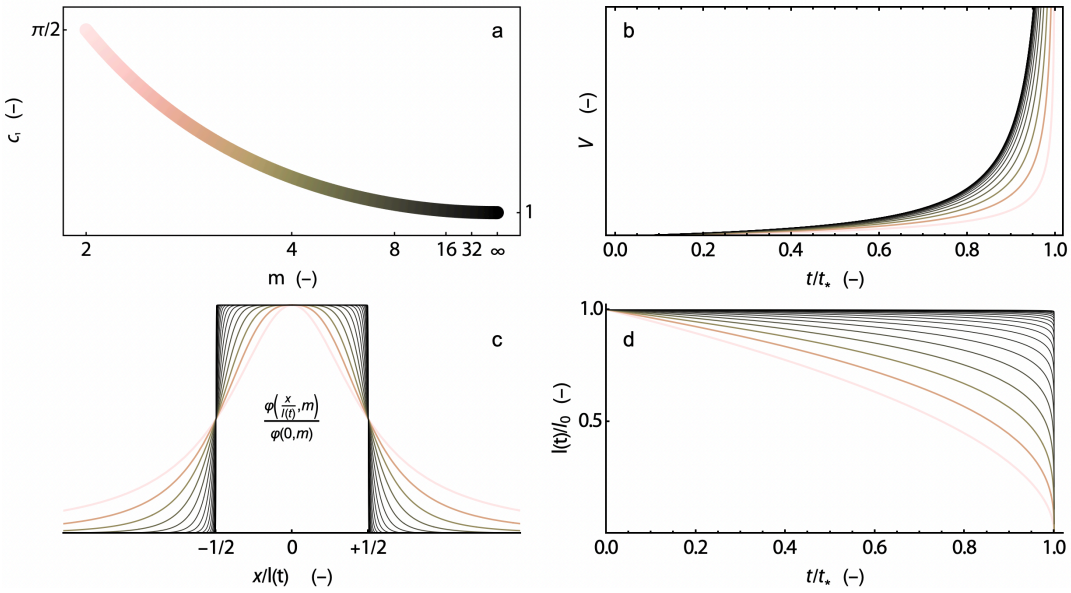

Figure 4. Asymptotic solutions during earthquake nucleation. a) Value of the dimensionless coefficient $c_1(m)$ versus distribution exponent m (see main text) represented on a reciprocal axis that ranges between 2 and ∞. The thick line acts also as a color bar approximately indicating the value of m at which the individual lines in the remaining figures are plotted. b) Dimensionless slip rate V versus dimensionless time t/t_* to the slip rate asymptote for different values $m \in \{2^{1.0}, 2^{1.5}, 2^{2.0} \ldots\}$. c) Nondimensionalized distributions $\varphi(x/l(t); m)/\varphi(0; m)$ versus dimensionless coordinate $x/l(t)$. d) Dynamic dimensionless length scale $l(t/t_*)/l_0$ versus dimensionless time t/t_*.

Assuming $\mathcal{M}(\square) = \square$, (21) can be shown to be solved by

$$V(t) \propto \left(1 - \frac{t}{t_*}\right)^{m^{-1}-1}, \tag{22a}$$

$$\varphi(t, x; m) := \left[l(t)c_1(m)\left(1 + \left|\frac{2x}{l(t)}\right|^m\right)\right]^{-1}, \qquad m \in \mathbb{R} \geq 2 \tag{22b}$$

$$l(t) \propto V(t)^{(1-m)^{-1}}, \tag{22c}$$

$$\gamma_0\gamma(t, x; m) := v_0 V(t)\varphi(t, x; m), \tag{22d}$$

$$c_1 = c_1(m) := \frac{\pi}{m}\csc\frac{\pi}{m}, \tag{22e}$$

in which the family of strain rate distributions $\varphi(t, x; m)$ generated by the exponent $m \in \mathbb{R} \geq 2$ (Figure 4c) evolves in accordance with a time-variable characteristic width $l(t)$ (Figure 4d). We note the following:

1. In the limit $m \to \infty$, the distribution $\varphi(t, x; m \to \infty)$ reduces to a uniform distribution on the constant interval $x \in [-l(0)/2, +l(0)/2]$. The parameter $c_1(m \to \infty) = 1$ (Figure 4a) and the solution (22a) reduces to the interfacial solution (20). However this limit also requires infinite mesh resolution or alternatively the definition of two mesh discontinuities, which defeats our intent.
2. In the other end-member case where $m = 2$, the distribution $\varphi(t, x; m = 2)$ is known as the Cauchy distribution, and its characteristic width $l(t) \propto V(t)^{-1}$. The parameter c_1 is chosen in such a way that the distribution $\varphi(t, x; m)$ integrates to one and the time of the singularity matches the prediction of the interfacial model. However, the trajectory of V *towards* the asymptote is unavoidably different (Figure 4b).
3. The functional form of $\varphi(t, x; m)$ is not valid outside of the limiting case $\dot{V} \propto V^2$ and is thus not stable over time, nor can it be enforced. Anelastic strain rate distributions that spontaneously establish over time in higher-dimensional simulations are expected to be of approximate Cauchy type (low exponent m) rather than of uniform type (high exponent m).

We therefore come to the qualitative conclusion that orders-of-magnitude increases of slip rate correspond to orders-of-magnitude increase of strain rate locality, placing insurmountable demands on mesh resolution. If left unconstrained, this process culminates

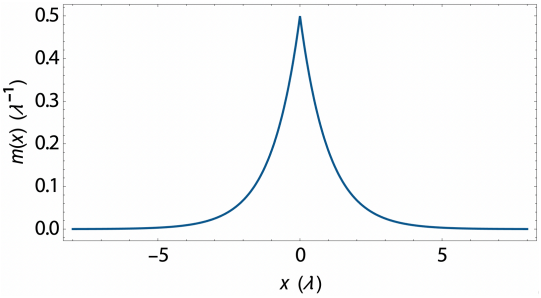

Figure 5. Plot of the mollifier $m(x) = \frac{1}{2\lambda} \exp\left(-\left|\frac{x}{\lambda}\right|\right)$.

in finite time to an infinite slip rate concentrated in an infinitesimally thin region of space – a plane.

3.3 Non-local equivalent strain rate

In order to counter this tendency towards complete localization to a plane we introduce the *non-local* operator $\mathcal{M}$, which *mollifies* its operand by means of convolution with the symmetric distribution $m(x)$:

$$\mathcal{M}(\gamma)(t,x) := \int_T m(\xi)\gamma(t, x-\xi)\mathrm{d}\xi. \tag{23}$$

We note that $\mathcal{M}(\gamma)(t,x)$ still satisfies the relation (6) given that the distribution m integrates to unity over its domain.

Evaluation of $\mathcal{M}(\gamma)$ requires the evaluation of the integral of γ over finite regions of space and therefore lacks the mathematical locality that is desirable for continuum models. We follow Peerlings et al. (1996) in constructing and truncating a Taylor series linearization of (23). We find that a mollifier $m(x) = \frac{1}{2\lambda} \exp\left(-\left|\frac{x}{\lambda}\right|\right)$ (Figure 5) leads to series coefficients that are even powers of the length scale λ:

$$\mathcal{M}(\gamma)(t,x) = \sum_{n=0}^{\infty} \lambda^{2n} \partial_x^{2n} \gamma$$

This infinite sum can be rolled up into a recursive or *implicit* definition of $\mathcal{M} = \mathcal{M}_{\mathrm{im}}$

$$\mathcal{M}_{\mathrm{im}}^{-1}(\bar{\gamma}) := \bar{\gamma} - \lambda^2 \partial_x^2 \bar{\gamma}, \tag{24}$$

which is to be solved for the auxiliary variable $\tilde{\gamma} = \mathcal{M}_{\mathrm{im}}(\gamma)$. Alternatively, the terms beyond $n = 1$ can be dropped, leading to the *explicit* definition of $\mathcal{M} = \mathcal{M}_{\mathrm{ex}}$

$$\mathcal{M}_{\mathrm{ex}}(\gamma) := \gamma + \lambda^2 \partial_x^2 \gamma. \tag{25}$$

We note that any truncation of the Taylor series into a linear combination of even derivatives of γ adheres to the integral property expressed in (6), even if these truncations lose correspondence to a concrete mollifier. The implicit definition (24) has numerically advantageous properties (Peerlings et al., 1996), but requires the potentially costly solution of an elliptic PDE. For this reason we will use the explicit definition (25) in this work and discuss the implicit definition (24) as an option for future implementations.

We generalize the second partial derivatives ∂_x^2 with respect to the x coordinate that appear in (24) and (25) to Laplacians ∇^2 in 3D, again assuming that variations of strain rate across the fault are much bigger than variations along it.

3.4 Non-locality as a localization limiter

We repeat the analysis of Section 3.2, but now use the explicit non-local operator $\mathcal{M}_{\text{ex}}$ in

$$\dot{\gamma} \propto c_1 \gamma \, \mathcal{M}_{\text{ex}}(\gamma)$$

and

$$\dot{\varphi} \propto V\big[\lambda_0 \, c_1 \, \varphi \, \mathcal{M}_{\text{ex}}(\varphi) - \varphi\big]. \tag{26}$$

We find that it is solved by

$$V(t) \propto \left(1 - \frac{t}{t_*}\right)^{-1}, \tag{27a}$$

$$\varphi(x) := \begin{cases} \frac{1}{\pi\,\lambda} \cos^2\left(\frac{1}{2}\frac{x}{\lambda}\right) & \forall\, x \in [-\pi\lambda, +\pi\lambda] \\ 0 & \forall\, x \notin [-\pi\lambda, +\pi\lambda] \end{cases}, \tag{27b}$$

$$\gamma_0\gamma(t,x) := v_0 V(t)\varphi(x), \tag{27c}$$

$$c_1 := 2\pi, \qquad \lambda_0 = \lambda, \tag{27d}$$

and observe that

1. The trajectory of V towards its asymptote can be made to exactly match the prediction (20) of the interfacial rate and state friction formulation (16a,b) by a suitable choice of the constant c_1.

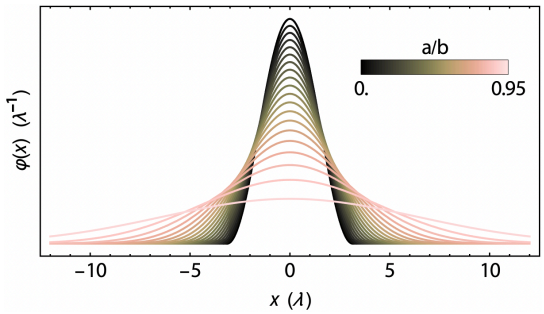

Figure 6. Steady-state distributions φ [cf. Eq. (29a)] of strain rate at various a/b versus distance x with respect to the fault core. The curve plotted for $a/b = 0$ also corresponds to the most extreme strain rate localization at any value of a/b (see main text).

2. The cosine-squared distribution with prescribed size λ is an attractive steady state during the process of earthquake nucleation. Further localization will not spontaneously occur. We interpret this distribution as striking a balance between the localizing tendency of "$\varphi^2 - \varphi$" and the delocalizing tendency of "$\varphi\nabla^2\varphi - \varphi$" (Equation (18) and Figure 3).

The implicit non-local operator $\mathcal{M}_{\text{ex}}$ in the same context does not have solutions that simultaneously satisfy $\dot{V} \propto V^2$ and $\dot{\varphi} = 0$ with non-degenerate distribution $\varphi = \varphi(x/\lambda)$.

3.5 Steady-state friction and deformation

As mentioned in Section 1.1, an important property of the state evolution law is the steady state $\Theta V = 1$. We wish to retain this steady state in the continuum equivalent (15) of the state evolution law, ideally at a steady strain rate distribution $\varphi(x)$, cf. (18). We are thus interested in a steady-state solution to

$$\dot{\varphi} \propto \lambda_0 \, c_1 \, \varphi \, \mathcal{M}_{\text{ex}}(\varphi) - \lambda_0^{\frac{a}{b}} \, c_2 \, \varphi^{1+\frac{a}{b}}, \tag{28}$$

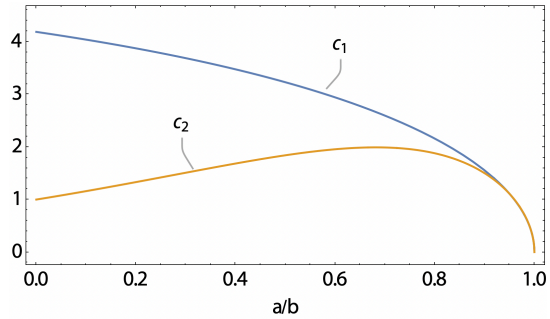

Figure 7. The coefficients $c_1\left(\frac{a}{b}\right)$ and $c_2\left(\frac{a}{b}\right)$, given by Eq. (30a) and (30b), are plotted versus the acceptable range of $\frac{a}{b}$.

which exists as long as $a, b > 0$ (standard requirements) and $a < b$ (a net weakening fault), and reads

$$\varphi\left(x; \tfrac{a}{b}\right) = \begin{cases} c_3\left(\frac{a}{b}\right) \frac{1}{\pi\lambda} \left[\cos\left(\frac{1}{2}\frac{x}{\lambda}\right)\right]^{2\left[1-\frac{a}{b}\right]^{-1}} & \forall\, x \in [-\pi\lambda, +\pi\lambda] \\ 0 & \forall\, x \notin [-\pi\lambda, +\pi\lambda] \end{cases} \tag{29a}$$

$$\lambda := \lambda_0 \left[1 - \frac{a}{b}\right]^{-1}. \tag{29b}$$

Faults that are net strengthening or neutral ($a > b$) do not possess a steady state of the strain rate distribution, instead delocalizing indefinitely.

The coefficients c_1 and c_2, and the newly introduced c_3, are given by

$$c_1\left(\tfrac{a}{b}\right) = \frac{1}{2}\pi^{\frac{3}{2}} \frac{\Gamma\left(1 + 2\left[1 - \frac{a}{b}\right]^{-1}\right)}{\Gamma\left(\frac{1}{2} + 2\left[1 - \frac{a}{b}\right]^{-1}\right)} c_3\left(\tfrac{a}{b}\right)^{-2} \tag{30a}$$

$$c_2\left(\tfrac{a}{b}\right) = \frac{1}{2}\pi^{\frac{1}{2}+\frac{a}{b}} \frac{\Gamma\left(2\left[1 - \frac{a}{b}\right]^{-1}\right)}{\Gamma\left(\frac{1}{2}\left[3 + \frac{a}{b}\right]\left[1 - \frac{a}{b}\right]^{-1}\right)} c_3\left(\tfrac{a}{b}\right)^{-1-\frac{a}{b}} \tag{30b}$$

$$c_3\left(\tfrac{a}{b}\right) = \frac{1}{2}\pi^{\frac{1}{2}} \frac{\Gamma\left(1 + \left[1 - \frac{a}{b}\right]^{-1}\right)}{\Gamma\left(\frac{1}{2} + \left[1 - \frac{a}{b}\right]^{-1}\right)}, \tag{30c}$$

with Γ the gamma function that has the property $\Gamma(n) = (n-1)! \forall n \in \mathbb{N}^+$. The coefficients c_1 and c_2 are plotted as functions of a/b in Figure 7. The solutions (29a) are plotted in Figure 6, in which we can see that steady-state anelastic strain rate distributions have finite width and are therefore numerically resolvable as long as the aforementioned requirements on a, b, and a/b are met.

3.6 Radiation damping

Finite time blow-up of the solution (Section 3.5) under quasi-static loading conditions is understood to be the consequence of the model's inability to radiate or dissipate liberated potential strain energy away from the fault. In what's called the *quasi-dynamic* or *radiation damping* approximation (Rice, 1993), a simple Newtonian sliding viscosity equal to half the material's shear impedance, here nondimensionalized as $\eta = \frac{1}{2}\frac{\mu}{\sigma}\frac{v_0}{c_s}$, is added to the friction law:

$$f = f_0 + a \log V + b \log \Theta + \eta V. \tag{31}$$

with $c_s = \sqrt{\mu/\rho}$ the shear wave speed and in turn μ the shear modulus and ρ the mass density.

We have no good way of replicating the behavior of the radiation damping term in the continuum model because introducing a term $\propto c_3 \lambda_0 \eta \gamma$ into the yield function changes the structure of (17a,b) to such an extent that in the high strain rate limit it no longer admits non-degenerate steady-state or self-similar distributions of anelastic strain. Instead this bulk viscous analog to radiation damping can be shown to promote unconstrained delocalization that is only arrested after a sufficiently large friction drop. This means that no constant of proportionality c_3 can be found that ensures a limit on slip rate to the extent that the term ηV does. We choose $c_3 = c_1$, which seems to give good agreement of maximum slip rate between continuum and interfacial models, at least for the parameter values tested.

It may be preferable to think of the term $\propto \eta \gamma$ as representing a residual viscosity (e.g. in extreme cases the viscosity of a melt) not subject to further weakening, and accept the limitation that the model with static momentum balance has a somewhat poorly constrained maximum slip rate.

The radiation-damped analog to the anelastic shear strain rate ODE (17a,b) is written for the newly introduced auxiliary field ζ:

$$a\,\dot{\zeta}(t,x) = b\,r_0\left(c_1\,\mathcal{M}_{\text{ex}} \circ \gamma \circ \zeta(t,x) - c_2\,\theta(\zeta,f)^{-1}\right) + \dot{f}(t), \tag{32a}$$

$$\theta(\zeta,f)^{-1} := (\gamma \circ \zeta(t,x))^{\frac{a}{b}} \exp\left(-\tfrac{1}{b}\left[f - f_0 - c_3\,\eta\,\gamma \circ \zeta(t,x)\right]\right), \tag{32b}$$

$$\gamma(\zeta) := \frac{a}{c_3\eta} W_0\left(\frac{c_3\eta}{a}\exp\zeta\right), \tag{32c}$$

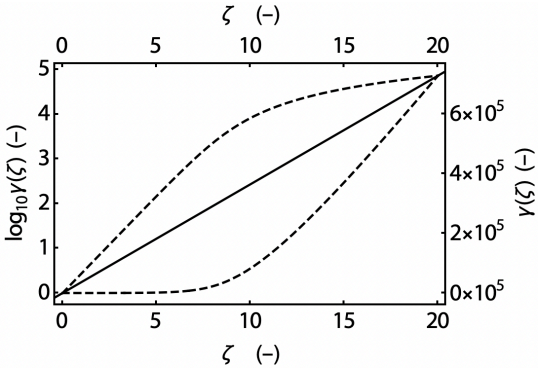

Figure 8. The function $\gamma(\zeta)$ defined in Eq. (32c) plotted on a logarithmic scale in the top left diagonal portion of the figure and on a linear scale in the opposing portion, highlighting the exponential behavior of $\gamma(\zeta)$ at low ζ and the linear behavior at high ζ.

with the principal branch $W_0(f)$ of the Lambert W function, which is an implicit transcendental function that is defined to be the solution to the equation $W_0 \exp W_0 = f$. See Figure 8 for a plot of the relation (32c). While we acknowledge that the use of implicit functions in physical descriptions is not ideal, from a practical standpoint there is little problem because reliable and fast algoritms for computing the common Lambert W function are included in many programming languages (Fritsch et al., 1973; Barry, Barry, & Culligan-Hensley, 1995; Barry, Culligan-Hensley, & Barry, 1995; Johansson, 2020).

4 Numerical models

Analytical predictions made in preceding sections are complemented here with numerical simulation results that better illustrate the complex time-dependent behavior of the system of equations, and allow us to establish the consequences of approximations and other model choices made in the process of formulating a local equivalent continuum rate and state friction law. As before we restrict the scope to 1D models across the shear band.

4.1 Equations

On the one-dimensional line $x \in [-L/2, +L/2]$ that deforms under simple shear, analogous to the transect T defined in Figure 1, we model the compact equation (32a,b) together with the one-dimensional static momentum balance equation $\partial\tau/\partial x = 0$ cf.

(8), elastic constitutive equation $\dot{\tau}(t,x) = \mu\,[\partial v(t,x)/\partial x - 2\gamma_0\gamma(t,x)]$ cf. (7) (with μ the shear modulus as before), yield equality $\tau(t) = f(t)\sigma$, and finally the boundary conditions $v(t, x = \mp L/2) = \mp V_p/2$ combined into the single ODE

$$\dot{f}(t) = \frac{\mu}{\sigma}\left[\frac{V_p}{L} - \frac{\mathcal{V}(\gamma \circ \zeta)}{L}\right], \tag{32d}$$

with V_p the 'plate' driving velocity that is applied antisymmetrically on both ends of the domain. This equation uses (6) that relates slip velocity to the integral (from $-L/2$ to $+L/2$) of anelastic shear strain rate γ, as well as (32c) for the relation between γ and the auxiliary field ζ.

The system is closed with natural boundary conditions $\partial\gamma/\partial x = 0$ on $\gamma(t, x = \mp L/2)$, and initial conditions $f(t = 0) = f_0 + \Delta f$ and $\gamma(t = 0, x) \propto \varphi_0(x) + \epsilon$, with $\varphi_0(x)$ deriving from the analytical prediction (29a) and ϵ a necessary but small homogeneous background value of anelastic shear strain rate. As long as this value is small enough ($\epsilon \ll V_p/L \ll v_0/L$) and the 'tails' of the strain rate distribution can be ignored, the required computational domain size may be based on the length scale λ and therefore cover only a potentially small fraction of the line $[-L/2, +L/2]$, greatly reducing the computational cost.

The solution procedure is outlined in Appendix Appendix A.

4.2 Continuum velocity stepping friction experiments

Velocity stepping laboratory experiments form the foundation on which rate and state friction was proposed (Dieterich, 1978). We compare numerical velocity stepping experiments of a traditional rate and state friction-governed interface with our proposed continuum equivalent under the same conditions. We impose a driving velocity that follows a smooth square wave function with respect to sliding distance δ:

$$\log_{10}\left(V_p(\delta)/v_0\right) = \frac{2}{\pi}\tan^{-1}\left(-\frac{1}{\nu}\cos\left(\frac{\pi\,\delta}{n\,d_c}\right)\right), \tag{34}$$

with $n = 20$ the amount of slip weakening distances d_c between velocity steps, and $\nu = 10^{-3}$ a parameter that controls the smoothness of the smooth square wave, with small numbers giving the squarest result (see Figure 9a). Some smoothness is important to maintain numerically stable results. The low and high driving velocities are $10^{-1}v_0$ and $10^{+1}v_0$ respectively. Other model parameters are given in Table 4.2. Inertial effects may be ignored (both in the original lab experiments and in our simulations) due to the low driv-

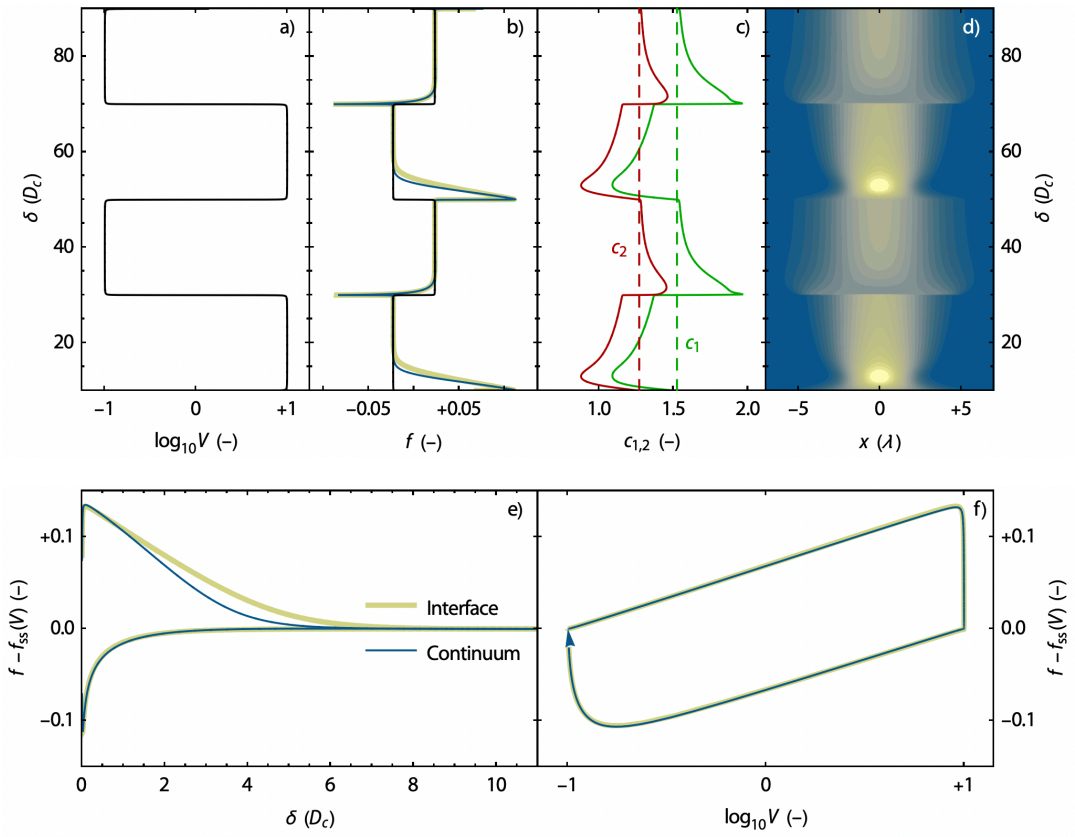

Figure 9. Results of the numerical velocity stepping experiments: a) dimensionless sliding velocity V closely tracking a smooth square wave signal (Eq. (34)); b) frictional response of the interfacial reference model (thick yellow line) and continuum model (dark blue line) with for reference the theoretical steady-state friction $f_{ss}(V)$ (black line); c) measured (solid lines; cf. Eq. (19a,b)) vs. used (dashed lines; cf. Eq. (30a,b)) values of the dimensionless coefficients $c_{1,2}$; d) distribution $\varphi = \gamma/V$ of anelastic shear strain rate with darker blue colors reflecting very low values and bright yellow colors reflecting high values, and contours distributed evenly on a linear scale. Reflecting the ultimate slip-dependence (rather than direct time-dependence) of the rate and state friction laws, curves in panels a)–d) are plotted against sliding distance δ (measured in critical slip distances d_c) on the vertical axis. Panel e) shows an enhanced view of the evolution of the instantaneous friction coefficient towards the steady-state value with slip distance δ, simultaneously showing a positive and negative step for both the interface and continuum models. Panel f) displays a full cycle of both models in terms of sliding velocity V and friction perturbation $f - f_{ss}$. It distinguishes interface and continuum models by line color and thickness as in panels b) and e).

Table 1. Parameter values used in the velocity stepping experiment

parameter	value	unit
a	$2\ 10^{-2}$	1
b	$3\ 10^{-2}$	1
d_c	10^{-5}	m
v_0	10^{-6}	m s^{-1}
ρ	10^{+3}	kg m^{-3}
μ	10^{+10}	Pa
c_s	$3.162...\ 10^{+3}$	m s^{-1}
η	$1.581...\ 10^{-6}$	1
σ	10^{+6}	Pa
d_c	$5.00\ 10^{-4}$	m
λ	$\sim 1.4\ 10^{-4}$	m
c_1	1.531...[a]	1
c_2	1.276...[a]	1
r_0	v_0/d_c	s^{-1}
ν	10^{-3}	1
n	20	1

Some rate and state parameter values are roughly based on Erickson et al. (2020).
[a]Using (30a) and (30b) with stated values of a and b.

ing velocity. Thus, the radiation damping viscosity η is set to zero and its effects not discussed here. In the limit $\eta \to 0^+$, (32c) simply reduces to $\gamma(\zeta) = \exp\zeta$ or equivalently $\zeta(\gamma) = \ln\gamma$, which reduces (32a,b) back to (17a,b).

Because the domain size L is very small and the system (32a–32d) therefore stiff, the friction tends to a stable steady state after being perturbed by a velocity step rather than developing a limit cycle. This also means that $\mathcal{V}(\gamma) \approx V_p(\delta)$. In fact, $\mathcal{V}(\gamma)$ and $V_p(\delta)$ are so close that their difference would not register in Figure 9a.

In the following, we use the terms time-dependence and slip dependence interchangeably but note that the rate of change of either is not constant from the perspective of the other. A slip-centric presentation of velocity-stepping results is commonplace in the earlier cited literature.

In Figure 9b (and its detailed view 9e) we explore the similarities and differences in slip dependence of the friction coefficient f between the interfacial and continuum velocity stepping experiments. Based on (18), we expect that a velocity step perturbs not just the magnitude of anelastic shear strain, but also its distribution (Figure 9d) and therefore the theoretical value of the dimensionless coefficients $c_{1,2}$ (Section 3.3; Equations (19a) and (19b); Figure 9c, solid lines) away from steady state. Since these coefficients are actually set to their constant steady-state predictions (Section 3.5; Equations (30a) and (30b); dashed lines in Figure 9c), we expect some differences in time or slip dependence to occur. Notably, a positive velocity step leads to a temporary localization of anelastic deformation and a corresponding decrease in the theoretical values of $c_{1,2}$ with respect to steady state. This leads to an overestimation of corresponding terms in the right-hand-side of (32a) and a faster decay towards the steady-state friction coefficient (Figure 9b,e) of the continuum model with respect to the interfacial model. A negative velocity step causes temporary delocalization (Figure 9d) and under-estimation of terms involving $c_{1,2}$ in (32a), but does not lead to any noticeable difference in friction coefficient evolution towards the steady state between the two models (Figure 9b,e). We conclude that those terms involving $c_{1,2}$ are negligible in this situation.

In Figure 9c and 9d it can also be seen that the *distribution* of anelastic shear strain rate tends more slowly to a steady state than the friction coefficient itself, and moreover that this decay is slower after a positive velocity step than after a negative one. In fact, the 40 d_c wavelength we use for the intput signal (34) is too small to enable a somewhat

complete evolution towards steady-state of the anelastic strain rate distribition during the high-velocity regime, but we have taken care that this does not meaningfully impact the results during the subsequent low-velocity regime.

Figure 9f shows that in V-f phase space where time- or slip-dependence is hidden, the response of the continuum and interface models are practically equivalent.

4.3 Continuum spring-slider experiments

Whereas velocity stepping experiments are useful for studying the way in which friction can be attracted to a steady state, permanently out-of-equilibrium 'limit cycle' behavior is more interesting in the study of earthquake dynamics. We achieve this behavior by increasing the size of the domain and the distance of the boundary conditions by many orders of magnitude ($L = 80$ km) with respect to the velocity stepping experiments. Thereby, we greatly reducing the effective stiffness of the medium as can be seen in (32d). This is analogous to performing the so-called spring-slider experiment in which a mass is dragged over a surface by a spring that is tensed at a constant rate to give an educational example of the apparent stick-slip behavior of seismogenic faults. Our goal in performing this experiment is to determine the degree to which the here proposed continuum friction laws match the predictions made by Dieterich and Ruina's interfacial laws and to describe the transient behavior of anelastic strain rate in the added dimension. Radiation damping plays an essential role in these experiments to close the limit cycle that otherwise extends to $V \to \infty$ (Section 3.6). As discussed in the same section, our continuum approximation to radiation damping precludes a steady-state anelastic strain rate distribution at high slip velocity, a fact that directly contradicts one of our precepts (Section 3.1). Therefore we anticipate a larger discrepancy between interface and continuum models here than was observed during the velocity stepping experiments.

Figure 10 depicts the evolution of quantities of interest over the course of multiple orbits of the solution as it converges to the limit cycle. The first important observation is that the continuum model still appears to exhibit spontaneous limit cycle behavior. Due to the appropriately chosen initial condition, the wind-up period is short and there is little variability between successive revolutions. As predicted (Section 3.5), the nucleation phase is marked by runaway amplitude increase of a quasi-steady strain

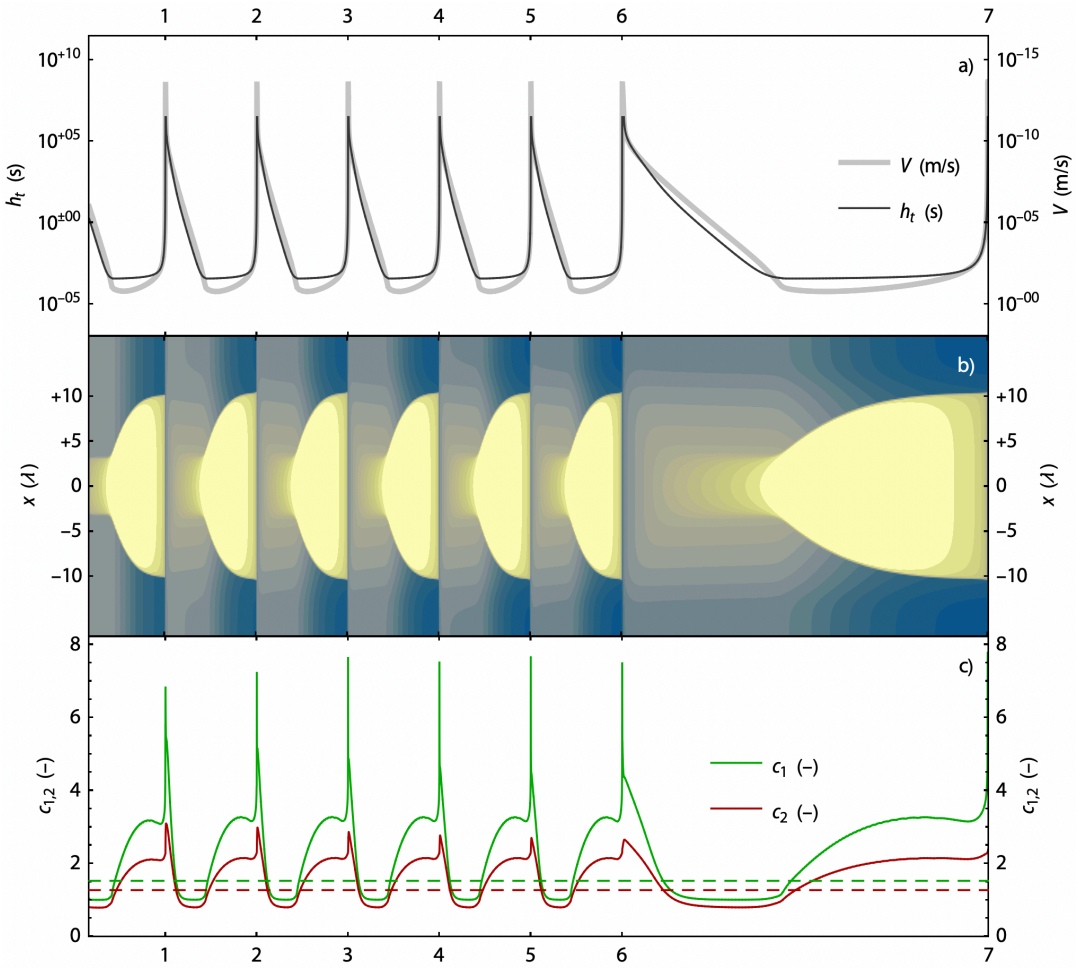

Figure 10. Evolution of the continuum seismic cycle simulation demonstrated by a succession of six events plotted against time step count on the horizontal axis, with a seventh event stretched by a factor four in order to better show some features of an individual event. The end of each event is punctuated by a slip velocity minimum and numbered 1 to 7 on the horizontal axes. Each revolution around the limit cycle takes 83.00 years and 107,500 time steps to complete. Panel a: time step size h_t (thin dark line; left axis; logarithmic scale) and slip velocity V (thick bright line, right axis; logarithmic scale). The former serves to aid the interpretation of the time-dependent progression of each event while the latter gives a sense of magnitude to the color scale of panel b. Axes are chosen so that together these curves express the imperfect reciprocal relation between time step and slip velocity. Panel b: logarithm of dimensionless anelastic shear strain rate versus signed distance x measured in number of length scales λ away from the shear zone center. Bright yellow colors indicate high strain rate; dark blue colors indicate low strain rate. Contours are drawn at equal intervals in log space. No explicit color scale is given because the magnitude of anelastic shear strain rate γ is tied to the value of λ in order to produce a length-scale independent history of sliding velocity V (panel a). Panel c: measured (solid lines; cf. Eq. (19a,b)) vs. used (dashed lines; cf. Eq. (30a,b)) values of the dimensionless parameters $c_{1,2}$.

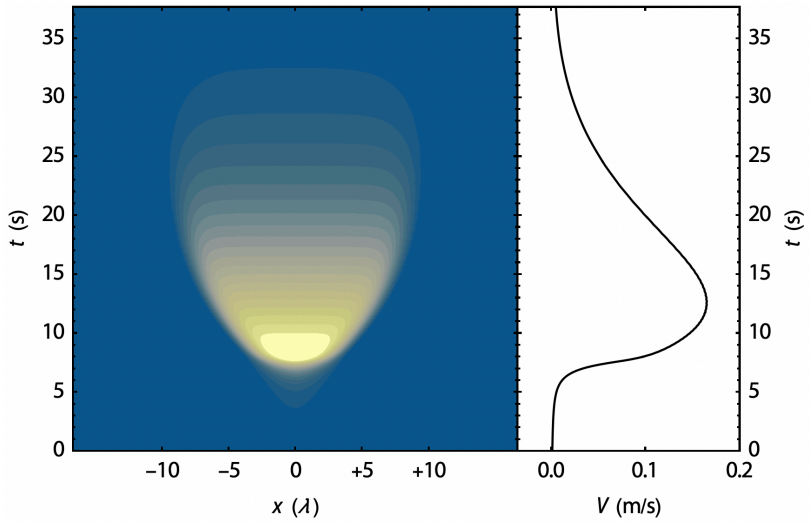

Figure 11. Focus on the last event of Figure 10, here visualized in panel a by strain rate on a linear color and contour scale and in panel b by slip rate V, both versus a linear time scale in seconds.

rate distribution (panel b). As before, we measure the dynamic value of the coefficients $c_{1,2}$ from the strain rate distribution following (19a,b), and compare to the analytically derived steady-state values (30a,b). In line with analytical predictions this phase is also accompanied by measured values of $c_{1,2}$ (panel c) that are slightly lower than the constant values that are prescribed (dashed lines in panel c). The opposite situation occurs throughout the remainder of each cycle where anelastic strain is more widely distributed. We can attribute this directly to the effect of 'radiation damping' or residual viscosity, which exerts a delocalizing influence. Figure 11 provides a more intuitive view of the short-lived delocalizing behavior of a strain pulse using linear rather than logarithmic scales. We emphasize that the ultimate extent of coseismic delocalization compared to the length scale λ is dependent on model and material parameters, among which the effective stiffness of the medium.

Large peaks in the measured values of $c_{1,2}$ shown in panel c of Figure 10 occur during the post- and interseismic phase, where anelastic shear strain rate is broadly and quite uniformly distributed. The values that are actually used (dashed lines) are way too small here, however, terms that do not involve these coefficients dominate the time-dependent behavior in this regime and the discrepancy's consequences are limited. This is largely confirmed in Figure 12, which tabulates the evolving scalar properties of a limit cycle

of the interfacial and continuum models of rate and state friction. In particular, the duration of the limit cycle (first column, panels a–c, time in years on the horizontal axis) is very similar even though the timing of the event in relation to its enclosing slip velocity minima is somewhat different.

Markedly different are the coseismic aspects of the limit cycle. Panel e of the middle column and panels h–j of the right column of Figure 12 clearly show this. The continuum model has a faster stress drop (panel h, $U' \propto f$) and a more symmetric slip rate response with time compared to the interfacial model. The limit cycle depicted in linear phase space (U', V) in Figure 12e shows a skewed triangular trajectory of the interfacial model and a more parabolic trajectory for the continuum model. It appears that the amount of stress drop that happens *before* peak slip rate in the interfacial model is small – about 10% of the total stress drop. At the same point in the continuum model the stress drop is already about one third of the total. The same limit cycle shown in Figure 12f in log-linear phase space $(U', log_{10}V)$ allows us to see that there are subtle differences between the limit cycles of the interfacial and continuum models over the whole range of slip velocities.

A further result of practical interest is the difference in adaptively chosen time step between the interfacial and continuum models. Figure 10a shows an approximate inverse relation between the time step of continuum model to the aggregate slip velocity $v_0 V$, which is unsurprising seeing how $d_c/[v_0 V(t)]$ gives a sensible local time scale for both interfacial and continuum rate and state friction models. The approximate inverse relation breaks down at large time scales where the corresponding time steps are truncated to a fixed value of 0.1 yr, and at large slip velocity and small time step, where the time scale of the continuum model becomes dominated by large spatial gradients and is generally much smaller than that of the interfacial model (Figures 12d,g,k). From these same plots it can be seen that, just like stress begins to drop in significant advance of an event, so too does the time step of the continuum model.

Finally, we remark that spatial resolution tests indicate that the quality of the solution is only influenced by the ratio of cell size h_x to λ, with acceptable results achieved when $h_x/\lambda \geq 10$. For the results presented here we have used $h_x/\lambda = 20$. Changing λ in proportion to h_x does nothing except to cause a wider or narrower but equally well resolved strain distribution and a virtually identical limit cycle.

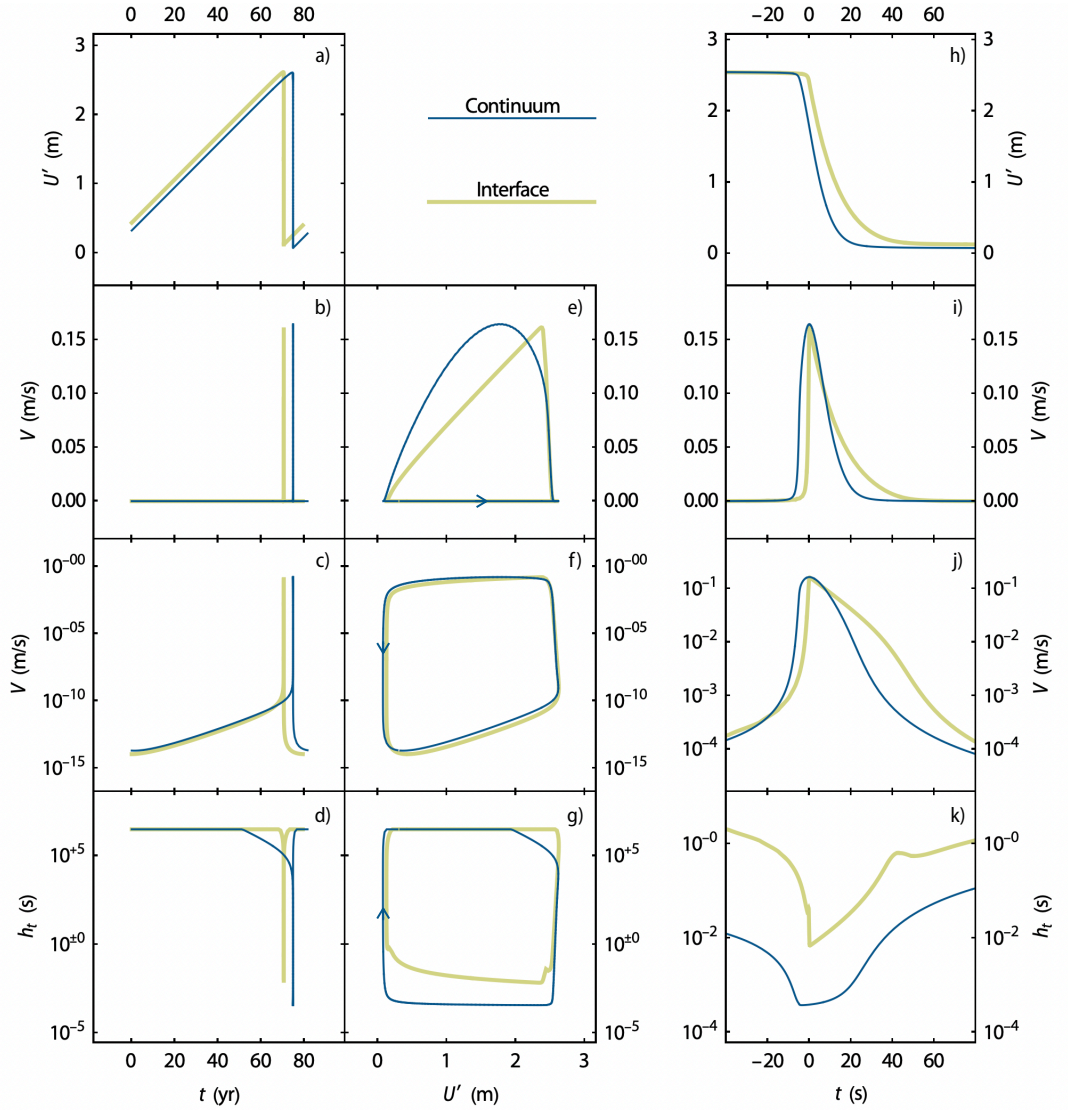

Figure 12. Panels a–g: behavior of the interfacial (bright yellow lines) and continuum (dark blue lines) models for a full limit cycle. Panels h–k: focus on the 100 seconds around an event. From left to right, horizontal axes are: time t in years, slip deficit U' in meters (linearly related to shear stress τ and friction coefficient f), and time t in seconds. Vertical axes from top to bottom are slip deficit U' in meters, slip velocity V in meters per second on a linear scale, the same on a logarithmic scale, and finally the time step size h_t in seconds on a logarithmic scale. Where limit cycles are shown (middle column, panels e–g), the cycle sense is indicated with small arrows.

5 Discussion

So far we have argued for a visco-elastic continuum rheology that resembles both the mathematical presentation as well as the resulting behavior of traditional interfacial rate and state friction as described by Dieterich and Ruina in their respective seminal puplications. Before addressing this work in the context of a much broader body of existing research and reflecting on critical assumptions made in this work and its extension towards the future, we briefly touch upon some alternate branches of our proposed theory that have been left out so far for the sake of clarity.

5.1 A primitive reformulation

In contrast to traditional models of plasticity, the continuum rate and state friction laws presented here possess a continuous time-dependence that makes the anelastic shear strain rate γ a predictable quantity rather than one that is to be solved by a constrained optimization algorithm (e.g. Simo & Taylor, 1985; Duretz et al., 2018). This is apparent from the ODE (17a,b) for γ, but is hidden in the Dieterich-Ruina form (14) and (15) of the continuum rate and state friction rheology. However, (17a,b) lacks the simplicity and elegance of the Dieterich-Ruina system and is difficult to interpret as the resultant of physical phenomena. By defining a set of primitive variables κ and ψ as

$$\begin{aligned} \dot{\kappa} &= r_\kappa \gamma \\ \dot{\psi} &= r_\psi \gamma^{a/b} \exp\left(-[f(\boldsymbol{\sigma}) - f_0]/b\right), \end{aligned} \tag{35}$$

with the respective reference rates $r_\kappa = r_0 c_1$ and $r_\psi = r_0 c_2$, we are able to write a more expressive formulation of (17a,b) as

$$\dot{\kappa} = r_\kappa \gamma = r_\kappa \left(\dot{\psi}/r_\psi\right)^{\frac{b}{a}} \exp\left([f(\boldsymbol{\sigma}) - f_0]/a\right) \tag{36}$$

$$\dot{\psi} = r_\psi \exp\left(\mathcal{M}_{\text{ex}}(\kappa) - \psi\right), \tag{37}$$

that still closes the momentum balance equation (8) and elasto-plastic constitutive equation (7) without algebraic constraints and thus without requiring a constrained optimization algorithm. Equation (37) can accommodate radiation damping using the Lambert W function analogous to the way it was discussed in Section 3.6.

We recognize $\kappa(t, x)$ as a dimensionless measure of accumulated anelastic shear strain and therefore assign to ψ the meaning of a 'virtual' or 'unrealized' strain. In spite of the

conceptual appeal of (36) and (37), it can be difficult to choose appropriate initial conditions for ψ.

5.2 Quadratic diffusion

Instead of the mollified term $\gamma\,\mathcal{M}(\gamma)$ that was introduced in (15) and used throughout the remainder of this work, we could have chosen to use $\mathcal{M}(\gamma^2)$:

$$a\,\dot{\gamma}_l = b\,r_0\left[c_1\mathcal{M}(\gamma^2) - c_2\gamma\,\theta(\gamma,f)^{-1}\right] + \gamma\dot{f}. \tag{38}$$

and the definition of c_1 is adapted to the new way of mollifying according to the same principles as laid out in Section 3.

This choice has advantages and disadvantages. A major disadvantage is that it does not seem to allow a return from (38) to the more elegant Dieterich-Ruina form (14) and (15), nor to the compact form (36)–(37) proposed in the preceding section. A clear advantage is that, taken together with the explicit non-local operator $\mathcal{M}_{\text{ex}}$, (38) produces a degenerate reaction-diffusion equation of generalized Fisher or Kolmogorov–Petrovsky–Piskunov (KPP) type (Fisher, 1937; Kolmogorov et al., 1937) that remains well-posed at and around $\gamma = 0$. This leads to shear zone solutions that can propagate into perfectly intact rock without taking recourse to an arbitrary small initial condition on γ. Furthermore, under quadratic diffusion, the function $\gamma(\zeta)$ used in the ODE form (32a–d) in Section 3.6 becomes algebraic instead of transcendentally implicit.

We have reproduced all results presented in Section 4 with this quadratic diffusion term and zero background anelastic strain rate too, and observe numerically smoother and better resolved shear fronts during the coseismic delocalization phase. These fronts also travel faster and further than in the linear diffusion case leading to a larger discrepancy between measured and used values of the coefficients $c_{1,2}$ and consequently larger deviations of the limit cycle with respect to that of the interfacial rate and state friction model. The generalized Fisher–KPP type equation has merited a large body of theoretical research (e.g. Sánchez-Garduño & Maini, 1994, 1995, 1997; Sherratt & Marchant, 1996; S. Petrovskii & Li, 2003; S. V. Petrovskii & Li, 2006; Gilding & Kersner, 2005; Y. Wu et al., 2006; Y. Li & Wu, 2008; Malaguti & Ruggerini, 2010; Broadbridge & Bradshaw-Hajek, 2016) and therefore some of the above mentioned empirical findings may be given a theoretical underpinning in future work.

5.3 Relation to regularized damage or plasticity models

As shown in Section 4.3 there are circumstances under which the transient viscous rheology proposed in this work promotes a spontaneous organization of periods and regions of negligible anelastic strain rate and those of significant anelastic strain rate. Even though a critical yield stress seems to be an emergent rather than an inherent property of the system of equations and initial and boundary conditions, it is tempting to think of this rheology as a smooth plasticity model.

We have shown (Section 3.5) that our model requires regularization by a non-local strain rate measure to avoid spurious mesh dependence. Non-locality in one form or another has been applied to combat ill-posedness in e.g. Bažant et al. (1984); Triantafyllidis and Aifantis (1986); Schreyer and Chen (1986); Pijaudier-Cabot and Bažant (1987); de Borst and Mühlhaus (1992); Peerlings et al. (1996); Jirásek (1998); Bažant and Jirásek (2002); Engelen et al. (2003); Jirásek and Rolshoven (2009b, 2009a); Burghardt et al. (2012); Lyakhovsky et al. (2011); Lyakhovsky and Ben-Zion (2014b, 2014a); Lyakhovsky et al. (2016); Kurzon et al. (2019, 2020); Kiefer et al. (2018); Abdallah et al. (2020).

The comprehensive non-local damage-breakage rheology of Lyakhovsky et al. (2011); Lyakhovsky and Ben-Zion (2014b, 2014a); Lyakhovsky et al. (2016), firmly rooted in thermodynamic theory and well-calibrated to match observations, is one of few damage theories developed to cover the full process of earthquake generation and healing (Lyakhovsky et al., 2016; Kurzon et al., 2019, 2020). As in our model, the nonlocality in theirs is of Gradient type, but differs subtly in that it acts on the damage parameter rather than on the anelastic shear strain rate. Another example is the Godunov-Peshkov-Romenski model (S. Godunov & Romenskii, 1972; S. K. Godunov & Romenskii, 2003; Resnyansky et al., 2003; Romenskii, 2007; Romenski et al., 2020), which was first used in Gabriel et al. (2021) to simulate dynamic rupture and off-fault damage generation. This model differs from our model and the aforementioned damage-breakage rheology in a fundamental way in that shear bands are produced by damage waves described by hyperbolic equations rather than parabolic equations of reaction-diffusion type.

Our model differs from phase-field models of fracture in some respects. By allowing the spatial distribution of anelastic shear strain rate to vary over time, we are unable to retrieve a phase field $d \in (0, 1)$. We do not consider this a disadvantage, but do acknowledge that this is mainly a consequence of our viscous approximation to ra-

diation damping (Section 3.6), which gives rise to a reaction term in (32) that can be considered as the derivative of a double-well potential. Multiple authors have commented critically on the use of double-well potentials in phase field methods (e.g. Kuhn et al., 2015; J.-Y. Wu, 2017).

Interestingly, our bulk viscous analog of radiation damping also appears as a regularization technique in some plasticity models (Needleman, 1988; Peirce et al., 1983; Duretz et al., 2019, 2020, 2021; de Borst & Duretz, 2020; Stathas & Stefanou, 2022). In these models the Kelvin–Voigt arrangement of yield strength and Newtonian viscosity truncates the steady-state anelastic strain rate that may be achieved, forcing a shear zone to have a finite width in order to slide at a certain macroscopic rate. In our model, the same viscosity also introduces an effective upper bound on anelastic strain rate attained during the limit cycle and causes subsequent delocalization, but we must still rely on the gradient regularization to combat the unbounded localization that would otherwise happen even at low strain rate. A detailed analysis of the efficacy of viscous regularization in dynamic problems is made in Stathas and Stefanou (2022).

5.4 Relation to other transient continuum rheologies

The framework here proposed can be seen as a generalization of the work of Herrendörfer et al. (2018); Preuss et al. (2019, 2020), who made the purely local substitution $v_0 V \to h_x \gamma_0 \gamma$ and used a Drucker-Prager elastoplastic model similar to the one set out in Section 2.4. Setting $\lambda, \eta = 0$ and assuming that strain rate fully localizes into a discrete Dirac function sampled every h_x, we find that the coefficients c_1 and c_2 become h_x/d_c and 1 respectively, substitution of which into (15) yields Herrendörfer's version of the aging law. Their model was first applied along a predefined staggered grid line in Herrendörfer et al. (2018), obeying the discrete Dirac distribution of anelastic shear strain automatically and effectively yielding a numerical method analogous to the stress glut method of Andrews (1999). Herrendörfer's model was subsequently applied in an unconstrained evolving continuum model in Preuss et al. (2019, 2020), but notwithstanding measures put in place that acknowledge the changing distribution of shear strain rate within a shear zone, their model ultimately lacks regularizations that remove mesh dependence.

As noted in Section 1.2 of the Introduction, a variety of continuum theories have been developed to explain the general rate- and state-like behavior of deformation in faults

and shear zones (e.g. Sleep, 1997; Braeck & Podladchikov, 2007; John et al., 2009; Thielmann et al., 2015; Thielmann, 2018; Rozel et al., 2011; Barbot, 2019; Niemeijer & Spiers, 2007; Chen & Spiers, 2016; Van den Ende et al., 2018; Roubíček, 2014; Lyakhovsky et al., 2011; Lyakhovsky & Ben-Zion, 2014b, 2014a; Lyakhovsky et al., 2016; Kurzon et al., 2019, 2020; Poulet et al., 2014; Veveakis et al., 2014; Alevizos et al., 2014; Rattez, Stefanou, & Sulem, 2018; Rattez, Stefanou, Sulem, Veveakis, & Poulet, 2018; Pozzi et al., 2021). We note again that an internal length scale in these models typically arises from the inclusion of a diffusion process (of e.g. temperature, pore pressure) but that may not always be adjusted to meet the constraints imposed by scale and computational power without changing the outcome of the model. In our proposed formulation an artificial diffusion process acts directly on the anelastic shear strain rate, resulting in a robust and controllable internal length scale.

In this last respect our work is preceded by nearly 25 years by Sleep (1997). In this work Sleep combined and extended earlier works (Sleep, 1995; Segall & Rice, 1995; Linker & Dieterich, 1992; Chester, 1994, 1995) in which rate and state friction was interpreted as the product of crack generation and healing, associated rheological weakening, and dissipative heating. This physical reasoning resulted in a model that contains only quantities that are either directly measurable or can be modeled by independent methods. This contrasts with our purely mathematical argumentation that serves to retain close correspondence to the original phenomenological description of rate and state dependent sliding on a frictional interface. Sleep (1997) neglects the fluxes and associated spatial gradients of the pore fluid but does include heat diffusion, which they note does however not play a significant role at the scale of their numerical experiments. Instead, resembling our approach, they impose an artificial length scale and forced strain distribution by explicit mollification of the anelastic shear strain rate with a Gaussian kernel. This leads to an aging law (Equation 53 in Sleep (1997)) that is structurally identical to our result (15). Like us, they find that strain localization can only occur when $a < b$, and that a rate-strengthening effect that activates at high strain rate leads to strain delocalization. With respect to Sleep's valuable contribution, in this work we provide a more complete argumentation for this type of spatial regularization and analysis of the resulting patterns of strain localization and delocalization over the seismic cycle.

5.5 Assumptions and their impact on future work

We proposed our model in a very general three-dimensional continuum mechanics framework, but for simplicity have considered only a small fault neighborhood in which in-plane variations of fault properties can be neglected (Section 2.2) so that the model becomes effectively one-dimensional. This same assumption was also taken in the numerical models that we have used in our analysis. The assumption is clearly violated around fault branches, at fault kinks or on rough faults, and near the fault or rupture tip. We note however that rate and state friction was proposed based on laboratory studies that also neglect these geometrical complexities. Although rate and state faults with branches and kinks are still largely non-standard in present-day numerical modeling studies, plenty of attention has been given to the critical nucleation patch and the structure of the cohesive zone near the rupture tip (e.g. Rice, 1993; Cocco & Bizzarri, 2002; Cocco et al., 2004; Rubin & Ampuero, 2005; Day et al., 2005; Lapusta & Liu, 2009; Viesca, 2016b, 2016a; Putelat et al., 2017). Given our main assumption, these features can only be accurately reproduced with our continuum formulation in higher-dimensional numerical models if the regularization length scale λ is significantly smaller than the length scales associated with the critical nucleation patch and the cohesive zone. In turn the cell size must be sufficient to resolve λ, and so we expect to need a grid resolution that is significantly higher than that of existing methods to simulate rate and state frictional interfaces. We have also seen that the diffusion process is associated with a smaller time scale, and thus, stricter time step constraints than the interfacial model. We note that both spatial and temporal resolution requirements already place challenging constraints on simulations of seismic and aseismic slip sequences (Erickson et al., 2020).

It seems prudent to first make a more detailed assessment of the computational demands and the techniques that may be required to meet the resolution requirements (e.g. adaptive mesh refinement, local adaptive time stepping). In the process of constructing higher-dimensional models one may first concentrate on the friendliest regions of parameter space, for example those that promote stable sliding, or undamped runaway localization as in Viesca (2016b, 2016a).

The temporal patterns of localization and delocalization that occur in our model yield testable predictions that neatly tie in to a recent surge in interest in similar patterns observed in the lab and in nature (e.g. Ben-Zion & Zaliapin, 2020; McBeck et al.,

2018; McBeck, Aiken, et al., 2020; McBeck, Ben-Zion, & Renard, 2020; McBeck et al., 2021). It could be the scope of future research to reinterpret anelastic strain rate in our model as a measure of the activity of a statistical distribution of cracks of various properties and compare to aforementioned lab and field observations. In this context, the delocalization that is in our models induced by a rate-limiting Kelvin viscosity is reminiscent of the growth of off-fault fracture networks during dynamic rupture (e.g. Templeton & Rice, 2008; K. Okubo et al., 2019; Gabriel et al., 2021).

6 Conclusions

In this work we have carefully constructed a coordinate-invariant and mesh-independent transient visco-elastic continuum rheology that behaves in a way that is consistent with rate and state friction on an interface. We have shown that inclusion of a diffusion-like spatial regularization ensures a limit to strain localization and thus guarantees mesh convergence. In a simplified 1D fault transect, important metrics of the seismic cycle – such as slip rate and friction – are independent of the diffusion length scale associated with the regularization. However, throughout this work we have assumed the regularization length scale to be small compared to the length scales associated with other features of interest, such as fault curvature or along-strike variations of slip rate and stress. Therefore, going forward, high-resolution 2D or 3D numerical models are required to apply this model to the study of the seismogenic behavior of emerging and evolving fault zone networks. Our continuum rheology resembles a reaction-diffusion equation for anelastic strain rate. Processes described by such equations are ubiquitous in nature, and it is tempting to compare temporal patterns of localization and delocalization produced by our model with natural observations.

Acknowledgments

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no. 823844 and the German Research Foundation (DFG projects grants no. GA 2465/2-1, GA 2465/3-1). None of the authors declare any competing interests or financial conflicts. All scripts used to generate the figures and their underlying data are available in the supplementary materials. We depend on the MIT-licenced 'Scientific Colour Maps' package (Crameri, 2021) for distortion-free representation of the model results – also for readers with color vision deficiencies (Crameri et al., 2020).

Appendix A Solution procedure

We discretize the Laplacian using a standard second-order accurate central difference stencil, with natural boundary conditions of the same accuracy implemented by staggering the fields with respect to the physical domain walls. The problem size is halved by exploiting symmetry across the shear zone. Integrals are evaluated using a midpoint rule. This gives the discrete system

$$\dot{\tilde{\boldsymbol{\zeta}}} = \mathbf{g}(\tilde{\boldsymbol{\zeta}}, \tilde{f})$$
$$\dot{\tilde{f}} = \mathrm{h}(\tilde{\boldsymbol{\zeta}}, \tilde{f}),$$

where tildes indicate approximate space-discrete quantities and numerical arrays are indicated in bold face. We form a symbolic rules for computing the Jacobian matrix $\mathbf{J}$ of this system of non-linear space-discrete equations,

$$\mathbf{J}(\tilde{\boldsymbol{\zeta}}, \tilde{f}) = \begin{bmatrix} \nabla_{\tilde{\boldsymbol{\zeta}}}\, \mathbf{g}(\tilde{\boldsymbol{\zeta}}, \tilde{f}) & \nabla_{\tilde{f}}\, \mathbf{g}(\tilde{\boldsymbol{\zeta}}, \tilde{f}) \\ \nabla_{\tilde{\boldsymbol{\zeta}}}\, \mathrm{h}(\tilde{\boldsymbol{\zeta}}, \tilde{f}) & \nabla_{\tilde{f}}\, \mathrm{h}(\tilde{\boldsymbol{\zeta}}, \tilde{f}) \end{bmatrix},$$

with the upper left block a dense $N \times N$ matrix with a dominant sparse band structure (N being the problem size), the lower right block a 1×1 empty matrix, and the off-diagonal blocks densely populated vectors of compatible shape. The system is then linearized as

$$\mathrm{d} \begin{bmatrix} \dot{\tilde{\boldsymbol{\zeta}}} \\ \dot{\tilde{f}} \end{bmatrix} = \mathbf{J}(\tilde{\boldsymbol{\zeta}}, \tilde{f})\, \mathrm{d} \begin{bmatrix} \tilde{\boldsymbol{\zeta}} \\ \tilde{f} \end{bmatrix}$$

and the smallest time scale $S_t(\tilde{\boldsymbol{\zeta}}, \tilde{f})$ to be resolved is computed as the reciprocal of the largest eigenvalue of the Jacobian matrix $\mathbf{J}$. The actual time step Δ_t is computed as some fixed fraction of S_t, optionally bounded by a maximum value and/or maximum growth rate to prevent time step overestimation as the time scale increases. The equations are discretized in time with forward and backward Euler schemes, both first-order accurate,

respectively as

$$\begin{bmatrix} \tilde{\boldsymbol{\zeta}}^{k+1} \\ \tilde{f}^{k+1} \end{bmatrix} = \begin{bmatrix} \tilde{\boldsymbol{\zeta}}^{k} \\ \tilde{f}^{k} \end{bmatrix} + \Delta_t \begin{bmatrix} \mathbf{g}(\tilde{\boldsymbol{\zeta}}^{k}, \tilde{f}^{k}) \\ \mathrm{h}(\tilde{\boldsymbol{\zeta}}^{k}, \tilde{f}^{k}) \end{bmatrix} \tag{A1}$$

$$\mathbf{f} = \begin{bmatrix} \tilde{\boldsymbol{\zeta}}^{k+1} \\ \tilde{f}^{k+1} \end{bmatrix} - \begin{bmatrix} \tilde{\boldsymbol{\zeta}}^{k} \\ \tilde{f}^{k} \end{bmatrix} - \Delta_t \begin{bmatrix} \mathbf{g}(\tilde{\boldsymbol{\zeta}}^{k+1}, \tilde{f}^{k+1}) \\ \mathrm{h}(\tilde{\boldsymbol{\zeta}}^{k+1}, \tilde{f}^{k+1}) \end{bmatrix} = \mathbf{0}. \tag{A2}$$

Our algorithm makes an explicit-in-time prediction using (A1) and evaluates the l_2 norm of the implicit-in-time residual $\mathbf{f}$. It includes the possibility to perform Newton-Raphson iterations using the Jacobian $\mathbf{J}$ to keep the residual bounded, although we find it to be more efficient to experimentally set the dimensionless time step Δ_t/S_t sufficiently small (e.g. 0.1) to never cause tolerances to be violated. This value is then taken to correspond to a stable time step size of the explicit problem.

We make our algorithms and scripts to produce figures available as Wolfram Mathematica (Wolfram Research, Inc., 2017) notebooks in the Supplementary Materials.

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