

Opinion dynamics with intermittent-influence leaders on the signed social network

Ziwen Shen, Guang He*, Xiaotai Wu, Yixin Zhu, Mingxuan Shen

Abstract—In this paper, the leader-follower architecture is constructed by combining intermittent-influence leaders with a signed social network. Unlike a typical network with leaders where leaders are supposed to continuously influence followers, in this article, the leaders intermittently influence followers. Furthermore, the number of influences is limited. We focus on how intermittent-influence leaders impact the evolution of followers' opinions. The relationship between followers' opinions and the number of leader broadcasts is analyzed in detail. Then, the number of broadcasts is regarded as the cost, and the changing trend of the revenue per broadcast is obtained. The results show that as the number of broadcasts increases, the revenue per broadcast decreases gradually. Finally, the concept of assimilation is introduced to weigh the costs and benefits, and the minimum number of broadcasts required for the leader to assimilate the followers is derived. Two examples are given to demonstrate the validity of the main conclusions.

Index Terms—opinion dynamics; signed graph; polarization; DeGroot model.

I. INTRODUCTION

In recent years, opinion dynamics, as an international hot research topic in the field of systems and control science, has been widely discussed [1], [2], [3], [4]. Opinion dynamics is concerned with a social network closely related to human life, in which agents in a network can be countries, social groups, or living individuals [5], [6]. The individuals' evolution of behaviors and opinion in social networks is specifically and deeply studied in opinion dynamics [7]. Opinion dynamics in society is scientifically modeled and analyzed, which can not only reveal the laws of development of human society and animal groups in nature (such as zero-sum games in futures, the migration of animal populations), but also facilitate the development of man-made complex networks such as the Internet and transportation networks [8], [9], [10].

To characterize the opinion evolution in social networks, a classical linear time-invariant model, also known as the DeGroot (DG) model, was proposed in [11]. In this model, each individual's opinion at this time is determined by the

weighted average of his own and his neighbors' opinion at the previous moment. In the past decades, the research on the consensus of the DG model has never stopped. For any individual in the network, all the remaining individuals are not necessarily its neighbors. Despite this, the network can achieve the opinion consensus as long as the network connectivity is strong enough. When each node has a self-loop, the network can achieve the opinion consensus if and only if the network has a spanning tree [12]. Later, in order to describe the interaction of opinions in more detail, a series of variants of the DG model such as the Friedkin-Johnsen (FJ) model [13] and the Hegselmann-Krause (HK) model [14] were proposed, and many outstanding results were obtained [15], [16], [17].

The DG model is widely used in reality, such as company board of directors and jury panels [18]. It is worth noting that in the classical DG model, only cooperative relationship between individuals is considered. However, whether in nature or human society, the confrontation (or distrust) exists widely [19], [20]. In nature, animals compete for food and territory. Meanwhile, plants compete for sunlight and water. For human society, people compete for a series of valuable things such as social resources and natural resources. Therefore, opinion dynamics with competitive relationship has attracted a lot of attention [21], [22], [23].

Typically, the competitive relationship between individuals can be modeled by negative ties, which makes the network topology corresponding to a social network with competition into a signed graph [22]. For a signed graph, different control protocols may lead to polarization, fluctuation, and neutrality [24]. In [20], the competition was introduced into the continuous-time DG model, and the necessary and sufficient conditions for opinion polarization were obtained when the signed graph was structurally balanced. However, since the structurally unbalanced network had a very complex structure, the structurally unbalanced networks were less discussed [25]. Although competition may be a source of inconsistency, the authors of [25] pointed out that opinions could achieve a consensus in a signed network. In [26], the sign-consensus of opinion was studied when opinion can not reach an agreement. In fact, the opinion evolution on the signed graph was more complicated than the unsigned graph due to the existence of competition [20]. Hence, more efforts should be made to explore the competition's impact on opinion evolution.

It should be pointed out that in all the opinion dynamics models mentioned above, each individual has exactly the same characteristics, i.e., while they influence others' opinions, their opinions are also influenced by others' opinions. But in real life, there are special individuals called opinion leaders who

Ziwen Shen (szwszw2022@163.com) is with the School of Mathematics and Finance, Anhui Polytechnic University, Wuhu, Anhui, 241000, China.

Guang He (hg-1211ok@163.com) is with the Key Laboratory of Advanced Perception and Intelligent Control of High-end Equipment, Ministry of Education and School of Mathematics and Finance, Anhui Polytechnic University, Wuhu, Anhui, 241000, China

Xiaotai Wu (aaxtwu@126.com) is with the School of Mathematics and Finance, Anhui Polytechnic University, Wuhu, Anhui, 241000, China.

Yixin Zhu (zhuyixin0920@163.com) is with the School of Mathematics and Finance, Anhui Polytechnic University, Wuhu, Anhui, 241000, China.

Mingxuan Shen (smx1011@163.com) is with the School of Mathematics and Finance, Anhui Polytechnic University, Wuhu, Anhui, 241000, China.

*Corresponding Author

are not influenced by other individuals [27]. Considering the existence of opinion leaders, the leader-follower framework was used to model opinion evolution [28], [29]. The leader-follower architecture divides individuals in a social network into leaders and followers. Leaders, as special individuals, influence the opinions of followers while keeping their own opinions unchanged. In other words, the opinions of all followers are influenced by the leaders and will gradually tend to the opinions of the leaders.

Recently, the lead-follower architecture has been widely discussed in social networks [30], [31]. The article [32] pointed out that opinion leaders can make the group's opinion converge faster. The opinion leaders were introduced to the fractional opinion formation model in [33] and sufficient conditions for all followers to converge to the leader's opinion were obtained. In [34], the authors got a sufficient condition to ensure that the followers' opinions move at the same speed as the dynamic leader's opinion (or the opposite of the dynamic leader's opinion). In the above-mentioned articles, leaders are continuously involved in the evolution of group opinions in a social network. This persistent influence truly characterizes the evolution of opinions in small groups. However, with the rapid development of communication technology and Internet technology, the way humans obtain and interact with information has undergone major changes. On the Internet, the influence of the leaders on the followers is often intermittent rather than continuous [35].

The intermittent-influence leaders are quite common on the Internet. For example, popular information exchange platforms such as Twitter or Weibo can be regarded as social networks. On this network, famous people such as celebrities can be regarded as leaders, and correspondingly their subscribers are regarded as followers. Famous people, through their tweets or blogs, influence their subscribers intermittently. In particular, the number of influences is limited. In this case, the follower's opinion often cannot converge to the leader's opinion, and there will be more complex phenomena in the signed social networks. In this article, these phenomena and the underlying factors that determine them are analyzed and discussed. This is undoubtedly an interesting thing.

The purpose of this study is to analyze the influence of intermittent-influence leaders on the evolution of followers' opinions in a signed social network. This intermittent influence of the leader on the followers is called broadcast. Without loss of generality, in this paper, we assume that followers have sufficient time to interact with opinions to achieve a modulus consensus of opinions after the leader's last broadcast. The main contributions of this paper are as follows.

First, we design intermittent influence leaders for a signed social network. Whereas, Liang et al. [35] considered leaders with intermittent influence for an unsigned network.

Second, since broadcasts are intermittent and limited, we generally think that the opinions modulus consensus of followers is related to the number of broadcasts and the time of broadcast. For single-leader and multi-leader situations, a sufficient and necessary condition and a sufficient condition are given respectively to ensure that the opinions of the modulus consensus of followers are only related to the number

of broadcasts and not to the time of broadcast. Also, an expression of the relationship between the opinions of followers' modulus consensus and the number of broadcasts is obtained.

Finally, as the number of broadcasts increases, the follower's opinion will gradually approach the leader's opinion (or the opposite of its opinion value), and the number of broadcasts can be regarded as the cost. Through analysis, we can conclude that as the number of broadcasts increases, the revenue of each broadcast decreases gradually. Then, after introducing the concept of assimilation, we discuss how to weigh the costs and benefits.

This paper is organized as follows. In section II, we first introduce the notions and the relevant knowledge of graph theory. Then we introduce the DG model with the competitive relationship. Finally, we conduct a brief introduction of the model studied in this paper. The main results and proofs are in Section III. Section IV provides two examples to verify our conclusions. Finally, we give our conclusion in section V.

II. PRELIMINARIES

2.1 Notions

In this article, the following rules regarding symbols are given. $\mathbb{R}^{n \times m}$ represents a real matrix with n rows and m columns, and \mathbb{R}^n represents a real space of n dimensions. For any matrix $W = [w_{ij}] \in \mathbb{R}^{n \times m}$, $\rho(W)$ denotes its spectral radius and every element in $|W|$ is the absolute value of each element in the matrix W . $W \geq 0$ means that every element in matrix W is not less than 0. For any column vector $\mathbf{b} = [b_1, \dots, b_n]^T \in \mathbb{R}^n$, $\text{diag}(\mathbf{b})$ represents a diagonal matrix whose main diagonal elements are the elements in the vector \mathbf{b} . $\mathbf{1}_n$ ($\mathbf{0}_n$) is the n -dimension column vector with all elements being 1 (0). I_n is an n -dimension identity matrix. \emptyset is used to represent the empty set. $\langle \wp \rangle$ is used to denote the number of elements in the set \wp . $\lceil m \rceil$ represents the first integer greater than m . $\Delta y(k)$ represents the forward difference of the function $y(k)$, i.e., $\Delta y(k) = y(k+1) - y(k)$.

2.2 Graph theory

Let $G(W) = G(\mathcal{V}, \mathcal{E}, W)$ represent a weighted directed graph, where the node set $\mathcal{V} = (v_1, \dots, v_N)$, the edge set $\mathcal{E} \in \mathcal{V} \times \mathcal{V}$, and $W = [w_{ij}] \in \mathbb{R}^{N \times N}$ represents the weighted adjacency matrix corresponding to this graph. An edge from v_j to v_i can be denoted as $e_{ji} = (v_j, v_i) \in \mathcal{E}$, it indicates that node i can receive information from node j . And $e_{jj} = (v_j, v_j)$ indicates that node j has a self-loop. $w_{ij} \neq 0$ if and only if $e_{ji} \in \mathcal{E}$. A directed path from v_i to v_j is a sequence of nodes starting at v_i and ending at v_j , where any node is distinct. In a directed graph, if there is a node v_o that has directed paths to all other nodes, then the node v_o is called the root node. A directed graph has a spanning tree if there is at least one root node in the graph. When all nodes in a directed graph are root nodes, the graph is said to be strongly connected.

2.3 DeGroot model with competition

Let us consider a social network consisting of N individuals. The interactions of these N individuals can be represented by a weighted directed graph $G(W)$. In this directed graph, the signs of edges can be negative, so $G(W)$ is a signed

graph. In the signed matrix W , $w_{ij} > 0$ ($w_{ij} < 0$) means that individual i trusts (distrusts) individual j , so there is a cooperative (competitive) relationship between individual i and individual j . Naturally, $w_{ij} = 0$ and $w_{ji} = 0$ indicate that there is no relationship between individual i and individual j . Without loss of generality, $|W|$ is assumed to be row random.

Let $z_i(k)$ represent the opinion of individual i at time k . $z_i(k)$ can be positive, negative, or zero. When $z_i(k)$ is positive (negative), it means that individual i holds a positive (negative) attitude on the topic at time k . When $z_i(k) = 0$, it means that the individual i is neutral on the topic at time k . The opinion of individual i at time k is affected by its own opinion and the opinion of its neighbors at time $k - 1$, i.e.,

$$z_i(k+1) = \sum_{j=1}^N w_{ij} z_j(k). \quad (1)$$

Let $z(k) = [z_1(k), z_2(k), \dots, z_N(k)] \in \mathbb{R}^N$, $W = [w_{ij}]$, then equation (1) can be written in the following compact form:

$$z(k+1) = Wz(k), \quad (2)$$

where $w_{ii} > 0$ for all $i = 1, 2, \dots, N$.

Definition 1. The signed network (2) can achieve modulus consensus if there exists $\gamma \in \mathbb{R}$ such that

$$\lim_{k \rightarrow \infty} |z_i(k)| = \lim_{k \rightarrow \infty} |z_j(k)| = \gamma, \quad (3)$$

for any $i, j = 1, 2, \dots, N$. In particular, if $\gamma \neq 0$, the signed social network (2) can achieve bipartite consensus.

In the signed social networks, the bipartite consensus is the polarization of individuals' opinions into two opposite groups. In order to deal with this kind of signed graph more conveniently, we also need the following definition and assumption.

Definition 2. A signed graph $G(W) = G(\mathcal{V}, \mathcal{E}, W)$ is $(\mathcal{V}', \mathcal{V}'')$ structurally balanced if its node set \mathcal{V} can be divided into two vertex sets \mathcal{V}' and \mathcal{V}'' , where $\mathcal{V}' \cup \mathcal{V}'' = \mathcal{V}$ and $\mathcal{V}' \cap \mathcal{V}'' = \emptyset$. The sign of the edges between the vertices in \mathcal{V}' (\mathcal{V}'') is positive, while the sign of the edges formed by the nodes in \mathcal{V}' and \mathcal{V}'' is negative. $G(W)$ is structurally balanced if and only if there exists a diagonal matrix $\Gamma = \text{diag}(\tau_1, \tau_2, \dots, \tau_N)$, $\tau_i = \pm 1$, $i = 1, 2, \dots, N$ such that $\Gamma W \Gamma \geq 0$ [20].

Assumptions 1. The network $G(W)$ is $(\mathcal{V}', \mathcal{V}'')$ structurally balanced and has a spanning tree. When there is a leader, the leader maintains a cooperative relationship with the followers in set \mathcal{V}' (\mathcal{V}'') and a competitive relationship with the followers in set \mathcal{V}'' (\mathcal{V}'). Similarly, when there are multiple leaders, multiple leaders are cooperative with the followers in set \mathcal{V}' (\mathcal{V}'') and competitive with the followers in set \mathcal{V}'' (\mathcal{V}').

Remark 1. When the network $G(W)$ composed of followers is $(\mathcal{V}', \mathcal{V}'')$ structurally balanced, the leader must maintain a cooperative relation with individuals in one set (\mathcal{V}' or \mathcal{V}'') and competitive relations with individuals in the other set, which is necessary to ensure that the social network containing leaders is structurally balanced. If not, the relationship between individuals in the network becomes extremely complicated and difficult to describe, and the evolution of opinion is full of

uncertainty due to a sea of different scenarios one needs to analyze.

When the $G(W)$ is $(\mathcal{V}', \mathcal{V}'')$ structurally balanced, its node set \mathcal{V} must can be divided into two vertex sets \mathcal{V}' and \mathcal{V}'' . There must be a diagonal matrix $\Gamma_2 = \text{diag}(\tau_1, \tau_2, \dots, \tau_N)$, $\tau_i = \pm 1$, $i = 1, 2, \dots, N$ such that $\Gamma_2 W \Gamma_2 \geq 0$. In particular, $\tau_i = 1$ if $v_i \in \mathcal{V}'$ and $\tau_i = -1$ if $v_i \in \mathcal{V}''$. $\Gamma_2 W \Gamma_2$ is a row random matrix, and the left and right eigenvectors corresponding to the eigenvalue of 1 are \mathbf{l} and $\mathbf{1}_N$, respectively. In addition, we have $\mathbf{l}^T \mathbf{1}_N = 1$. For this social network to achieve modulus consensus, we assume that $G(W)$ has a spanning tree. Let $g(k) = \Gamma_2 z(k)$. Then we have

$$\begin{aligned} g(k+1) &= \Gamma_2 z(k+1) \\ &= \Gamma_2 W z(k) \\ &= \Gamma_2 W \Gamma_2 g(k). \end{aligned} \quad (4)$$

Obviously, $G(\Gamma_2 W \Gamma_2)$ has a spanning tree and the elements of its main diagonal are all greater than 0. In this case, 1 is the only maximum modulus eigenvalue of $\Gamma_2 W \Gamma_2$ and its algebraic multiplicity is 1 [12]. Naturally, we have $\lim_{k \rightarrow \infty} (\Gamma_2 W \Gamma_2)^k = \mathbf{1}_N \mathbf{l}^T$ [36]. Then, $\lim_{k \rightarrow \infty} g(k) = \mathbf{1}_N \mathbf{l}^T g(0)$ and $\lim_{k \rightarrow \infty} z(k) = \Gamma_2 \mathbf{1}_N \mathbf{l}^T g(0)$. Let $\alpha = \mathbf{l}^T g(0)$, we have $\lim_{k \rightarrow \infty} g(k) = \mathbf{1}_N \alpha$ and $\lim_{k \rightarrow \infty} z(k) = \Gamma_2 \mathbf{1}_N \alpha$. It can be seen from the above equations that when there are no leaders participating on a signed social network, if $\alpha \neq 0$, the opinions of individuals in a social network are polarized into two distinct groups α and $-\alpha$.

When the $G(W)$ is structurally unbalanced and strongly connected, $\rho(W) < 1$ [25]. Naturally, $\lim_{k \rightarrow \infty} W^k = \mathbf{0}$, where $\mathbf{0}$ is a matrix with all elements being 0. At this time, for any initial value $z(0)$, we have $\lim_{k \rightarrow \infty} z(k) = \mathbf{0}_N$. Eventually, all individuals remain neutral on the topic.

2.4 Model formulation

In this section, we introduce the intermittent-influence leaders into the DeGroot model with competition, and propose a leader-follower architecture on a signed social network. This type of leaders is an abstraction of some of the stars on the Internet platform. Stars can influence subscribers through text and video, while their own opinions are not influenced by subscribers. In short, subscribers are not neighbors of stars and stars are neighbors of subscribers.

For the convenience of description, the behavior of the leader influencing the followers is called broadcast, and these intermittent moments when the leader influences the followers are called broadcast moments. The opinions of followers evolve independently of the leaders at most moments, which are known as silence moments.

The influence factor $b_i \in [-1, 1]$ represents the influence of the leader on the follower v_i at the broadcast moment. When v_i receives the influence of the leader, the influence of the rest of v_i 's neighbors on v_i is weakened, so the influence of v_i 's neighbors on the individual v_i becomes $(1 - |b_i|)$ times the original influence. Without loss of generality, the broadcast factor is assumed to be finite, i.e.,

$$\sum_{j=1}^N b_j = b_s. \quad (5)$$

When there is only one leader, we use v_l and x_l to represent the leader and the opinion of leader, respectively. Let $\mathbf{b} = [b_1, b_2, \dots, b_N]^T$ and $B = \text{diag}(|\mathbf{b}|)$. At this time, the state vector in Equation (2) is augmented to $\bar{z}(k) = [z_l, z^T(k)]^T$, and the adjacency matrix is augmented to T (F) at the broadcast moments (silence moments). The specific forms of T and F are as follows:

$$\begin{aligned} T &= \begin{bmatrix} 1 & 0 \\ \mathbf{b} & (I - B)W \end{bmatrix} \\ F &= \begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix}. \end{aligned} \quad (6)$$

When there are multiple leaders, we use superscripts to number the leaders for convenience. Suppose there are p leaders, denoted by $v_l^1, v_l^2, \dots, v_l^p$, respectively. Correspondingly their opinions are denoted as $z_l = [z_l^1, \dots, z_l^p]^T$. For all $d = 1, 2, \dots, N$, $\mathbf{b}^d = [b_1^d, b_2^d, \dots, b_N^d]^T$ represent the influence of the leader v_l^d on N followers. $B^d = \text{diag}(|\mathbf{b}^d|)$. At this time, the opinion vector in Equation (2) is augmented to $\bar{z}(k) = [z_l^T, z^T(k)]^T \in \mathbb{R}^{N+p}$, and the adjacency matrices at the broadcast moments and silence moments are respectively augmented to the following forms:

$$\begin{aligned} T &= \begin{bmatrix} I_p & 0 \\ \mathbf{b}^1, \dots, \mathbf{b}^p & (I - \sum_{d=1}^p B^d)W \end{bmatrix} \\ F &= \begin{bmatrix} I_p & 0 \\ 0 & W \end{bmatrix}. \end{aligned} \quad (7)$$

Since the broadcast moments are intermittent and limited, without loss of generality, the set of broadcast moments is represented by $\wp = \{t_1, \dots, t_k\}$, and the number of broadcasts is denoted as $\langle \wp \rangle$. Let \bar{z}^* denote the final opinion vector when leaders participate in the opinions' evolution of social network. It is supposed that after the last broadcast, the followers can still have sufficient discussions. Naturally, the evolution of a social network consisting of leaders and followers can be expressed as:

$$\bar{z}^* = F^* \left(\prod_{h=1}^{k-1} T F^{t_{h+1}-t_h-1} \right) T F^{t_1-1} \bar{z}(0), \quad (8)$$

where

$$F^* = \lim_{k \rightarrow \infty} F^k. \quad (9)$$

Remark 2. F^* represents a phenomenon in which the followers' opinions evolve independently of the leaders after the last broadcast. This phenomenon fits our real life. After browsing the information on Weibo, Twitter and other network platforms, individuals often share, communicate and discuss information online or offline with friends, relatives, colleagues, etc.

III. MAIN RESULTS

Inspired by the fact that internet famous use text or video to influence their subscribers intermittently on Twitter or other network platforms, we introduce intermittent-influence leaders into the DG model with competitive relationship. With the

participation of these particular leaders, the evolution of the follower's opinions has different special properties than those described in the existing literature. This article will focus on these special properties and the determinants behind them.

Since the number of broadcasts is limited, the followers' opinions cannot completely converge to the leader's opinion (or its opposite value), but only tend to the leader's opinion (or its opposite value) to a certain extent. It is worth noting that the degree of this tendency is related to the broadcast moments and the number of broadcasts. Therefore, we use mathematical expressions to give the relationship between the two in structurally balanced and structurally unbalanced networks, respectively. As the number of broadcasts increases, so does the corresponding control cost. Trends in the change in marginal revenue per broadcast are analyzed. And after introducing the concept of assimilation, we give our unique insights on how to weigh the cost and benefit.

Theorem 1. Assume that Assumption 1 is satisfied. If there is only one leader and $\langle \wp \rangle$ is given, then opinion evolution of followers is independent of the set of broadcast moments \wp if and only if $|b_i| = \hat{b}$ for $\forall i = 1, 2, \dots, N$. Furthermore, if $\langle \wp \rangle = k$, the final opinion vector of follower has the following form:

$$z^* = \Gamma_2 \mathbf{1}_N \{ [1 - (1 - \hat{b})^k] \varpi z_l + (1 - \hat{b})^k \alpha \}. \quad (10)$$

Proof: Since the Assumption 1 is satisfied, there must be a matrix

$$\Gamma = \begin{bmatrix} \varpi & 0 \\ 0 & \Gamma_2 \end{bmatrix}, \quad (11)$$

where $\varpi = 1$ or -1 , such that $\Gamma T \Gamma \geq 0$ and $\Gamma F \Gamma \geq 0$. In particular, $\varpi = 1$ if the leader is cooperative with the followers in set \mathcal{V}' and competitive with the followers in set \mathcal{V}'' , $\varpi = -1$ if the leader is cooperative with the followers in set \mathcal{V}'' and competitive with the followers in set \mathcal{V}' . Let $\tilde{T} = \Gamma T \Gamma$ and $\tilde{F} = \Gamma F \Gamma$, then we have

$$\begin{aligned} \tilde{T} &= \begin{bmatrix} \varpi & 0 \\ 0 & \Gamma_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \mathbf{b} & (I - B)W \end{bmatrix} \begin{bmatrix} \varpi & 0 \\ 0 & \Gamma_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ \Gamma_2 \mathbf{b} \varpi & (I - B) \Gamma_2 W \Gamma_2 \end{bmatrix} \geq 0, \end{aligned} \quad (12)$$

$$\begin{aligned} \tilde{F} &= \begin{bmatrix} \varpi & 0 \\ 0 & \Gamma_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} \varpi & 0 \\ 0 & \Gamma_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \Gamma_2 W \Gamma_2 \end{bmatrix} \geq 0. \end{aligned} \quad (13)$$

Let $\bar{g}(k) = \Gamma \bar{z}(k)$, accordingly, Equation (8) has the following variant:

$$\bar{g}^* = \tilde{F}^* \left(\prod_{h=1}^{k-1} \tilde{T} \tilde{F}^{t_{h+1}-t_h-1} \right) \tilde{T} \tilde{F}^{t_1-1} \bar{g}(0), \quad (14)$$

where

$$\bar{g}(0) = \Gamma \bar{z}(0) = \begin{bmatrix} \varpi & 0 \\ 0 & \Gamma_2 \end{bmatrix} \begin{bmatrix} z_l \\ z(0) \end{bmatrix} = \begin{bmatrix} \varpi z_l \\ \Gamma_2 z(0) \end{bmatrix}, \quad (15)$$

$$\tilde{F}^* = \lim_{k \rightarrow \infty} \tilde{F}^k = \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{1}_N l^T \end{bmatrix}. \quad (16)$$

A necessary and sufficient condition for followers' opinions to be independent of the broadcast moments is that T and F are commutative. Further, T and F are commutative equivalent to \tilde{T} and \tilde{F} are commutative since $TF = \Gamma\tilde{T}\Gamma\tilde{F}\Gamma = \Gamma\tilde{F}\Gamma\tilde{T}\Gamma = F\tilde{T}$. Then, one has

$$\begin{aligned}\tilde{T}\tilde{F} &= \begin{bmatrix} 1 & 0 \\ \Gamma_2\mathbf{b}\varpi & (I-B)\Gamma_2W\Gamma_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ 0 & \Gamma_2W\Gamma_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ \Gamma_2\mathbf{b}\varpi & (I-B)\Gamma_2W^2\Gamma_2 \end{bmatrix},\end{aligned}\quad (17)$$

$$\begin{aligned}\tilde{F}\tilde{T} &= \begin{bmatrix} 1 & 0 \\ 0 & \Gamma_2W\Gamma_2 \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \Gamma_2\mathbf{b}\varpi & (I-B)\Gamma_2W\Gamma_2 \end{bmatrix} \\ &= \begin{bmatrix} 1 & 0 \\ \Gamma_2W\Gamma_2\Gamma_2\mathbf{b}\varpi & \Gamma_2W\Gamma_2(I-B)\Gamma_2W\Gamma_2 \end{bmatrix}.\end{aligned}\quad (18)$$

Sufficiency: If $|b_i| = \hat{b}$ for $i = 1, 2, \dots, N$, then by Equation (12), $\Gamma_2\mathbf{b}\varpi$ can be represented by $\hat{b}\mathbf{1}_N$. Correspondingly, $(I-B)$ can be represented by $(1-\hat{b})I$. Naturally, one can get

$$\begin{aligned}\Gamma_2W\Gamma_2\Gamma_2\mathbf{b}\varpi &= \Gamma_2W\Gamma_2\hat{b}\mathbf{1}_N = \hat{b}\Gamma_2W\Gamma_2\mathbf{1}_N \\ &= \hat{b}\mathbf{1}_N = \Gamma_2\mathbf{b}\varpi\end{aligned}\quad (19)$$

$$\begin{aligned}\Gamma_2W\Gamma_2(I-B)\Gamma_2W\Gamma_2 &= \Gamma_2W\Gamma_2(1-\hat{b})I\Gamma_2W\Gamma_2 \\ &= (1-\hat{b})\Gamma_2W^2\Gamma_2 \\ &= (I-B)\Gamma_2W^2\Gamma_2.\end{aligned}\quad (20)$$

This shows that $\tilde{F}\tilde{T} = \tilde{T}\tilde{F}$. At this point, the evolution of followers' opinions is independent of the sequence of broadcast moments φ .

Necessity: $\tilde{F}\tilde{T} = \tilde{T}\tilde{F}$ implies that $\Gamma_2W\Gamma_2\Gamma_2\mathbf{b}\varpi = \Gamma_2\mathbf{b}\varpi$. If $\mathbf{b} = \mathbf{0}_N$, then $b_i = 0$ for $i = 1, 2, \dots, N$. If $\mathbf{b} \neq \mathbf{0}_N$, it means that $\Gamma_2\mathbf{b}\varpi$ is the corresponding right eigenvector when the eigenvalue of $\Gamma_2W\Gamma_2$ is 1. Since $\Gamma_2W\Gamma_2$ is row random, the corresponding right eigenvector when its eigenvalue is 1 is $\mathbf{1}_N$, and $\Gamma_2\mathbf{b}\varpi$ is linearly related to $\mathbf{1}_N$. So, $\Gamma_2\mathbf{b}\varpi = \hat{b}\mathbf{1}_N$, $|b_i| = \hat{b}$ for any $i = 1, 2, \dots, N$.

At this point, \tilde{T} and \tilde{F} are commutative. When the number of broadcasts $\langle \varphi \rangle = k$, it can be concluded that

$$\begin{aligned}\tilde{T}^k &= \begin{bmatrix} 1 & 0 \\ \hat{b}\mathbf{1}_N & (1-\hat{b})\Gamma_2W\Gamma_2 \end{bmatrix}^k \\ &= \begin{bmatrix} 1 & 0 \\ [1-(1-\hat{b})^k]\mathbf{1}_N & (1-\hat{b})^k\Gamma_2W^k\Gamma_2 \end{bmatrix}.\end{aligned}\quad (21)$$

Regardless of how the broadcast moments are chosen, the final opinion vector is as follows:

$$\begin{aligned}\bar{g}^* &= \tilde{F}^*\tilde{T}^k\bar{g}(0) \\ &= \tilde{F}^*\tilde{T}^k\Gamma\bar{z}(0) \\ &= \begin{bmatrix} 1 & 0 \\ 0 & \mathbf{1}_N l^T \end{bmatrix} \begin{bmatrix} 1 & 0 \\ \hat{b}\mathbf{1}_N & (1-\hat{b})\Gamma_2W\Gamma_2 \end{bmatrix}^k \begin{bmatrix} \varpi z_l \\ \Gamma_2 z(0) \end{bmatrix} \\ &= \begin{bmatrix} \varpi z_l \\ \varpi z_l[1-(1-\hat{b})^k]\mathbf{1}_N + (1-\hat{b})^k\mathbf{1}_N l^T \Gamma_2 z(0) \end{bmatrix} \\ &= \begin{bmatrix} \varpi z_l \\ \varpi z_l[1-(1-\hat{b})^k]\mathbf{1}_N + (1-\hat{b})^k\mathbf{1}_N l^T g(0) \end{bmatrix} \\ &= \begin{bmatrix} \varpi z_l \\ \varpi z_l[1-(1-\hat{b})^k]\mathbf{1}_N + \alpha(1-\hat{b})^k\mathbf{1}_N \end{bmatrix}.\end{aligned}\quad (22)$$

Then,

$$\begin{aligned}\bar{z}^* &= \Gamma\bar{g}^* \\ &= \begin{bmatrix} \varpi & 0 \\ 0 & \Gamma_2 \end{bmatrix} \begin{bmatrix} \varpi z_l \\ \varpi z_l[1-(1-\hat{b})^k]\mathbf{1}_N + \alpha(1-\hat{b})^k\mathbf{1}_N \end{bmatrix} \\ &= \begin{bmatrix} \varpi z_l \\ \varpi z_l[1-(1-\hat{b})^k]\Gamma_2\mathbf{1}_N + \alpha(1-\hat{b})^k\Gamma_2\mathbf{1}_N \end{bmatrix}.\end{aligned}\quad (23)$$

In the end, the opinion vector of followers is as follows:

$$z^* = \Gamma_2\mathbf{1}_N\{[1-(1-\hat{b})^k]\varpi z_l + (1-\hat{b})^k\alpha\}.\quad (24)$$

Remark 3. In Equation (24), α or $-\alpha$ is the final opinion value of followers when no leader is involved. The constant z_l is the leader's opinion value. When $\varpi = 1$, the modulus of each follower's opinion is:

$$|[1-(1-\hat{b})^k]z_l + (1-\hat{b})^k\alpha|.\quad (25)$$

When $\varpi = -1$, similarly, the modulus of each follower's opinion is:

$$|[1-(1-\hat{b})^k]z_l + (1-\hat{b})^k\alpha|.\quad (26)$$

It can be seen from the above results that the modulus consensus opinion of followers is only related to the number of broadcasts and has nothing to do with the broadcast moments. $[1-(1-\hat{b})^k]z_l$ represents the influence of the leader on the followers. The more broadcast times, the greater the influence of the leader on the followers. When the number of broadcasts is close to 0, $\lim_{k \rightarrow 0}(1-\hat{b})^k = 1$. Naturally, $z^* = \Gamma_2\mathbf{1}_N\alpha$, and the evolution of the followers' opinions is independent of the leader, i.e., each follower's opinion is either α or $-\alpha$. When the number of broadcasts tends to infinity, $\lim_{k \rightarrow \infty}(1-\hat{b})^k = 0$. After that, $z^* = \Gamma_2\mathbf{1}_N\varpi z_l$. In this case, each follower's opinion is either z_l or $-z_l$.

Next, we will specifically study which individuals in the followers tend to z_l and which tend to $-z_l$ as the number of broadcasts k tends to infinity.

Corollary 1. It is assumed that the conditions and assumptions of Theorem 1 are satisfied. When the number of broadcasts tends to infinity, if $\varpi = 1$ ($\varpi = -1$), the opinions of followers in \mathcal{V}' (\mathcal{V}'') converge to z_l , and the opinions of followers in the set \mathcal{V}'' (\mathcal{V}') converge to $-z_l$.

Proof: By equation (24), $\lim_{k \rightarrow \infty} z^* = \Gamma_2\mathbf{1}_N\varpi z_l$. When $\varpi = 1$, $\lim_{k \rightarrow \infty} z^* = \Gamma_2\mathbf{1}_N z_l$. Naturally, the opinion of followers in set \mathcal{V}' converges to z_l , while the opinion of followers in set \mathcal{V}'' converges to $-z_l$. When $\varpi = -1$, similar to the above discussion, it can be concluded that the opinion of followers in set \mathcal{V}' converges to z_l , and the opinion of followers in set \mathcal{V}'' converges to $-z_l$. So far, the proof of this Corollary is completed.

Remark 4. Corollary 1 points out that in a structurally balanced network, when the number of broadcasts tends to infinity, the followers in the camp that cooperated with the leader would tend to the leader's opinion, while the followers in the other camp would tend to the opposite value of the leader's opinion. This conclusion is consistent with the Theorem 1 in [34] and the Theorem 2 in [37].

Combining Theorem 1 and Corollary 1, when $\varpi = 1$, the more the number of broadcasts, the closer the followers'

opinion in \mathcal{V}' is to the leader's opinion z_l , and the closer the followers' opinion in \mathcal{V}'' is to $-z_l$. The less the number of broadcasts, the closer the followers' opinion in \mathcal{V}' is to the opinion α when no leader is involved, and the closer the followers' opinion in \mathcal{V}'' is to $-\alpha$. Similarly, when $\varpi = -1$, the more the number of broadcasts, the closer the opinion of followers in \mathcal{V}' and \mathcal{V}'' is to $-z_l$ and z_l , respectively. The less the number of broadcasts, the closer the followers' opinion in \mathcal{V}' and \mathcal{V}'' is to α and $-\alpha$, respectively.

Theorem 2. Suppose there are multiple leaders and Assumption 1 is satisfied. When $\langle \varphi \rangle$ is given, if $\forall d = 1, 2, \dots, p$, $|b_i^d| = |b_j^d| = \hat{b}^d$ for $\forall i, j \in \mathcal{V}$, the modulus consensus opinion of followers is independent of the set of broadcast moments φ . Further, if $\langle \varphi \rangle = k$, the final opinion vector of follower has the following form:

$$z^* = \Gamma_2 \mathbf{1}_N \left\{ \frac{1 - (1 - \hat{b}')^k}{\hat{b}'} \check{\mathbf{b}}^T \Gamma_1 z_l + (1 - \hat{b}')^k \alpha \right\}, \quad (27)$$

where $\check{\mathbf{b}} = [\hat{b}^1, \hat{b}^2, \dots, \hat{b}^p]^T$ and $\hat{b}' = \sum_{d=1}^p \hat{b}^d$.

Proof: Similar to the proof of Theorem 1, we have

$$\begin{aligned} \tilde{T} &= \Gamma T \Gamma \\ &= \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix} \begin{bmatrix} I_p & 0 \\ \mathbf{b}^1, \dots, \mathbf{b}^p & (I - \sum_{d=1}^p B^d)W \end{bmatrix} \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix} \\ &= \begin{bmatrix} I_p & 0 \\ \Gamma_2(\mathbf{b}^1, \dots, \mathbf{b}^p)\Gamma_1 & \Gamma_2(I - \sum_{d=1}^p B^d)W\Gamma_2 \end{bmatrix} \geq 0, \end{aligned} \quad (28)$$

$$\begin{aligned} \tilde{F} &= \Gamma F \Gamma \\ &= \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & W \end{bmatrix} \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix} \\ &= \begin{bmatrix} I_p & 0 \\ 0 & \Gamma_2 W \Gamma_2 \end{bmatrix} \geq 0, \end{aligned} \quad (29)$$

where $\Gamma_1 = I_p$ or $-I_p$, $\Gamma_1 = I_p$ if multiple leaders are cooperative with the followers in set \mathcal{V}' and competitive with the followers in set \mathcal{V}'' , $\Gamma_1 = -I_p$ if multiple leaders are cooperative with the followers in set \mathcal{V}'' and competitive with the followers in set \mathcal{V}' . When $\forall d = 1, 2, \dots, p$, $|b_i^d| = |b_j^d| = \hat{b}^d$ for $\forall i, j \in \mathcal{V}$, let $\check{\mathbf{b}} = [\hat{b}^1, \hat{b}^2, \dots, \hat{b}^p]^T$ and $\hat{b}' = \sum_{d=1}^p \hat{b}^d$, then $\Gamma_2(\mathbf{b}^1, \dots, \mathbf{b}^p)\Gamma_1$ and $\Gamma_2(I - \sum_{d=1}^p B^d)W\Gamma_2$ can be represented by $\mathbf{1}_N \check{\mathbf{b}}^T$ and $(1 - \hat{b}')\Gamma_2 W \Gamma_2$, respectively. At this time,

$$\begin{aligned} \tilde{T}\tilde{F} &= \begin{bmatrix} I_p & 0 \\ \mathbf{1}_N \check{\mathbf{b}}^T & (1 - \hat{b}')\Gamma_2 W \Gamma_2 \end{bmatrix} \begin{bmatrix} I_p & 0 \\ 0 & \Gamma_2 W \Gamma_2 \end{bmatrix} \\ &= \begin{bmatrix} I_p & 0 \\ \mathbf{1}_N \check{\mathbf{b}}^T & (1 - \hat{b}')\Gamma_2 W^2 \Gamma_2 \end{bmatrix}, \end{aligned} \quad (30)$$

$$\begin{aligned} \tilde{F}\tilde{T} &= \begin{bmatrix} I_p & 0 \\ 0 & \Gamma_2 W \Gamma_2 \end{bmatrix} \begin{bmatrix} I_p & 0 \\ \mathbf{1}_N \check{\mathbf{b}}^T & (1 - \hat{b}')\Gamma_2 W \Gamma_2 \end{bmatrix} \\ &= \begin{bmatrix} I_p & 0 \\ \Gamma_2 W \Gamma_2 \mathbf{1}_N \check{\mathbf{b}}^T & (1 - \hat{b}')\Gamma_2 W^2 \Gamma_2 \end{bmatrix}. \end{aligned} \quad (31)$$

Since $\Gamma_2 W \Gamma_2$ is row-random, $\Gamma_2 W \Gamma_2 \mathbf{1}_N = \mathbf{1}_N$. So, $\Gamma_2 W \Gamma_2 \mathbf{1}_N \check{\mathbf{b}}^T = \mathbf{1}_N \check{\mathbf{b}}^T$, i.e., $\tilde{T}\tilde{F} = \tilde{F}\tilde{T}$. Naturally, $TF = \Gamma \tilde{T} \Gamma \tilde{F} \Gamma = \Gamma \tilde{T} \tilde{F} \Gamma = \Gamma \tilde{F} \tilde{T} \Gamma = FT$. In this case, the

modulus consensus opinion of followers is independent of the broadcast moments φ . If $\langle \varphi \rangle = k$, Equation (14) has the following form:

$$\begin{aligned} \bar{g}^* &= \tilde{F}^* \tilde{T}^k \bar{g}(0) \\ &= \tilde{F}^* \tilde{T}^k \Gamma \bar{z}(0) \\ &= \begin{bmatrix} I_p & 0 \\ 0 & \mathbf{1}_N l^T \end{bmatrix} \begin{bmatrix} I_p & 0 \\ \mathbf{1}_N \check{\mathbf{b}}^T & (1 - \hat{b}')\Gamma_2 W \Gamma_2 \end{bmatrix}^k \begin{bmatrix} \Gamma_1 z_l \\ \Gamma_2 z(0) \end{bmatrix} \\ &= \begin{bmatrix} I_p & 0 \\ 0 & \mathbf{1}_N l^T \end{bmatrix} \begin{bmatrix} I_p & 0 \\ \phi_1 \mathbf{1}_N \check{\mathbf{b}}^T & \phi_2 \Gamma_2 W^k \Gamma_2 \end{bmatrix} \begin{bmatrix} \Gamma_1 z_l \\ \Gamma_2 z(0) \end{bmatrix} \\ &= \begin{bmatrix} I_p & 0 \\ 0 & \mathbf{1}_N l^T \end{bmatrix} \begin{bmatrix} \Gamma_1 z_l \\ \phi_1 \mathbf{1}_N \check{\mathbf{b}}^T \Gamma_1 z_l + \phi_2 \Gamma_2 W^k \Gamma_2 \Gamma_2 z(0) \end{bmatrix} \\ &= \begin{bmatrix} \Gamma_1 z_l \\ \phi_1 \mathbf{1}_N \check{\mathbf{b}}^T \Gamma_1 z_l + \phi_2 \mathbf{1}_N l^T \Gamma_2 z(0) \end{bmatrix} \\ &= \begin{bmatrix} \Gamma_1 z_l \\ \phi_1 \mathbf{1}_N \check{\mathbf{b}}^T \Gamma_1 z_l + \phi_2 \mathbf{1}_N \alpha \end{bmatrix}, \end{aligned} \quad (32)$$

where $\phi_1 = \frac{1 - (1 - \hat{b}')^k}{\hat{b}'}$, $\phi_2 = (1 - \hat{b}')^k$.

Then,

$$\begin{aligned} \bar{z}^* &= \Gamma \bar{g}^* \\ &= \begin{bmatrix} \Gamma_1 & 0 \\ 0 & \Gamma_2 \end{bmatrix} \begin{bmatrix} \Gamma_1 z_l \\ \frac{1 - (1 - \hat{b}')^k}{\hat{b}'} \mathbf{1}_N \check{\mathbf{b}}^T \Gamma_1 z_l + (1 - \hat{b}')^k \mathbf{1}_N \alpha \end{bmatrix} \\ &= \begin{bmatrix} z_l \\ \frac{1 - (1 - \hat{b}')^k}{\hat{b}'} \Gamma_2 \mathbf{1}_N \check{\mathbf{b}}^T \Gamma_1 z_l + (1 - \hat{b}')^k \Gamma_2 \mathbf{1}_N \alpha \end{bmatrix}. \end{aligned} \quad (33)$$

In the end, the opinion vector of followers is as follows:

$$z^* = \Gamma_2 \mathbf{1}_N \left\{ \frac{1 - (1 - \hat{b}')^k}{\hat{b}'} \check{\mathbf{b}}^T \Gamma_1 z_l + (1 - \hat{b}')^k \alpha \right\}. \quad (34)$$

Remark 5. When $\Gamma_1 = I_p$, $\check{\mathbf{b}}^T \Gamma_1 z_l = \check{\mathbf{b}}^T z_l = \sum_{j=1}^p \hat{b}^j z_l^j$, the modulus of each follower's opinion can be expressed as

$$|[1 - (1 - \hat{b}')^k] \sum_{j=1}^p \frac{\hat{b}^j}{\hat{b}'} z_l^j + (1 - \hat{b}')^k \alpha|. \quad (35)$$

When $\Gamma_1 = -I_p$, $\check{\mathbf{b}}^T \Gamma_1 z_l = -\check{\mathbf{b}}^T z_l = -\sum_{j=1}^p \hat{b}^j z_l^j$, the modulus of each follower's opinion can be expressed as

$$|-[1 - (1 - \hat{b}')^k] \sum_{j=1}^p \frac{\hat{b}^j}{\hat{b}'} z_l^j + (1 - \hat{b}')^k \alpha|. \quad (36)$$

Comparing (35) and (36) with (25) and (26) respectively, it is not difficult to find that the multi-leader situation can be equivalently regarded as the single-leader situation with $x_l = \sum_{j=1}^p \hat{b}^j x_l^j$ and the influence factor is \hat{b}' . At this time, the modulus consensus opinion of followers is jointly influenced by all leaders.

In the above discussion, we have studied the case that the signed graph is structurally balanced. For the structurally unbalanced signed networks, we have the following result.

Theorem 3. When $\langle \varphi \rangle$ is given, if $G(W)$ is structurally unbalanced and strongly connected, eventually all followers remain neutral on the topic.

Proof: When $G(W)$ is structurally unbalanced and strongly connected, $\rho(W) < 1$. Naturally, $\lim_{k \rightarrow \infty} W^k = \mathbf{0}$. After that,

$$\begin{aligned}
F^* &= \lim_{k \rightarrow \infty} F^k \\
&= \lim_{k \rightarrow \infty} \begin{bmatrix} I_p & 0 \\ 0 & W \end{bmatrix}^k \\
&= \begin{bmatrix} I_p & 0 \\ 0 & 0 \end{bmatrix}.
\end{aligned} \tag{37}$$

Substituting F^* into Equation (8), we can get $\lim_{k \rightarrow \infty} \bar{z}(k) = [z_l^T, \mathbf{0}_N^T]^T \in \mathbb{R}^{N+p}$. At this point, no matter how many leaders there are in the network or the number of broadcasts, all followers remain neutral on the topic. This result is in line with our real life. When there are multiple interest groups in an enterprise, different interest groups check each other, so that the interests are not biased towards a single camp.

At this point, we have investigated the evolution of the opinion of followers for the structurally balanced and unbalanced networks, respectively. Next, we focus on the marginal effect of each broadcast by the leader. As the number of broadcasts in structurally unbalanced networks has no effect on the evolution of followers' opinions, so the marginal effect is specifically analyzed only in the structurally balanced network. For this, we have the following results.

Theorem 4. Assume that Assumption 1 is satisfied. Suppose there is only one leader and $|b_i| = \hat{b}$ for $\forall i = 1, 2, \dots, N$. As the number of the leader's broadcasts increases, the revenue of one single broadcast on the followers' opinions gradually decreases.

Proof: When $\varpi = 1$, the opinion of followers in \mathcal{V}' lies between the opinion z_l of the leader and the opinion α of the followers without the participation of the leader. The opinion of followers in \mathcal{V}'' is between $-z_l$ and $-\alpha$. At this time, combined with Equation (25), we can use $|[1 - (1 - \hat{b})^k]z_l + (1 - \hat{b})^k\alpha - \alpha|$ to represent the deviation of the opinion. The larger is the deviation of this opinion, the closer the opinion of followers in \mathcal{V}' is to z_l , and the closer the opinion of followers in \mathcal{V}'' is to $-z_l$. Therefore, this deviation can be used to measure the benefits of broadcasts. Without loss of generality, the number of broadcasts k is used to denote the cost. After that, we can define a cost-performance function $y(k)$ whose specific form is as follows:

$$\begin{aligned}
y(k) &= \frac{|[1 - (1 - \hat{b})^k]z_l + (1 - \hat{b})^k\alpha - \alpha|}{k} \\
&= \frac{1 - (1 - \hat{b})^k}{k} |z_l - \alpha|.
\end{aligned} \tag{38}$$

Taking forward difference of $y(k)$, we can get

$$\begin{aligned}
\Delta y(k) &= y(k+1) - y(k) \\
&= \frac{(k\hat{b} + 1)(1 - \hat{b})^k - 1}{k(k+1)} |z_l - \alpha|.
\end{aligned} \tag{39}$$

Let $x(k) = (k\hat{b} + 1)(1 - \hat{b})^k - 1$, then

$$\Delta x(k) = -\hat{b}^2(k+1)(1 - \hat{b})^k. \tag{40}$$

Since k is a positive integer and $0 < \hat{b} < 1$, $\Delta x(k) < 0$. So, the maximum value of the monotonically decreasing function $x(k)$ is $x(1) = (\hat{b} + 1)(1 - \hat{b}) - 1 = -\hat{b}^2 < 0$.

Naturally, $\Delta y(k) < 0$, the cost-performance function $y(k)$ is monotonically decreasing. When $\varpi = -1$, combined with Equation (26), deviation of the opinion can be denoted as $|-[1 - (1 - \hat{b})^k]z_l + (1 - \hat{b})^k\alpha - \alpha|$. At this point, the cost-performance function is $y(k) = \frac{|-[1 - (1 - \hat{b})^k]z_l + (1 - \hat{b})^k\alpha - \alpha|}{k}$. Similarly, we can conclude that the cost-performance function is still decreasing.

Theorem 4 states that when the network includes a leader, the marginal revenue per broadcast decreases gradually as the number of broadcasts increases. In fact, this phenomenon also applies to multiple leaders. When extending the single-leader situation to the multiple-leader situation, we just need to replace \hat{b} and z_l with \hat{b}' and $\sum_{j=1}^p (\hat{b}'/b') z_l^j$, respectively. Later, the same conclusion can be drawn.

Remark 6. For a cooperative social network, it was obtained that the marginal revenue of broadcasting decreases gradually with the increase of the number of broadcasts [35]. Theorem 4 shows that it is still true for a structurally balanced social network with cooperation and competition.

From the above discussion, it can be concluded that in a structurally balanced leader-follower network, when the leader maintains a cooperative relationship with the followers in set \mathcal{V}' (\mathcal{V}'') and a competitive relationship with the followers in set \mathcal{V}'' (\mathcal{V}'), with the increase of the number of broadcasts, the opinion of the followers in \mathcal{V}' (\mathcal{V}'') will gradually approach that of the leader. but the marginal revenue per broadcast is decreasing. The number of broadcasts represents the cost, and without loss of generality, it is assumed that the cost of each broadcast is the same. Naturally, how to weigh the cost and the benefit of broadcasts is an intriguing topic. To this end, we introduce the following concept of assimilation.

In a leader-follower network, the leader's opinion is assumed to be z_l . Taking the opinion z_l as the center, the constant ε is the neighborhood radius. Followers are said to be assimilated by the leader if their opinions are located within the neighborhood. In particular, the neighborhood radius ε is called the assimilation limit. The minimum number of broadcasts required to assimilate followers is inextricably linked to the assimilation limit. Next, under the premise that ε is the assimilation limit, the minimum number of broadcasts required for assimilation is specifically analyzed.

Corollary 2. It is assumed that the conditions and assumptions of Theorem 4 are satisfied. Then, the following conclusions hold.

- 1) If $\varpi = 1$, the minimum number of broadcasts required by the leader to assimilate the followers in \mathcal{V}' is $\lceil \frac{\ln \varepsilon - \ln |\alpha - z_l|}{\ln(1 - \hat{b})} \rceil$.
- 2) If $\varpi = -1$, the minimum number of broadcasts required by the leader to assimilate the followers in \mathcal{V}'' is $\lceil \frac{\ln \varepsilon - \ln |\alpha + z_l|}{\ln(1 - \hat{b})} \rceil$.

Proof: Item1): When $\varpi = 1$, assume the minimum number of broadcasts required by the leader to assimilate all followers is m , then we have

$$\begin{aligned}
|[1 - (1 - \hat{b})^m]z_l + (1 - \hat{b})^m\alpha - z_l| &< \varepsilon \\
|[1 - (1 - \hat{b})^{m-1}]z_l + (1 - \hat{b})^{m-1}\alpha - z_l| &\geq \varepsilon.
\end{aligned} \tag{41}$$

Afterwards,

$$\begin{aligned} (1 - \hat{b})^m |\alpha - z_l| &< \varepsilon \\ (1 - \hat{b})^{m-1} |\alpha - z_l| &\geq \varepsilon \end{aligned} \quad (42)$$

Taking the logarithm of both sides of the above two inequalities, we have

$$\begin{aligned} m \ln(1 - \hat{b}) &< \ln \varepsilon - \ln |\alpha - z_l| \\ (m - 1) \ln(1 - \hat{b}) &\geq \ln \varepsilon - \ln |\alpha - z_l| \end{aligned} \quad (43)$$

Finally,

$$\begin{aligned} m &> \frac{\ln \varepsilon - \ln |\alpha - z_l|}{\ln(1 - \hat{b})} \\ m &\leq \frac{\ln \varepsilon - \ln |\alpha - z_l|}{\ln(1 - \hat{b})} + 1 \end{aligned} \quad (44)$$

Naturally, $m = \lceil \frac{\ln \varepsilon - \ln |\alpha - z_l|}{\ln(1 - \hat{b})} \rceil$.

Item2): When $\varpi = -1$, assume the minimum number of broadcasts required by the leader to assimilate all followers is m , then we have

$$\begin{aligned} | - [1 - (1 - \hat{b})^m] z_l + (1 - \hat{b})^m \alpha + z_l | &< \varepsilon \\ | - [1 - (1 - \hat{b})^{m-1}] z_l + (1 - \hat{b})^{m-1} \alpha + z_l | &\geq \varepsilon \end{aligned} \quad (45)$$

Similar to the processing of Item 1), we have

$$\begin{aligned} m &> \frac{\ln \varepsilon - \ln |\alpha + z_l|}{\ln(1 - \hat{b})} \\ m &\leq \frac{\ln \varepsilon - \ln |\alpha + z_l|}{\ln(1 - \hat{b})} + 1 \end{aligned} \quad (46)$$

Naturally, $m = \lceil \frac{\ln \varepsilon - \ln |\alpha + z_l|}{\ln(1 - \hat{b})} \rceil$.

IV. NUMERICAL EXAMPLES

In this section, we will give several examples to observe the evolution of the systems state in order to verify our conclusions.

Example 1: Consider a social network \mathfrak{S}_0 with four followers, and the associated adjacency matrix W is given by:

$$W = \begin{bmatrix} 1 & 0 & 0 & 0 \\ 0.6 & 0.4 & 0 & 0 \\ -0.4 & 0 & 0.6 & 0 \\ 0 & -0.8 & 0 & 0.2 \end{bmatrix}.$$

Obviously, $G(W)$ has a spanning tree and is structurally balanced. Individuals 1 and 2 belong to set \mathcal{V}' , and individuals 3 and 4 belong to set \mathcal{V}'' . At this time, through simple calculation, we can obtain that the corresponding left eigenvector is $l = [1, 0, 0, 0]^T$ when the eigenvalue of matrix $\Gamma W T$ is 1. Assume that the initial value of the four followers is $z(0) = [0.2, 0.6, -0.7, 0.32]^T$. Naturally, when the leader is absent, each follower's opinion is $\alpha = 0.2$ or $-\alpha = -0.2$.

After that, we construct a leader-follower network \mathfrak{S}_1 by introducing the leader into network \mathfrak{S}_0 . Suppose the leader maintains a cooperative relationship with the follower in the set \mathcal{V}' and its opinion is $z_l = 1$. Suppose $\mathbf{b} = [0.3, 0.3, -0.3, -0.3]^T$. At this time, all the conditions of Theorem 1 are satisfied.

k	1	2	3	4
y(k)	0.24	0.204	0.1752	0.152

TABLE I

According to the conclusion of Theorem 1, the modulus of each follower's opinion is

$$|1 - (0.7)^k + (0.7)^k \times 0.2|.$$

When the number of broadcasts $k = 1$ and $k = 2$, the modulus of each follower's opinion is 0.44 and 0.608, respectively. When the number of broadcasts k is 3 and 4, the modulus of each follower's opinion is 0.7256 and 0.80792, respectively. These results are shown in Fig. 1. In Fig. 1, the black lines

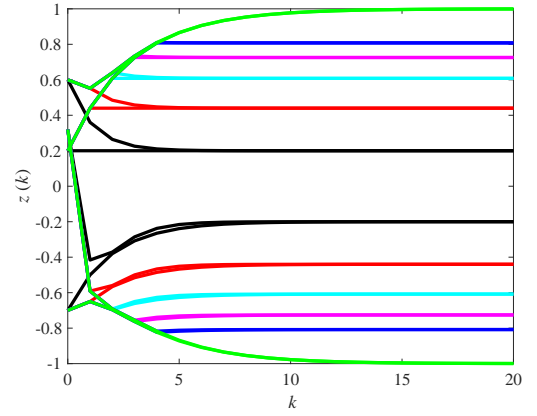


Fig. 1. The opinion evolution for four followers in Example 1.

represent the evolution of followers' opinions when no leader is involved (α and $-\alpha$), and the green lines represent the evolution of followers' opinions when the leader continues to influence followers (z_l and $-z_l$). The red lines, the cyan lines, the pink lines, and the blue lines represent the evolution of the followers' opinions when the leader broadcasts 1, 2, 3, and 4 times, respectively. It is not difficult to see that with the increase of the number of broadcasts, the opinions of followers in \mathcal{V}' gradually tend to 1, while those of followers in \mathcal{V}'' gradually tend to -1 . Next, we consider the marginal revenue per broadcast in network \mathfrak{S}_1 . When the number of broadcasts is 1, 2, 3, and 4, the corresponding marginal revenue is clearly represented in Table I. Obviously, the marginal revenue per broadcast decreases gradually as the number of broadcasts increases.

Example 2: Consider a network \mathfrak{S}_2 with four followers, and the adjacency matrix associated with it is as follows:

$$W = \begin{bmatrix} 0.2 & 0 & -0.2 & 0.6 \\ 0.3 & 0.4 & 0 & -0.3 \\ 0 & 0.5 & 0.5 & 0 \\ 0 & 0 & 0.3 & 0.7 \end{bmatrix}.$$

Clearly, $G(W)$ is strongly connected and aperiodic and structurally unbalanced. After that, we introduce three leaders into the network \mathfrak{S}_2 to construct a leader-follower network \mathfrak{S}_3 . $\mathbf{b}^1 = [-0.1, 0.1, 0.1, 0.1]^T$, $\mathbf{b}^2 = [-0.2, 0.2, -0.2, 0.2]^T$,

$\mathbf{b}^3 = [0.3, 0.3, -0.3, 0.3]^T$. At this time, all the conditions of Theorem 3 are satisfied. When the number of broadcasts $k = 1, 2, 3, 4$, the evolution of the followers' opinions is shown in Fig. 2. The red lines, the cyan lines, the pink lines, and the blue lines represent the evolution of the followers' opinions when the leader broadcasts 1, 2, 3, and 4 times, respectively. In Fig. 2, regardless of the number of broadcasts (the number of broadcasts is finite), the four individuals end up being neutral on the topic.

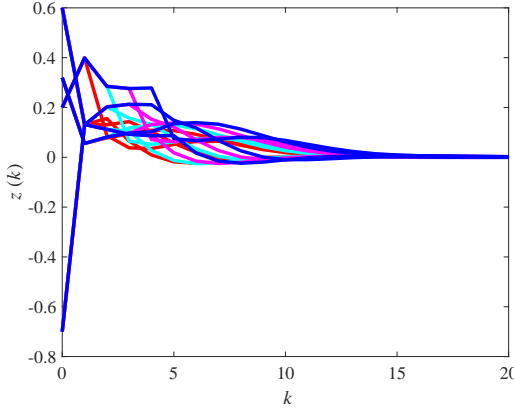


Fig. 2. The opinion evolution for four followers in Example 2.

V. CONCLUSIONS

This paper investigates the influence of intermittent-influence leaders on followers' opinions in a signed social network. First, for a structurally balanced network, we analyze the relationship between the followers' final opinions and the number of broadcasts, and extend the single-leader case to the multiple-leader case. Second, in a structurally unbalanced network, it is concluded that all followers remain neutral on the topic regardless of the number of broadcasts. Finally, by analyzing the cost-performance function for a structurally balanced network, the fact that the marginal revenue per broadcast decreases gradually as the number of broadcasts increases is obtained. Therefore, the concept of assimilation is introduced to solve the problem of how to weigh the costs and benefits, and the minimum number of broadcasts required by leaders to assimilate followers has been calculated. Finally, two examples are used to verify our conclusions.

ACKNOWLEDGEMENTS This work was supported in part by the National Natural Science Foundation of China under grant 61873294, 12271003, in part by the Scientific Research Project of Colleges and Universities of Anhui under Grant KJ2021A0513. The authors declare no conflicts of interest.

REFERENCES

- [1] J. Liu, X. Chen, T. Basar, and M. A. Belabbas, "Exponential convergence of the discrete- and continuous-time altafini models," *IEEE Transactions on Automatic Control*, vol. 62, no. 12, pp. 6168–6182, 2017.
- [2] L. Zhang, Y. Zhang, Z. Shang, Y. Su, F. Yan, and Z. Wu, "A generation and distribution system of clock signal source for signal acquisition system," *Engineering Reports*, vol. 4, no. 6, pp. 1–16, 2022.
- [3] Y. Tang, F. Qian, H. Gao, and J. Kurths, "Synchronization in complex networks and its application—a survey of recent advances and challenges," *Annual Reviews in Control*, vol. 38, no. 2, pp. 184–198, 2014.
- [4] H. Liang, H. Su, X. Wang, and M. Z. Q. Chen, "Swarming of heterogeneous multi-agent systems with periodically intermittent control," *Neurocomputing*, vol. 207, pp. 213–219, 2016.
- [5] S. E. Parsegov, A. V. Proskurnikov, R. Tempo, and N. E. Friedkin, "Novel multidimensional models of opinion dynamics in social networks," *IEEE Transactions on Automatic Control*, vol. 62, no. 5, pp. 2270–2285, 2017.
- [6] M. Ye, J. Liu, L. Wang, B. D. O. Anderson, and M. Cao, "Consensus and disagreement of heterogeneous belief systems in influence networks," *IEEE Transactions on Automatic Control*, vol. 65, no. 11, pp. 4679–4694, 2020.
- [7] P. Jia, A. MirTabatabaei, N. E. Friedkin, and F. Bullo, "Opinion dynamics and the evolution of social power in influence networks," *SIAM Review*, vol. 57, no. 3, pp. 367–397, 2015.
- [8] Y. Tang, X. Jin, Y. Shi, and W. Du, "Event-triggered attitude synchronization of multiple rigid body systems with velocity-free measurements," *Automatica*, vol. 143, p. 110460, 2022.
- [9] L. Wang, Y. Tian, and J. Du, "Opinion dynamics in social networks (in chinese)," *Scientia Sinica (Informationis)*, vol. 48, pp. 3–23, 2018.
- [10] M. A. El-Dabah, S. Kamel, M. A. Y. Abido, and B. Khan, "Optimal tuning of fractional-order proportional, integral, derivative and tilt-integral-derivative based power system stabilizers using runge kutta optimizer," *Engineering Reports*, vol. 4, no. 6, pp. 1–12, 2022.
- [11] M. H. DeGroot, "Reaching a consensus," *Journal of the American Statistical Association*, vol. 69, no. 345, pp. 118–121, 1974.
- [12] W. Ren and R. W. Beard, "Consensus seeking in multiagent systems under dynamically changing interaction topologies," *IEEE Transactions on Automatic Control*, vol. 50, no. 5, pp. 655–661, 2005.
- [13] N. E. Friedkin and E. C. Johnsen, "Social influence networks and opinion change," *Advances in Group Processes*, vol. 16, pp. 1–29, 1999.
- [14] R. Hegselmann and U. Krause, "Opinion dynamics and bounded confidence models, analysis, and simulation," *Journal of Artificial Societies and Social Simulation*, vol. 5, no. 3, p. 2, 2002.
- [15] J. Ghaderi and R. Srikant, "Opinion dynamics in social networks with stubborn agents: Equilibrium and convergence rate," *Automatica*, vol. 50, no. 12, pp. 3209–3215, 2014.
- [16] D. Bindel, J. Kleinberg, and S. Oren, "How bad is forming your own opinion?" *Games and Economic Behavior*, vol. 92, pp. 248–265, 2015.
- [17] B. Chazelle and C. Wang, "Inertial hegselmann-krause systems," *IEEE Transactions on Automatic Control*, vol. 62, no. 8, pp. 3905–3913, 2017.
- [18] B. D. O. Anderson and M. Ye, "Recent advances in the modelling and analysis of opinion dynamics on influence networks," *International Journal of Automation and Computing*, vol. 16, no. 2, pp. 129–149, 2019.
- [19] Y. Zhu, S. Li, J. Ma, and Y. Zheng, "Bipartite consensus in networks of agents with antagonistic interactions and quantization," *IEEE Transactions on Circuits and Systems II: Express Briefs*, vol. 65, no. 12, pp. 2012–2016, 2018.
- [20] C. Altafini, "Consensus problems on networks with antagonistic interactions," *IEEE Transactions on Automatic Control*, vol. 58, no. 4, pp. 935–946, 2013.
- [21] A. V. Proskurnikov, A. S. Matveev, and M. Cao, "Opinion dynamics in social networks with hostile camps: Consensus vs. polarization," *IEEE Transactions on Automatic Control*, vol. 61, no. 6, pp. 1524–1536, 2016.
- [22] A. Flache and M. W. Macy, "Small worlds and cultural polarization," *Journal of Mathematical Sociology*, vol. 35, no. 1-3, pp. 146–176, 2011.
- [23] W. Xia, M. Cao, and K. H. Johansson, "Structural balance and opinion separation in trust-mistrust social networks," *IEEE Transactions on Control of Network Systems*, vol. 3, no. 1, pp. 46–56, 2016.
- [24] F. Liu, D. Xue, S. Hirche, and M. Buss, "Polarizability, consensusability, and neutralizability of opinion dynamics on cooperative networks," *IEEE Transactions on Automatic Control*, vol. 64, no. 8, pp. 3339–3346, 2019.
- [25] G. He, H. Ruan, Y. Wu, and J. Liu, "Opinion dynamics with competitive relationship and switching topologies," *IEEE Access*, vol. 9, pp. 3016–3025, 2021.
- [26] Y. Jiang, H. Zhang, and J. Chen, "Sign-consensus over cooperative-antagonistic networks with switching topologies," *International Journal of Robust and Nonlinear Control*, vol. 28, no. 18, pp. 6146–6162, 2018.
- [27] Y. Dong, M. Zhan, G. Kou, Z. Ding, and H. Liang, "A survey on the fusion process in opinion dynamics," *Information Fusion*, vol. 43, pp. 57–65, 2018.
- [28] Y. Dong, Z. Ding, L. Martinez, and F. Herrera, "Managing consensus based on leadership in opinion dynamics," *Information Sciences*, vol. 397, pp. 187–205, 2017.

- [29] F. Dietrich, S. Martin, and M. Jungers, "Control via leadership of opinion dynamics with state and time-dependent interactions," *IEEE Transactions on Automatic Control*, vol. 63, no. 4, pp. 1200–1207, 2018.
- [30] Y. Zhao and G. Kou, "Bounded confidence-based opinion formation for opinion leaders and opinion followers on social networks," *Studies in Informatics and Control*, vol. 23, no. 2, pp. 153–162, 2014.
- [31] R. Dabarerera, K. Premaratne, M. N. Murthi, and D. Sarkar, "Consensus in the presence of multiple opinion leaders: Effect of bounded confidence," *IEEE Transactions on Signal and Information Processing over Networks*, vol. 2, no. 3, pp. 336–349, 2016.
- [32] S. Liu, C. Jiang, Z. Lin, Y. Ding, R. Duan, and Z. Xu, "Identifying effective influencers based on trust for electronic word-of-mouth marketing: A domain-aware approach," *Information Sciences*, vol. 306, pp. 34–52, 2015.
- [33] R. Almeida, A. B. Malinowska, and T. Odziejewicz, "Optimal leader-follower control for the fractional opinion formation model," *Journal of Optimization Theory and Applications*, vol. 182, no. 3, pp. 1171–1185, 2019.
- [34] H. Liang, F. Yuan, Z. Zhou, and H. Su, "Opinion separation in leader-follower cooperative social networks," *Neurocomputing*, vol. 434, pp. 90–97, 2021.
- [35] Z. Zhao, L. Shi, T. Li, J. Shao, and Y. Cheng, "Opinion dynamics of social networks with intermittent-influence leaders," *IEEE Transactions on Computational Social Systems*, pp. 1–10, 2022.
- [36] R. A. Horn and C. R. Johnson, *Matrix Analysis*. Cambridge, U.K.: Cambridge Univ. Press, 2012.
- [37] L. Pan, H. Shao, and D. Li, "Peer selection in opinion dynamics on signed social networks with stubborn individuals," *Neurocomputing*, vol. 477, pp. 104–113, 2022.
- [38] H. Liang, Y. Yang, and X. Wang, "Opinion dynamics in networks with heterogeneous confidence and influence," *Physica A: Statistical Mechanics and its Applications*, vol. 392, no. 9, pp. 2248–2256, 2013.
- [39] Z. Xu, J. Liu, and T. Başar, "On a modified degroot-friedkin model of opinion dynamics," in *2015 American Control Conference (ACC)*, 2015, pp. 1047–1052.
- [40] P. Dandekar, A. Goel, and D. T. Lee, "Biased assimilation, homophily, and the dynamics of polarization," *Proceedings of the National Academy of Sciences*, vol. 110, no. 15, pp. 5791–5796, 2013.
- [41] E. Girejko, L. Machado, A. B. Malinowska, and N. Martins, "Krause's model of opinion dynamics on isolated time scales," *Mathematical Methods in the Applied Sciences*, vol. 39, no. 18, pp. 5302–5314, 2016.
- [42] S. Motsch and E. Tadmor, "Heterophilious dynamics enhances consensus," *SIAM Review*, vol. 56, no. 4, pp. 577–621, 2014.
- [43] L. Shi, Y. Cheng, J. Shao, X. Wang, and H. Sheng, "Leader-follower opinion dynamics of signed social networks with asynchronous trust/distrust level evolution," *IEEE Transactions on Network Science and Engineering*, vol. 9, no. 2, pp. 495–509, 2022.