# Note on "An existence result with numerical solution of nonlinear fractional integral equations"

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**Abstract**

In this note, we mention that the recent iterative numerical method in [Kazemi, Manochehr, Amar Deep, and Juan Nieto, "An existence result with numerical solution of nonlinear fractional integral equations," Mathematical Methods in the Applied Sciences (2023)] needs some corrections. We give a counterexample to one of their main statements (Theorem 5.1). The iterative method used is untrue, and we will correct it. With an example, it can be seen that using the midpoint rule will not be suitable and accurate in fractional cases ($0 < \tau < 1$); instead, using Jacobi's quadrature rule can work well. Also, to show the validity of our numerical method, a non-linear example is considered.

MSC: 60H20, 47H10.

Keywords and phrases: fixed point theorem, fractional integral equations, Jacobi's quadrature rule

## 1 Introduction and preliminaries

Let us briefly introduce some preliminaries and notations:

- $(\mathfrak{B}, ||\cdot||)$: real Banach space.
- $B_\rho$, $\partial B_\rho$: open ball of radius $\rho > 0$ with center 0, the boundary of $B_\rho$, respectively.
- $C(I)$ is a set of real valued continues functions on $I := [0, b] \subset \mathbb{R}$ with the uniform norm

$$\| x \| = \sup\{|x(s)|, s \in I\}, \qquad x \in C(I). \tag{1}$$

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In [12], $T : C(I) \to C(I)$ was defined as

$$Tz(s) = \zeta\left(s, \Psi_1(z)(s), \Psi_2(z)(s), \Phi_1(z)(s)\right), \quad s \in I, z \in B_\rho(E), s \in I := [0, b], \tag{2}$$

in which

$$\begin{cases} \Psi_1(z)(s) = f(s, z(\alpha(s))), s \in I, \\ \Psi_2(z)(s) = u(s, z(\beta(s))), s \in I, \\ \Phi_1(z)(s) = \dfrac{1}{\Gamma(\tau)} \displaystyle\int_0^{\theta(s)} \frac{p(s, \xi, z(\gamma(\xi)))}{(\theta(s) - \xi)^{1-\tau}} d\xi, s \in I, 0 < \tau \leq 1. \end{cases} \tag{3}$$

The goal of article [12] is to examine in order to determine fixed points of equation (2) and a subjected iterative method to solve numerically it if the conditions are considered appropriate for all continuous functions in (3) (refer to [12, conditions (1)-(3)]). Consequently, Kazemi et al. [12] demonstrated the existence of a fixed point for $T$ using Petryshyn's theorem. For its numerical solution, they also suggested an iterative approach based on the midpoint rule. We shall demonstrate the falsity of Theorem 5.1 in [12] in this note, among others. We further show that, generally speaking, the recursive connection described in [12] is untrue since sentences in the recursion sequence cannot be calculated from previously unknown phrases; the method can only be used in a specific situation, and the midpoint rule is not good at all to find numerically a fixed point of equation (2) in fractional cases ($0 < \tau < 1$); instead, using Jacobi's quadrature rule works well.

## 2 Numerical iterative method

In Section 5, Kazemi et al. [12] introduced a numerical method that is based on an iterative procedure such that the integral term in (4) is obtained by estimating the midpoint rule as follows: Consider a mesh $\Delta$ with nodal points $\{s_i\}_{i=0}^n$ on $s \in [0, b]$ as follows:

$$\Delta : 0 = s_0 < s_1 < s_2 < ... < s_{n-1} < s_n = b,$$

where $s_i = ih, i = 0, ...n$, $h = \frac{b}{n}$ and the operator $T$ on the ball $B_\rho$:

$$Tz(s) = \zeta\left(s, f(s, \Psi_1(z)(s), \Phi_1(z)(s)\right). \tag{4}$$

Notice that the term $\Psi_2(z)$ does not exist in (4) since the method can be similarly applied to (2). They considered the Picard sequence $z_{M+1} = Tz_M$ as follows:

$$z_{M+1}(s_i) = (Tz_M)(s_i) = \zeta\left(s_i, f(s_i, z_M(\alpha(s_i))), \frac{1}{\Gamma(\tau)} \int_0^{\theta(s_i)} \frac{p(s_i, \xi, z_M(\gamma(\xi)))}{(\theta(s_i) - \xi)^{1-\tau}} d\xi\right), M = 1, 2, ..., \tag{5}$$

with arbitrary initial point $z_0(s) = g(s), s \in [0, b]$ and used the midpoint quadrature rule to approximate the integrals term in (5) for points $\xi_j = jh', j = 0, 1, ..., n$, where $h' = \frac{\theta(s_i)}{n}$.

$$\int_0^{\theta(s_i)} \frac{p(s_i, \xi, z(\gamma(\xi)))}{(\theta(s_i) - \xi)^{1-\tau}} d\xi \simeq \frac{\theta(s_i)}{n} \sum_{j=0}^{n-1} \frac{p\big(s_i, \xi_j + \frac{h'}{2}, z(\gamma(\xi_j + \frac{h'}{2}))\big)}{\big(\theta(s_i) - \big(\xi_j + \frac{h'}{2}\big)\big)^{1-\tau}}. \tag{6}$$

The combination of (5) and (6) leads to the following recursive procedure:

$$\bar{z}_0(s_i) = g(s_i),$$
$$\bar{z}_M(s_i) = \zeta\left(s_i, f(s_i, \bar{z}_{M-1}(\alpha(s_i))), \frac{h'}{\Gamma(\tau)} \sum_{j=0}^{n-1} \frac{p\big(s_i, \xi_j + \frac{h'}{2}, \bar{z}_{M-1}(\gamma(\xi_j + \frac{h'}{2}))\big)}{\big(\theta(s_i) - (\xi_j + \frac{h'}{2})\big)^{1-\tau}}\right), \tag{7}$$

where $h = \frac{b}{n}$, $s_i = ih, i = 0, \ldots, n$.
Kazemi et al. [12, Theorem 5.1] showed that the iterative method (5) is convergent on the ball $B_\rho$, which is not correct. More precisely, here the unknown is $z$, which is the fixed point of equation (5), and they showed that the Picard sequence $z_{M+1}$ converges to $z$. In the next section, we discuss this theorem. Then they used the iterative sequence (7) to solve numerically integral equation (4). It is clear that in a recursive relation, the sentences of each step are obtained from previously known (or approximated) terms, but notice that the above iterative formula $z_M(s_i), s_i = ih, i = 0, ...n$, where $h = \frac{b}{n}$ is approximated from unknown terms $z_{M-1}(\alpha(s_i))$ and $z_{M-1}(\gamma(\xi_j + \frac{h'}{2}))\big)$, which is not applicable in practice (compare with (10) and (14) below). More precisely, let us start with the initial value $\bar{z}_0(s) = g(s), s \in I$ ($g \in C(I)$ is arbitrary, e.g., consider $g = 0$), then we get

$$\bar{z}_1(s_i) = \zeta\left(s_i, f(s_i, \bar{z}_0(\alpha(s_i))), \frac{h'}{\Gamma(\tau)} \sum_{j=0}^{n-1} \frac{p\big(s_i, \xi_j + \frac{h'}{2}, \bar{z}_0(\gamma(\xi_j + \frac{h'}{2}))\big)}{\big(\theta(s_i) - (\xi_j + \frac{h'}{2})\big)^{1-\tau}}\right), i = 0, ..., n.$$

Now, in the iterative procedure, your data ("only") are $\bar{z}_1(s_i), i = 0, ..., n$ (see also the algorithm in [12]). In the next step, you have to compute:

$$\bar{z}_2(s_i) = \zeta\left(s_i, f(s_i, \bar{z}_1(\alpha(s_i))), \frac{h'}{\Gamma(\tau)} \sum_{j=0}^{n-1} \frac{p\big(s_i, \xi_j + \frac{h'}{2}, \bar{z}_1(\gamma(\xi_j + \frac{h'}{2}))\big)}{\big(\theta(s_i) - (\xi_j + \frac{h'}{2})\big)^{1-\tau}}\right), i = 0, ..., n.$$

How can we possibly compute $\bar{z}_2(s_i)$ from the unknown terms $\bar{z}_1(\alpha(s_i))$ and $\bar{z}_1(\gamma(\xi_j + \frac{h'}{2}))$? Notice that your known data "only" are $\bar{z}_1(s_i), i = 0, ..., n$, so the iterative procedure (7) cannot be applied in practice.
In the following, we correct this problem. Before introducing the method, it is important to mention that this method can only be used for Fredholm integral equations. In what follows, we have to assume that $\alpha(t) = \gamma(t) = t, \theta(t) = b$ in (5). More precisely, we want to solve numerically the following:

$$z(s) = (Tz)(s) = \zeta\left(s, f(s, z(s)), \frac{1}{\Gamma(\tau)} \int_0^b \frac{p(s, \xi, z(\xi)))}{(b - \xi)^{1-\tau}} d\xi\right), \ s \in I, 0 < \tau \leq 1. \tag{8}$$

In general, numerical integration is primarily concerned with the computation of numerical values that approximate the values of definite integrals. In many cases, this value is computed using the quadrature rule formula:

$$\int_0^b w(\xi)\phi(\xi)d\xi = \sum_{k=1}^{n} \omega_k \phi(\xi_k) + E_n(f), \tag{9}$$

where $0 \leq \xi_1 \leq \cdots \leq \xi_n \leq b$ are (preassigned or not) abscissas in $[0, b]$. This formulation has important applications in numerical analysis, where the weights $\omega_i$ and nods $\xi_i$ are known in advance. Many of these methods are assigned to nods inside the integration interval and when $w(\xi) = 1$, such as Newton-Cotes rule, midpoint rule and so on. It can be combined with (8) and (9) to obtain an iterative numerical formula in nods $\xi_i, i = 1, \ldots, n$ as follows:

$$\bar{z}_M(\xi_i) = \zeta\left(\xi_i, f(\xi_i, \bar{z}_{M-1}(\xi_i)), \frac{1}{\Gamma(\tau)} \sum_{k=1}^{n} \omega_k \frac{p(\xi_i, \xi_k, \bar{z}_{M-1}(\xi_k))}{(b - \xi_k)^{1-\tau}}\right), i = 0, \ldots, n, \tag{10}$$

where $\bar{z}_0(\xi_i) = g(\xi_i)$ be an arbitrary initial value, and for $0 < \tau < 1$ it cannot be possible to use the numerical integral method (9) such that $\xi_n = b$ because of the singularity in (10). Let us examine this method in some examples. First, Example 5.3 in [12] is not correct; we can see this below.

**Example 2.1.** Kazemi et al. [12] solved numerically the following nonlinear fractional integral equation:

$$z(s) = \psi(s) + \frac{e^{-s\sqrt{s}} z^2(s)}{1 + s^2} + \frac{e^{-s}}{5\Gamma(\frac{1}{3})} \int_0^1 \frac{\xi \sin(s) + \frac{1}{3} z^2(\xi)}{(1 - \xi)^{\frac{2}{3}}} d\xi, \qquad s \in [0, 1], \tag{11}$$

where

$$\psi(s) = 1 + \sqrt{s} - e^{-s\sqrt{s}} - \frac{2\sqrt{3} e^{-s}}{75\pi\Gamma(\frac{5}{6})} \left(\sqrt{\pi^3} + \frac{405}{56}\Gamma(\frac{2}{3})\Gamma(\frac{5}{6}) \left(\sin(s) + \frac{49}{54}\right)\right),$$

and with the exact solution $z(s) = 1 + \sqrt{s}$. Let

$$\begin{aligned} g(s) &= \frac{e^{-s\sqrt{s}} z^2(s)}{1 + s^2} + \frac{e^{-s}}{5\Gamma(\frac{1}{3})} \int_0^1 \frac{\xi \sin(s) + \frac{1}{3} z^2(\xi)}{(1 - \xi)^{\frac{2}{3}}} d\xi, \\ &= \frac{e^{-s\sqrt{s}} z^2(s)}{1 + s^2} + \frac{e^{-s}}{5\Gamma(\frac{1}{3})} \times \frac{9e^{-s} \sin(s)}{4} + \frac{e^{-s}}{5\Gamma(\frac{1}{3})} \int_0^1 \frac{\frac{1}{3} z^2(\xi)}{(1 - \xi)^{\frac{2}{3}}} d\xi, \qquad s \in [0, 1], \end{aligned}$$

then we get

$$\psi(0) = -\frac{2\sqrt{3}}{75\pi\Gamma(\frac{5}{6})} \left(\sqrt{\pi^3} + \frac{405}{56}\Gamma(\frac{2}{3})\Gamma(\frac{5}{6})\frac{49}{54}\right) \simeq -0.2032,$$

and

$$g(0) = 1 + \frac{1}{5\Gamma(\frac{1}{3})} \int_0^1 \frac{\frac{1}{3} z^2(\xi)}{(1 - \xi)^{\frac{2}{3}}} d\xi \simeq 1.3326.$$

So $z(0) \neq g(0) + \psi(0)$. Also, if one puts $z$ on (11) then one can plot the right and left-hand sides of (11) by Matlab; Fig.1 shows the results.

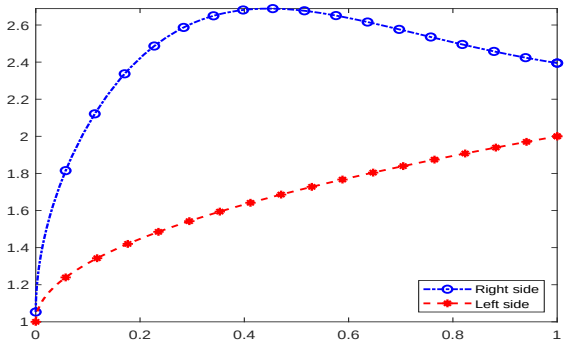



Figure 1: Graphs of left and right hand side of (11)

## 2.1 Using midpoint rule

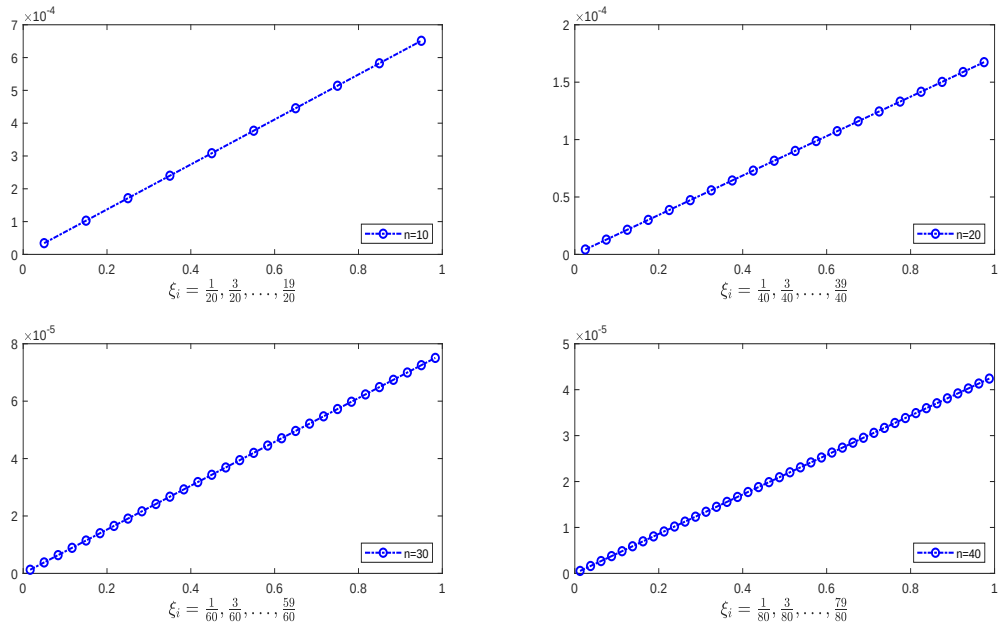
In the following examples, put $\|e_n\|_{\max} = \max\{e_n(\xi_i), i = 0, 1, ..., n\}$, where $e_n(\xi_i) = \mid z(\xi_i) - \bar{z}_M(\xi_i) \mid$, $z$ is the exact solution and $\bar{z}_M$ is obtained from Picard sequence (10) and the termination condition in Matlab codes were $\max\{|e_{n+1}(\xi_i) - e_n(\xi_i)|, i = 0, 1, ..., n\} < 10^{-20}$. Since Example 5.3 in [12] is not correct, we consider other cases. Notice that this method is not applicable for the Volterra integral equation, such as Example 5.2 of [12]. Since in general it cannot be proven that the Picard sequence converges (see Section 3), we assume that equation (8) has a unique solution $z$ in $C(I)$ and that the Picard sequence $z_{M+1} = T(z_M)$ converges to $z$. Let us consider the non-singular integral equation (when in (10) we have $\tau = 1$) and apply the iterative Picard sequence (10) with the midpoint rule, where in (10) we have $w_i = \frac{b}{n}$ and $\xi_i = \frac{2i-1}{2n}b, i = 1, \ldots, n$.

**Example 2.2** (see [4, 13])**.** Consider the following non-singular Fredholm integral equation:

$$z(s) = -s^2 - \frac{s}{3}(2\sqrt{2} - 1) + 2 + \int_0^1 st\sqrt{z(t)}dt,$$

where $s \in [0, 1]$. The exact solution is given by $z(s) = 2 - s^2$. For $n = 10, 20, 30, 40$, Fig. 2 summarizes the results of $e_n(s)$.

Figure 2: The graph of error functions $e_n(s), s = \frac{1}{2n}, \frac{3}{2n}, \ldots, \frac{2n-1}{2n}$ for $n = 10, 20, 30, 40$



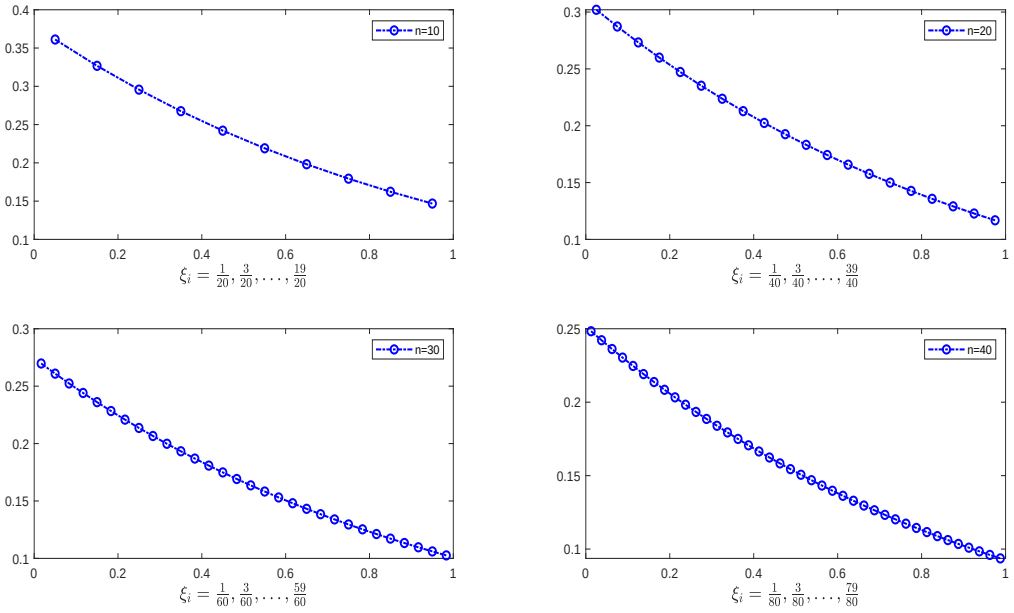

But the midpoint rule does not work well for the singular Fredholm integral equation. Let us consider a simple example as follows:

**Example 2.3.** Consider the following singular linear Fredholm integral equation

$$z(s) = s - \frac{3\mathrm{e}^{-s}}{4\Gamma(\frac{1}{3})} + \frac{1}{\Gamma(\frac{1}{3})}\int_0^1 \frac{\mathrm{e}^{-s}}{3(1-t)^{\frac{2}{3}}} z(t)dt, \tag{12}$$

where $s \in [0, 1]$. The exact solution is $z(s) = s$. For $n = 10, 20, 30, 40$, Fig. 3 summarizes the results of $e_n(s)$.

Figure 3: The graph of error functions $e_n(s), s = \frac{1}{2n}, \frac{3}{2n}, \ldots, \frac{2n-1}{2n}$ for $n = 10, 20, 30, 40$




The reason for this inaccuracy, even for a simple example, is that the mid-point rule does not work well for integral approximating of non-smooth functions such as $(1-x)^\alpha, -1 < \alpha < 1$ (see, for instance, [9, Sec. 5.8.2] and [3]).

## 2.2 Using Gaussian quadrature

In this subsection, we propose Jacobi's quadrature rule instead of the midpoint rule for the fractional Fredholm integral equation. It is known that Gaussian quadrature is a very powerful tool for approximating integrals. One of the most important of these methods is Jacobi's rule, which can be applied for approximating integrals of non-smooth functions such as $(1-x)^\alpha, -1 < \alpha < 1$. Jacobi's weights are defined as

$$w(x) = (1-x)^\alpha (1+x)^\beta, \qquad (\alpha, \beta > -1), \qquad x \in [-1, 1]. \tag{13}$$

It is well known that $\xi_i, i = 1, \ldots, n$ in (9) are the zeros of Jacobi's polynomials of degree $n$ (change of variable $y = \frac{x+1}{2}$ transfers zeros to interval $[0, 1]$, and the interval $[-1, 1]$ changes to $[0, 1]$) . More details about this rule may be found in [8]. In this method, in equation (8), consider $w(x) = (1-x)^{-(1-\tau)}$, where in (13) we have $\alpha = -(1-\tau), \beta = 0$, then equation (8) is solved numerically by

$$\bar{z}_M(\xi_i) = \zeta \left( \xi_i, f(\xi_i, \bar{z}_{M-1}(\xi_i)), \frac{1}{\Gamma(\tau)} \sum_{k=1}^{n} \omega_k p(\xi_i, \xi_k, \bar{z}_{M-1}(\xi_k)) \right), i = 1, \ldots, n. \tag{14}$$

where $\bar{z}_0(\xi_i) = g(\xi_i)$ be arbitrary initial value and $\xi_i, i = 1, \ldots, n$ are the zeros of Jacobi's polynomial of degree $n$. Since Jacobi's polynomials are orthogonal and (9) is exact for polynomials of degree less than $n$ (see [8]), $w_i$ can be calculated from the following system:

$$\sum_{i=1}^{n} w_i(\xi_i)^j = \int_0^1 (1-x)^{-(1-\tau)} x^j dx, j = 0, \ldots, n-1.$$

**Example 2.4.** Let us solve equation (12) from Jacobi's quadrature rule. An easy Matlab code can calculate the zeros of Jacobi's polynomials and coefficients $w_i$ for $\alpha = -(1-\tau) = -\frac{2}{3}, \beta = 0$ and $n = 10, 20$. For example, for $n = 10$ we have

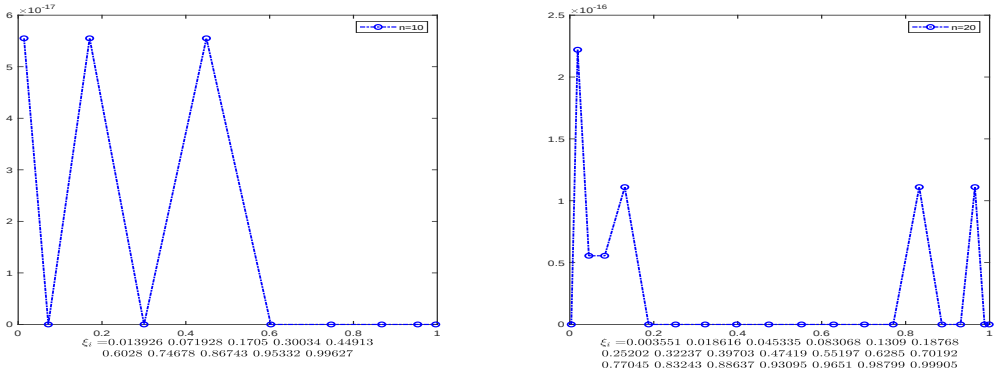
$$\begin{aligned}\xi_i =& 0.013926464, 0.071927637, 0.170501691, 0.300335271, 0.449132428,\\ & 0.602798417, 0.7467766, 0.86742754, 0.953318687, 0.996269028.\end{aligned}$$

and

$$\begin{aligned}w_i =& 0.035907187, 0.083596575, 0.131441854, 0.179613411, 0.228671065,\\ & 0.279851231, 0.335897, 0.403596938, 0.505034731, 0.816390047.\end{aligned}$$

Fig.4 shows the errors $e_n(\xi_i)$ in using equation (14)

Figure 4: The graph of errors $e_n(\xi_i)$ for $n = 10, 20$




To show the validity of Jacobi's method, let us consider a non-linear example too.

**Example 2.5.** Let us solve equation

$$z(t) = 1 + \sqrt{s} - \frac{e^{-s}\left(\pi + \frac{10}{3}\right)}{15\Gamma(\frac{1}{2})} + \frac{1}{15\Gamma(\frac{1}{2})} \int_0^1 \frac{e^{-s}}{(1-t)^{\frac{1}{2}}} z^2(t) dt \tag{15}$$

from Jacobi's quadrature rule (14), where the exact solution is $z(t) = 1 + \sqrt{s}$. An easy Matlab code can calculate the zeros of Jacobi's polynomials and coefficients $w_i$ for $\alpha = -(1-\tau) = -\frac{1}{2}, \beta = 0$ and $n = 10, 20$. For example, for $n = 10$ we have

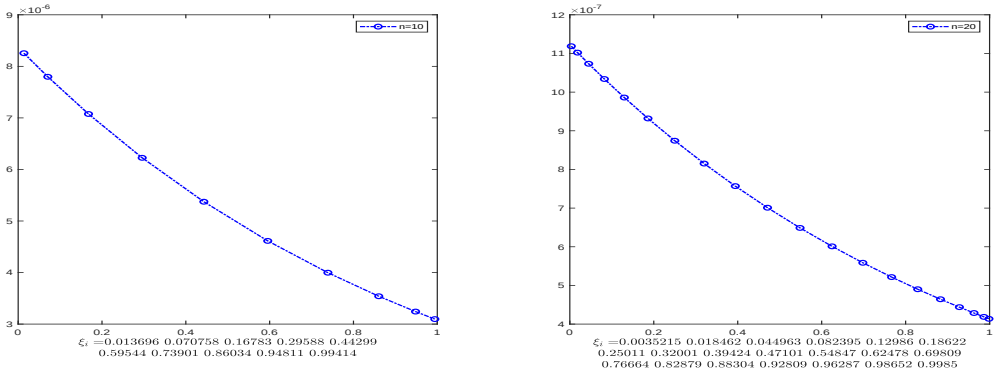
$$\begin{aligned}\xi_i =& 0.013695585, 0.070758123, 0.167828348, 0.295882708, 0.442988685,\\ & 0.595435715, 0.739014906, 0.860343759, 0.948113606, 0.994143692,\end{aligned}$$

and

$$w_i = 0.035228014, 0.08120286, 0.125344097, 0.166553483, 0.20386024,$$
$$0.236389064, 0.263377277, 0.284192219, 0.298345973, 0.305506774.$$

where $\xi_i$ are the zeros of Jacobi's polynomials. Fig.5 shows the errors $e_n(\xi_i)$ in using equation (14).

Figure 5: The graph of errors $e_n(\xi_i)$ for $n = 10, 20$




**Remark 2.6.** Notice that it can be proven that both equations (12) and (15) are contractions with the metric induced by norm (1), so they have a unique solution.

# 3 Counterexamples

It is known that the problem of solving nonlinear functional equations, e.g., nonlinear integral equations, can be formulated using a fixed point finding for a given nonlinear equation [10, 11]. In this section, we discuss the Leray- Schauder type map and a counterexample for [12, Theorem 5.1].

## 3.1 Leray- Schauder type map and existence results

There are many existence fixed point theorems investigated by numerous mathematicians after the Leray-Schauder theorem that can be applied to finding solutions to implicit nonlinear functional equations such as nonlinear integral equations, boundary value problems, etc.; for instance, see [1, 5, 6, 16, 18, 19]. One of the most famous of them is the following:

**Theorem 3.1** (Leray-Schauder theorem [14])**.** Let $\mathfrak{B}$ be a Banach space, $C$ a bounded, open subset of $\mathfrak{B}$ and $0 \in C$. Suppose that $T : \overline{C} \to \mathfrak{B}$ is a continuous, condensing map, and assume that

$$T(x) \neq \lambda x \text{ for } x \in \partial A \text{ and } \lambda \in (1, +\infty), \tag{L-S}$$

holds. Then $T$ has a fixed point in $\overline{C}$.

The relation (L-S) is called the Leray-Schauder boundary condition, and we say $T$ is a Leray-Schauder type map if $T$ satisfies the conditions of theorem 3.1. Also, this result is due to Petryshyn [17] for $C = B_\rho$ with the following boundary condition, which is obviously equivalent to (L-S),

$$\text{if } \ T(z) = \lambda z, \quad \text{for some } z \in \partial B_\rho, \quad \text{then } \lambda \leq 1. \tag{P}$$

Conditions (1)-(2) in [12, Theorem 3.1] imply that $T$ is a continuous and condensing map, respectively, and conditions (3) imply that

$$T(B_\rho) \subseteq B_\rho, \tag{16}$$

which is a stronger condition than (P) (see also [15, Theorem 2.1]). Sometimes it is better to consider condition (L-S) instead of (16):

**Example 3.2.** Consider $T : B_\rho \to E$ with $\rho > 0$ and

$$T(z)(s) = 1 + 2(s-1)\int_0^1 z(t)dt \tag{17}$$

It is easy to check conditions (1)-(2) in [12, Theorem 3.1] hold, where $\zeta(s, u_1, u_2, u_3) = 1 + u_3$ and $u_3 = 2(s-1)\int_0^1 z(t)dt$, then we have $-2\rho \leq u_3 \leq 2\rho$ and

$$\sup\{|\zeta(s, u_1, u_2, u_3)|; s \in [0,1], u_3 \in [-2\rho, 2\rho]\} \leq 1 + 2\rho \not\leq \rho, \qquad \forall \rho > 0.$$

So, it cannot be possible to verify condition (3) of [12, Theorem 3.1]. We show that (L-S) holds. Let $z = \lambda T(z)$ for some $z \in \partial B_1$ and $\lambda \in (1, +\infty)$. From (17) we get

$$\int_0^1 T(z)(s)ds = \int_0^1 [1 + 2(s-1)a]ds = 1 - a = \lambda a \to a = \frac{1}{1+\lambda} < 1/2$$

where $a = \int_0^1 z(t)dt$, equality $1 - a = \lambda a$ implies that $a > 0$ and inequality $\lambda a = 1 - a < 1/2$ is a contradiction, since we have $1/2 < a$ and $a < \lambda a = 1 - a < 1/2$, thus $T$ has a fixed point in $B_1$. It is easy to check that the fixed point is $z(s) = s, s \in [0, 1]$.

## 3.2 Leray-Schauder type map and counterexamples

The Banach contraction principle implies that every contraction $T$ is a Picard operator on a metric space, i.e., the Picard sequence $z_n = T^n(z_0)$, for all $z_0 \in X$, converges to a unique fixed point of $T$ (in numerical analysis, this technique is called the successive approximation method). Pay attention; Leray-Schauder type maps do not make a claim about the convergence of the Picard sequence. Usually, in these cases, mathematicians have other suggestions for using iteration procedures such as Kirk, Krasnoselskij, Mann, and Ishikawa, e.t.c., iteration sequences (see [7, Section 6] and [2, Section 8]) to generate successive approximation sequences and find fixed points of various classes of mappings in normed linear spaces. Unfortunately, Picard sequences do not converge in general, even on the real axis [19, Section 2]. Also, fixed points of Leray-Schauder type map $T$ can be non-unique [20, Chapter 15], and the Picard sequences may not converge (even point-wise) in Banach space $C(E)$ (see also [18]), specially when $T$ is a Frehoem integral equation, e.g.:

**Example 3.3.** Consider $T$ in Example 3.2. Let $z_0(s) = 0, s \in [0,1]$ then we have

$$z_1(s) = 1, z_2(s) = 2s - 1, z_3(s) = 1, z_4(s) = 2s - 1, \ldots.$$

This shows that $\lim_{n\to+\infty} z_n(s)$ does not exist for every $s \in [0,1)$.

Kazemi et al. [12] in Theorem 5.1 assert that "under the assumptions of [12, Theorem 3.1], the iterative method (5) is convergent on the ball $B_\rho$".
It is possible to construct integral equations (especially, Fredholm integral equations) such that conditions [12, Theorem 3.1-(1)-(3)] hold and Picard sequence (5) does not converge even point-wise in $C(I)$, e.g.:

**Example 3.4.** Consider $T : B_\rho \to E$ with $1 \leq \rho$ and

$$T(z)(s) = -\frac{1}{4} + \frac{|z\,(s)|}{2} + \frac{1}{4}\int_0^1 \sqrt{|z\,(t)|}dt,$$

or

$$T(z)(s) = -\frac{1}{4} - \frac{z\,(s)}{2} + \frac{1}{4}\int_0^1 \sqrt{|z\,(t)|}dt.$$

It is easy to check conditions (1)-(2) in [12, Theorem 3.1], where $\zeta(s,u_1,u_2) = -\frac{1}{4}+\frac{1}{2}|u_1|+u_2$ or $\zeta(s,u_1,u_2) = -\frac{1}{4} - \frac{1}{2}u_1 + u_2$ and $u_1 = z(s), u_2 = \frac{1}{4}\int_0^1 \sqrt{|z\,(t)|}dt$. In both cases, we have

$$\sup\left\{|\zeta(s,u_1,u_2)|; s \in [0,1], u_1 \in [-\rho,\rho], u_2 \in \left[-\frac{1}{4}\sqrt{\rho}, \frac{1}{4}\sqrt{\rho}\right]\right\} \leq \frac{1}{4} + \frac{1}{2}\rho + \frac{1}{4}\sqrt{\rho} \leq \rho,$$

thus, condition (3) of [12, Theorem 3.1] is satisfied too. Let $z_0(s) = 0, s \in [0,1]$ then we have

$$z_1(s) = -\frac{1}{4}, z_2(s) = 0, z_3(s) = -\frac{1}{4}, \ldots.$$

Thus, $\lim_{n\to+\infty} z_n(s)$ does not exist for every $s \in [0,1]$.

At the end, let us mention some unjustified statements and some errors in the proof of [12, Theorem 5.1]:

- The proof begins with this sentence: By complete induction on $M \in \mathbb{N}$, we assume that for all $m \leq M - 1$,

  $$|z_m(s) - z(s)| \leq \sigma, \quad s \in [0,b]. \tag{18}$$

  Relation (18) is the hypothesis of induction, so from induction steps it must be proved that $|z_M(s) - z(s)| \leq \sigma, s \in [0,b]$, which is not seen anywhere in the proof. Then, without showing this, they have set $M \to \infty$ and $\sigma \to 0$.

- They showed that the following inequality holds:

$$
\begin{aligned}
|z_M(s) - z(s)| &= |(Tz_{M-1})(s) - (Tz)(s)| \\
&\vdots \\
&\leq k_1 |f(s, z_{M-1}(\alpha(s)) - f(s, z(\alpha(s))| \\
&+ \frac{k_3}{\Gamma(\tau)} \int_0^{\theta(s)} \frac{|p(s,\xi,z_{M-1}(\gamma(\xi))) - p(s,\xi,z(\gamma(\xi)))|}{(\theta(s)-\xi)^{1-\tau}} d\xi, \qquad (19) \\
&\vdots \\
&\leq (k_1 k_4)^M |z_0(\alpha^M(s)) - z(\alpha^M(s))| + \frac{1}{1-k_1 k_4} \frac{k_3}{\Gamma(\tau+1)} D^\tau \omega(p,\sigma). \qquad (20)
\end{aligned}
$$

  Please check [12] for more details on this inequality. The comparison of relations (19) and (20) shows that modulus of continuity $\omega(p,\sigma)$ in this inequality is on the third component, while they already in the proof of Theorem 3.1 were defined as

$$\omega(p,\sigma) = \sup\{|p(s,\xi,z) - p(\bar{s},\xi,z)| : |s-\bar{s}| \leq \sigma, s, \bar{s} \in I_b, \xi \in [0,D], z \in [-\rho,\rho]\}, \quad (21)$$

  where the modulus of continuity $\omega(p,\sigma)$ in (21) is on the first component.

## Availability of data and materials

No data were used to support this study.

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