# Invariant measure of the backward Euler method for stochastic differential equations driven by $\alpha$-stable process

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**Abstract**

The backward Euler method is employed to approximate the invariant measure of a class of stochastic differential equations(SDEs) driven by $\alpha$-stable processes. The existence and uniqueness of the numerical invariant measure is proved. Then the numerical invariant measure is shown to converge to the underlying invariant measure. Numerical examples are provided to demonstrate the theoretical results.

**Key words**:$\alpha$-stable processes, stochastic differential equations, the backward Euler method, invariant measure.

## 1 Introduction

In recent years, asymptotic behaviour in the distribution sense of stochastic differential equations (SDEs) driven by the $\alpha$-stable processes have been attracting increasingly attention. Wang in [13] studied the exponential ergodicity of some SDEs with the additive $\alpha$-stable processes noise. Li and

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Ma investigated the $\alpha$-stable extended CIR model [10]. Zhang, Zhang and Tong discussed the ergodicity of a population model driven by the $\alpha$-stable processes [15].

The $\alpha$-stable processes belong to the more general processes, Lévy processes. However, compared to those Lévy processes with the finite measure, main difficulties brought in by the $\alpha$-stable process include the non-existence of the $p$th moment when $p \geq \alpha$. For detailed introductions of $\alpha$-stable processes, Lévy processes and corresponding SDEs, we refer the readers to [1, 12].

In this paper, we consider

$$dx(t) = \theta(x(t))dt + \varphi(x(t))dW(t) + \kappa dL(t).$$

It should be noted that there are many numerical methods to approximate the invariant measure of SDEs driven only by Browian motion (i.e. $\kappa = 0$), see [7, 9, 14].

To our best knowledge, there are few works on numerical approximations to invariant measure of SDEs driven by the $\alpha$-stable processes. Therefore, we fill in this gap by studying the numerical invariant measure generated by the backward Euler method. Although the classical Euler-Maruyama method is easy to use in computation and implementation, the numerical solutions of SDEs with super-linear coefficients may diverge to infinity in finite time. To tackle this drawback, we choose the backward Euler method.

The backward Euler method is a popular method that has been broadly investigated for SDEs driven by the Brownian motion only or jump processes with the finite Poisson measure, see [2, 4, 5, 11].

The main contributions of this paper are twofold.

- The existence and uniqueness of the invariant measure of the numerical solution generated by the backward Euler method are proved.
- The numerical invariant measure is shown to be convergent to the underlying one.

The rest of this paper is constructed in the following way. Section 2 contains some notations and mathematical preliminaries. The main results and proofs are presented in Section 3. Numerical examples are provided in Section 4.

## 2 Mathematical Preliminaries

Throughout this paper, Let $(\Omega, \mathcal{F}, \{\mathcal{F}_t\}_{t\geq t_0}, \mathbb{P})$ be a complete probability space with a filtration $\{\mathcal{F}_t\}_{t\geq t_0}$ satisfying the usual conditions that it is right continuous and increasing while $F_{t_0}$ contains all $\mathbb{P}$-null sets. Let $|x|$ denotes its Euclidean norm in $\mathbb{R}^d$. Let $W(t)$ be the a scalar Brownian motion defined on the probability space. The results derived in this paper can be extended to the case of the multi-dimensional Brownian motion straightforwardly.

A random variable $X$ is said to follow a stable distribution, denote by $X \sim \mathbb{S}_\alpha(\sigma, \beta, \mu)$, if it has characteristic function of the following form:

$$\phi_X(u) = \mathbb{E}\exp\{iuX\} = \begin{cases} \exp\{-\sigma^\alpha |u|^\alpha \left(1 - i\beta \operatorname{sgn}(u) \tan\dfrac{\alpha\pi}{2}\right) + i\mu u\}, \text{ if } \alpha \neq 1 \\ \exp\{-\sigma|u| \left(1 + i\beta \dfrac{2}{\pi} \operatorname{sgn}(u) \log|u|\right) + i\mu u\}, \text{ if } \alpha = 1, \end{cases}$$

where $\alpha \in (0, 2], \sigma \in (0, \infty), \beta \in [-1, 1]$, and $\mu \in (-\infty, \infty)$ are the index of stability, the scale, skewness, and location parameters, respectively. When $\mu = 0$, we say $X$ is strictly $\alpha$-stable($\alpha \neq 1$). If in addition $\beta = 0$, we call $X$ is symmetric $\alpha$-stable. We refer to [12] for more details on stable distributions.

$L(t)$ is a scalar $\alpha$-stable process ($\alpha \in (0, 2)$). There are equivalent definitions of the $\alpha$-stable processes, such as by using the Lévy-Khinchine formula or by using the Lévy-Itô decomposition (See for example Page 5 of [12]). Here, we use the following one as it is convenient for the simulation. A stochastic process $L(t)$ is called the strict $\alpha$-stable process if

- L(0)=0, a.s.;
- For any $m \in N$ and $0 \leq t_1 < t_2 < \cdots < t_m \leq T$, the random variables $L(t_0), L(t_1) - L(t_0), L(t_2) - L(t_1), \cdots, L(t_m) - L(t_{m-1})$ are independent;
- For any $0 \leq s < t < \infty$, $L(t) - L(s)$ follows $\mathbb{S}_\alpha((t-s)^{1/\alpha}, \beta, 0)$, where $\mathbb{S}_\alpha(\sigma, \beta, \mu)$ is a four-parameter stable distribution.

We investigate the SDEs driven by both the Brownian motion and the $\alpha$-stable processes of the form

$$dx(t) = \theta(x(t))dt + \varphi(x(t))dW(t) + \kappa dL(t) \tag{1}$$

with initial value $x(0) = x_0$, where $\theta, \varphi : \mathbb{R}^d \mapsto \mathbb{R}^d$ and $\kappa$ is a constant.

For any $k \in (0,2]$, define a metric $\mathbb{Z}_k(x,y)$ on $\mathbb{R}^d$ by

$$\mathbb{Z}_k(x,y) = |x-y|^k, x, y \in \mathbb{R}^d.$$

For $k \in (0,2]$, the Wasserstein distance between $\omega \in \mathcal{P}(\mathbb{R}^d)$ and $\omega^{'} \in \mathcal{P}(\mathbb{R}^d)$ is defined by

$$W_k(\omega, \omega^{'}) = \inf \mathbb{E}(\mathbb{Z}_k(x,y)),$$

where the infimum is taken over all pairs of random variables $x$ and $y$ on $\mathbb{R}^d$ with respect to the laws $\omega$ and $\omega'$.

Denote the transition probability kernel induced by the underlying solution, $x(t)$, by $\bar{\mathbb{P}}_t(\cdot,\cdot)$, with the notation $\delta_x\bar{\mathbb{P}}_t$ emphasizing the initial value $x$. Recall that a probability measure, $\pi(\cdot) \in \mathcal{P}(\mathbb{R}^d)$, is called an invariant measure of $x(t)$, if

$$\pi(B) = \int_{\mathbb{R}^d} \bar{\mathbb{P}}_t(x,B)\pi(dx)$$

holds for any $t \geq 0$ and any Borel set $B \subset \mathbb{R}^d$.

Now we impose the following the assumptions in this paper.

**Assumption 2.1** *There exists a constant $M_1 > 0$ such that*

$$\langle x-y, \theta(x)-\theta(y)\rangle \leq -M_1|x-y|^2$$

*for any $x, y \in \mathbb{R}^d$.*

**Assumption 2.2** *There exists a constant $M_2 > 0$ such that*

$$|\varphi(x)-\varphi(y)|^2 \leq M_2|x-y|^2$$

*for any $x, y \in \mathbb{R}^d$.*

**Assumption 2.3** *There exist constants $L_1 > 0$ and $\beta_1 > 0$ such that*

$$\langle x, \theta(x)\rangle \leq -L_1|x|^2 + \beta_1$$

*for any $x \in \mathbb{R}^d$.*

**Assumption 2.4** *There exist positive constants $L_2$ and $\beta_2$ such that*

$$|\varphi(x)|^2 \leq L_2|x|^2 + \beta_2$$

*for any $x \in \mathbb{R}^d$.*

**Lemma 2.5** *Let $L(t) \sim \mathbb{S}_\alpha(t^{1/\alpha}, \beta, 0)$ with $0 < \alpha < 2$ and $\beta = 0$ in the case $\alpha = 1$. Then, for every $q \in (0, \alpha)$, there is a constant $K_1$ such that (see the page 18 of [12])*

$$\mathbb{E}|L(t)|^q \leq K_1 t^{q/\alpha}.$$

The backward Euler method for (1) is defined by

$$Y_{i+1} = Y_i + \theta(Y_{i+1})h + \varphi(Y_i)\Delta W_i + \kappa \Delta L_i \tag{2}$$

with $Y_0 = x(0)$. where $\Delta W_i = W(t_{i+1}) - W(t_i)(t_i = ih)$ is the Brownian motion increment following the normal distribution with the $\Delta W_i \sim (0, h)$ and $\Delta L_i = L(t_{i+1}) - L(t_i)$ follows the stable distribution $\mathbb{S}_\alpha(h^{1/\alpha}, \beta, 0)$ for $i = 1, 2, 3, \cdots$

**Lemma 2.6** *Let Assumption 2.1 - 2.2 hold, the backward Euler method solution is well defined.*

*Proof.* It is useful to write (2) as

$$Y_{i+1} - \theta(Y_{i+1})h = Y_i + \varphi(Y_i)\Delta W_i + \kappa \Delta L_i$$

Define a function $G : \mathbb{R}^d \to \mathbb{R}^d$ by $G(x) = x - \theta(x)h$, then

$$\begin{aligned}
\langle x - y, G(x) - G(y)\rangle =& \langle x - y, x - \theta(x)h - (y - \theta(y)h)\rangle \\
=& (x - y)^2 - \langle x - y, \theta(x) - \theta(y)\rangle h \\
\geq& (x - y)^2 + M_1(x - y)^2 h \\
=& (x - y)^2(1 + M_1 h) \\
>& 0.
\end{aligned}$$

Then $G$ has its inverse function $G^{-1} : \mathbb{R}^d \to \mathbb{R}^d$, the backward Euler method solution can be represented as

$$Y_{i+1} = G^{-1}(Y_i + \varphi(Y_i)\Delta W_i + \kappa \Delta L_i).$$

Thus, the backward Euler method solution is well defined.

For any $x \in \mathbb{R}^d$ and any Borel set $B \subset \mathbb{R}^d$, define the one-step and the $i$-step transition probability kernels for the numerical solutions, respectively, by

$$\mathbb{P}(x, B) := \mathbb{P}(Y_1 \in B \big| Y_0 = x) \quad \text{and} \quad \mathbb{P}_i(x, B) := \mathbb{P}(Y_i \in B \big| Y_0 = x).$$

If $\Pi_h(\cdot) \in \mathcal{P}(\mathbb{R}^d)$ satisfies

$$\Pi_h(B) = \int_{\mathbb{R}^d} \mathbb{P}_i(x, B)\Pi_h(dx)$$

for any $i \geq 0$ and any Borel set $B \subset \mathbb{R}^d$, then $\Pi_h(\cdot)$ is called the numerical invariant measure of $Y_i$.

# 3 Main Results

## 3.1 The existence and uniqueness of the numerical invariant measure

We state our main theorem first and postpone the proof to the end of Section 3.1.

**Theorem 3.1** *Assume that Assumption 2.1 to 2.4 hold, then there exists a constant $h^{**} > 0$ small enough such that for any given $h \in (0, h^{**})$, the numerical solution $\{Y_i\}_{i\geq 0}$ generated by the backward Euler method has a unique invariant measure $\Pi_h$.*

To prove this theorem, we need the following two lemmas. The first one is the $p$-moment uniform boundedness of the numerical solution and the second one is that two numerical solutions starting from two different initial values will get arbitrary close in the mean square sense when the time variable gets large.

**Lemma 3.2** *Given Assumptions 2.3 and 2.4 hold, there exists $h^* \in (0, 1)$ such that for any $h \in (0, h^*)$, the numerical solution generated by the backward Euler method (2) is uniformly bounded, i.e.*

$$\mathbb{E}|Y_i|^p \leq C_1^i \mathbb{E}|Y_0|^p + \frac{C_2(1 - C_1^i)}{1 - C_1},$$

*for $p \in (0, 1)$, $i = 1, 2, ...,$ where*

$$C_1 := \frac{1 + (hL_2)^{p/2}}{(1 + 2hL_1)^{p/2}} \quad \text{and} \quad C_2 := \frac{\Big((2\beta_1)^{p/2} + \beta_2^{p/2} + \kappa^p K_1\Big) h^{p/2}}{(1 + 2hL_1)^{p/2}}.$$

*Proof.* Multiplying scalarly both sides of (2) with $Y_{i+1}$ yields

$$|Y_{i+1}|^2 = \langle Y_{i+1}, \theta(Y_{i+1})h\rangle + \langle Y_{i+1}, Y_i + \varphi(Y_i)\Delta W_i + \kappa \Delta L_i\rangle.$$

Applying Assumption 2.3 and the elementary, $\langle a, b\rangle \leq 0.5(|a|^2 + |b|^2)$ for $a, b \in \mathbb{R}^d$, we obtain

$$|Y_{i+1}|^2 \leq \left(\frac{1}{2} - hL_1\right)|Y_{i+1}|^2 + h\beta_1 + \frac{1}{2}|Y_i + \varphi(Y_i)\Delta W_i + \kappa\Delta L_i|^2.$$

For any fixed $p \in (0,1)$, taking the power of $p/2$ on both sides gives

$$\begin{aligned}\left(\frac{1}{2} + hL_1\right)^{p/2}|Y_{i+1}|^p &\leq \left(h\beta_1 + \frac{1}{2}|Y_i + \varphi(Y_i)\Delta W_i + \kappa\Delta L_i|^2\right)^{p/2} \\ &\leq (h\beta_1)^{p/2} + \left(\frac{1}{2}\right)^{p/2}|Y_i + \varphi(Y_i)\Delta W_i + \kappa\Delta L_i|^p, \end{aligned} \tag{3}$$

where the elementary inequality

$$\left(\sum_{k=1}^m a_k\right)^p \leq \sum_{k=1}^m |a_k|^p \tag{4}$$

for $a_k \in \mathbb{R},\ k = 1, 2, ..., m$ and $p \in (0,1)$ is used for the second inequality.

Thanks to Assumption 2.4, Lemma 2.5 and the elementary inequality (4), taking expectations on both sides of (3) results in

$$\mathbb{E}|Y_{i+1}|^p \leq \frac{1 + (hL_2)^{p/2}}{(1+2hL_1)^{p/2}}\mathbb{E}|Y_i|^p + \frac{\left((2\beta_1)^{p/2} + \beta_2^{p/2} + \kappa^p K_1\right)h^{p/2}}{(1+2hL_1)^{p/2}}, \tag{5}$$

where the facts, $\mathbb{E}|\Delta W_i|^p \leq h^{p/2}$ and $h^{p/2} > h^{p/\alpha}$ for $h \in (0,1)$, are used.

Since $L_1 > 0$, by choosing $h^* \in (0,1)$ sufficiently small we can have that for any $h \in (0, h^*)$

$$C_1 := \frac{1 + (hL_2)^{p/2}}{(1+2hL_1)^{p/2}} \in (0,1).$$

Now set $\varepsilon = 2L_1 - L_2$, we can choose $p$ sufficiently small for $(hL_2)^{\frac{p}{2}} < \frac{1}{2}pL_2h$. Then we have

$$(1+u)^{\frac{p}{2}} = 1 + \frac{p}{2}u + \frac{p(p-2)}{8}u^2 + \frac{p(p-2)(p-4)}{48}u^3. \tag{6}$$

Using (6),we have

$$(1+2hL_1)^{\frac{p}{2}} \geq 1 + phL_1 + ch^2 > 0. \tag{7}$$

By further reducing $h$, if necessary, so that

$$ch > \frac{1}{4}p\varepsilon, \qquad |p(L_1 + \frac{1}{4}\varepsilon)h| \leq \frac{1}{2}, \tag{8}$$

Using (7) and (8), we get

$$\frac{1+(hL_2)^{p/2}}{(1+2hL_1)^{p/2}} \leq \frac{1+\frac{1}{2}pL_2h}{1+p(L_1+\frac{1}{4}\varepsilon)h}.$$

Note that for any $u \in [-\frac{1}{2}, \frac{1}{2}]$,

$$\frac{1}{1+u} \leq 1 - u + 2u^2. \tag{9}$$

By further reducing $h$, if necessary, so that

$$2(hL_1 + \frac{1}{4}\varepsilon h)^2 + \frac{1}{2}pL_2h \times (-phL_1 - \frac{1}{4}p\varepsilon h + 2h^2(pL_1 + \frac{1}{4}p\varepsilon)^2) < \frac{3}{4}\varepsilon h.$$

We may ensure that

$$\begin{aligned}
\frac{1+(hL_2)^{p/2}}{(1+2hL_1)^{p/2}} <& [1+\frac{1}{2}pL_2h][1-phL_1-\frac{1}{4}p\varepsilon h + 2h^2(pL_1+\frac{1}{4}p\varepsilon)^2] \\
<& 1 + \frac{1}{2}pL_2h - phL_1 - \frac{1}{4}p\varepsilon h + 2(phL_1 + \frac{1}{4}\varepsilon h)^2 + \frac{1}{2}pL_2h \\
& \times(-phL_1 - \frac{1}{4}p\varepsilon h + 2h^2(pL_1 + \frac{1}{4}p\varepsilon)^2) \\
<& 1 + \frac{1}{2}phL_2 - phL_1 + p\varepsilon h \\
<& 1.
\end{aligned}$$

From (5), we derive

$$\mathbb{E}|Y_i|^p \leq C_1^i \mathbb{E}|Y_0|^p + \frac{C_2(1-C_1^i)}{1-C_1},$$

where

$$C_2 := \frac{\left((2\beta_1)^{p/2} + \beta_2^{p/2} + \kappa^p K_1\right) h^{p/2}}{(1+2hL_1)^{p/2}}.$$

Therefore, the assertion holds.

**Lemma 3.3** *Given Assumptions 2.1 and 2.2, there exists $h^{'} \in (0,1)$ such that for any $h \in (0, h^{'})$ the numerical solution generated by the backward Euler method* (2) *is uniformly bounded,*

$$\lim_{i\to\infty} \mathbb{E}|Y_i^x - Y_i^y| = 0,$$

*where $Y_i^x$ and $Y_i^y$ are numerical solution with two different intial values $x$ and $y$, but the driving noise terms are the same.*

*Proof.* Multiplying scalarly both sides of (2) with $Y_{i+1}$ yields

$$\begin{aligned}|Y_{i+1}^x - Y_{i+1}^y|^2 =& \langle Y_{i+1}^x - Y_{i+1}^y, \theta(Y_{i+1}^x) - \theta(Y_{i+1}^y)\rangle h \\ &+\langle Y_{i+1}^x - Y_{i+1}^y, Y_i^x - Y_i^y + (\varphi(Y_i^x) - \varphi(Y_i^y))\Delta W_i\rangle.\end{aligned}$$

Applying Assumption 2.1 and the elementary, $\langle a, b\rangle \leq 0.5(|a|^2 + |b|^2)$ for $a, b \in \mathbb{R}^d$, we obtain

$$|Y_{i+1}^x - Y_{i+1}^y|^2 \leq \left(\frac{1}{2} - hM_1\right)|Y_{i+1}^x - Y_{i+1}^y|^2 + \frac{1}{2}|Y_i^x - Y_i^y + (\varphi(Y_i^x) - \varphi(Y_i^y))\Delta W_i|^2.$$

$$(1 + 2hM_1)\,|Y_{i+1}^x - Y_{i+1}^y|^2 \leq |Y_i^x - Y_i^y + (\varphi(Y_i^x) - \varphi(Y_i^y))\Delta W_i|^2.$$

Hence we obtain

$$|Y_{i+1}^x - Y_{i+1}^y|^2 \leq \frac{1}{1 + 2hM_1}|Y_i^x - Y_i^y + (\varphi(Y_i^x) - \varphi(Y_i^y))\Delta W_i|^2$$

For any fixed $p \in (0, 1)$, taking the power of $p/2$ on both sides and then using (6) gives

$$|Y_{i+1}^x - Y_{i+1}^y|^p \leq \frac{1}{(1 + 2hM_1)^{\frac{p}{2}}}|Y_i^x - Y_i^y + (\varphi(Y_i^x) - \varphi(Y_i^y))\Delta W_i|^p. \tag{10}$$

Using Assumption 2.2 and the elementary inequality (4), taking expectations on both sides of (10) results in

$$\mathbb{E}|Y_{i+1}^x - Y_{i+1}^y|^p \leq \frac{1 + (hM_2)^{p/2}}{(1 + 2hM_1)^{p/2}}\mathbb{E}|Y_i^x - Y_i^y|^p,$$

where the facts, $\mathbb{E}|\Delta W_i|^p \leq h^{p/2}$ and $h^{p/2} > h^{p/\alpha}$ for $h \in (0, 1)$, are used.

$$C_3 := \frac{1 + (hM_2)^{p/2}}{(1 + 2hM_1)^{p/2}} \in (0, 1).$$

In the same way as in the proof of Lemma 3.1, We set $\varepsilon = 2M_1 - M_2$ and choose sufficiently small $h^{'}$ and $p^*$ such that for any $p \in (0, p^*)$ and $h \in (0, h^{'})$, we can show that

$$\mathbb{E}(|Y_i^x - Y_i^y|^p) \leq C_3^i \mathbb{E}(|Y_0^x - Y_0^y|^p).$$

Therefore, the assertion holds.

Now we are ready to prove Theorem 3.1.

*Proof of Theorem 3.1.* For each integer $n \geq 1$ and any Borel set $B \subset \mathbb{R}^d$, define the measure

$$\omega_n(B) = \frac{1}{n} \sum_{i=0}^{n} \mathbb{P}(Y_i \in B).$$

Lemma 3.2 together with the Chebyshev inequality yields that the measure sequence $\{\omega_n\}_{n\geq 1}$ is tight. Then, a subsequence that converges to an invariant measure can be extracted. This proves the existence of the numerical invariant measure.

Assume $\Pi_h^x$ and $\Pi_h^y$ are invariant measure of $Y_i^x$ and $Y_i^y$, then

$$W_k(\Pi_h^x, \Pi_h^y) = W_k(\Pi_h^x \mathbb{P}_i, \Pi_h^y \mathbb{P}_i) \leq \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \Pi_h^x(dx) \Pi_h^y(dy) W_k(\delta_x \mathbb{P}_i, \delta_y \mathbb{P}_i).$$

From Lemma 3.3, we have

$$W_k(\delta_x \mathbb{P}_i, \delta_y \mathbb{P}_i) \leq \left(C_3^i \mathbb{E}|x - y|^p\right)^{k/p} \to 0, \text{ as } i \to \infty.$$

Therefore, we have

$$\lim_{i\to\infty} W_k(\Pi_h^x, \Pi_h^y) = 0,$$

which indicates the uniqueness of the invariant measure.

## 3.2 Convergence of the numerical invariant measure to the true one

**Theorem 3.4** *Given Assumptions 2.1 to 2.4, for any given $h \in (0, h^{**})$, $h^{**} := h^{'} \wedge h^*$, there exists a constant $C_3$ such that*

$$W_k(\pi, \Pi_h) \leq C_3 h^k,$$

*where $k \in (0, 2]$.*

*Proof.* Note that for any $k \in (0,2]$

$$W_k(\delta_x \bar{\mathbb{P}}_{ih}, \pi) \le \int_{\mathbb{R}^d} \pi(dy)\ W_k(\delta_x \bar{\mathbb{P}}_{ih}, \delta_y \bar{\mathbb{P}}_{ih}),$$

and

$$W_k(\delta_x \mathbb{P}_{ih}, \Pi_h) \le \int_{\mathbb{R}^d} \Pi_h(dy)\ W_k(\delta_x \mathbb{P}_{ih}, \delta_y \mathbb{P}_{ih}).$$

Due to the existence and uniqueness of the invariant measure for the underlying SDE (2)[13] and Theorem 3.1, for the given $h \in (0, h^{**})$, one can choose $i$ sufficiently large such that

$$W_k(\delta_x \bar{\mathbb{P}}_{ih}, \pi) \le \frac{C_3}{3} h^k \quad \text{and} \quad W_k(\delta_x \mathbb{P}_{ih}, \Pi_h) \le \frac{C_3}{3} h^k.$$

In addition, for the chosen $i$, by the strong convergence rate of the backward method [6], we have that

$$W_k(\delta_x \bar{\mathbb{P}}_{ih}, \delta_x \mathbb{P}_{ih}) \le \frac{C_3}{3} h^k.$$

Therefore, the proof is completed by the triangle inequality.

# 4 Numerical Examples

**Example 4.1** *We consider the $\alpha$-stable Ornstein-Uhlenbeck(OU) process*

$$dx(t) = -ax(t)dt + bdL(t), \quad \textit{with} \quad x(0) = 8.$$

Set $\alpha = 1.8$ and $a = b = 2$. We simulate numerical solution on 1000 paths by the backward Euler method. In this case, the invariant measure is $\mathbb{S}_\alpha\left(b\left(\frac{1}{\alpha a}\right)^{1/\alpha}, 0, 0\right)$ (See [3]).

In order to measure the difference between empirical distributions of numerical solution and explicit solution more clearly, the Kolmogorov-Smirnov test (K-S test) [8] is used. It can be seen clearly from the left plot in Figure 1 that the empirical distributions at $t = 0.1$, $t = 0.3$ and $t = 0.8$ are quite different from the true one. But when time gets large, for example $t = 2$, the difference between the empirical and true distributions is quite close.

From the right plot in Figure 1, the difference in distribution of numerical solutions and true solutions decrease as time increasing. The reason that the difference seems not tend to zero is due to the number of sample paths and the step-size of the backward Euler method.

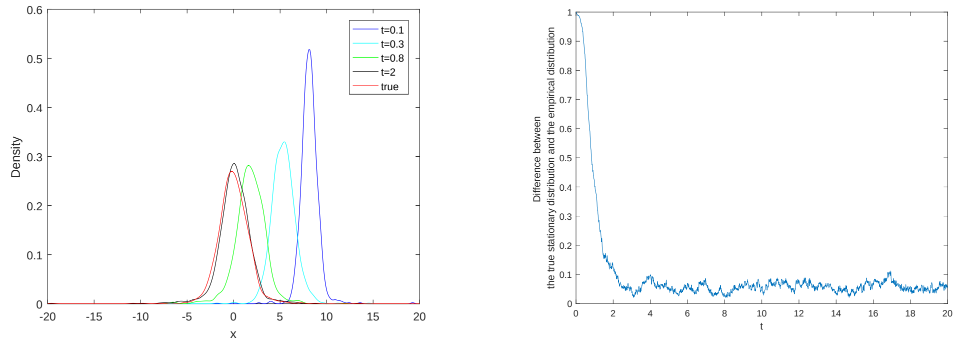


Figure 1: Left: The empirical distributions and the true one. Right: The difference in distribution between numerical solutions and the true solution.


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