Supplement to: A Conceptual Model of Polar Overturning Circulations

Thomas W. N. Haine*

Earth & Planetary Sciences, Johns Hopkins University, Baltimore, Maryland

Corresponding author: Thomas W. N. Haine, Thomas.Haine@jhu.edu

ABSTRACT

Supplementary material on theory and numerical methods.

S1. Model Solution

Adding the mass, salt, and heat equations together yields (from (12) and moving the known parameters to the right hand side):

$$\begin{pmatrix} \rho_2 & \rho_3 & \rho_i \\ \rho_2 S_2 & \rho_3 S_3 & \rho_i S_i \\ c_p \rho_2 T_f & c_p \rho_3 T_3 & -\rho_i L' \end{pmatrix} \begin{pmatrix} U_2 \\ U_3 \\ U_i \end{pmatrix} = \begin{pmatrix} \mathcal{F} - \rho_1 U_1 \\ -\rho_1 U_1 S_1 \\ \mathcal{Q} - c_p \rho_1 U_1 T_1 \end{pmatrix}. \tag{S1}$$

This system is three equations in five unknowns $\{U_2, U_3, U_i, T_3, S_3\}$. The five independent parameters are $\{\mathcal{F}, \mathcal{Q}, U_1, U_1 T_1, U_1 S_1\}$. However, the entrainment model specifies T_3 as a function of Φ only from (18) (as T_a is known from (13)). The entrainment model also specifies S_3 as a function of Φ and S_s from (19). And (16), (17) and (19), (21) boil down to two more (nonlinear) equations for Φ and S_s in terms of U_3 and S_3. The system is therefore closed. It is solved as follows:

1. Select trial values for Φ and $S_s \leq {S_s}^{\max}$, and compute T_3, S_3 from (18) and (19).

2. Compute ρ_s and ρ_3 from (16) and (20) using the Gibbs Seawater (GSW) Oceanographic Toolbox (IOC, SCOR, and IAPSO 2010). Skip statically unstable cases from (26).

3. Solve (S1) for $\{U_2, U_3, U_i\}$, and only retain solutions that satisfy the sign constraints (23) and (24).

4. Compare the trial Φ with the theoretical value from (17) and (21), and only retain good fits (within a threshold $\delta\Phi$).

These steps produce sets of candidate $\{U_2, U_3, U_i, S_s\}$ solutions that satisfy the entrainment model within $\delta\Phi$. For fixed parameters $\{\mathcal{F}, \mathcal{Q}, U_1, U_1 T_1, U_1 S_1\}$ there may be zero or many solutions. If there are many solutions, they sample the continuum of solutions that exist due to continuous variations of Φ and S_s (which is discretely sampled in step 1).

For these candidate solutions to be valid, they must also satisfy the shelf system of equations. From (12) they read:

$$\begin{pmatrix} \rho_1 & \rho_i \\ \rho_1 S_1 & \rho_i S_i \\ c_p \rho_1 T_1 & -\rho_i L' \end{pmatrix} \begin{pmatrix} u_1 \\ u_i \end{pmatrix} = \begin{pmatrix} \mathcal{F}_s - \rho_s u_s \\ -\rho_s u_s S_s \\ \mathcal{Q}_s - c_p \rho_s u_s T_f \end{pmatrix}. \tag{S2}$$

Given a candidate u_s solution from (21), this is an overdetermined linear system for two variables $\{u_1, u_i\}$ with three equations. In addition the constraints $u_1 > 0$ and $u_i < U_i$ apply from (23). We seek $0 \le \mathcal{Q}_s \le \mathcal{Q}$ and $\mathcal{F} \le \mathcal{F}_s \le 0$ values that satisfy this problem. There may be zero, one, or a one-parameter infinity of solutions (where $\mathcal{Q}_s$ and $\mathcal{F}_s$ tradeoff).

This system is solved in two further steps. For each candidate $\{U_2, U_3, U_i, S_s\}$ solution:

5. Write one constraint on $\{\mathcal{Q}_s, \mathcal{F}_s\}$ by requiring that the right hand side of (S2) lies in the range of the matrix

$$\mathbf{E}_s = \begin{pmatrix} \rho_1 & \rho_i \\ \rho_1 S_1 & \rho_i S_i \\ c_p \rho_1 T_1 & -\rho_i L' \end{pmatrix}. \tag{S3}$$

Call the range vector associated with the singular value of this matrix $\mathbf{s} = (s_1, s_2, s_3)^T$, which depends on the parameters in $\mathbf{E}_s$. Therefore,

$$s_1 \left(\mathcal{F}_s - \rho_s u_s\right) - s_2 \rho_s u_s S_s + s_3 \left(\mathcal{Q}_s - c_p \rho_s u_s T_f\right) = 0, \tag{S4}$$

which is a linear relationship between $\mathcal{Q}_s$ and $\mathcal{F}_s$. Consider it in the form

$$\mathcal{F}_s = \frac{\rho_s u_s}{s_1} \begin{pmatrix} 1 & S_s & T_f \end{pmatrix} \mathbf{s} - \frac{s_3}{s_1} \mathcal{Q}_s, \tag{S5}$$

where

$$\frac{s_3}{s_1} = \frac{S_1 - S_i}{L' S_1 + c_p T_1 S_i} \tag{S6}$$

is positive so that increasing positive Q_s means increasing negative $\mathcal{F}_s$. We require $0 \le Q_s \le Q$ and $\mathcal{F} \le \mathcal{F}_s \le 0$. To test these inequalities, compute the $\mathcal{F}_s(Q_s)$ values from (S5) at the $Q_s \in \{0, Q\}$ limits. This allows calculation of the $\{\mathcal{F}_s, Q_s\}$ pairs, call them $\{\mathcal{F}_s{}^a, Q_s{}^a\}$, that bound the line segment of acceptable values, if they exist.

6. If they do exist, compute the corresponding $\{u_1, u_i\}$ solutions from

$$\begin{pmatrix} u_1 \\ u_i \end{pmatrix} = \left(\mathbf{E}_s^T \mathbf{E}_s\right)^{-1} \mathbf{E}_s^T \begin{pmatrix} \mathcal{F}_s{}^a - \rho_s u_s \\ -\rho_s u_s S_s \\ Q_s{}^a - c_p \rho_s u_s T_f \end{pmatrix}. \tag{S7}$$

The line segment $\{\mathcal{F}_s{}^a, Q_s{}^a\}$ maps to a line segment in $\{u_1, u_i\}$ space. Finally, the inequalities on $\{u_1, u_i\}$ from (23) and (25) must be satisfied. Again, that yields, in general, zero, one, or an infinite number of $\{Q_s, \mathcal{F}_s\}$ pairs that qualify. Corresponding to every valid $\{Q_s, \mathcal{F}_s\}$ pair is a unique $\{u_1, u_i\}$ pair. If there is an infinite number, then it is a one parameter infinity.

This strategy finds solutions that exactly satisfy the conservation equations and inequalities, but that approximate the entrainment parameterization formula (17), within threshold $\delta\Phi$. For each set of parameter values, there are zero, one, or many valid solutions for $\{U_2, U_3, U_i, S_s\}$. For each one of these solutions, there are zero, one, or an infinite number of $\{u_1, u_i\}$ solutions.

S2. Theory for tradeoff between entrainment and shelf circulation

The entrainment/shelf salinity tradeoff is described as follows (from (17) and (21)):

$$\Phi = 1 - \gamma \frac{|u_s|^{1/3}}{(\rho_s - \rho_a)^{2/3}},$$
$$U_3 = \frac{u_s}{1 - \Phi}. \tag{S8}$$

Now $\rho_s - \rho_a \approx \rho_0 \beta \Delta S_s$, where $\Delta S_s = S_s - S_a{}^*$, where $S_a{}^*$ is the salinity of the water at the freezing temperature with the same density as the aW: $\rho_a = \rho(T_f, S_a{}^*)$. So for fixed U_3 (which is an

approximation; see below):

$$\Phi = 1 - \frac{\gamma^{3/2}}{\rho_0 \beta \Delta S_s} |U_3|^{1/2}, \tag{S9}$$

which is (28). In practice, U_3 varies a little from this theoretical curve because S_3 and T_3 depend on Φ and S_s too. Fig. 7 shows this variation is relatively weak.

S3. Theoretical arguments for degeneracy of forcing parameters

Here we justify the statement that acceptable model solutions for U_2 (for example) in the space of flux parameters, $\{\mathcal{F}, \mathcal{Q}, U_1, U_1T_1, U_1S_1\}$ (for fixed ϕ), typically lie along the line $\mathcal{Q} + L\mathcal{F} - \rho_0 c_p U_1 T_1$. This degeneracy is a great simplification because it points to the importance of a single compound parameter, rather than the full five dimensional parameter space. It thereby provides insight into the dominant processes in the model. Two lines of reasoning are shown:

First, consider the following approximation: The seawater freezing temperature, sea ice temperature, and sea ice salinity are zero $T_f = T_i = S_i = 0$ (which implies that $L' = L$) and the densities are identical $\rho_1 = \rho_2 = \rho_3 = \rho_i = \rho_0$. The equations for the system (S1) as a whole now read:

$$\begin{pmatrix} 1 & 1 & 1 \\ S_2 & S_3 & 0 \\ 0 & 0 & -L \end{pmatrix} \begin{pmatrix} U_2 \\ U_3 \\ U_i \end{pmatrix} \approx \begin{pmatrix} \mathcal{F}/\rho_0 - U_1 \\ -U_1 S_1 \\ \mathcal{Q}/\rho_0 - c_p U_1 T_1 \end{pmatrix}. \tag{S10}$$

In the final equation assume that $|c_p U_3 T_3| \ll |L U_i|$ too. This approximation is not wholly supported by the heat budgets shown in Figs. 4 and 5, although the AW term is smaller than the sea ice term in both cases (see below). Physically, the approximation means that the freezing rate to form sea ice U_i is proportional to the difference between ocean heat loss $\mathcal{Q}$ and AW heat flux $c_p \rho_0 U_1 T_1$. It is important because it reduces the rank of (S10) by one. Therefore, $U_i \approx \left(\mathcal{Q} - c_p \rho_0 U_1 T_1\right) / (\rho_0 L)$

and

$$\begin{pmatrix} 1 & 1 \\ S_2 & S_3 \end{pmatrix} \begin{pmatrix} U_2 \\ U_3 \end{pmatrix} \approx \begin{pmatrix} \mathcal{F}/\rho_0 - U_1 + \mathcal{Q}/(\rho_0 L) - c_p U_1 T_1/L \\ -\rho_0 U_1 S_1 \end{pmatrix}. \tag{S11}$$

The volume budget therefore reads

$$U_1 + U_2 + U_3 \approx \mathcal{F}/\rho_0 + \mathcal{Q}/(\rho_0 L) - c_p U_1 T_1/L \tag{S12}$$

The inverse of the matrix in (S11) is

$$\frac{1}{S_3 - S_2} \begin{pmatrix} S_3 & -1 \\ -S_2 & 1 \end{pmatrix} \tag{S13}$$

and therefore the OW flux U_2 is

$$U_2 \approx \frac{\left[\mathcal{Q} + L\mathcal{F} - \rho_0 c_p U_1 T_1\right] + \rho_0 L\,(S_1/S_3 - 1)\,U_1}{\rho_0 L\,(1 - S_2/S_3)}. \tag{S14}$$

The second numerator term, which is proportional to $(S_1/S_3 - 1)\,U_1$, is smaller in practice than the first term in brackets, which dominates the numerator. Therefore, this formula says that the OW flux U_2 is essentially proportional to $\mathcal{Q} + L\mathcal{F} - \rho_0 c_p U_1 T_1$.

Second, the exact formula for $U_2 \equiv \mathcal{N}/\mathcal{D}$ is

$$U_2 = \frac{(1 - S_i/S_3)\,\mathcal{Q} + \left(L' + c_p T_3 S_i/S_3\right)\mathcal{F} + L'\rho_1\,(S_1/S_3 - 1)\,U_1 + c_p\rho_1\left[(S_1/S_3 - S_i/S_3)\,T_3 + (S_i/S_3 - 1)\,T_1\right]U_1}{\rho_2\left\{L'\,(1 - S_2/S_3) + c_p\left[(S_i/S_3 - S_2/S_3)\,T_3 + (1 - S_i/S_3)\,T_f\right]\right\}}, \tag{S15}$$

which results from solving (S1). Notice that the flux parameters appear exclusively in the numerator $\mathcal{N}$ as a linear combination of $\mathcal{F}, \mathcal{Q}, U_1, U_1T_1$ and U_1S_1, from the right hand side of (S1). This property depends on the matrix form of the equations. Vanishing of U_2 (loss of the estuarine cell) is therefore controlled by $\mathcal{N}$ vanishing. The coefficients in $\mathcal{N}$ depend on T_3, S_3, however, which vary, and the denominator $\mathcal{D}$ also depends on them. This is the reason that solving exactly for U_2 is hard (sections S1 and S5). Now consider what can be said about $\mathcal{N}$ and U_2 without knowing T_3

and S_3: Approximate S_3 with S_1 and drop the uncertain T_3 terms to give:

$$\begin{aligned}\mathcal{N} \approx \mathcal{N}^* &\equiv (1 - S_i/S_1)\,\mathcal{Q} + L'\mathcal{F} + \rho_1 c_p\,(S_i/S_1 - 1)\,T_1 U_1,\\ &\approx \mathcal{Q} + L\mathcal{F} - \rho_0 c_p U_1 T_1,\\ &\approx \rho_i L' U_1\,(1 + U_2/U_1 + U_3/U_1),\end{aligned} \tag{S16}$$

which is (29)–(31). Plotting the distribution of U_2 against $\mathcal{N}^*$ for various forcing parameters tests these assumptions by quantifying what can be said about U_2 without knowing T_3 and S_3, as in Fig. 8.

S4. Theory for parametric locations of PW crises and OW emergency

Consider first the edge of the salt crisis hatched region in Fig. 6 near experiment 2, indicating $U_2 = 0$. From (S15) that $\mathcal{Q}$ limit, $\mathcal{Q}^+$, is

$$\mathcal{Q}^+ = \max\left\{\frac{\left(L' + c_p T_3 S_i/S_3\right)\mathcal{F} + L'\rho_1\,(S_1/S_3 - 1)\,U_1 + c_p\rho_1\,[(S_1/S_3 - S_i/S_3)\,T_3 + (S_i/S_3 - 1)\,T_1]\,U_1}{S_i/S_3 - 1}\right\}, \tag{S17}$$

which is controlled by the variation of T_3, S_3. For the salt crisis, $T_3 = T_f$ and $S_3 = S_s^-$, where S_s^- is the shelf salinity for vanishing entrainment $\Phi = 0$, given implicitly by $\rho(T_f, S_s^-) = |U_1|^{1/2}\gamma^{3/2} + \rho(T_a, S_a)$ from (17), (21), and $U_2 = 0$. Therefore,

$$\mathcal{Q}^+ = \frac{\left(L' S_s^- + c_p T_f S_i\right)(\rho_1 U_1 - \mathcal{F}) - \rho_1\left(L' + c_p T_f\right) U_1 S_1}{S_s^- - S_i} + c_p\rho_1 U_1 T_1. \tag{S18}$$

All the terms in this expression can be computed with knowledge of the system parameters. See Fig. 9.

Consider next the parametric location of the OW emergency in Fig. 6, given by $\mathcal{Q}^*$, for which $U_3 = 0$. By solving (S1) for $U_3 = 0$:

$$\mathcal{Q}^* = \frac{\left(L' S_2 + c_p T_f S_i\right)(\rho_1 U_1 - \mathcal{F}) - \rho_1\left(L' + c_p T_f\right) U_1 S_1}{S_2 - S_i} + c_p\rho_1 U_1 T_1. \tag{S19}$$

Notice the similarity with (S18) and that all these terms can be computed with knowledge of the system parameters. See Fig. 9.

Consider finally the edge of the heat crisis in Fig. 6, similar to experiment 1, given by $\mathcal{Q}^-$. Like the salt crisis

$$\mathcal{Q}^- = \frac{\left(L'S_3 + c_pT_3S_i\right)\left(\rho_1U_1 - \mathcal{F}\right) - \rho_1\left(L' + c_pT_3\right)U_1S_1}{S_3 - S_i} + c_p\rho_1U_1T_1. \tag{S20}$$

Predicting the values of T_3 and S_3 without solving the model is tricky, however. Fig. 9 shows three hierarchical approaches to estimate T_3 and S_3: (i) The patch shows upper and lower bounds for $\mathcal{Q}^-$ (plotted using the related $\mathcal{N}^*$ parameter in Fig. 9). These bounds derive from the range of possible (T_3, S_3) values for which the OW density equals the AW density. Furthermore, the $\mathcal{Q}^-$ upper bound cannot exceed the $\mathcal{Q}^*$ value associated with the OW emergency (because the OW emergency and the heat crisis exclude one another). These criteria guarantee static stability of AW and OW, regardless of the details of the entrainment process or the shelf water properties. In this sense, they make minimal assumptions. (ii) The group of points labelled "$\mathcal{Q}^-$ heat crisis mid 2" use (T_3, S_3) values that assume $\rho(T_3 = T_a, S_3) = \rho(T_1, S_1)$. The assumption $T_3 = T_a$ corresponds to no upper limit on S_s or therefore on ρ_s. (iii) The group of points labelled "$\mathcal{Q}^-$ heat crisis mid 1" estimates the maximum entrainment with

$$\Phi_{\max} \approx 1 - \frac{\gamma^{3/2}|U_1|^{1/2}}{\rho(T_f, {S_s}^{\max}) - \rho_a}, \tag{S21}$$

from (17) assuming that $u_s \approx U_3 \approx -U_1$. Given $\Phi_{\max}$, T_3 and S_3 follow from (18) and (19).

S5. Theory for selection of OW properties T_3, S_3 from budget equations

The OW properties T_3, S_3 are essential for understanding solutions to the model, and specifically the controls on the strength of the estuarine cell U_2. Here we extend sections S3 and S4 to discuss

how these properties are selected, in particular, how the basin-wide budget (S1) interacts with the sign constraints on the volume fluxes (23) and (24).

Consider again the formula for $U_2 = \mathcal{N}/\mathcal{D}$ (S15):

$$S_3\mathcal{N} = (S_3 - S_i)\mathcal{Q} + \left(L'S_3 + c_pT_3S_i\right)\mathcal{F} + L'\rho_1(S_1 - S_3)U_1 + c_p\rho_1\left[(S_1 - S_i)T_3 + (S_i - S_3)T_1\right]U_1,$$
$$\frac{S_3}{\rho_2}\mathcal{D} = L'(S_3 - S_2) + c_p\left[(S_i - S_2)T_3 + (S_3 - S_i)T_f\right]. \tag{S22}$$

Both the numerator $\mathcal{N}$ and denominator $\mathcal{D}$ depend on T_3 and S_3, as previously discussed. Therefore, consider the functions $\mathcal{N}(T_3, S_3)$ and $\mathcal{D}(T_3, S_3)$ in the temperature/salinity (TS) plane. We know that $-U_1 \lessapprox U_2 < 0$ for valid solutions, so the signs of $\mathcal{N}$ and $\mathcal{D}$ must be different and $\mathcal{D}$ cannot be very small. Valid T_3, S_3 points are therefore constrained by the $\mathcal{N}(T_3, S_3) = 0$ and $\mathcal{D}(T_3, S_3) = 0$ contours because $\mathcal{N}(T_3, S_3) = 0$ defines the locus of $U_2 = 0$ points and $\mathcal{D}(T_3, S_3) = 0$ defines the locus of $|U_2| = \infty$ points in TS space. What are these loci of points? For $\mathcal{N}(T_3, S_3) = 0$,

$$T_3 = \frac{\mathcal{Q}S_i - \rho_1\left(L'S_1 + c_pS_iT_1\right)U_1 + S_3\left[\rho_1U_1\left(L' + c_pT_1\right) - \mathcal{Q} - L'\mathcal{F}\right]}{c_p\left[\mathcal{F}S_i + \rho_1(S_1 - S_i)U_1\right]}, \tag{S23}$$

which is a line. The line intercepts the freezing temperature at salinity S^*

$$S^* = \frac{c_pT_f\left[\mathcal{F}S_i + \rho_1(S_1 - S_i)U_1\right] + \rho_1\left(L'S_1 + c_pS_iT_1\right)U_1 - \mathcal{Q}S_i}{\rho_1U_1\left(L' + c_pT_1\right) - \mathcal{Q} - L'\mathcal{F}}, \tag{S24}$$

and has slope

$$\frac{\rho_1U_1\left(L' + c_pT_1\right) - \mathcal{Q} - L'\mathcal{F}}{c_p\left[\mathcal{F}S_i + \rho_1(S_1 - S_i)U_1\right]} \approx \frac{L}{c_pS_1}. \tag{S25}$$

This line is entirely determined by the flux parameters $\{\mathcal{Q}, \mathcal{F}, U_1, T_1U_1, S_1U_1\}$ and constants. Think of it as having (nearly) fixed slope and an intercept S^* that changes with the flux parameters: it slides left and right in TS space, for instance as $\mathcal{Q}$ decreases and increases. Figure S1 illustrates this behaviour. For $\mathcal{D}(T_3, S_3) = 0$,

$$T_3 = \frac{-L'S_2 - c_pT_fS_i + S_3\left(L' + c_pT_f\right)}{c_p(S_2 - S_i)}, \tag{S26}$$

which again is a line. The line intercepts the freezing temperature at salinity S_2 and has slope

$$\frac{L' + c_p T_f}{c_p (S_2 - S_i)} \approx \frac{L}{c_p S_2}. \tag{S27}$$

This line is entirely determined by S_2 and constants. Think of it as being fixed to the PW properties (T_f, S_2) with (nearly) fixed positive slope (nearly) parallel to the $\mathcal{N} = 0$ line. For valid U_2 solutions, OW properties T_3 and S_3 cannot lie in the TS gap between the $\mathcal{N} = 0$ and $\mathcal{D} = 0$ lines because in this gap the signs of $\mathcal{N}$ and $\mathcal{D}$ are the same and $U_2 \not< 0$. Also, as OW properties approach the $\mathcal{D} = 0$ line, for instance as $S^* \to S_2$, U_2 becomes very sensitive to small changes in T_3 and S_3.

Consider also the formulae for U_3 and U_i from solving (S1):

$$-\frac{\rho_3 \mathcal{D}}{\rho_2} U_3 = (S_2 - S_i)\, \mathcal{Q} + \left(L' S_2 + c_p T_f S_i\right) \mathcal{F} + L' \rho_1 (S_1 - S_2)\, U_1 + c_p \rho_1 \left[(S_1 - S_i)\, T_f + (S_i - S_2)\, T_1\right] U_1 \tag{S28}$$

$$-\frac{\rho_i \mathcal{D}}{\rho_2} U_i = (S_3 - S_2)\, \mathcal{Q} + c_p \left(T_3 S_2 - T_f S_3\right) \mathcal{F} - c_p T_f \rho_1 (S_1 - S_3)\, U_1 + c_p \rho_1 \left[(S_1 - S_2)\, T_3 + (S_2 - S_3)\, T_1\right] U_1. \tag{S29}$$

Notice that the expression for U_3 resembles (S22) with S_3 replaced by S_2 and T_3 replaced by T_f in the numerator. This formula only depends on T_3 and S_3 via the denominator $\mathcal{D}$, unlike (S22). Similarly, the expression for U_i resembles (S22) with S_i replaced by S_2 and L' replaced by $-c_p T_f$ in the numerator. The denominator $\mathcal{D}$ is unchanged, however, because it is proportional to the determinant of the matrix $\mathbf{E}$ in (S1):

$$|\mathbf{E}| = -\rho_1 \rho_3 S_3 \mathcal{D} = \rho_2 \rho_3 \rho_i \left\{L' (S_2 - S_3) + c_p \left[(S_2 - S_i)\, T_3 + (S_i - S_3)\, T_f\right]\right\}. \tag{S30}$$

Therefore, the arguments above about the $\mathcal{D} = 0$ line apply unchanged. The requirement that $U_i < 0$ implies the existence of a third line in the TS plane corresponding to the $\mathcal{N} = 0$ line (S23):

$$T_3 = \frac{\mathcal{Q} S_2 + c_p \rho_1 \left(T_f S_1 - S_2 T_1\right) U_1 + S_3 \left[c_p \rho_1 U_1 \left(T_1 - T_f\right) - \mathcal{Q} + c_p T_f \mathcal{F}\right]}{c_p \left[\mathcal{F} S_2 + \rho_1 (S_1 - S_2)\, U_1\right]} \tag{S31}$$

(again, changing $S_i \to S_2$ and $L' \to -c_p T_f$). This line depends on the flux parameters and S_2. Its slope can change sign, but it passes through the PW properties (T_f, S_2), like the $\mathcal{D} = 0$ line. The requirement that $U_i < 0$ means that valid (T_3, S_3) points cannot lie between the $\mathcal{D} = 0$ and this second $\mathcal{N} = 0$ line. Thus, the joint requirements that $U_i < 0$ and $U_2 < 0$ require that (T_3, S_3) points cannot lie between the two $\mathcal{N} = 0$ lines defined by (S23) and (S31). (Because the $\mathcal{D} = 0$ line lies between the two $\mathcal{N} = 0$ lines.) Physically, this idea means that the OW temperature T_3 must be relatively close to the AW temperature or freezing, and not in between. The requirement that $U_3 < 0$ adds no new constraint because its numerator does not depend on (T_3, S_3): It implies an inequality on S_2, but this restriction is automatically satisfied.

These constraints combine with three other straightforward constraints to restrict possible OW properties, namely $\rho_3 \geq \rho_1$, and $T_f \leq T_3 \leq T_a$. They are shown schematically in Fig. S1. The intersection of all these restrictions are plotted as the brown patches in Figs. 3, 4, 5, and S1. In practice, the solutions lie close to the $\mathcal{N} = 0$ line given by (S23). Therefore, the selection of the strong (weak) shelf circulation mode with cold (warm) OW requires positive (negative) $S^* - S_2$. The additional requirement to satisfy the entrainment model (green patches, see section S6) selects a sub-region of these patches as valid solutions (plotted as the OW points in the solution figures).

S6. Theory for selection of OW properties T_3, S_3 from entrainment model

Consider T_3, S_3 constraints from the entrainment model. From (16)–(19) and (21),

$$\begin{aligned} 1 - \Phi &= \frac{\gamma^{3/2} |U_3|^{1/2}}{\rho_s - \rho_a}, \\ &= \frac{T_a - T_3}{T_a - T_f}, \\ &= \frac{S_a - S_3}{S_a - S_s}. \end{aligned} \tag{S32}$$

Therefore,

$$T_3 = T_a - \left(T_a - T_f\right) \frac{\gamma^{3/2}|U_3|^{1/2}}{\rho_s - \rho_a},$$
$$S_3 = S_a + (S_s - S_a) \frac{\gamma^{3/2}|U_3|^{1/2}}{\rho_s - \rho_a}. \tag{S33}$$

The common fraction is proportional to the departure in OW properties from aW properties. Its maximum conceivable value is $\gamma^{3/2}|U_1|^{1/2}/\left(\rho_1 - \rho_a\right) = 1 - \Phi_{min}$, which is associated with the minimum value of entrainment fraction Φ_{min}. This minimum entrainment value can be greater than zero. Therefore, the lower limit for T_3 is $T_a - \left(T_a - T_f\right)(1 - \Phi_{min})$. For S_3 we seek the maximum value of $(S_s - S_a)\,\gamma^{3/2}|U_3|^{1/2}/\left(\rho_s - \rho_a\right)$. An estimate of this maximum value is $\gamma^{3/2}|U_1|^{1/2}/\rho_0\beta$, because $\rho_s - \rho_a \approx \rho_0\beta\,(S_s - S_a{}^*) > \rho_0\beta\,(S_s - S_a)$ (from section S2, $\rho_a = \rho(T_f, S_a{}^*)$). Therefore, the upper limit for S_3 is approximately $S_a + \gamma^{3/2}|U_1|^{1/2}/\rho_0\beta$.

These limits on T_3 and S_3 are computed and plotted in Figs. 3, 4, and 5. For warmer/fresher aW, the range of possible OW properties shrinks, because OW cannot be less dense than AW and also because Φ_{min} decreases as ρ_a decreases. Note that the minimum value of Φ may not give a viable solution if the budget equations (S1) cannot also be satisfied. In other words, Φ_{min} and the T_3, S_3 limits from (S33), are necessary, but not sufficient to specify the OW properties.

References

IOC, SCOR, and IAPSO, 2010: The international thermodynamic equation of seawater – 2010: Calculation and use of thermodynamic properties. Tech. rep., Intergovernmental Oceanographic Commission, Manuals and Guides No. 56, UNESCO (English). 196pp.

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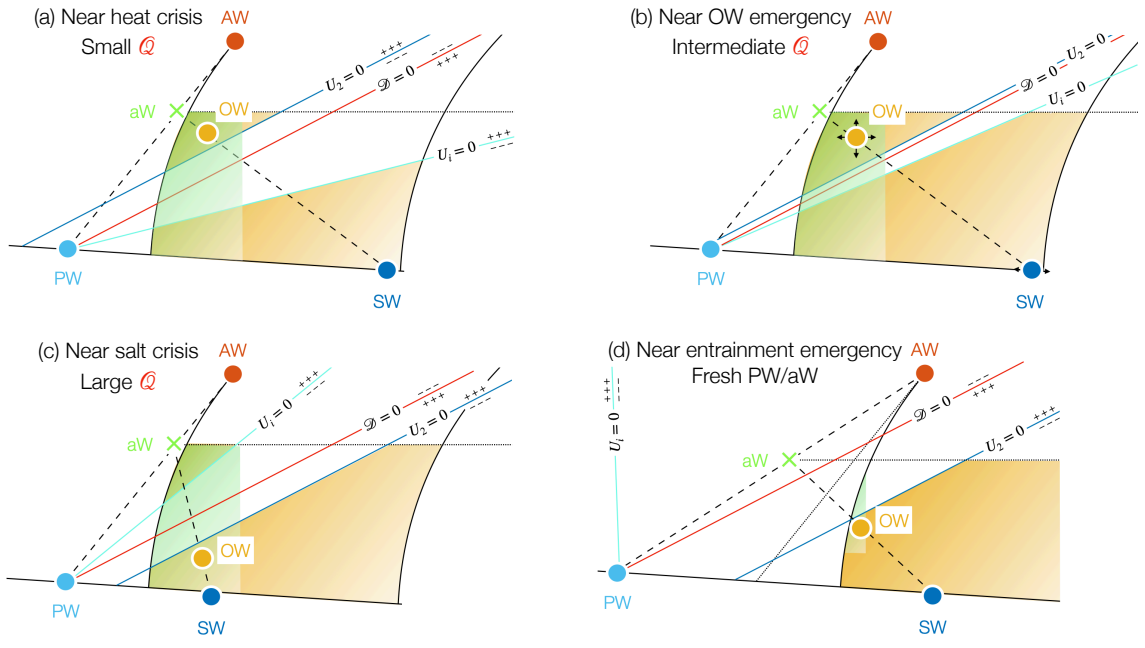

FIG. S1. Schematic temperature salinity diagrams of the processes affecting OW properties for different limiting cases (see also Fig. 12). (a) Heat crisis, (b) OW emergency, (c) Salt crisis, and (d) Entrainment emergency. The three lines are the contours of $U_2 = 0$ (blue), $U_i = 0$ (cyan), and denominator $\mathcal{D} = 0$ (red). Plus and minus signs indicate the signs either side of these zero contours. The shaded brown patches show possible solutions that satisfy the mass, freshwater, and heat budgets, plus the sign constraints on U_2, U_3, and U_i. See Fig. 3 and section S5 for further explanation. The shaded green patches show possible solutions that satisfy the entrainment model. See section S6 for further explanation. In practice, viable solutions lie in the green and brown patches near the blue $U_2 = 0$ line.