Supporting Information for "Reaction-diffusion waves in hydro-mechanically coupled porous solids"

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S1. Linear stability analysis

The proposed system of reaction-cross-diffusion equations (equation 9 and 10 in the main text) describing the porous material behavior post yield are high-order nonlinear partial differential equations, for which no analytical solutions can be obtained. To conduct the linear stability analysis, we first consider a set of solutions described by a small perturbation (denoted with *) around the steady state ($\tilde{p}_{s0}$, $\tilde{p}_{f0}$)=(0, 0):

$$\tilde{p}_s(\tilde{x},\tilde{t}) = \tilde{p}_{s0}(\tilde{x},\tilde{t}) + \tilde{p}_s^*(\tilde{x},\tilde{t}), \tag{1}$$

$$\tilde{p}_f(\tilde{x},\tilde{t}) = \tilde{p}_{f0}(\tilde{x},\tilde{t}) + \tilde{p}_f^*(\tilde{x},\tilde{t}), \tag{2}$$

The perturbation satisfies the following linearized version of the cross-diffusion equations given by:

$$\frac{\partial \tilde{p}_s^*}{\partial \tilde{t}} = \tilde{D}_M \frac{\partial^2 \tilde{p}_s^*}{\partial \tilde{x}^2} + \tilde{d}_H \frac{\partial^2 \tilde{p}_s^*}{\partial \tilde{x}^2} + \tilde{a}_{11}\tilde{p}_s^* + \tilde{a}_{12}\tilde{p}_f^* \tag{3}$$

$$\frac{\partial \tilde{p}_f^*}{\partial \tilde{t}} = \tilde{d}_M \frac{\partial^2 \tilde{p}_s^*}{\partial \tilde{x}^2} + \tilde{D}_H \frac{\partial^2 \tilde{p}_s^*}{\partial \tilde{x}^2} + \tilde{a}_{21}\tilde{p}_s^* + \tilde{a}_{22}\tilde{p}_f^* \tag{4}$$

where $\tilde{a}_{11} = \frac{\partial \tilde{R}_1}{\partial \tilde{p}_s}\Big|_{\tilde{p}_s=\tilde{p}_{s0}}$, $\tilde{a}_{12} = \frac{\partial \tilde{R}_1}{\partial \tilde{p}_f}\Big|_{\tilde{p}_f=\tilde{p}_{f0}}$, $\tilde{a}_{21} = \frac{\partial \tilde{R}_2}{\partial \tilde{p}_s}\Big|_{\tilde{p}_s=\tilde{p}_{s0}}$, $\tilde{a}_{22} = \frac{\partial \tilde{R}_2}{\partial \tilde{p}_f}\Big|_{\tilde{p}_f=\tilde{p}_{f0}}$ are the first order derivatives of the normalized reaction terms.

By applying a space Fourier transform to the above equations, the perturbation can be expressed as:

$$\tilde{p}_s^*(\tilde{x},\tilde{t}) = \tilde{p}_s^\star \exp(ik\tilde{x} + s_k\tilde{t}) \tag{5}$$

$$\tilde{p}_f^*(\tilde{x},\tilde{t}) = \tilde{p}_f^\star \exp(ik\tilde{x} + s_k\tilde{t}) \tag{6}$$

where k denotes the wavenumber in space while s_k is the growth rate of the perturbation. By substituting Eq. (5) and Eq. (6) into Eq. (3) and Eq. (4), the applied perturbation translates into:

$$\begin{bmatrix} s_k + k^2\tilde{D}_M - \tilde{a}_{11} & k^2\tilde{d}_H - \tilde{a}_{12} \\ k^2\tilde{d}_M - \tilde{a}_{21} & s_k + k^2\tilde{D}_H - \tilde{a}_{22} \end{bmatrix} \begin{bmatrix} \tilde{p}_s^\star \\ \tilde{p}_f^\star \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix} \tag{7}$$

which leads to the following condition:

$$det \begin{bmatrix} s_k + k^2\tilde{D}_M - \tilde{a}_{11} & k^2\tilde{d}_H - \tilde{a}_{12} \\ k^2\tilde{d}_M - \tilde{a}_{21} & s_k + k^2\tilde{D}_H - \tilde{a}_{22} \end{bmatrix} = 0 \tag{8}$$

From Eq. (8), we derive a characteristic equation of s_k:

$${s_k}^2 - \mathrm{tr}_k s_k + \Delta_k = 0 \tag{9}$$

where $\mathrm{tr}_k = (\tilde{a}_{11} + \tilde{a}_{22}) - k^2(\tilde{D}_M + \tilde{D}_H)$ and $\Delta_k = \tilde{a}_{11}\tilde{a}_{22} - \tilde{a}_{12}\tilde{a}_{21} + k^4(\tilde{D}_M\tilde{D}_H - \tilde{d}_M\tilde{d}_H) - k^2(\tilde{a}_{11}\tilde{D}_H + \tilde{a}_{22}\tilde{D}_M - \tilde{a}_{21}\tilde{d}_H - \tilde{a}_{12}\tilde{d}_M)$. Thus, the solution of Eq. (8) is expressed as

$$s_k = \frac{\mathrm{tr}_k \pm \sqrt{{\mathrm{tr}_k}^2 - 4\Delta_k}}{2} \tag{10}$$

Based on material stability theory, the system becomes unstable in the Lyapunov sense if there exists $\mathrm{Re}(s_k) > 0$ since the perturbation would increase with time in this case. Moreover, if s_{k_c} is a real number upon the occurrence of an instability (i.e. $s_{k_c} \geq 0$ for the critical wavenumber k_c), the system undergoes a saddle-node bifurcation or the so-called Turing bifurcation, along with the previous stable nodes in the phase space changing to the unstable saddle. However, if s_{k_c} is a pure complex number upon the occurrence of instability, the system undergoes a Hopf bifurcation as the previous stable focus in the phase space changes to an unstable one. Based on the above derivation, we present in the main manuscript a detailed discussion of these typical types of instabilities as well as a newly discovered quasisoliton wave type in relation to reaction-diffusion waves in the context of poromechanics.