A New Numerical Approach to Gardner Kawahara Equation in Magneto-Acoustic Waves in Plasma Physics

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Abstract

The basic idea of this article is to investigate the numerical solutions of Gardner Kawahara equation, a particular case of extended Korteweg-de Vries (KdV) equation, by means of finite element method. For this purpose, a collocation finite element method based on trigonometric quintic B-spline basis functions is presented. The standard finite difference method is used to discretize time derivative and Crank-Nicolson approach is used to obtain more accurate numerical results. Several numerical examples are presented and discussed to exhibit the feasibility and capability of the finite element method and trigonometric B-spline basis functions. More specifically, the error norms L_2 and L_∞ are reported for numerous time and space discretization numbers in tables. Graphical representations of the solutions which describe motion of wave are presented.

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1 Introduction

Most of the natural principles and laws in the real worls are modelled by differential equations and only a few of them could be solved analytically. Thus obtaining numerical solutions for those equations has become more important. Gardner-Kawahara equation belongs to the class of nonlinear Partial Differential Equations (PDEs) and has gained much more popularity as it models several physical phenomena in the nature. In fact, Gardner-Kawahara equation is a particular form of extended KdV equation. The equation can be widely encountered in shallow water waves having surface tension as well as in plasma physics. In terms of physical interpretation, the Gardner-Kawahara equation expalins the solitary wave propagation in media [9].In the study of interfacial waves between two immiscible fluids, there are such situation when the double critical conditions can occur, i.e. when both the coefficients of quadratic nonlinearity and third-order dispersion vanish simultaneously. In the near-critical situation the basic governing equation is the Gardner-Kawahar equation. In the present article, numerical solutions of the nonlinear Gardner-Kawahara equation in the following form

$$u_t + \kappa u_x + \lambda u\ u_x - \alpha u^2 u_x + \mu u_{xxx} + \beta u_{xxxxx} = 0$$

are going to be sought for. Where κ, λ, α, μ and β are positive real parameters. The problem is subject to the following initial-boundary conditions

$$u\,(x,0) = f(x),\ x \in [a,b] \tag{1.1}$$

$$\begin{aligned} u\,(a,t) &= g_1(t),\ u\,(b,t) = g_2(t),\ t \geq 0 && (1.2) \\ u_x\,(a,t) &= g_2(t),\ u_x\,(b,t) = g_4(t),\ t \geq 0 && (1.3) \\ u_{xx}\,(a,t) &= g_5(t),\ u_{xx}\,(b,t) = g_6(t),\ t \geq 0 && (1.4) \end{aligned}$$

in which $f(x)$ is a given smooth function, $g_1(t)$ and $g_2(t)$ are prescribed functions to be given in Numerical computation and analysis section.First of all, by using the conversion. $u_{xxx} = v$ in the equation, it will be converted into a coupled system of equations. Then quintic B-spline basis functions are going to be used for approximate solutions.

In order to solve the nonlinear Gardner-Kawahara equation both analytically and numerically, several researchers have used various methods and techniques. In recent years, a great many of authors have been pushing forward several theories with different styles and perspectives. Among others, Hussein and Taha [4] have efficiently and successfully used (G'/G)-expansion method to find new solitary wave solutions of the Gardner-Kawahara equation. Swain et al. [9] have investigated the equation using the Lie symmetry method. First they have reduced the equation into a nonlienar ordinary differential equation form and then obtained the exact solution in the explicit form by different significant methods. Khusnutdinova et al. [6] have obtained soliton solutions to the fifth-order Korteweg-de Vries equation and their applications to surface and internal water waves. Wazwaz [10] has found out soliton solutions for the fifth-order KdV equation and also the Kawahara equation with time-dependent coefficients. Zahra et al. [11] have proposed an effective scheme based on quartic B-spline for the solution of Gardner equation and Harry Dym equation. Kurkina et al. [7] have derived a new model equation describing weakly nonlinear internal waves at the interface between two thin layers of different density for the specific relationships between the densities, layer thickness and surface tension between the layers. For this purpose, they derived and dubbed the Gardner-Kawahara equation representing a natural generalization of the well-known Kortweg-de Vries (KdV) equation containing the cubic nonlinear term as well as fifth-order dispersion term.They also investigated solitary wave solutions numerically and categorized in terms of two dimensionless parameters, the wave speed and fifth-order dispersion and concluded that the derived equation may be applicable to wave description in other media.

2 Structure of the method

The governing equation describing the long internal waves having small amplitude at the interface in a two-layer fluid is widely known KdV equation (see, e.g., [1–3] and those references therein).

$$u_t + cu_x + \alpha u\ u_x + \beta_1 u_{xxx} = 0 \tag{2.5}$$

But, under certain conditions, KdV equation degenerates due to the fact that some of its coefficients vanish. Thus a generalization becomes necessary epecially when the density interface is located near the half of the depth of the fluid. If this is the case, then the coefficient of the quadratic nonlinearity tends to be anomalously small, and one should consider the next order nonlinear term in order to balance the dispersion effect. This condition leads to Gardner equation.

$$u_t + cu_x + \alpha u\ u_x - \alpha_1 u^2 u_x + \beta_1 u_{xxx} = 0 \tag{2.6}$$

Both the Kdv and Gardner equations are definably integrable; that is, they possess soliton solutions.There are also situations when the dispersion coefficient vanishes. When this happens, the next-order dispersion should be considered. This leads to the fifth-order KdV equation with quadratic nonlinearity.

$$u_t + cu_x + \alpha u\ u_x + \beta_1 u_{xxx} + \beta_2 u_{xxxxx} = 0 \tag{2.7}$$

Although this equation was derived by Kakutani and Ono in 1969 [5], it is currently known as the Kawahara equation. In the meantime, in the case of internal waves in two-layer fluid with strong surface tension at the interface the double-critical situation is also possible when both the coefficients of quadratic nonlinearity and the tird order dispersion become so small that the next order are required. In the vicinity of the double critical situation the governing equation, namely GK equation is generally given in the following form

$$u_t + \kappa u_x + \lambda u\, u_x - \alpha u^2 u_x + \mu u_{xxx} + \beta u_{xxxxx} = 0 \tag{2.8}$$

together with the appropriate physical initial and boundary conditions, in which t is time , x is the space coordinate and $\kappa, \lambda, \alpha, \mu$ and β are predefined parameters. For the considered problems, the appropriate initial and boundary conditions will be taken from the exact solution.

For numerical evaluation of an initial-boundary value problem by a finite elements method, let us divide the solution region $(x, t) \in [a, b] \times [0, T]$ of Gardner-Kawahara equation given by Eq. (2.8) in space direction into N equal consecutive sub-space intervals as $a = x_0 < x_1 < \cdots < x_{N-1} < x_N = b$, $\Delta x = h = x_{i+1} - x_i$, $i = 0(1)N - 1$ by nodal points $\{x_i\}_{i=0}^{N}$ and in temporal direction into M equal consecutive sub-time intevals as $0 = t_0 < t_1 < \cdots < t_{M-1} < t_M = T$, $\Delta t = k = t_{j+1} - t_j$, $j = 0(1)M - 1$ by nodal points $\{t_j\}_{j=0}^{M}$ where T stands for final time.For the numerical discretization of Eq. (2.8), first of all, we have split it as follows

$$\begin{aligned} v &= u_{xxx} \\ u_t + \kappa u_x + \lambda u\, u_x - \alpha u^2 u_x + \mu v + \beta u_{xx} &= 0 \end{aligned}$$

For the nonlinear terms, the following Rubin-Graves type linearization is applied

$$\begin{aligned} (u\, u_x)^{n+1} &= u^{n+1} u_x^n + u^n u_x^{n+1} - u^n u_x^n \\ (u^2\, u_x)^{n+1} &= u^{n+1}\, u^n\, u_x^n + u^n\, u^{n+1}\, u_x^n + u^n\, u^n\, u_x^{n+1} - 2u^n\, u^n\, u_x^n \end{aligned}$$

For the numerical discretization of Eq. (2.8), using the first-order forward finite difference approximation for the derivative in time and the Crank-Nicolson type formulation for all space derivatives, we first obtain the following semi-discretized equation

$$\frac{v_{j+1} + v_j}{2} = \frac{(u_{xxx})_{j+1} + (u_{xxx})_j}{2}$$

$$\frac{u_{j+1}-u_j}{k}+\kappa\frac{(u_x)_{j+1}+(uu_{xj})_j}{2}+\lambda\frac{(u\ u_x)_{j+1}+(u\ u_x)_j}{2}$$
$$-\alpha\frac{(u^2u_x)_{j+1}+(u^2u_x)_j}{2}+\mu\frac{v_{j+1}+v_j}{2}+\beta\frac{(u_{xx})_{j+1}+(u_{xx})_j}{2} \quad = \quad 0 \qquad (2.9)$$

The trigonometric quintic B-splines $T_m^5(x)$, $(m=-1(1)N+1)$, at the knots x_m are defined over the interval $[a,b]$ by [8]

$$T_i^5(x)=\frac{1}{\theta}\begin{cases}
p^5(x_{i-3}) & ,x_{i-3}\le x<x_{i-2} \\
-p^4(x_{i-3})p(x_{i-1}) & \\
-p^3(x_{i-3})p(x_i)p(x_{i-2}) & ,x_{i-2}\le x<x_{i-1} \\
-p^2(x_{i-3})p(x_{i+1})p^2(x_{i-2}) & \\
-p(x_{i-3})p(x_{i+2})p^3(x_{i-2})-p(x_{i+3})p^4(x_{i-2}) & \\
p^3(x_{i-3})p^2(x_i) & \\
+p^2(x_{i-3})p(x_{i+1})p(x_{i-2})p(x_i) & \\
+p^2(x_{i-3})p^2(x_{i+1})p(x_{i-1}) & \\
+p(x_{i-3})p(x_{i+2})p^2(x_{i-2})p(x_i) & \\
+p(x_{i-3})p(x_{i+2})p(x_{i-2})p(x_{i+1})p(x_{i-1}) & ,x_{i-1}\le x<x_i \\
+p(x_{i-3})p^2(x_{i+2})p^2(x_{i-1}) & \\
+p(x_{i-3})p^3(x_{i-2})p(x_i) & \\
+p(x_{i+3})p^2(x_{i-2})p(x_{i+1})p(x_{i-1}) & \\
+p(x_{i+3})p(x_{i-2})p(x_{i+2})p^2(x_{i-1}) & \\
+p^2(x_{i+3})p^3(x_{i-1}) & \\
-p^2(x_{i-3})p^3(x_{i+1}) & \\
-p(x_{i-3})p(x_{i+2})p(x_{i-2})p^2(x_{i+1}) & \\
-p(x_{i-3})p^2(x_{i+2})p(x_{i-1})p(x_{i+1}) & \\
-p(x_{i-3})p^3(x_{i+2})p(x_i) & \\
-p(x_{i+3})p^2(x_{i-2})p^2(x_{i+1}) & ,x_i\le x<x_{i+1} \\
-p(x_{i+3})p(x_{i-2})p(x_{i+2})p(x_{i-1})p(x_{i+1}) & \\
-p(x_{i+3})p(x_{i-2})p^2(x_{i+2})p(x_{i-1}) & \\
-p^2(x_{i+3})p^2(x_{i-1})p(x_{i+1}) & \\
-p^2(x_{i+3})p(x_{i-1})p(x_{i+2})p(x_i) & \\
-p^3(x_{i+3})p^2(x_i) & \\
p(x_{i-3})p^4(x_{i+2})+p(x_{i+3})p(x_{i-2})p^3(x_{i+2}) & ,x_{i+1}\le x<x_{i+2} \\
+p^2(x_{i+3})p(x_{i-1})p^2(x_{i+2}) & \\
+p^3(x_{i+3})p(x_i)p(x_{i+2})+p^4(x_{i+3})p(x_{i+1}) & \\
-p^5(x_{i+3}) & ,x_{i+2}\le x<x_{i+3} \\
0 & ,\text{ otherwise}
\end{cases}$$

(2.10)

in which

$$p(x_i) = \sin\left(\frac{x - x_i}{2}\right), \theta = \sin\left(\frac{h}{2}\right)\sin(h)\sin\left(\frac{3h}{2}\right)\sin(2h)\sin\left(\frac{5h}{2}\right), i = 0, ..., N$$

The set of trigonometric quintic B-splines $\left\{T^5_{-2}(x), T^5_{-1}(x), T^5_0(x), \ldots, T^5_{N+1}(x)\right\}$ forms a basis for the smooth functions defined over [a,b]. Therefore, an approximation solution $u_N(x,t)$ and $v_N(x,t)$ can be written in terms of the trigonometric cubic B-splines as trial functions:

$$u(x,t) \approx u_N(x,t) = \sum_{i=-2}^{N+1} T^5_i(x)\delta_i(t) \tag{2.11}$$

$$v(x,t) \approx v_N(x,t) = \sum_{i=-2}^{N+1} T^5_i(x)\sigma_i(t) \tag{2.12}$$

where $\delta_i(t)$'s and $\sigma_i(t)$'s are unknown, time dependent quantities to be determined from the boundary and trigonometric quintic B-spline collocation conditions. Each trigonometric quintic B-spline covers six elements so that each element $[x_i, x_{i+1}]$ is covered by six trigonometric quintic B-splines. For this problem, the finite elements are identified with the interval $[x_i, x_{i+1}]$. Using the nodal values $u_i, u_i^{'}, u_i^{''}, u_i^{'''}$ and $u_i^{(4)}$are given in terms of the parameter δ_i by:

$$\begin{aligned}
u_i &= u(x_i) = a_1\delta_{i-2} + a_2\delta_{i-1} + a_3\delta_i + a_2\delta_{i+1} + a_1\delta_{i+2} \\
u_i' &= u'(x_i) = b_1\delta_{i-2} + b_2\delta_{i-1} - b_2\delta_{i+1} - b_1\delta_{i+2} \\
u_i'' &= u''(x_i) = c_1\delta_{i-2} + c_2\delta_{i-1} + c_3\delta_i + c_2\delta_{i+1} + c_1\delta_{i+2} \\
u_i''' &= u'''(x_i) = d_1\delta_{i-2} + d_2\delta_{i-1} - d_2\delta_{i+1} - d_1\delta_{i+2} \\
u_i^{(4)} &= u^{(4)}(x_i) = e_1\delta_{i-2} + e_2\delta_{i-1} + e_3\delta_i + e_2\delta_{i+1} + e_1\delta_{i+2}
\end{aligned} \tag{2.13}$$

and again using the nodal values $v_i, v_i^{'}, v_i^{''}, v_i^{'''}$ and $v_i^{(4)}$are given in terms of the parameter σ_i by:

$$\begin{aligned}
v_i &= v(x_i) = a_1\sigma_{i-2} + a_2\sigma_{i-1} + a_3\sigma_i + a_2\sigma_{i+1} + a_1\sigma_{i+2} \\
v_i' &= v'(x_i) = b_1\sigma_{i-2} + b_2\sigma_{i-1} - b_2\sigma_{i+1} - b_1\sigma_{i+2} \\
v_i'' &= v''(x_i) = c_1\sigma_{i-2} + c_2\sigma_{i-1} + c_3\sigma_i + c_2\sigma_{i+1} + c_1\sigma_{i+2} \\
v_i''' &= v'''(x_i) = d_1\sigma_{i-2} + d_2\sigma_{i-1} - d_2\sigma_{i+1} - d_1\sigma_{i+2} \\
v_i^{(4)} &= v^{(4)}(x_i) = e_1\sigma_{i-2} + e_2\sigma_{i-1} + e_3\sigma_i + e_2\sigma_{i+1} + e_1\sigma_{i+2}
\end{aligned} \tag{2.14}$$

where

$$
\begin{aligned}
a_1 &= \frac{\sin^5\left(\frac{h}{2}\right)}{\theta} \\
a_2 &= \frac{2\sin^5\left(\frac{h}{2}\right)\cos\left(\frac{h}{2}\right)\left(16\cos^2\left(\frac{h}{2}\right)-3\right)}{\theta}, \\
a_3 &= \frac{2\left(1+48\cos^4\left(\frac{h}{2}\right)-16\cos^2\left(\frac{h}{2}\right)\right)\sin^5\left(\frac{h}{2}\right)}{\theta}, \\
b_1 &= -\frac{5\sin^4\left(\frac{h}{2}\right)\cos\left(\frac{h}{2}\right)}{2\theta}, \\
b_2 &= -\frac{5\sin^4\left(\frac{h}{2}\right)\cos^2\left(\frac{h}{2}\right)\left(8\cos^2\left(\frac{h}{2}\right)-3\right)}{\theta}, \\
c_1 &= \frac{5\sin^3\left(\frac{h}{2}\right)\left(5\cos^2\left(\frac{h}{2}\right)-1\right)}{4\theta}, \\
c_2 &= \frac{5\sin^3\left(\frac{h}{2}\right)\cos\left(\frac{h}{2}\right)\left(-15\cos^2\left(\frac{h}{2}\right)+3+16\cos^4\left(\frac{h}{2}\right)\right)}{2\theta}, \\
c_3 &= -\frac{5\sin^3\left(\frac{h}{2}\right)\left(16\cos^2\left(\frac{h}{2}\right)-5\cos^2\left(\frac{h}{2}\right)+1\right)}{2\theta}, \\
d_1 &= -\frac{5\sin^2\left(\frac{h}{2}\right)\cos\left(\frac{h}{2}\right)\left(25\cos^2\left(\frac{h}{2}\right)-13\right)}{8\theta}, \\
d_2 &= -\frac{5\sin^2\left(\frac{h}{2}\right)\cos^2\left(\frac{h}{2}\right)\left(8\cos^4\left(\frac{h}{2}\right)-35\cos^2\left(\frac{h}{2}\right)+15\right)}{4\theta}, \\
e_1 &= \frac{5\left(125\cos^4\left(\frac{h}{2}\right)-114\cos^2\left(\frac{h}{2}\right)+13\right)\sin\left(\frac{h}{2}\right)}{16\theta}, \\
e_2 &= -\frac{5\sin\left(\frac{h}{2}\right)\cos\left(\frac{h}{2}\right)\left(176\cos^6\left(\frac{h}{2}\right)-137\cos^4\left(\frac{h}{2}\right)-6\cos^2\left(\frac{h}{2}\right)+15\right)}{8\theta}, \\
e_3 &= \frac{5\left(92\cos^6\left(\frac{h}{2}\right)-117\cos^4\left(\frac{h}{2}\right)+62\cos^2\left(\frac{h}{2}\right)-13\right)\left(-1+4\cos^2\left(\frac{h}{2}\right)\right)\sin\left(\frac{h}{2}\right)}{8\theta},
\end{aligned}
$$

When the values of u and v and their respective derivatives given by Eqs.(2.13) and (2.14) are used in Eq. (2.9), an algebraic system in the following form is obtained

$$
\begin{aligned}
-d_1\delta_{i-2}^{n+1} - d_2\delta_{i-1}^{n+1} + d_2\delta_{i+1}^{n+1} + d_1\delta_{i+2}^{n+1} + a_1\sigma_{i-2}^{n+1} + a_2\sigma_{i-1}^{n+1} + a_3\sigma_{i}^{n+1} + a_2\sigma_{i+1}^{n+1} + a_1\sigma_{i+2}^{n+1} \\
= d_1\delta_{i-2}^{n} + d_2\delta_{i-1}^{n} - d_2\delta_{i+1}^{n} - d_1\delta_{i+2}^{n} - a_1\sigma_{i-2}^{n} - a_2\sigma_{i-1}^{n} - a_3\sigma_{i}^{n} - a_2\sigma_{i+1}^{n} - a_1\sigma_{i+2}^{n}
\end{aligned}
\tag{2.15}
$$

and

$$\begin{aligned}
&\delta_{i-2}^{n+1}\left(a_1+\tfrac{k}{2}\left(a_1\Lambda_1+b_1\Lambda_2\right)\right)+\delta_{i-1}^{n+1}\left(a_2+\tfrac{k}{2}\left(a_2\Lambda_1+b_2\Lambda_2\right)\right)+\delta_{i}^{n+1}\left(a_3+\tfrac{k}{2}a_3\Lambda_1\right)\\
&+\delta_{i+1}^{n+1}\left(a_2+\tfrac{k}{2}\left(a_2\Lambda_1+b_2\Lambda_2\right)\right)+\delta_{i+2}^{n+1}\left(a_1+\tfrac{k}{2}\left(a_1\Lambda_1+b_1\Lambda_2\right)\right)\\
&\qquad=\delta_{i-2}^{n}\left(a_1-\tfrac{k}{2}b_1\Lambda_3\right)+\delta_{i-1}^{n}\left(a_2-\tfrac{k}{2}b_2\Lambda_3\right)+\delta_{i}^{n}\left(a_3\right)+\delta_{i+1}^{n}\left(a_2+\tfrac{k}{2}b_2\Lambda_3\right)\\
&\qquad+\delta_{i+2}^{n}\left(a_1+\tfrac{k}{2}b_1\Lambda_3\right)
\end{aligned} \tag{2.16}$$

where

$$\begin{aligned}
\Lambda_1 &= \lambda u_x - 2\alpha u u_x\\
\Lambda_2 &= \kappa + \lambda u - \alpha u^2\\
\Lambda_3 &= \kappa + \alpha u^2.
\end{aligned}$$

As it is seen from the ssytems of equations given above, each of the systems given by Eqs. (2.15) and (2.16) consists of (N+1) equations for $i = 0, 1, 2, ...N$.But the system in Eq. (2.15) contains (N+9) unknowns and Eq.(2.16)contains (N+5) unknowns. In order to obtain a solvable coupled system, the unknowns δ_{-2}, δ_{-1}, δ_{N+1}, δ_{N+2}, σ_{-2}, σ_{-1}, σ_{N+1}, σ_{N+2} are elimanted from Eq.(2.15) and the unknowns $\delta_{-2}, \delta_{-1}, \delta_{N+1}, \delta_{N+2}$ are eliminated from Eq.2.16.This elimination process is carried out by applying the boundary conditions given by Eq.(1.2) to the first and the last rows of the system of equations in Eqs.(2.15) ve (2.16). Finally, the systems of Eqs.(2.15) and (2.16) can be written in the (2N+2)x(2N+2)-type matrix form as follows

$$\begin{bmatrix} A & B \\ C & D \end{bmatrix}\begin{bmatrix} \delta^{n+1} \\ \sigma^{n+1} \end{bmatrix} = \begin{bmatrix} \Phi_1 \\ \Phi_2 \end{bmatrix}\begin{bmatrix} \delta^{n} \\ \sigma^{n} \end{bmatrix} \tag{2.17}$$

Here, A and B are the coefficients corresponding to the unknowns δ and σin the algebraic system constructed for the first equation of Gardner-Kawahara equations sytem which is converted into coupled form by means of auxiliary variable. In a similar way, C and D are the coefficients of the second equation of the constructed coupled algebraic system of equations. Φ_1 and Φ_2 are known values found on the right hand side of the algebraic equations system, δ^n and σ^n represent the coefficients matrices.

In fact, the matrix given by Eq.(2.17) is an iterative system. In order to start the iteration, the initial values of δ^0 and σ^0 for the value of $n = 0$ are needed. When looked at from this perspective, it is obvious that the initial condition given by Eq. (1.1) is enough to solve this problem. When the initial condition is written for $i = 0, 1, 2, ...N$, it is certain that again a system of equations of the form $(N + 1) \times (N + 5)$ is encountered

$$
\begin{aligned}
&u(a,0)=a_1\delta_{-2}+a_2\delta_{-1}+a_3\delta_0+a_2\delta_1+a_1\delta_2=f(a),\\
&u(x_1,0)=a_1\delta_{-1}+a_2\delta_0+a_3\delta_1+a_2\delta_2+a_1\delta_3=f(x_1),\\
&\vdots\\
&u(x_{N-1},0)=a_1\delta_{N-3}+a_2\delta_{N-3}+a_3\delta_{N-1}+a_2\delta_N+a_1\delta_{N+1}=f(x_{N-1}),\\
&u(x_N,0)=a_1\delta_{N-2}+a_2\delta_{N-1}+a_3\delta_N+a_2\delta_{N+1}+a_1\delta_{N+2}=f(x_N).
\end{aligned}
$$

The elimination process for this system is also dealt with the help of boundary conditions. Once a solvable system has been obtained. Thus, the first procedure has been applied in order to obtain the desired solution at the final time T by solving the system of equations given by Eq. (2.17). During the solution process, firstly, for the time discretization forward finite difference scheme and then for the space discritization finite element collocation method based on trigonometric quintic B-spline basis functions are going to be implemented.

3 Numerical computation and analysis

In the present section, three different test problems for controlling the numerical simulations have been handled and solved by the proposed numerical scheme. For all computations, the MATLAB software is used. To testify the accuracy and efficiency of the scheme, we have calculated L_2 and L_∞ error norms for the numerical $u_{num}(x,t)$ and analytical $u_{exact}(x,t)$ solutions. In this section, we are going to present new numerical results for Gardner-Kawahara equation using finite element method based on trigonometric quintic B-spline basis. In subsection 4.1, we start with numerical investigation of first example and in sub-section 4.2, we present new numerical results for second example for the problem given in Eq (2.8). The programs used here have been coded in Matlab R2018a. The interval of the problems are discussed in $[x_L, x_R]$ as $[0,10]$ and the final time is selected as $T=5$. Number of time and space discretization are changed for all examples. Numerical results are compared with exact ones and the results produced in this article are measured via the absolute error given by the following formulas

$$
L_2=\sum_{j=0}^{N}\sqrt{\Big(\left(u_{exact}(x,t)\right)_j-\left(u_{num}\right)_j(x,t)\Big)^2}\quad (x,t)\in[x_L,x_R]\times[0,T]
$$

$$
L_\infty=\max\left|u_{exact}(x,t)-u_{num}(x,t)\right|
$$

3.1 Example 1:

Consider the following Gardner-Kawahara equation given [9]

$$u_t(x,t) + \kappa u_x(x,t) + \lambda u(x,t)\, u_x(x,t) - \alpha\left(u(x,t)\right)^2 u_x(x,t) + \mu u_{xxx}(x,t) + \beta u_{xxxxx}(x,t) = 0 \tag{3.18}$$

with initial condition

$$u(x,t) = \vartheta\left(\tanh(x)\right)^2,\ \ x_L \le x \le x_R$$

and boundary conditions

$$u(x_L,t) = \vartheta\left(\tanh(x_L - ct)\right)^2$$

$$u(x_R,t) = \vartheta\left(\tanh(x_R - ct)\right)^2$$

where $[x_L, x_R] = [0, 10]$ and the parameters seen in the equation (3.18) are taken as $\kappa = 1, \alpha = 2, \vartheta = 1$.

The other parameters are calculated using these values as follows

$$\lambda = \frac{16\alpha\vartheta}{15}, \qquad \mu = \frac{\alpha\vartheta^2}{45}, \qquad \beta = \frac{\alpha\vartheta^2}{360}$$

Additionally, the parameter seen in the initial condition is computed as

$$c = \frac{5\alpha + \alpha\vartheta^2}{5}$$

The exact solution of this problem is given by

$$u(x,t) = \vartheta\left(\tanh(x - ct)\right)^2 \tag{3.19}$$

The initial and boundary conditions of the equation are directly taken from the analytical solution. (3.19).

The proposed numerical scheme is applied to this problem and the computed approximate results for different values of the time step size k and partition number N at some values of T on the solution domain $[0, 10]$ are displayed in tables. In Table 1, the numerical results are presented for $\Delta t = 0.1, 0.05, 0.025$ and $h = 1/N = 1/100, 1/200, 1/400, 1/800$ where N is partition number of interval $[0, 10]$. It is observed from the Table that the high accuracy is achieved. In other words, the error is not spread with increasing the time. In Table 2, the numerical results at the point

Table 1: The error norms L_2 and L_∞ for $N = 100, 200, 400, 800$ and $\Delta t = 0.1, 0.005$, 0.025, 0.00125 at $t_f = 5$ on $x \in [0, 10]$ of Example 1.

$t_f = 5$	$\Delta t = 0.1$		$\Delta t = 0.05$	
N	L_2	L_∞	L_2	L_∞
100	3.50033×10^{-2}	2.82159×10^{-2}	1.00019×10^{-2}	7.55899×10^{-3}
200	3.43956×10^{-2}	2.81899×10^{-2}	1.00573×10^{-2}	7.75841×10^{-3}
400	3.39665×10^{-2}	2.79280×10^{-2}	1.00021×10^{-2}	7.76200×10^{-3}
800	3.37323×10^{-2}	2.78452×10^{-2}	9.95381×10^{-3}	7.74103×10^{-3}

	$\Delta t = 0.025$		$\Delta t = 0.00125$	
	L_2	L_∞	L_2	L_∞
100	2.24511×10^{-3}	1.71609×10^{-3}	4.55957×10^{-4}	2.91257×10^{-4}
200	2.49505×10^{-3}	1.89270×10^{-3}	1.06954×10^{-4}	7.10721×10^{-5}
400	2.53888×10^{-3}	1.93589×10^{-3}	2.24935×10^{-5}	1.56145×10^{-5}
800	2.53948×10^{-3}	1.94299×10^{-3}	2.65972×10^{-6}	2.47563×10^{-6}

(x_j, t_n) are reported for $\Delta t = 0.0001$ and $h = 1/800$ and compared with the exact ones. The Absolute errors $|u_{num} - u_{exact}|$ in the Table 2 shows that finite element method based on trigonometric quintic splines leads for achieving the high accuracy. It is seen from those tables that the numerical solutions obtained by the proposed scheme are in very good agreement with the analytical ones, and L_2 and L_∞ errors are reasonably small enough. This fact is a clear evidence for the accuracy and reliability of the numerical scheme. Figure 1 shows wave profiles of numerical solutions It is apparently seen from the figure that the obtained numerical solutions also verify the continuity with the correct physical behavior of the problem. 4

3.2 Example 2:

As the second example,, the Gardner-Kawahara equation (3.18) is considered where the parameters and the exact solution are as follows. When the parameters $a = 1$, $\alpha = 0.1$, $\lambda = 0.1$, $\vartheta_0 = -0.001$ are taken and the remaining parameters are calculated as follows

$$\mu = \frac{8\lambda\vartheta_0 - 2\alpha\vartheta_0^2}{36}, \qquad \beta = \frac{\alpha\vartheta_0^2}{360}, \qquad c = \frac{45a + 15\lambda\vartheta_0 - 8\alpha\vartheta_0^2}{45}$$

over the solution domain $[x_L, x_R] = [-10, 10]$.The exact solution of this problem is given by

Table 2: Some nodal values of $U(x,t)$ for $N = 800$ and $\Delta t = 0.0001$ at $t_f = 5$ on $x \in [0, 10]$ of Example 1.

$\Delta t = 0.0001$	$N = 800$	$t_f = 5$	
x	u_N	u	$\lvert u_N - u \rvert$
0.0125	0.9999929044	0.9999965897	3.6853×10^{-6}
1.00	0.9999782695	0.9999754235	2.8461×10^{-6}
2.00	0.9998195042	0.9998184168	1.0875×10^{-6}
3.00	0.9986613945	0.9986590493	2.3452×10^{-6}
4.00	0.9901351172	0.9901339628	1.1544×10^{-6}
5.00	0.9293507812	0.9293491751	1.6061×10^{-6}
6.00	0.5800286973	0.5800256584	3.0389×10^{-6}
7.00	0.0000037425	0.0000000000	3.7425×10^{-6}
8.00	0.5800247405	0.5800256584	9.178×10^{-7}
9.00	0.9293523536	0.9293491751	3.1784×10^{-6}
9.9875	0.9898817862	0.9898854692	3.6830×10^{-6}

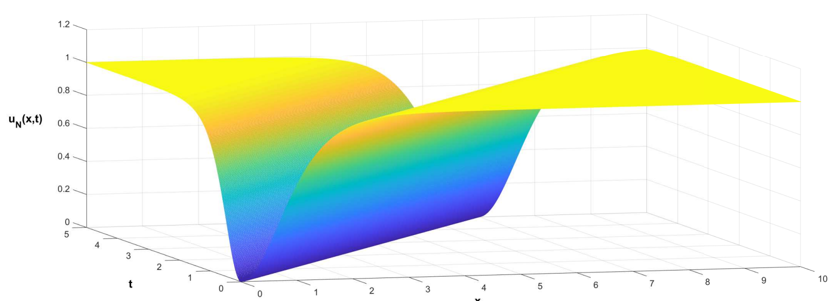

Figure 1: Numerical simulations of Problem I for values of $N = 400$, $\Delta t = 0.01$, $t_f = 5$ over $[0, 10]$.

Table 3: The error norms L_2 and L_∞ for $N = 100, 200, 400, 800$ and $\Delta t = 0.05$, 0.025, 0.0025, 0.00125, 0.005, 0.0025 at $t_f = 5$ on $x \in [-10, 10]$ of Example 2.

$t_f = 5$	$\Delta t = 0.05$		$\Delta t = 0.025$	
N	L_2	L_∞		
100	4.30729×10^{-6}	4.33615×10^{-6}	1.09531×10^{-6}	1.11416×10^{-6}
200	4.28870×10^{-6}	4.27028×10^{-6}	1.07519×10^{-6}	1.06383×10^{-6}
400	4.28846×10^{-6}	4.28469×10^{-6}	1.07495×10^{-6}	1.06331×10^{-6}
800	4.28846×10^{-6}	4.29290×10^{-6}	1.07495×10^{-6}	1.06679×10^{-6}
	$\Delta t = 0.0025$		$\Delta t = 0.00125$	
	L_2	L_∞	L_2	L_∞
100	6.53519×10^{-8}	8.59348×10^{-8}	6.21053×10^{-8}	8.03733×10^{-8}
200	1.10242×10^{-8}	1.10836×10^{-8}	3.01022×10^{-9}	3.24820×10^{-9}
400	1.07621×10^{-8}	1.06229×10^{-8}	2.69316×10^{-9}	2.66066×10^{-9}
800	1.07586×10^{-8}	1.06441×10^{-8}	2.68971×10^{-9}	2.66106×10^{-9}

Table 4: The error norms L_2 and L_∞ for $N = 400$ and $\Delta t = 0.001$ at $t_f = 1, 2, 3, 4, 5$ on $x \in [-10, 10]$ of Example 2.

	$\Delta t = 0.001$	$N = 400$
t_f	L_2	L_∞
1	3.44977×10^{-10}	3.40962×10^{-10}
2	6.89955×10^{-10}	6.81975×10^{-10}
3	1.03493×10^{-9}	1.02304×10^{-9}
4	1.37991×10^{-9}	1.36415×10^{-9}
5	1.72489×10^{-9}	1.70521×10^{-9}

$$u(x,t) = \vartheta \left(1 - \tanh^2(x - ct)\right) \tag{3.20}$$

The presented numerical scheme is applied to the second test problem and the calculated approximate results for different values of the time step size k and partition number N at some values of T on the solution domain $[-10, 10]$ are displayed in tables. In Tables 3-4, some values of the error norms L_2 and L_∞ computed by the presented method of Example 2 are listed. It can be easily seen from the each table that the error norms obtained by the present method are sufficiently small. This is also an evidence for the effectiveness of the presented method.

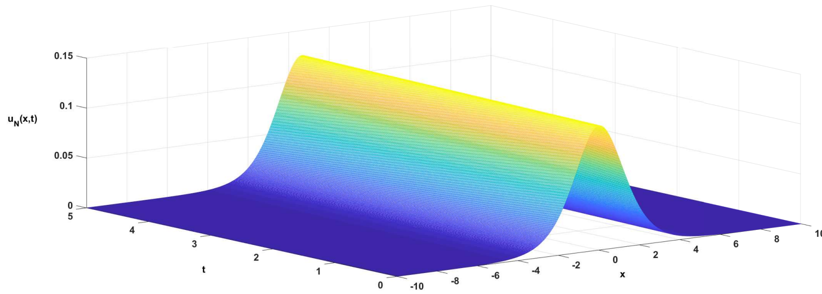

Figure 2: Numerical simulations of Problem III for values of $N = 400$, $\Delta t = 0.01$, $t_f = 5$, $k = 0.6$, $\lambda = 1$, $\beta = 0.01$, $\alpha = 3$ over $[-10, 10]$.

3.3 Example 3

As the last example, the Gardner-Kawahara equation (3.18) is considered where the parameters and the exact solution is as follows. When the parameters are taken as $a = 1$, $\alpha = 3$, $k = 0.6$, $\lambda = 1$, $\beta = 0.001$ the other parameters are calculated as follows

$$Amp = \frac{6\sqrt{10\beta}k^2}{\sqrt{\alpha}}, \qquad \mu = \frac{1}{2}\left[-40k^2\beta + \frac{\sqrt{10\beta}\lambda}{\sqrt{\alpha}}\right] \qquad c = a - 64k^4\beta + \frac{2\sqrt{10\beta}k^2\lambda}{\sqrt{\alpha}}.$$

over the solution domain $[x_L, x_R] = [-10, 10]$.For this test problem the exact solution is

$$u(x,t) = Amp \sec h^2(kx - ct) \tag{3.21}$$

The newly given numerical scheme is also applied to the last test problem and the computed approximate results for different values of the time step size k and partition number N at some values of T on the solution domain $[-10, 10]$ are displayed in tables. From the tables 5-6, it is obviously seen that the error norms are again sufficiently small enough. This is also an evidence for the correctness and reliability of the present method. Figure 2 shows wave profiles of numerical solutions It is apparently seen from the figure that the obtained numerical solutions also verify the continuity with the correct physical behavior of the problem. Figure 3 illustrates the error values between the wave profile of both analytical and numerical solutions. It is easy to see from the figure that both solutions are so close to each other that the errors are so small.

Table 5: The error norms L_2 and L_∞ for $N = 100, 200, 400, 800$ and $\Delta t = 0.1, 0.05, 0.025,\ 0.0125,\ 0.01, 0.00625$ at $t_f = 5$ on $x \in [0, 10]$ of Example 3.

$t_f = 5$	$\Delta t = 0.1$		$\Delta t = 0.05$	
N	L_2	L_∞	L_2	L_∞
100	1.02377×10^{-6}	9.22495×10^{-7}	6.25711×10^{-7}	4.80110×10^{-7}
200	1.00544×10^{-6}	6.80180×10^{-7}	2.46280×10^{-7}	1.74076×10^{-7}
400	1.01344×10^{-6}	6.88123×10^{-7}	2.75748×10^{-7}	1.88348×10^{-7}
800	1.00766×10^{-6}	6.81887×10^{-7}	2.62404×10^{-7}	1.79857×10^{-7}
	$\Delta t = 0.025$		$\Delta t = 0.0125$	
	L_2	L_∞	L_2	L_∞
100	6.45687×10^{-7}	5.12302×10^{-7}	6.65059×10^{-7}	5.35246×10^{-7}
200	6.40611×10^{-8}	5.54092×10^{-8}	5.15831×10^{-8}	3.67079×10^{-8}
400	8.46955×10^{-8}	5.22023×10^{-8}	2.54306×10^{-8}	1.53132×10^{-8}
800	8.42129×10^{-8}	5.21506×10^{-8}	3.74952×10^{-8}	1.77597×10^{-8}
	$\Delta t = 0.01$		$\Delta t = 0.00625$	
	L_2	L_∞	L_2	L_∞
100	6.68433×10^{-7}	5.38796×10^{-7}	6.73074×10^{-7}	5.42629×10^{-7}
200	5.60563×10^{-8}	3.57945×10^{-8}	6.37202×10^{-8}	3.79268×10^{-8}
400	1.32595×10^{-8}	8.49295×10^{-9}	4.92957×10^{-9}	3.44549×10^{-9}
800	2.82291×10^{-8}	1.26192×10^{-8}	1.42320×10^{-8}	5.76886×10^{-9}
	$\Delta t = 0.005$		$\Delta t = 0.0025$	
	L_2	L_∞	L_2	L_∞
100	6.74534×10^{-7}	5.43796×10^{-7}	6.77105×10^{-7}	5.45541×10^{-7}
200	6.74859×10^{-8}	3.93914×10^{-8}	7.34651×10^{-8}	4.16391×10^{-8}
400	4.45864×10^{-9}	3.40237×10^{-9}	1.10974×10^{-8}	6.05392×10^{-9}
800	1.14687×10^{-8}	5.23753×10^{-9}	3.24496×10^{-9}	1.39197×10^{-9}

Table 6: The error norms L_2 and L_∞ for $N = 100, 200, 400, 800$ and $\Delta t = 0.001$, 0.0001 at $T = 5$ on $x \in [0, 10]$ of Example 3.

$tf = 5$	$\Delta t = 0.001$		$\Delta t = 0.0001$	
N	L_2	L_∞	L_2	L_∞
100	6.78548×10^{-7}	5.46391×10^{-7}	6.79372×10^{-7}	5.46813×10^{-7}
200	7.72424×10^{-8}	4.27762×10^{-8}	7.94907×10^{-8}	4.34046×10^{-8}
400	1.55816×10^{-8}	7.14367×10^{-9}	1.82022×10^{-8}	7.87596×10^{-9}
800	1.71302×10^{-9}	1.16066×10^{-9}	4.38535×10^{-9}	1.92288×10^{-9}

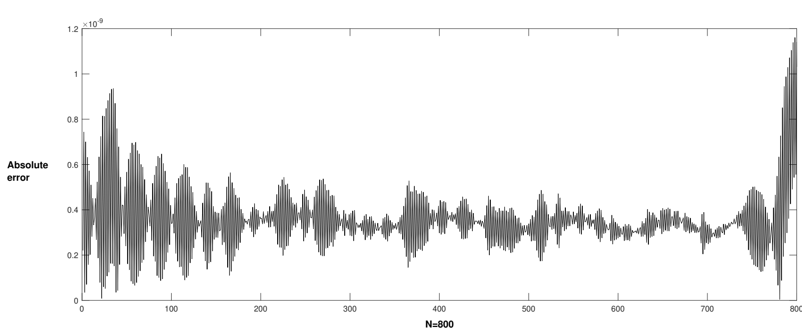

Figure 3: Error graphics of Problem III for values of $N = 800$, $\Delta t = 0.001$, $t_f = 5$, $k = 0.6$, $\lambda = 1$, $\beta = 0.01$, $\alpha = 3$ over $[-10, 10]$.

4 Conclusion

In this study, the proposed scheme resulting an implicit linear algebraic system has been successfully applied to obtain the approximate solutions of the Gardner-Kawahara equation. The error norms L_2 and L_∞ of the presented scheme are calculated. The three numerical experiments showed that the approximate solutions are in very good agreement with the analytical ones, and also the error norms are adequately small. The obtained results support that the numerical accuracy of the scheme is in consistency with its theoretical value and that the scheme is also unconditionally stable. In conclusion, the present numerical scheme, which can be easily implemented, produces accurate and reliable results. As a future work, the method can be successfully used to find approximate solutions of such combined partial differential equations that play an important role in describing nonlinear wave propagation encountered in physics and applied mathematics.

Competing interests

The authors have no competing interests

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