

CMMSE: POWERS OF CATALAN GENERATING FUNCTIONS FOR BOUNDED OPERATORS

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ABSTRACT. In this paper we consider the Catalan triangle numbers $(B_{n,k})_{n \geq 1, 1 \leq k \leq n}$ and $(A_{n,k})_{n \geq 1, 1 \leq k \leq n+1}$ to define powers of Catalan generating function $C(T)$ where T is a linear and bounded operator on a Banach space X . When the operator $4T$ is of power-bounded operator, the Catalan generating function is given by the Taylor series

$$C(T) := \sum_{n=0}^{\infty} C_n T^n,$$

where $c = (C_n)_{n \geq 0}$ is the Catalan sequence. Note that the operator $C(T)$ is a solution of the quadratic equation $TY^2 - Y + I = 0$. We obtain new formulae which involves Catalan triangle numbers $(B_{n,k})_{n \geq 1, 1 \leq k \leq n}$ and $(A_{n,k})_{n \geq 1, 1 \leq k \leq n+1}$. As element in the Banach algebra $\ell^1(\mathbb{N}^0, \frac{1}{4^n})$, we describe the spectrum of c^{*j} for $j \geq 1$, and the expression of c^{-*j} in terms of Catalan polynomials. In the last section, we give some particular examples to illustrate our results and some ideas to continue this research in the future.

1. INTRODUCTION

The well-known Catalan numbers $(C_n)_{n \geq 0}$ is given by the combinatorial formula

$$C_n = \frac{1}{n+1} \binom{2n}{n}, \quad n \geq 0,$$

They may be defined recursively by $C_0 = 1$ and

$$(1.1) \quad C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}, \quad n \geq 1,$$

and first terms in this sequence are 1, 1, 2, 5, 14, 42, 132, ... They appear in a wide range of physical problems: counts the number of ways to triangulate a regular polygon with $n+2$ sides; or, the number of ways that

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$2n$ people seat around a circular table are simultaneously shaking hands with another person at the table in such a way that none of the arms cross each other, see for example [13, 15].

The generating function of the Catalan sequence $c = (C_n)_{n \geq 0}$ is defined by

$$(1.2) \quad C(z) := \sum_{n=0}^{\infty} C_n z^n = \frac{1 - \sqrt{1 - 4z}}{2z}, \quad z \in D(0, \frac{1}{4}) := \{z \in \mathbb{C} \mid |z| < \frac{1}{4}\}.$$

This function satisfies the quadratic equation $zy^2 - y + 1 = 0$. These equations are frequently used in the study of, for example, physical or biological phenomena.

The main aim in [10] is to consider the quadratic equation

$$(1.3) \quad TY^2 - Y + I = 0,$$

in the set of linear and bounded operators, $\mathcal{B}(X)$ on a Banach space X , where I is the identity on the Banach space, and $T, Y \in \mathcal{B}(X)$. Formally, some solutions of this vector-valued quadratic equations are expressed by

$$Y = \frac{1 \pm \sqrt{1 - 4T}}{2T},$$

which involves the (non-trivial) problems of the square root of operator $1 - 4T$ and the inverse of operator T .

In this paper, we concern about the powers of $(C(T))^n$ for $n \in \mathbb{Z}$ and it is organized as follows. In the second section we consider the Catalan triangle sequences $(B_{n,k})_{n \geq 1, 1 \leq k \leq n}$ and $(A_{n,k})_{n \geq 1, 1 \leq k \leq n+1}$. We prove new formulae for these numbers (Lemma 2.2) and their asymptotic estimation (Lemma 2.3). We treat polynomials and generating formulae for these Catalan triangle numbers, see Definition 2.4 and Theorem 2.7.

In third section, we consider the Banach algebra $(\ell^1(\mathbb{N}^0, \frac{1}{4^n}), \| \cdot \|_{1, \frac{1}{4^n}}, *)$, where

$$\|a\|_{1, \frac{1}{4^n}} := \sum_{n=0}^{\infty} \frac{|a(n)|}{4^n} < \infty, \quad (a * b)(n) = \sum_{j=0}^n a(n-j)b(j), \quad n \geq 0,$$

where $a, b \in \ell^1(\mathbb{N}^0, \frac{1}{4^n})$. We consider Catalan triangle sequences $(a_k)_{k \geq 1}$, $(b_k)_{k \geq 1} \subset \ell^1(\mathbb{N}^0, \frac{1}{4^n})$ (Definition 3.1). These sequences are powers of the Catalan sequence c in $\ell^1(\mathbb{N}^0, \frac{1}{4^n})$ (Proposition 3.2); we describe their spectrum set in Proposition 3.3. An original and motivating results connects c^{-*k} and Catalan polynomials in Theorem 3.7.

The powers of the Catalan generating operator $C(T)$ are studied in forth section. We transfer our results from the algebra $\ell^1(\mathbb{N}^0, \frac{1}{4^n})$ to $\mathcal{B}(X)$ via the algebra homomorphism Φ ,

$$\Phi(a)x := \sum_{n \geq 0} a_n T^n(x), \quad a = (a_n)_{n \geq 0} \in \ell^1(\mathbb{N}^0, \frac{1}{4^n}), \quad x \in X,$$

Note that $\Phi(c) = C(T)$, $\Phi(b_k) = (C(T))^{2k}$ and $\Phi(a_k) = (C(T))^{2k-1}$ for $k \geq 1$. We describe $(C(T))^{-j}$ in terms of Catalan polynomials; we estimate their norms and describe $\sigma((C(T))^j)$ for $j \in \mathbb{Z}$ in Theorem 4.1.

In the last section we illustrate our results with some concrete operators T in the equation (1.3). We consider the Euclidean space \mathbb{C}^2 and matrices

$$T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}, \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}, \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix}.$$

We solve the equation (1.3) and calculate $(C(T))^j$ for these matrices and $j \in \mathbb{Z}$. We also check $(C(a))^j$ for some particular values of $a \in \ell^1(\mathbb{N}^0, \frac{1}{4^n})$ and $j \geq 1$. Finally we present some ideas to continue this research.

2. SOME NEWS RESULTS ABOUT CATALAN TRIANGLE NUMBERS

Catalan triangle numbers $(B_{n,k})_{n \geq 1, 1 \leq k \leq n}$ were introduced in [11]. These combinatorial numbers $B_{n,k}$ are the entries of the following Catalan triangle:

$$(2.1) \quad \begin{array}{c|cccccc} n \setminus k & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ \hline 1 & 1 & & & & & & \\ 2 & 2 & 1 & & & & & \\ 3 & 5 & 4 & 1 & & & & \\ 4 & 14 & 14 & 6 & 1 & & & \\ 5 & 42 & 48 & 27 & 8 & 1 & & \\ 6 & 132 & 165 & 110 & 44 & 10 & 1 & \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array}$$

which are given by

$$(2.2) \quad B_{n,k} := \frac{k}{n} \binom{2n}{n-k}, \quad n, k \in \mathbb{N}, \quad k \leq n.$$

Notice that $B_{n,1} = C_n$ and $B_{n,n} = 1$ $n \geq 1$.

In the last years, Catalan triangle (2.1) has been studied in detail. These numbers $(B_{n,k})_{n \geq k \geq 1}$ have been analyzed in many ways. For instance, symmetric functions have been used in [1], recurrence relations in [12], or in [5] the Newton interpolation formula, which is applied to conclude divisibility properties of sums of products of binomial coefficients.

Other combinatorial numbers $A_{n,k}$ defined as follows

$$(2.3) \quad A_{n,k} := \frac{2k-1}{2n+1} \binom{2n+1}{n+1-k}, \quad n, k \in \mathbb{N}, \quad k \leq n+1,$$

appear as the entries of this other Catalan triangle,

$$(2.4) \quad \begin{array}{c|ccccccc} n \setminus k & 1 & 2 & 3 & 4 & 5 & 6 & \dots \\ \hline 0 & 1 & & & & & & \\ 1 & 1 & 1 & & & & & \\ 2 & 2 & 3 & 1 & & & & \\ 3 & 5 & 9 & 5 & 1 & & & \\ 4 & 14 & 28 & 20 & 7 & 1 & & \\ 5 & 42 & 90 & 75 & 35 & 9 & 1 & \\ 6 & 132 & 297 & 275 & 154 & 54 & 11 & 1 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \end{array}$$

which is considered in [8]. Notice that $A_{n,1} = C_n$ and $C_{2n+1,n-k+1} = A_{n,k}$ for $k \leq n+1$.

The entries $B_{n,k}$ and $A_{n,k}$ of the above two particular Catalan triangles satisfy the recurrence relations

$$(2.5) \quad B_{n,k} = B_{n-1,k-1} + 2B_{n-1,k} + B_{n-1,k+1}, \quad k \geq 2,$$

and

$$(2.6) \quad A_{n,k} = A_{n-1,k-1} + 2A_{n-1,k} + A_{n-1,k+1}, \quad k \geq 2.$$

The generating function of the Catalan sequence $(C_n)_{n \geq 0}$ is defined by

$$(2.7) \quad C(z) := \sum_{n=0}^{\infty} C_n z^n = \frac{1 - \sqrt{1-4z}}{2z}, \quad z \in D(0, \frac{1}{4}) := \{z \in \mathbb{C} \mid |z| < \frac{1}{4}\}.$$

Note that $C(\frac{1}{4}) = 2$.

Theorem 2.1. Take $z \in D(0, \frac{1}{4})$.

(i) For $\lambda \neq C(z)$,

$$\frac{1}{\lambda - C(z)} = \frac{\lambda z - 1 + zC(z)}{\lambda^2 z - \lambda + 1}.$$

(ii) For $w \in D(0, \frac{1}{4})$ and $w \neq \frac{z}{(1+z)^2}$,

$$\frac{C^2(w)}{1 - zwC^2(w)} = \frac{C(w) - (z+1)}{w(1+z)^2 - z}.$$

Proof. (i) Note that

$$(\lambda - C(z))(\lambda z - 1 + zC(z)) = z\lambda^2 - \lambda + C(z) - zC^2(z) = z\lambda^2 - \lambda + 1,$$

for $\lambda \neq C(z)$.

(ii) By item (i), we get that

$$\begin{aligned} \frac{C^2(w)}{1 - zwC^2(w)} &= \frac{C^2(w)}{z} \frac{1}{\frac{1+z}{z} - C(w)} = C^2(w) \frac{w(1+z) - z + wzC(w)}{w(1+z)^2 - z} \\ &= \frac{C(w) - 1}{w} \frac{w(1+z) - z + wzC(w)}{w(1+z)^2 - z} = \frac{C(w) - (z+1)}{w(1+z)^2 - z}, \end{aligned}$$

where we have applied again the equality $wC^2(w) - C(w) + 1 = 0$. \square

The generating functions of the Catalan triangle sequences are defined by

$$(2.8) \quad \sum_{n=k}^{\infty} B_{n,k} z^n = z^k C^{2k}(z) = (C(z) - 1)^k, \quad k \geq 1,$$

$$(2.9) \quad \sum_{n=k}^{\infty} A_{n,k+1} z^n = z^k C^{2k+1}(z) = C(z)(C(z) - 1)^k, \quad k \geq 0,$$

for $z \in D(0, \frac{1}{4})$ ([15, Exercise A.32(a)]). Since

$$\lim_{z \rightarrow \frac{1}{4}} C(z) = 2, \quad \lim_{z \rightarrow -\frac{1}{4}} C(z) = 2(\sqrt{2} - 1),$$

see for example [15, Exercise A.66], a direct application of Abel's theorem allows us to prove the following result.

Lemma 2.2. *Given $k \geq 1$,*

$$\begin{aligned} \sum_{n=k}^{\infty} B_{n,k} \frac{1}{4^n} &= 1, & \sum_{n=k}^{\infty} B_{n,k} \frac{(-1)^n}{4^n} &= (2\sqrt{2} - 3)^k, \\ \sum_{n,k \geq 1} B_{n,k} \frac{1}{4^{n+k}} &= \frac{1}{3}, & \sum_{n,k \geq 1} B_{n,k} \frac{(-1)^n}{4^{n+k}} &= \frac{8\sqrt{2} - 13}{41}, \\ \sum_{n=k}^{\infty} A_{n,k+1} \frac{1}{4^n} &= 2, & \sum_{n=k}^{\infty} A_{n,k+1} \frac{(-1)^n}{4^n} &= 2(\sqrt{2} - 1)(2\sqrt{2} - 3)^k, \\ \sum_{n,k \geq 0} A_{n,k+1} \frac{1}{4^{n+k}} &= \frac{8}{3}, & \sum_{n,k \geq 0} A_{n,k+1} \frac{(-1)^n}{4^{n+k}} &= \frac{8}{41}(5\sqrt{2} - 3). \end{aligned}$$

Proof. We apply formulae (2.8) and (2.9) in the points $z = \frac{1}{4}$ and $-\frac{1}{4}$. \square

In the next lemma, we extend the asymptotic estimation for Catalan numbers

$$C_n \sim \frac{4^n}{\sqrt{\pi n^{\frac{3}{2}}}}, \quad n \rightarrow \infty,$$

([15, Exercise A.64]) to Catalan triangle numbers.

Lemma 2.3. *Given $k \geq 1$,*

$$\begin{aligned} B_{n,k} &\sim \frac{4^n}{\sqrt{\pi}} \frac{k}{n^{\frac{3}{2}}}, & n \rightarrow \infty, \\ A_{n,k} &\sim \frac{4^n}{\sqrt{\pi}} \frac{2k-1}{n^{\frac{3}{2}}}, & n \rightarrow \infty. \end{aligned}$$

Proof. We use the well-known Stirling formula $n! \sim e^{-n} n^n \sqrt{2\pi n}$ to show both equivalences. \square

Now we define polynomials taking into count the rows of the Catalan triangle numbers.

Definition 2.4. Given $n \geq 0$, we define the polynomials

$$P_n(z) := \sum_{j=0}^n B_{n+1,j+1} z^j, \quad Q_n(z) := \sum_{j=0}^{n+1} A_{n+1,j+1} z^j.$$

The first values of these families of polynomials are given by

$$\begin{aligned} P_0(z) &= 1, & Q_0(z) &= 1 + z \\ P_1(z) &= 2 + z, & Q_1(z) &= 2 + 3z + 1 \\ P_2(z) &= 5 + 4z + z^2, & Q_2(z) &= 5 + 9z + 5z^2 + z^3 \\ P_3(z) &= 14 + 14z + 6z^2 + z^3, & Q_3(z) &= 14 + 28z + 20z^2 + 7z^3 + z^4 \end{aligned}$$

Theorem 2.5. (i) *The only solution of the recurrence system*

$$\begin{cases} R_0(z) = 1, \\ zR_n(z) + C_n = (z+1)^2 R_{n-1}(z), \quad n \geq 1, \end{cases}$$

is the polynomial sequence $(P_n)_{n \geq 0}$ given in Definition 2.4.

(ii) *The only solution of the recurrence system*

$$\begin{cases} R_0(z) = 1 + z, \\ zR_n(z) + C_n = (z+1)^2 R_{n-1}(z), \quad n \geq 1, \end{cases}$$

is the polynomial sequence $(Q_n)_{n \geq 0}$ given in Definition 2.4.

Proof. It is enough to check that the sequence $(P_n)_{n \geq 0}$ satisfies the recurrence relation. Similarly the polynomial sequence $(Q_n)_{n \geq 0}$ does. By the recurrence relation 2.5, we get

$$\begin{aligned} P_{n+1}(z) &= \sum_{j=0}^{n+1} B_{n+2,j+1} z^j = \sum_{j=0}^{n+1} (B_{n+1,j} + 2B_{n+1,j+1} + B_{n+1,j+2}) z^j \\ &= z \sum_{j=1}^{n+1} B_{n+1,j} z^{j-1} + 2 \sum_{j=0}^{n+1} B_{n+1,j+1} z^j + \frac{1}{z} \sum_{j=0}^{n+1} B_{n+1,j+2} z^{j+1} \\ &= (z+2)P_n(z) + \frac{1}{z} \left(\sum_{j=0}^n B_{n+1,j+1} z^j - B_{n+1,1} \right) \\ &= \frac{(z+1)^2}{z} P_n(z) - \frac{C_{n+1}}{z}, \end{aligned}$$

and we conclude the equality. \square

Remark 2.6. The sequences of polynomials $(P_n)_{n \geq 0}$ and $(Q_n)_{n \geq 0}$ are useful to prove equalities for Catalan triangles numbers and other sequences of

integer numbers. For example, taking $z = 1$ in Theorem 2.5, we prove easily that

$$\sum_{k=1}^n B_{n,k} = \frac{n+1}{2} C_n, \quad \sum_{k=1}^{n+1} A_{n,k} = (n+1) C_n, \quad n \geq 1,$$

see an alternative proof in [11, Proposition 3.1]; and for $z = -1$, we get that

$$\sum_{k=1}^n (-1)^k B_{n,k} = -C_{n-1}, \quad \sum_{k=1}^{n+1} (-1)^k A_{n,k} = 0, \quad n \geq 1.$$

see, for example [9, Theorem 2.1 and 2.2] and references therein.

For $z = \frac{1}{4}$, we get that

$$\sum_{k=1}^n B_{n,k} \left(\frac{1}{4}\right)^k = \frac{a(n)}{4^n}, \quad \sum_{k=1}^{n+1} A_{n,k} \left(\frac{1}{4}\right)^k = \frac{b(n)}{4^{n+1}},$$

where $(a(n))_{n \geq 1}$ is the integer sequence A194725 and $(b(n))_{n \geq 0}$ is A130970 given in the The On-Line Encyclopedia of Integer Sequences by N.J.A. Sloane, [14].

For $z = \frac{-1}{4}$, we obtain that

$$\sum_{k=1}^n B_{n,k} \left(\frac{-1}{4}\right)^k = -\frac{d(n)}{(-4)^n}, \quad \sum_{k=1}^{n+1} A_{n,k} \left(\frac{-1}{4}\right)^k = -\frac{e(n)}{4^{n+1}},$$

where $(d(n))_{n \geq 1}$ is the integer sequence A051550 and $(e(n))_{n \geq 0}$ is A132863 given in [14].

In the next theorem, we obtain the bivariate generating function for polynomial $(P_n)_{n \geq 0}$ and $(Q_n)_{n \geq 0}$ given in Definition 2.4.

Theorem 2.7. For $n \geq 0$,

$$\begin{aligned} P(z, w) &:= \sum_{n \geq 0} P_n(z) w^n = \frac{C(w) - (z+1)}{w(1+z)^2 - z}, \\ Q(z, w) &:= \sum_{n \geq 0} Q_n(z) w^n = \frac{(C(w) - (z+1))(z+1)}{w(1+z)^2 - z} = P(z, w)(z+1). \end{aligned}$$

Proof. We take $z, w \in \mathbb{C}$ such that the bivariate generating function for polynomial $(P_n)_{n \geq 0}$ converges. Then

$$\begin{aligned} P(z, w) &= \sum_{n \geq 0} P_n(z) w^n = \sum_{n \geq 0} \sum_{j=0}^n B_{n+1,j+1} z^j w^n = \sum_{j \geq 0} z^j \sum_{n=j}^{\infty} B_{n+1,j+1} w^n \\ &= \sum_{j \geq 0} z^j w^j C^{2j+2}(w) = \frac{C^2(w)}{1 - zwC^2(w)} = \frac{C(w) - (z+1)}{w(1+z)^2 - z}, \end{aligned}$$

where we have applied the equation (2.8), and Theorem 2.1 (ii).

Similarly,

$$\begin{aligned}
Q(z, w) &= \sum_{n \geq 0} Q_n(z) w^n = \sum_{n \geq -1} \sum_{j=0}^{n+1} A_{n+1, j+1} z^j w^n - \frac{1}{w} \\
&= \sum_{j \geq 0} z^j \sum_{n=j-1}^{\infty} A_{n+1, j+1} w^n - \frac{1}{w} = \sum_{j \geq 0} z^j w^{j-1} C^{2j+1}(w) - \frac{1}{w} \\
&= \frac{1}{w} \frac{C(w) - 1 + zwC^2(w)}{1 - zwC^2(w)} = \frac{(1+z)C^2(w)}{1 - zwC^2(w)} \\
&= \frac{(C(w) - (z+1))(z+1)}{w(1+z)^2 - z} = P(z, w)(z+1),
\end{aligned}$$

where we have applied the equation (2.9), and Theorem 2.1 (ii). \square

Remark 2.8. Note that for $|w| \leq \frac{1}{4}$ and $|z| < 1$, functions $P(z, w)$ and $Q(z, w)$ are well-defined due to

$$|P(z, w)| \leq \sum_{n \geq 0} |P_n(z)| \frac{1}{4^n} = 4 \sum_{j \geq 0} |z|^j = 4 \frac{1}{1 - |z|}.$$

Formulae given in Theorem 2.7 extend several known generating formula, for example, for Catalan numbers

$$\begin{aligned}
P(0, w) &= \sum_{n \geq 0} P_n(0) w^n = \sum_{n \geq 0} B_{n+1, 1} w^n = \sum_{n \geq 0} C_{n+1} w^n = \frac{C(w) - 1}{w}, \\
Q(0, w) &= \sum_{n \geq 0} Q_n(0) w^n = \sum_{n \geq 0} A_{n+1, 1} w^n = \sum_{n \geq 0} C_{n+1} w^n = \frac{C(w) - 1}{w}.
\end{aligned}$$

Other generating functions for integer natural sequences, see Remark 2.6, are also obtained.

3. SEQUENCES OF CATALAN TRIANGLE NUMBERS

In this section, we consider the weight Banach algebra $\ell^1(\mathbb{N}^0, \frac{1}{4^n})$. This algebra is formed by sequence $a = (a(n))_{n \geq 0}$ such that

$$\|a\|_{1, \frac{1}{4^n}} := \sum_{n=0}^{\infty} \frac{|a(n)|}{4^n} < \infty,$$

and the product is the usual convolution $*$ defined by

$$(a * b)(n) = \sum_{j=0}^n a(n-j)b(j), \quad a, b \in \ell^1(\mathbb{N}^0, \frac{1}{4^n}).$$

We write $a^{*0} = a$ and $a^{*n} = a \dots^{n-1} a$ for $n \in \mathbb{N}$.

The canonical base $\{\delta_j\}_{j \geq 0}$ is formed by sequences such that $(\delta_j)(n) := \delta_{j,n}$ is the known delta Kronecker. Note that $\delta_1^{*n} = \delta_n$ for $n \in \mathbb{N}$. This

Banach algebra has identity element, δ_0 , its spectrum set is the closed disc $\overline{D(0, \frac{1}{4})}$ and its Gelfand transform is given by the Z -transform

$$Z(a)(z) := \sum_{n=0}^{\infty} a(n)z^n, \quad z \in \overline{D(0, \frac{1}{4})}.$$

It is straightforward to check that $Z(\delta_n)(z) = z^n$ for $n \geq 0$ (see, for example, [7]).

We recall that the resolvent set of $a \in \ell^1(\mathbb{N}^0, \frac{1}{4^n})$, denoted as $\rho(a)$, is defined by

$$\rho(a) := \{\lambda \in \mathbb{C} : (\lambda\delta_0 - a)^{-1} \in \ell^1(\mathbb{N}^0, \frac{1}{4^n})\},$$

and the spectrum set of a is denoted by $\sigma(a)$ and given by $\sigma(a) := \mathbb{C} \setminus \rho(a)$.

The Catalan numbers may be defined recursively by $C_0 = 1$ and

$$(3.1) \quad C_n = \sum_{i=0}^{n-1} C_i C_{n-1-i}, \quad n \geq 1.$$

We write $c = (C_n)_{n \geq 0}$ and then $\|c\|_{1, \frac{1}{4^n}} = 2$ and $C(z) = Z(c)(z)$ for $z \in D(0, \frac{1}{4})$. We may interpret the equality (3.1) in terms of convolution product in the following closed form

$$\delta_1 * c^{*1} - c + \delta_0 = 0,$$

where we deduce that

$$(3.2) \quad c^{-1} = \delta_0 - \delta_1 * c.$$

Definition 3.1. Given the Catalan triangle numbers $(B_{n,k})_{n,k}$ and $(A_{n,k})_{n,k}$ considered in Section 2, we define the Catalan triangle sequences a_k and b_k by

$$a_k(n) := A_{n+k-1,k}, \quad b_k(n) := B_{n+k,k}, \quad n \geq 0,$$

for $k \geq 1$. Note that $a_1(n) = A_{n,1} = C_n$ and $b_1(n) = B_{n+1,1} = C_{n+1}$ for $n \geq 0$.

Proposition 3.2. For $k \geq 1$, consider the sequences a_k and b_k given in Definition 3.1. Then

(i) $a_k, b_k \in \ell^1(\mathbb{N}^*, \frac{1}{4^n})$ and

$$\|a_k\|_{1, \frac{1}{4^n}} = 2^{2k-1}, \quad \|b_k\|_{1, \frac{1}{4^n}} = 2^{2k}.$$

(ii) $Z(a_k)(z) = (C(z))^{2k-1}$ and $Z(b_k)(z) = (C(z))^{2k}$ for $z \in D(0, \frac{1}{4})$.

(iii) $a_k = c^{*(2k-2)}$ and $b_k = c^{*(2k-1)}$.

Proof. The item (i) is a consequence of Lemma 2.2. To check (ii), note that

$$\begin{aligned} Z(a_k)(z) &= \sum_{n=0}^{\infty} A_{n+k-1,k} z^n = z^{-k+1} \sum_{m=k-1}^{\infty} A_{m,k} z^m = C^{2k-1}(z), \\ Z(b_k)(z) &= \sum_{n=0}^{\infty} B_{n+k,k} z^n = z^{-k} \sum_{m=k}^{\infty} B_{m,k} z^m = C^{2k}(z), \end{aligned}$$

where we have applied fomulae (2.8) and (2.9). The item (iii) is a straightforward consequence of (ii). \square

Proposition 3.3. *The spectra of the Catalan triangle sequences $(a_k)_{k \geq 1}$ and $(b_k)_{k \geq 1}$ in the algebra $\ell^1(\mathbb{N}^0, \frac{1}{4^n})$ are given by*

$$\sigma(a_k) = \left(\overline{C(D(0, \frac{1}{4}))} \right)^{2k-1}, \quad \sigma(b_k) = \left(\overline{C(D(0, \frac{1}{4}))} \right)^{2k},$$

for $k \geq 1$. Their boundary is given by

$$\begin{aligned} \partial(\sigma(a_k)) &= \left\{ 2^{2k-1} e^{-i(2k-1)\theta} \left(1 - \sqrt{2|\sin(\frac{\theta}{2})|} e^{\frac{i(\pi-\theta)}{4}} \right)^{2k-1} : \theta \in (-\pi, \pi) \right\}, \\ \partial(\sigma(b_k)) &= \left\{ 2^{2k} e^{-i2k\theta} \left(1 - \sqrt{2|\sin(\frac{\theta}{2})|} e^{\frac{i(\pi-\theta)}{4}} \right)^{2k} : \theta \in (-\pi, \pi) \right\}. \end{aligned}$$

Proof. As the algebra $\ell^1(\mathbb{N}^0, \frac{1}{4^n})$ has identity and $\sigma(c) = \overline{C(D(0, \frac{1}{4}))}$ ([10, Proposition 3.2]), we apply [7, Theorem 3.4.1] and Proposition 3.2 (ii) to get both first equalities, i.e,

$$\begin{aligned} \sigma(a_k) &= \overline{Z(a_k)(D(0, \frac{1}{4}))} = \left(\overline{C(D(0, \frac{1}{4}))} \right)^{2k-1}, \\ \sigma(b_k) &= \overline{Z(a_k)(D(0, \frac{1}{4}))} = \left(\overline{C(D(0, \frac{1}{4}))} \right)^{2k}, \end{aligned}$$

for $k \geq 1$. As

$$\partial(\sigma(c)) = \left\{ 2e^{-i\theta} \left(1 - \sqrt{2|\sin(\frac{\theta}{2})|} e^{\frac{i(\pi-\theta)}{4}} \right) : \theta \in (-\pi, \pi) \right\},$$

see [10, Proposition 3.2], we obtain second equalities from previous ones. \square

Remark 3.4. In the Figure 1, we plot the sets $\partial(\sigma(c))$, $\partial(\sigma(b_1))$ and $\partial(\sigma(a_2))$.

Catalan polynomials is defined by the following linear recurrence relation

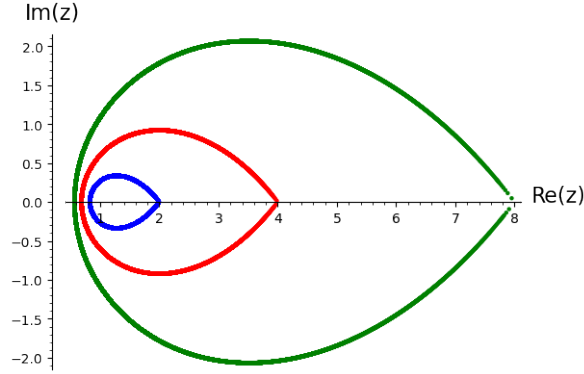
$$(3.3) \quad \mathcal{P}_{k+2}(z) = \mathcal{P}_{k+1}(z) - z\mathcal{P}_k(z), \quad k \geq 2,$$

and the starting values $\mathcal{P}_0(z) = \mathcal{P}_1(z) = 1$. The first values obtained are $\mathcal{P}_2(z) = 1 - z$, $\mathcal{P}_3(z) = 1 - 2z$ and $\mathcal{P}_4(z) = 1 - 3z + z^2$. The closed form of \mathcal{P}_k is given by the formula

$$\mathcal{P}_k(z) = \frac{(1 + \sqrt{1 - 4z})^{k+1} - (1 - \sqrt{1 - 4z})^{k+1}}{2^{k+1}\sqrt{1 - 4z}},$$

for $k \geq 0$. The bivariate generating function is

$$\frac{1}{1 - t + zt^2} = \sum_{k \geq 0} \mathcal{P}_k(z)t^k,$$

FIGURE 1. Sets $\partial(\sigma(c))$, $\partial(\sigma(b_1))$ and $\partial(\sigma(a_2))$.

see these and other properties in [6]. Other interesting property of Catalan polynomials is the following

$$\frac{d\mathcal{P}_k(z)}{dz} = \frac{-1}{2^{k-1}} \sum_{l=0}^{k-2} (l+2)2^l \mathcal{P}_l(z), \quad k \geq 2,$$

([2, Identity II]) which implies that the sign of coefficients are alternative.

In the next results, we use the usual notation $P(\delta_1)$ where

$$P(\delta_1) := \sum_{k=0}^n a_k \delta_1^{*k} = \sum_{k=0}^n a_k \delta_k$$

and P is the polynomial, $P(z) = \sum_{k=0}^n a_k z^{*k}$.

Lemma 3.5. *Take the Catalan sequence polynomials $(\mathcal{P}_k)_{k \geq 0}$. Then $\mathcal{P}_k(\delta_1) \in \ell^1(\mathbb{N}^0, \frac{1}{4^n})$, $\|\mathcal{P}_0(\delta_1)\|_{1, \frac{1}{4^n}} = 1$ and*

$$\|\mathcal{P}_k(\delta_1)\|_{1, \frac{1}{4^n}} = \mathcal{P}_k\left(\frac{-1}{4}\right) = \frac{\alpha_k}{4^{k-1}}, \quad k \geq 1,$$

where $\alpha_1 = 1$, $\alpha_2 = 5$ and $\alpha_k = 4(\alpha_{k-1} + \alpha_{k-2})$ for $k \geq 3$.

Proof. It is clear that $\mathcal{P}_k(\delta_1) \in \ell^1(\mathbb{N}^0, \frac{1}{4^n})$ and $\|\mathcal{P}_0(\delta_1)\|_{1, \frac{1}{4^n}} = 1$. As the sign of coefficients in polynomials $(\mathcal{P}_k)_{k \geq 0}$ are alternative, we have that

$$\begin{aligned} \|\mathcal{P}_k(\delta_1)\|_{1, \frac{1}{4^n}} &= \sum_{j=0}^k a_j \left(\frac{-1}{4}\right)^j = \mathcal{P}_k\left(\frac{-1}{4}\right) \\ &= \frac{(1 + \sqrt{2})^{k+1} - (1 - \sqrt{2})^{k+1}}{\sqrt{2}2^{k+1}} = \frac{\alpha_k}{4^{k-1}}, \end{aligned}$$

where the integer sequence $(\alpha_k)_{k \geq 1}$ is numbered as A086347 in [14] and treated in detail there. \square

Remark 3.6. The first values of the sequence $(\alpha_k)_{k \geq 1}$ are 1, 5, 24, 116, 560.... This sequence is an example of generalized Fibonacci numbers $g(k) = cg(k-1) + dg(k-2)$ for $k \geq 2$ and seed values $g(0) = a$ and $g(1) = b$ ($a, b, c, d \in \mathbb{N}$.)

Theorem 3.7. For $k \geq 1$,

$$(c^{*k})^{-1} = \mathcal{P}_k(\delta_1) + (-c * \delta_1) * \mathcal{P}_{k-1}(\delta_1).$$

Moreover $\|(c * c)^{-1}\|_{1, \frac{1}{4^n}} = \frac{3}{2}$ and $\|(c^{*k})^{-1}\|_{1, \frac{1}{4^n}} \leq \frac{1}{4^k} (\alpha_{k+1} + 4\alpha_{k-1})$ for $k \geq 2$, where $(\alpha_k)_{k \geq 1}$ are defined in Lemma 3.5.

Proof. Note that $c^{-1} = \delta_0 - \delta_1 * c$, see formula (3.2) and then

$$\begin{aligned} (c * c)^{-1} &= c^{-1} * c^{-1} = \delta_0 - 2\delta_1 * c + \delta_1 * (\delta_1 * c * c) \\ &= \delta_0 - \delta_1 - \delta_1 * c = \mathcal{P}_1(\delta_1) + (-c * \delta_1) * \mathcal{P}_0(\delta_1), \end{aligned}$$

where we have applied that $\delta_1 * c^{*1} = c - \delta_0$. By induction, we have that

$$\begin{aligned} (c^{*(k+1)})^{-1} &= c^{-1} * (c^{*k})^{-1} = (\delta_0 - \delta_1 * c) * (\mathcal{P}_k(\delta_1) + (-c * \delta_1) * \mathcal{P}_{k-1}(\delta_1)) \\ &= \mathcal{P}_k(\delta_1) - \delta_1 * c * \mathcal{P}_k(\delta_1) - \delta_1 * \mathcal{P}_{k-1}(\delta_1) \\ &= \mathcal{P}_{k+1}(\delta_1) + (-c * \delta_1) * \mathcal{P}_k(\delta_1), \end{aligned}$$

where we have applied the recurrence relation (3.3).

Finally, we apply Lemma 3.5 to get

$$\|(c^{*k})^{-1}\|_{1, \frac{1}{4^n}} \leq \|\mathcal{P}_k(\delta_1)\|_{1, \frac{1}{4^n}} + \frac{1}{2} \|\mathcal{P}_{k-1}(\delta_1)\|_{1, \frac{1}{4^n}} = \frac{1}{4^k} (\alpha_{k+1} + 4\alpha_{k-1})$$

for $k \geq 2$. □

4. POWERS OF CATALAN GENERATING FUNCTIONS FOR BOUNDED OPERATORS

In this section, we consider the particular case that T is a linear and bounded operator on the Banach space X , $T \in \mathcal{B}(X)$, such that

$$(4.1) \quad \sup_{n \geq 0} \|4^n T^n\| := M < \infty,$$

i.e., $4T$ is a power-bounded operator. In this case $\sigma(T) \subset \overline{D(0, \frac{1}{4})}$. Under the condition (4.1), we define the Catalan generating function, $C(T)$, by

$$(4.2) \quad C(T) := \sum_{n \geq 0} C_n T^n,$$

see [10, Section 5]. The bounded operator $C(T)$ may be consider as the image of the Catalan sequence $c = (C_n)_{n \geq 0}$ in the algebra homomorphism $\Phi : \ell^1(\mathbb{N}^0, \frac{1}{4^n}) \rightarrow \mathcal{B}(X)$ where

$$\Phi(a)x := \sum_{n \geq 0} a_n T^n(x), \quad a = (a_n)_{n \geq 0} \in \ell^1(\mathbb{N}^0, \frac{1}{4^n}), \quad x \in X,$$

i.e., $\Phi(c) = C(T)$. The Φ algebra homomorphism (also called functional calculus) has been considered in several papers, two of them are [3, Section 2] and more recently [4, Section 5.2]. Note that

$$\|\Phi(a)\| \leq \sup_{n \geq 0} \|4^n T^n\| \|a\|_{1, \frac{1}{4^n}}, \quad a \in \ell^1(\mathbb{N}^0, \frac{1}{4^n}).$$

In particular, the map Φ allows to define the following operators

$$\begin{aligned} \Phi(\delta_n) &= T^n, \quad n \geq 0, \\ \Phi\left(\frac{1}{\lambda} p_\lambda\right) &= (\lambda - T)^{-1}, \quad |\lambda| > \frac{1}{4}, \\ \Phi(P) &= \sum_{k=0}^n a_k T^k, \end{aligned}$$

where P is the compact support sequence, $P(z) = \sum_{k=0}^n a_k \delta_k$.

Theorem 4.1. *Given $T \in \mathcal{B}(X)$ such that $4T$ is power-bounded and $c = (C_n)_{n \geq 0}$ the Catalan sequence. Then*

- (i) *The powers $(C(T))^{2k-1} = \Phi(a_k)$ and $(C(T))^{2k} = \Phi(b_k)$ for $k \geq 1$, and*

$$\|(C(T))^j\| \leq (C(\|T\|))^j, \quad j \geq 1.$$

- (ii) *The operator $C(T)$ is invertible, $(C(T))^{-1} = I - TC(T)$,*

$$(C(T))^{-(j+1)} = \mathcal{P}_j(T) - TC(T)\mathcal{P}_{j-1}(T) \quad j \geq 1,$$

$$\|C(T)^{-1}\| \leq 1 + \frac{1}{2} \sup_{n \geq 0} \|4^n T^n\|, \quad \|C(T)^{-2}\| \leq \frac{3}{2} \sup_{n \geq 0} \|4^n T^n\| \text{ and}$$

$$\|(C(T))^{-(j+1)}\| \leq \frac{1}{4^j} \sup_{n \geq 0} \|4^n T^n\| (\alpha_{j+1} + 4\alpha_{j-1}), \quad j \geq 2,$$

where $(\alpha_j)_{j \geq 1}$ are defined in Lemma 3.5.

- (iii) *Take $(P_n)_{n \geq 0}$ and $(Q_n)_{n \geq 0}$ polynomials given in Definition 2.4. Then*

$$\begin{aligned} \sum_{n \geq 0} P_n(z) T^n &= \frac{C(T) - (z+1)}{T(1+z)^2 - z}, \\ \sum_{n \geq 0} Q_n(z) T^n &= \frac{(C(T) - (z+1))(z+1)}{T(1+z)^2 - z}, \end{aligned}$$

for $|z| < 1$.

- (iv) *The spectral mapping theorem holds for $(C(T))^n$, i.e., $\sigma((C(T))^n) = C^n(\sigma(T))$ for $n \in \mathbb{Z}$.*

Proof. (i) From (4.2), $\Phi(c) = C(T) \in \mathcal{B}(X)$ as we have commented above. By Proposition 3.2 (iii), we have

$$\begin{aligned} (C(T))^{2k-1} &= (\Phi(c))^{2k-1} = \Phi(c^{*(2k-2)}) = \Phi(a_k), \\ (C(T))^{2k} &= (\Phi(c))^{2k} = \Phi(c^{*(2k-1)}) = \Phi(b_k), \end{aligned}$$

for $k \geq 1$. By Proposition 3.2 (ii), we get

$$\begin{aligned}\|(C(T))^{2k-1}\| &= \|\Phi(a_k)\| \leq \sum_{j \geq 0} a_k(j) \|T\|^j = (C(\|T\|))^{2k-1}, \\ \|(C(T))^{2k}\| &= \|\Phi(b_k)\| \leq \sum_{j \geq 0} b_k(j) \|T\|^j = (C(\|T\|))^{2k},\end{aligned}$$

for $k \geq 1$ and we conclude the proof of (i).

(ii) As the homomorphism Φ is continuous, we apply the formula (3.2) to get

$$C(T)(I - TC(T)) = \Phi(c)(\Phi(\delta_0 - \delta_1 * c)) = \Phi(c - \delta_1 * c^1) = \Phi(\delta_0) = I.$$

In fact $(C(T))^{-1} = \Phi(c^{-1})$ and

$$(C(T))^{-(j+1)} = \Phi((c^{-1})^{*j}) = \Phi(c^{*j})^{-1} = \mathcal{P}_j(T) - TC(T)\mathcal{P}_{j-1}(T), \quad j \geq 1,$$

where we have applied Theorem 3.7 and Φ is an algebra homomorphism. The estimation of $\|(C(T))^{-(j+1)}\|$ follows also from Theorem 3.7.

(iii) For $|z| < 1$, and Theorem 2.7, we have

$$\begin{aligned}\sum_{n \geq 0} P_n(z) T^n &= \sum_{n \geq 0} \sum_{j=0}^n B_{n+1,j+1} z^j T^n = \sum_{j \geq 0} z^j \sum_{n=j}^{\infty} B_{n+1,j+1} T^n \\ &= \sum_{j \geq 0} z^j T^j C^{2j+2}(T) = \frac{C^2(T)}{1 - zTC^2(T)} = \frac{C(T) - (z+1)}{T(1+z)^2 - z}.\end{aligned}$$

Similarly we check that $\sum_{n \geq 0} Q_n(z) T^n = \frac{(C(T) - (z+1))(z+1)}{T(1+z)^2 - z}$ for $|z| < 1$.

(iv) Since $4T$ is power bounded, the spectral mapping theorem for $C^n(T)$ may found in [3, Theorem 2.1] and then $\sigma((C(T))^n) = C^n(\sigma(T))$ for $n \in \mathbb{Z}$. \square

Remark 4.2. As $\sigma(T) \subset \overline{D(0, \frac{1}{4})}$, we apply Proposition 3.3 to conclude that

$$\sigma(C^n(T)) \subset \overline{C^n(D(0, \frac{1}{4}))}, \quad n \in \mathbb{Z}.$$

5. EXAMPLES, APPLICATIONS AND FINAL COMMENTS

In this section we present some particular examples of operators T for which we solve the equation (1.3), calculate $C(T)$ and $(C(T))^k$ for $k \in \mathbb{Z}$. In the subsection 5.1, we consider the Euclidean space \mathbb{C}^2 and some matrices T . To resolve this matrix equation, we need to solve a system of four quadratic equations. We also calculate $(C(T))^n$ for these matrices. In subsection 5.2 we check $C(a)$ for some $a \in \ell^1(\mathbb{N}^0, \frac{1}{4^n})$. Finally we present some ideas to continue this research in subsection 5.3.

5.1. Matrices on \mathbb{C}^2 . We consider the Euclidean space \mathbb{C}^2 and the operator $T = \begin{pmatrix} \lambda & 0 \\ 0 & \mu \end{pmatrix}$, with $0 \neq \lambda, \mu \in \mathbb{C}$. For $\lambda = \mu$, the solution is presented in [10, Subsection 6.1]. For $\lambda \neq \mu$, the solution of (1.3) is given by

$$Y = \begin{pmatrix} \frac{1 \pm \sqrt{1-4\lambda}}{2\lambda} & 0 \\ 0 & \frac{1 \pm \sqrt{1-4\mu}}{2\mu} \end{pmatrix},$$

where the allowed signs are all four combinations. In the case that $|\lambda|, |\mu| \leq \frac{1}{4}$, note that

$$(C(T))^j = \begin{pmatrix} (C(\lambda))^j & 0 \\ 0 & (C(\mu))^j \end{pmatrix}.$$

for $j \in \mathbb{Z}$.

Now we study the case $T = \begin{pmatrix} 0 & \lambda \\ \lambda & 0 \end{pmatrix}$ with $\lambda \in \mathbb{C} \setminus \{0\}$. When $|\lambda| \leq \frac{1}{4}$, we get that

$$C(T) = \begin{pmatrix} C_e(\lambda) & C_o(\lambda) \\ C_o(\lambda) & C_e(\lambda) \end{pmatrix},$$

where functions C_e and C_o are functions given by

$$\begin{aligned} C_e(z) &:= \sum_{n=0}^{\infty} C_{2n} z^{2n} = \frac{\sqrt{1+4z} - \sqrt{1-4z}}{4z}, \\ C_o(z) &:= \sum_{n=0}^{\infty} C_{2n+1} z^{2n+1} = \frac{2 - \sqrt{1+4z} - \sqrt{1-4z}}{4z}, \end{aligned}$$

see [10, Section 6.1]. As

$$\begin{aligned} \begin{pmatrix} a & b \\ b & a \end{pmatrix}^{2n} &= \begin{pmatrix} a^{2n} + \binom{2n}{2} a^{2(n-2)} b^2 + \dots + b^{2n} \\ \left(\binom{2n}{1} a^{2n-1} b + \dots + \binom{2n}{1} a b^{2n-1} \right) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &+ \begin{pmatrix} \left(\binom{2n}{1} a^{2n-1} b + \dots + \binom{2n}{1} a b^{2n-1} \right) \\ \left(\binom{2n}{1} a^{2n-1} b + \dots + \binom{2n}{1} a b^{2n-1} \right) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \\ \begin{pmatrix} a & b \\ b & a \end{pmatrix}^{2n+1} &= \begin{pmatrix} a^{2n+1} + \dots + \binom{2n+1}{2n} a b^{2n} \\ \left(\binom{2n+1}{1} a^{2n} b + \dots + b^{2n+1} \right) \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \\ &+ \begin{pmatrix} \left(\binom{2n+1}{1} a^{2n} b + \dots + b^{2n+1} \right) \\ \left(\binom{2n+1}{1} a^{2n} b + \dots + b^{2n+1} \right) \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \end{aligned}$$

we may also obtain $(C(T))^n$ using Theorem 4.1 and get new generating formulae for Catalan triangle numbers.

Theorem 5.1. *Take $n \geq 0$ and $z \in \overline{D(0, \frac{1}{4})}$. Then*

$$\sum_{k=n}^{\infty} B_{2k-n,n} z^{2k} = z^{2n} \left(C_e^{2n}(z) + \binom{2n}{2} C_e^{2(n-2)}(z) C_o^2(z) + \dots + C_o^{2n}(z) \right),$$

$$\begin{aligned}
\sum_{k=n}^{\infty} B_{2k+1-n,n} z^{2k+1} &= z^{2n} \left(\binom{2n}{1} C_e^{2n-1}(z) C_o(z) + \dots + \binom{2n}{1} C_e(z) C_o^{2n-1}(z) \right), \\
\sum_{k=n}^{\infty} A_{2k-1-n,n} z^{2k} &= z^{2n} \left(C_e^{2n-1}(z) + \dots + \binom{2n-1}{2n-2} C_e(z) C_o^{2n-2}(z) \right), \\
\sum_{k=n}^{\infty} A_{2k-n,n} z^{2k+1} &= z^{2n} \left(\binom{2n-1}{1} C_e^{2n-1}(z) C_o(z) + \dots + C_o^{2n-1}(z) \right),
\end{aligned}$$

Finally we study the case $T = \begin{pmatrix} \lambda & \mu \\ 0 & \lambda \end{pmatrix}$ with $\lambda, \mu \in \mathbb{C} \setminus \{0\}$. The solutions of (1.3) are given by

$$Y = \begin{pmatrix} a & \frac{\mu(a-1)}{\lambda(1-2\lambda a)} \\ 0 & a \end{pmatrix}$$

where a is a solution of the quadratic Catalan equation $\lambda a^2 - a + 1 = 0$. In the case that $|\lambda| \leq \frac{1}{4}$, we get that

$$C(T) = \begin{pmatrix} C(\lambda) & \frac{\mu((C(\lambda)-1)}{\lambda(1-2\lambda C(\lambda))} \\ 0 & C(\lambda) \end{pmatrix},$$

and

$$(C(T))^j = \begin{pmatrix} (C(\lambda))^j & n(C(\lambda))^{j-1} \frac{\mu((C(\lambda)-1)}{\lambda(1-2\lambda C(\lambda))} \\ 0 & (C(\lambda))^j \end{pmatrix},$$

for $j \geq 1$. As $(C(T))^{-1} = \frac{1}{(C(\lambda))^2} \begin{pmatrix} C(\lambda) & -\frac{\mu((C(\lambda)-1)}{\lambda(1-2\lambda C(\lambda))} \\ 0 & C(\lambda) \end{pmatrix}$, we get that

$$(C(T))^{-j} = \frac{1}{(C(\lambda))^{2j}} \begin{pmatrix} (C(\lambda))^j & -n(C(\lambda))^{j-1} \frac{\mu((C(\lambda)-1)}{\lambda(1-2\lambda C(\lambda))} \\ 0 & (C(\lambda))^j \end{pmatrix},$$

for $j \geq 1$.

5.2. Catalan operators on ℓ^p . We consider the space of sequences $\ell^p(\mathbb{N}^0, \frac{1}{4^n})$ where

$$\|a\|_{p, \frac{1}{4^n}} := \left(\sum_{n=0}^{\infty} \frac{|a_n|^p}{4^{np}} \right)^{\frac{1}{p}} < \infty,$$

for $1 \leq p < \infty$ and $\ell^\infty(\mathbb{N}^0, \frac{1}{4^n})$ the space of sequences embedded with the norm

$$\|a\|_{\infty, \frac{1}{4^n}} := \sup_{n \geq 0} \frac{|a_n|}{4^n} < \infty.$$

Note that $\ell^1(\mathbb{N}^0, \frac{1}{4^n}) \hookrightarrow \ell^p(\mathbb{N}^0, \frac{1}{4^n}) \hookrightarrow \ell^\infty(\mathbb{N}^0, \frac{1}{4^n})$.

Now we consider sequences $c, (a_k), (b_k) \in \ell^1(\mathbb{N}^0, \frac{1}{4^n})$, the Catalan triangle sequences given in Definition 3.1 and convolution operators $C(f) := c * f$,

$C^{2k}(f) = b_k * f$ and $C^{2k-1}(f) = a_k * f$ for $f \in \ell^p(\mathbb{N}^0, \frac{1}{4^n})$ and $k \geq 1$ with $1 \leq p \leq \infty$. By Theorem 4.1 (iv), we get that

$$\sigma(C^n) = C^n(\sigma(\delta_1)) = \overline{C^n(D(0, \frac{1}{4}))}, \quad n \geq 1.$$

Note that the set $\sigma(C^n)$ independent on p and coincides with the spectrum of the power of Catalan sequence c in $\ell^1(\mathbb{N}^0, \frac{1}{4^n})$ (Proposition 3.3).

5.3. A future research. Given $a, b \neq 0 \in \mathbb{C}$, the quadratic equation

$$(5.1) \quad \frac{bz}{2}y^2 - y + \frac{a}{2b} = 0,$$

has two solutions given by

$$y = \frac{1 \pm \sqrt{1 - za}}{bu}.$$

We define $C^{a,b}(z) := \frac{1 - \sqrt{1 - za}}{bz}$; note that $C^{a,b}(z) = \frac{a}{2b}C(\frac{az}{4})$ and

$$C^{a,b}(z) = \sum_{n \geq 0} \frac{a^{n+1}}{2^{2n+1}b} C_n z^n.$$

It would be natural to consider a vector-valued of equation (5.1) for $a, b, z \in \mathcal{B}(X)$.

REFERENCES

- [1] X. Chen and W. Chu, *Moments on Catalan numbers*, J. Math. Anal. Appl., 349 (2), (2009), 311–316.
- [2] J.A. Clapperton, P.J. Larcombe and E.J. Fennessey *Some new identities for Catalan polynomials*, Utilitas Mathematica., 80 (2009), 1–8.
- [3] N. Dungey, *Subordinated discrete semigroups of operators*, Trans. Amer. Math. Soc., 363 (4), (2011), 1721–1741.
- [4] A. Gomilko and Y. Tomilov, *On discrete subordination of power bounded and Ritt operators*, Indiana. Univ. Math. J., 67 (2), (2018), 781–829.
- [5] V. J. W. Guo and J. Zeng: *Factors of binomial sums from Catalan triangle*. J. Number Theory, **130** (2010), no. 1, 172–186.
- [6] A.F. Jarvis, P.J. Larcombe and E.J. Fennessey *Some Factorization and Divisibility Properties of Catalan Polynomials*, Bulletin of the ICA. 71 (2014), 36–56.
- [7] R. Larsen, *Banach Algebras: An Introduction*. Marcel Dekker, New York, 1973.
- [8] P.J. Miana and N. Romero, *Moments of combinatorial and Catalan numbers*, J. Number Theory, 130 (8), (2010), 1876–1887.
- [9] P.J. Miana and N. Romero, *Moments of Catalan triangle numbers*, in *Number Theory and Its Applications*, Intechopen, 2020.
- [10] P.J. Miana and N. Romero, *Catalan generating functions for bounded operators*, Preprint, 2022.
- [11] L. W. Shapiro, *A Catalan triangle*, Discrete Math., 14 (1976), 83–90.
- [12] A. Slavík: *Identities with squares of binomial coefficients*, Ars Combinatoria, **113** (2014), 377–383.
- [13] N. Sloane, *A Handbook of Integer Sequences*, Academic Press, New Jersey, 1973.
- [14] N. Sloane, <https://oeis.org/>
- [15] R. P. Stanley, *Catalan Numbers*, Cambridge University Press, Cambridge, 2015.

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