Received: 00 Month 0000 | Revised: 00 Month 0000 | Accepted: 00 Month 0000
DOI: xxx/xxxx

ARTICLE TYPE

On the discontinuous solutions of explicit neutral delay differential equations

Ali Davari*[1] | Mohammad Maleki[2]

[1] Department of Mathematics, Khansar Campus, University of Isfahan, Isfahan, Iran
[2] Department of Mathematics,Shahid Mohajer branch, Technical and Vocational University, Isfahan, Iran

Correspondence
*Ali Davari, Email: a_davari@sci.ui.ac.ir

Summary

This paper considers explicit neutral delay differential equations (NDDE) with piecewise continuous initial functions. We explain how the discontinuities in the solutions arise and present a perturbing scheme, in combination with an adaptive Legendre–Gauss–Radau collocation method, to deal with this type of problems computationally. The pointwise and mean convergence of the continuous solution of the perturbed NDDE to the discontinuous solution of the original NDDE are proved. Our new method for discontinuous NDDEs and the rigorous theoretical analysis provided are particularly important since explicit NDDEs have received little attention in the literature. Numerical results are given to show that the proposed method can be implemented in an efficient and accurate manner.

KEYWORDS:
Adaptive collocation method; Convergence; Discontinuous solutions; Neutral delay systems; Perturbed initial condition; Spectral accuracy.

1 | INTRODUCTION

Delay differential equations (DDE) are useful in modeling of population dynamics, the spread of infection diseases, two-body problems of electrodynamics, blood cell production models, etc,[1,2,3,4,5,6,7]. The lags can represent gestation times, incubation periods, transport delays, or they can simply lump complicated biological processes together, accounting only for the time required for these processes to occur. Given the fact that the time delay could cause a stable equilibrium to become unstable and cause the populations to fluctuate, DDEs exhibit much more complicated dynamics than ODEs.

Consider the following DDE with multiple time dependent delays

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}U(t) = f\Big(&t, U(t), U(t-\tau_1(t)), \dots, U(t-\tau_m(t)),\\ &\frac{\mathrm{d}}{\mathrm{d}t}U(t-\tau_{m+1}(t)), \dots, \frac{\mathrm{d}}{\mathrm{d}t}U(t-\tau_{m+n}(t))\Big), \quad t_0 \leqslant t \leqslant T,\end{aligned} \tag{1}$$

and initial functions

$$U(t) = \phi(t), \quad \frac{\mathrm{d}}{\mathrm{d}t}U(t) = \frac{\mathrm{d}}{\mathrm{d}t}\phi(t), \quad t_{-1} \leqslant t < t_0, \tag{2}$$

where f and ϕ are given functions which satisfy certain conditions with certain properties, T is a positive constant, $\tau_q(t) \geqslant 0$, $1 \leqslant q \leqslant m+n$ are time dependent delays and $t_{-1} = \inf_{t_0 \leqslant t \leqslant T}\{t - \tau_q(t)\}_q$. The existence and uniqueness of the solution to the model (1)–(2) are described in detail in[8,9,10]. The model (1) with derivative delay terms is called *explicit* neutral delay differential

AMS subject classifications: 34K40, 65L04, 65L06, 65L20, 65L60, 65L70, 41A05.

equation (NDDE), otherwise it is called retarded differential equation (RDE). Another well-known and widely studied class of NDDEs is *implicit* NDDEs, a form frequently called Hale's form [11].

In the context of population dynamics, the model problem (1) can be obtained, e.g., from the balance laws of the age-structured population dynamics, assuming that the birth rates and death rates, as functions of age, are piecewise constant. The delay arises naturally from biology as the age-at-maturity of individuals. This modeling approach has also applications in population dynamics of isolated populations [12], interplay of predators and prey [13], and tumor modeling [14].

It is well known that the DDEs solutions can behave quite differently from ODEs. For instance, the DDEs characteristic equation, in contrast to ODEs, may have infinitely many roots and a DDE can have solutions that oscillate rapidly. Discontinuity of NDDEs' solutions makes the bifurcation analyses and numerical treatment of such equations, much more complicated than those of RDEs. The solution of RDEs becomes smoother when the integration proceeds in successive subintervals, but it is not necessarily the case for NDDEs [2,3].

Many numerical methods are proposed for solving DDEs using the Runge–Kutta method. These methods are developed based on Taylor's expansions or quadrature formulas [1,15,16,17,18,19,20,21,22,23,24,25]. Another class of efficient methods for DDEs is the class of spectral methods. A spectral method employs global orthogonal polynomials as trial functions and it provides exceedingly accurate numerical results for smooth solutions [26,27]. So far, there are few numerical methods with the spectral accuracy for DDEs and specifically for NDDEs [28,29,30,31,32,33,34,35].

In [29], we presented an adaptive Legendre–Gauss-Radau (LGR) collocation method to solve RDEs and NDDEs with constant or time-dependent delays. In the current work, we present a modified version of this method for approximating discontinuous solution of explicit NDDEs with a discontinuous initial function. A perturbed continuous problems is first derived by perturbing the initial function. Then, a hybrid perturbation–collocation scheme is developed. The pointwise and mean convergence of the perturbed continuous solution to the discontinuous solution of the original are proved.

The remainder of the article is organized as follows: In Section 2, some properties of shifted Legendre polynomials are reviewed. Section 3, describes the distinctions between NDDEs with continuous and discontinuous initial function. I Section 4, the theoretical results regarding pointwise and mean convergence properties of the perturbed problem are given. The hybrid perturbation–collocation method is described in Section 5. Section 6 is for some numerical results and justifies our theoretical analysis. Finally, conclusions are given in Section 7.

2 | PROPERTIES OF SHIFTED LEGENDRE POLYNOMIALS

This section is devoted to giving some mathematical preliminaries required for our subsequent development. The shifted Legendre polynomial of degree n in the interval $I = [a, b]$ is defined by

$$L_{I,n}(t) = L_n\left(\frac{2t}{b-a} - \frac{b+a}{b-a}\right), \quad n = 0, 1, 2, \dots$$

where $L_n(t)$ is the standard Legendre polynomial of degree n [26].

In particular,

$$L_{I,0}(t) = 1, \quad L_{I,1}(t) = \frac{2t}{b-a} - \frac{b+a}{b-a}, \quad L_{I,2}(t) = \frac{1}{2}\left(3(\frac{2t}{b-a} - \frac{b+a}{b-a})^2 - 1\right)$$

The shifted Legendre polynomials satisfy the following three term recurrence relation

$$L_{I,n+1}(t) = \frac{2n+1}{n+1}\left(\frac{2t}{b-a} - \frac{b+a}{b-a}\right) L_{I,n}(t) - \frac{n}{n+1} L_{I,n-1}(t), \quad n = 1, 2, 3, \dots .$$

We then have

$$L_{I,n}(a) = (-1)^n, \qquad L_{I,n}(b) = 1. \tag{3}$$

and

$$\frac{\mathrm{d}}{\mathrm{dt}} L_{I,n}(t) = \frac{n+1}{b-a} J^{(1,1)}_{I,n-1}(t). \tag{4}$$

in which, we have $L_{I,n}(t) = J^{(0,0)}_{I,n}(t)$ and $J^{(\alpha,\beta)}_{I,n}$ is the shifted Jacobi polynomial of degree n.

The set of polynomials $\{L_{I,n}(t)\}$ is a complete orthogonal system in the space $L^2(I)$, namely,

$$\int_a^b L_{I,m}(t) L_{I,n}(t)\, \mathrm{d}t = \frac{b-a}{2n+1}\delta_{m,n},$$

where $\delta_{m,n}$ is the Kronecker function symbol. Then, the formal series of a function $u \in L^2(I)$ in terms of the system $\{L_{I,n}(t)\}$ and the expansion coefficients are defined as

$$u(t) = \sum_{n=0}^{\infty} u_{I,n} L_{I,n}(t), \qquad u_{I,n} = \frac{2n+1}{b-a} \int_a^b u(t) L_{I,n}(t)\,\mathrm{d}t. \tag{5}$$

We denote by t_j, $0 \leqslant j \leqslant N$, the nodes of the standard LGR interpolation on the interval $[-1, 1)$. (The nodes of the standard LGR interpolation on the interval $[-1, 1)$ is denoted by t_j, $0 \leqslant j \leqslant N$.) In particular, $t_0 = -1$ and $t_N < 1$. The corresponding Christoffel numbers are w_j, $0 \leqslant j \leqslant N$.(w_j, $0 \leqslant j \leqslant N$ are the corresponding Christoffel numbers .)

Then the nodes of the shifted LGR interpolation on the interval $[a, b)$ are the distinct zeros of $L_{I,N}(t) + L_{I,N+1}(t)$, denoted by $t_{I,j}$, $0 \leqslant j \leqslant N$. In particular, $t_{I,0} = a$. Clearly, the nodes $t_{I,j}$ can be obtained by shifting the nodes t_j and the corresponding Christoffel numbers are $w_{I,j} = \frac{b-a}{2} w_j$, $0 \leqslant j \leqslant N$.

Let $\mathcal{P}_N$ be the set of polynomials of degree at most N. Thanks to the property of the standard LGR quadrature, it follows that for any $p \in \mathcal{P}_{2N}$ on I,

$$\begin{aligned} \int_a^b p(t)\,\mathrm{d}t &= \frac{b-a}{2} \int_{-1}^{1} p\left(\frac{b-a}{2}t + \frac{b+a}{2}\right) \mathrm{d}t \\ &= \frac{b-a}{2} \sum_{j=0}^{N} w_j p\left(\frac{b-a}{2}t_j + \frac{b+a}{2}\right) = \sum_{j=0}^{N} w_{I,j} p(t_{I,j}). \end{aligned} \tag{6}$$

Let $\langle u, v\rangle_I$ and $\|u\|_I$ be the inner product and the norm of space $L^2(I)$, respectively. We also define the following discrete inner product and norm,

$$\langle u, v\rangle_{I,N} = \sum_{j=0}^{N} w_{I,j} u(t_{I,j}) v(t_{I,j}), \qquad \|u\|_{I,N} = \sqrt{\langle u, u\rangle_{I,N}}. \tag{7}$$

Due to (6), for any $pg \in \mathcal{P}_{2N}$ and $g \in \mathcal{P}_N$,

$$\langle g, p\rangle_I = \langle g, p\rangle_{I,N}, \qquad \|g\|_I = \|g\|_{I,N}. \tag{8}$$

For any $U \in C(I)$, the shifted LGR interpolation $\mathcal{I}_N U(t) \in \mathcal{P}_N$ is determined uniquely by

$$\mathcal{I}_N U(t_{I,j}) = U(t_{I,j}), \quad 0 \leqslant j \leqslant N. \tag{9}$$

Because of (8), for any $g \in \mathcal{P}_N$,

$$\langle \mathcal{I}_N U, g\rangle_I = \langle \mathcal{I}_N U, g\rangle_{I,N} = \langle U, g\rangle_{I,N}. \tag{10}$$

This shows that the interpolant $\mathcal{I}_N U$ is the orthogonal projection of u upon $\mathcal{P}_N$ on I with respect to the discrete inner product (7). The interpolation $\mathcal{I}_N U(t)$ in the interval I can be expanded as

$$\mathcal{I}_N U(t) = \sum_{n=0}^{N} \widetilde{u}_{I,n} L_{I,n}(t), \tag{11}$$

and with the aid of (5) and (10) we obtain

$$\widetilde{u}_{I,n} = \frac{2n+1}{b-a} \langle \mathcal{I}_N U, L_{I,n}\rangle_I = \frac{2n+1}{b-a} \langle U, L_{I,n}\rangle_{I,N}, \quad 0 \leqslant n \leqslant N. \tag{12}$$

3 | DISTINCTION BETWEEN CONTINUOUS AND DISCONTINUOUS INITIAL FUNCTIONS

In[29] we developed an adaptive Legendre-Gauss-Radau collocation method by discretizing the DDE (1)–(2) with a continuous initial function $\phi(t)$ and employing a dynamic set of mesh points. In the next subsection, we review this method briefly. For the sake of simplicity, in the model (1)-(2) we assume that $m = n = 1$. Thus, we consider the NDDE

$$\frac{\mathrm{d}}{\mathrm{d}t} U(t) = f\left(t, U(t), U(t-\tau_1(t)), \frac{\mathrm{d}}{\mathrm{d}t} U(t-\tau_2(t))\right), \quad t_0 \leqslant t \leqslant T, \tag{13}$$

$$U(t) = \phi(t), \quad \frac{\mathrm{d}}{\mathrm{d}t} U(t) = \frac{\mathrm{d}}{\mathrm{d}t} \phi(t), \quad t_{-1} \leqslant t < t_0. \tag{14}$$

Next, we explain a perturbation scheme to deal with discontinuous initial functions.

3.1 | Continuous initial function

Assume that $\phi(t) \in C^l$ with $l \geqslant 0$, and $\frac{\mathrm{d}^{l+1}}{\mathrm{d}t^{l+1}}\phi(t)$ is piecewise continuous. By this we mean that $\frac{\mathrm{d}^{l+1}}{\mathrm{d}t^{l+1}}\phi(t)$ is continuous on the interval $[t_{-1}, t_0]$, except at a finite number of points $\{t_0-\sigma_s\}_{s=1}^{S}$ at which $\frac{\mathrm{d}^{l+1}}{\mathrm{d}t^{l+1}}\phi(t)$ has jump discontinuity with continuity from the right. It is well known that the accuracy of the numerical solution of the problem (13)–(14) is highly dependent on the considered mesh due to loss of regularity of the solution even when the functions f, τ_q and ϕ are smooth[29,25]. Therefore, it is necessary to define a dynamic mesh, $\boldsymbol{M}$, that includes the set of breaking points. For constant and time-dependent delays considered in this work, the set of breaking points can be constructed in advance. To this end we start with $\boldsymbol{B} = \{t_0, t_0 - \sigma_1, \dots, t_0 - \sigma_S\}$. Then, assuming each delay is monotonic, we solve each scalar equation

$$t - \tau_q(t) = \zeta, \quad q = 1, 2,$$

by an appropriate iterative solver, where ζ is a computed breaking point. If the obtained solutions are less than $\boldsymbol{T}$ we add them to the set $\boldsymbol{B}$. In the first step of the method, we set $\boldsymbol{M} = \boldsymbol{B}$ and in the subsequent steps it may be updated based on a certain mesh refinement algorithm described below.

Suppose that we have sorted the set of mesh points as $\boldsymbol{M} = \{\zeta_k\}_k$. Let $\boldsymbol{I}^{(k)} = [\zeta_{k-1}, \zeta_k)$, $h_k = \zeta_k - \zeta_{k-1}$, $U_k(t)$ be the smooth local solution of the problem (13)–(14) on the subinterval $\boldsymbol{I}^{(k)}$ and $t_{I^{(k)},j} := \hat{t}_{k,j}$, $0 \leqslant j \leqslant N_k$ be the shifted LGR quadrature points in the subinterval $\boldsymbol{I}^{(k)}$. By considering the $(k-1)$-dimensional multi-index $\widetilde{N} := (N_1, \dots, N_{k-1})$, we define the function $\Psi_{\widetilde{N}} U$ as

$$\Psi_{\widetilde{N}} U(t) = \begin{cases} \phi(t), & t_{-1} \leqslant t \leqslant t_0, \\ \mathcal{I}_{N_i} U_i(t), & t \in \boldsymbol{I}^{(i)}, \quad 1 \leqslant i \leqslant k-1. \end{cases} \tag{15}$$

In the k^{th} step, the LGR collocation method for solving (13)–(14) is to seek $U_k^{N_k}(t) \in \mathcal{P}_{N_k}(\boldsymbol{I}^{(k)})$, such that

$$\begin{cases} \frac{\mathrm{d}}{\mathrm{d}t} U_k^{N_k}(\hat{t}_{k,j}) = f\left(\hat{t}_{k,j}, U_k^{N_k}(\hat{t}_{k,j}), u^{\widetilde{N}}(\hat{t}_{k,j} - \tau_1(\hat{t}_{k,j})), \frac{\mathrm{d}}{\mathrm{d}t} u^{\widetilde{N}}(\hat{t}_{k,j} - \tau_2(\hat{t}_{k,j}))\right), & 1 \leqslant j \leqslant N_k, \\ U_k^{N_k}(\zeta_{k-1}) = u^{\widetilde{N}}(\zeta_{k-1}), \end{cases} \tag{16}$$

where $u^{\widetilde{N}}(t)$ is the piecewise polynomial approximation of $\Psi_{\widetilde{N}} U(t)$ obtained in the preceding steps. Note that, the possible jump discontinuities in the first derivative of $U(t)$ at the breaking points is not an issue for approximating the solution of NDDEs, because the LGR scheme avoids collocation at breaking points.

We next describe the numerical implementation for (16). We expand the collocation solution as

$$U_k^{N_k}(t) = \sum_{n=0}^{N_k} \widetilde{u}_{I^{(k)},n}^{N_k} L_{I^{(k)},n}(t), \quad t \in \boldsymbol{I}^{(k)}. \tag{17}$$

Since $U_k^{N_k}(t) L_{I^{(k)},n}(t) \in \mathcal{P}_{2N_k}$, by integrating it over the interval $\boldsymbol{I}^{(k)}$ and using (5) and (8) it can be verified that

$$\begin{aligned} \widetilde{u}_{I^{(k)},n}^{N_k} &= \frac{2n+1}{h_k} \langle U_k^{N_k}, L_{I^{(k)},n} \rangle_{I^{(k)}} = \frac{2n+1}{h_k} \langle U_k^{N_k}, L_{I^{(k)},n} \rangle_{I^{(k)},N_k} \\ &= \frac{2n+1}{h_k} \sum_{j=0}^{N_k} U_k^{N_k}(\hat{t}_{k,j}) L_{I^{(k)},n}(\hat{t}_{k,j}) w_{I^{(k)},j}, \quad 0 \leqslant n \leqslant N_k. \end{aligned} \tag{18}$$

Then, by virtue of (4), we deduce that

$$\frac{\mathrm{d}}{\mathrm{d}t} U_k^{N_k}(t) = \frac{1}{h_k} \sum_{n=1}^{N_k} (n+1) \widetilde{u}_{I^{(k)},n}^{N_k} J_{I^{(k)},n-1}^{(1,1)}(t), \quad t \in \boldsymbol{I}^{(k)}. \tag{19}$$

Furthermore, using (3), a direct calculation shows $L_{I^{(k)},n}(\zeta_{k-1}) = (-1)^n$. Therefore, we have from (16) and (17) with $t = \zeta_{k-1}$ that

$$\sum_{n=0}^{N_k} (-1)^n \widetilde{u}_{I^{(k)},n}^{N_k} = u^{\widetilde{N}}(\zeta_{k-1}). \tag{20}$$

Consequently, we use (17)–(20) to obtain from (16) for $1 \leqslant j \leqslant N_k$ that

$$\begin{cases} \dfrac{1}{h_k} \displaystyle\sum_{n=1}^{N_k} (n+1)\widetilde{u}^{N_k}_{I^{(k)},n} J^{(1,1)}_{I^{(k)},n-1}(\hat{t}_{k,j}) = f\left(\hat{t}_{k,j}, U^{N_k}_k(\hat{t}_{k,j}), u^{\widetilde{N}}(\hat{t}_{k,j} - \tau_1(\hat{t}_{k,j})), \dfrac{\mathrm{d}}{\mathrm{d}t} u^{\widetilde{N}}(\hat{t}_{k,j} - \tau_2(\hat{t}_{k,j}))\right), \\ \displaystyle\sum_{n=0}^{N_k} (-1)^n \widetilde{u}^{N_k}_{I^{(k)},n} = u^{\widetilde{N}}(\zeta_{k-1}). \end{cases} \tag{21}$$

When the function f is nonlinear, we first use certain iteration process to solve (21) and obtain $\widetilde{u}^{N_k}_{I^{(k)},n}$, $0 \leqslant n \leqslant N_k$. Finally, we add $U^{N_k}_k(t)$ as a new piece to $u^{\widetilde{N}}(t)$ and we use $U^{N_k}_k(\zeta_k)$ as the appoximate initial value to be used in the next step. For the step size control strategy regarding this method, refer to [29].
The spectral accuracy of numerical solutions can be confirmed by comparing $u^{N_k}_k(t)$ with the interpolation approximation $I_{N_k} U_k(t)$. The following theorem establishes the convergence rate of the multistep collocation scheme (16) for NDDEs.

Theorem 1. If f in the scheme (16) satisfies the Lipschitz condition with a Lipschitz constant $\gamma \geqslant 0$ as $|f(t, y_1, y_2, y_3) - f(t, z_1, z_2, z_3)| \leqslant \gamma \left(|y_1 - z_1| + |y_2 - z_2| + |y_3 - z_3|\right)$ and α_k with $k \geqslant 1$ be a positive constant such that $\sqrt{24} h_k \gamma \leqslant \alpha_k < 1$, then for any $U_k \in H^r(I^{(k)})$ and integer $r \geqslant 2$, we have

$$\left\| U_k - u^{N_k}_k \right\|^2_{I^{(k)}} \leqslant c_{\alpha_k} \sum_{i=1}^{k} \sum_{l=min\{r,N_i+1\}}^{r} N_i^{3-2r} h_i^{2l-2} \left\| U_i^{(l)} \right\|^2_{I^{(i)}}, \tag{22}$$

$$\left| U_k(\zeta_k) - u^{N_k}_k(\zeta_k) \right|^2 \leqslant c_{\alpha_k} \sum_{i=1}^{k} \sum_{l=min\{r,N_i+1\}}^{r} N_i^{3-2r} h_i^{2l-3} \left\| U_i^{(l)} \right\|^2_{I^{(i)}}, \tag{23}$$

where c_{α_k} is a positive constant depending only on α_k and r.

Proof. Ref to [29]

3.2 | Discontinuous initial function

In [11] the authors illustrate and explain how discontinuities in the solution of *implicit* NDDEs arise. Also, an efficient strategy for approximating discontinuous solutions of a type of differential-algebraic equations is discussed in [25].

As discussed in [11], this discontinuity in the initial function propagates and causes the solution itself to be discontinuous, too.

Let the initial function $\phi(t)$ in the explicit NDDE (13) be continuously differentiable on $[t_{-1}, t_0]$ with the exception of the distinct ordered points $\{t_0 - \sigma_1, \dots, t_0 - \sigma_S\} \subset [t_{-1}, t_0]$, at each of which $\phi(t)$ is discontinuous but continuous from the right. The lack of smoothness in $\phi(t)$ propagates forward to the sequence of points of the set B. Similarly to [36], we give the following definition.

Definition 1. *A possibly discontinuous function $U(t) \equiv U(\phi; t)$, $t \in [t_{-1}, T)$ is a solution of the explicit NDDE (13) with the above mentioned initial function if*
(i) $U(t) = \phi(t)$ for $t \in [t_{-1}, t_0]$;
(ii) $U(t)$ satisfies (13) for $[t_0, T)\backslash B$ $\left(\text{i.e., } \frac{\mathrm{d}}{\mathrm{d}t}U(t) \text{ represents the conventional two-sided derivative for all } t \in [t_0, T)\backslash B\right)$;
(iii) at those points of B which (13) is not satisfied we interpret $\frac{\mathrm{d}}{\mathrm{d}t}U(t)$ as the one-sided right derivative.

Authors of [11] have proved the existence and uniqueness of discontinuous solutions of NDDEs in Hale's form. As far as we know, similar results for explicit NDDEs is not yet available in the literature and such a study is beyond the scope of this paper. So, we assume that the problem considered here has a unique discontinuous solution.

To approximate the discontinuous solution of the NDDE (13) and to govern the size of the jumps at the breaking points, we combine the perturbing scheme proposed in [11] with the adaptive collocation scheme explained before. We replace the discontinuous $\phi(t)$ by a sufficiently smooth approximation $\phi_\delta(t)$ such that $U(\phi_\delta; t)$ is continuous and provides a good approximation $U_\delta(t) := U(\phi_\delta; t) \approx U(t)$ for each $t \in [t_{-1}, T)$. For simplicity of statement, we suppose that $S = 1$, i.e., $\phi(t)$ has only one jump discontinuity at a point $t_0 - \sigma \in [t_{-1}, t_0]$. Evidently, this discontinuity in $\phi(t)$ propagates forward to the sequence of breaking points. For $S \geqslant 2$ the results can be extended with slight modifications.

Now, choose a proper parameter $\delta \in (0,1)$ and consider the perturbed initial function $\phi_\delta(t)$ that equals to $\phi(t)$ except in the small interval $(t_0-\sigma-\delta, t_0-\sigma)$ in which $\phi_\delta(t)$ is the cubic polynomial satisfying

$$\phi_\delta(t_0-\sigma-\delta) = \phi(t_0-\sigma-\delta), \qquad \phi_\delta(t_0-\sigma) = \lim_{t\to t_0-\sigma^+} \phi(t),$$

and

$$\frac{\mathrm{d}}{\mathrm{d}t}\phi_\delta(t_0-\sigma-\delta) = \frac{\mathrm{d}}{\mathrm{d}t}\phi(t_0-\sigma-\delta), \qquad \frac{\mathrm{d}}{\mathrm{d}t}\phi_\delta(t_0-\sigma) = \lim_{t\to t_0-\sigma^+} \frac{\mathrm{d}}{\mathrm{d}t}\phi(t).$$

Clearly, $\phi_\delta(t) \in C^1[t_{-1}, t_0]$ though its derivative is large over the interval $(t_0-\sigma-\delta, t_0-\sigma)$ for small δ. Moreover, it is straightforward to show that

$$\lim_{\delta\to 0}\phi_\delta(t) = \phi(t), \qquad \lim_{\delta\to 0}\frac{\mathrm{d}}{\mathrm{d}t}\phi_\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\phi(t),$$

pointwise for all $t \in [t_{-1}, t_0-\sigma) \cup [t_0-\sigma, t_0)$. This convergence is clearly not uniform. However, we have

$$\lim_{\delta\to 0}\left\|\phi_\delta(t) - \phi(t)\right\|_{L^1[t_{-1},t_0]} = 0,$$

i.e., convergence in the mean; but

$$\lim_{\delta\to 0}\left\|\frac{\mathrm{d}}{\mathrm{d}t}\phi_\delta(t) - \frac{\mathrm{d}}{\mathrm{d}t}\phi(t)\right\|_{L^1[t_{-1},t_0]} = J_0,$$

where J_0 is the size of the jump at $t = t_0-\sigma$.

4 | THEORETICAL RESULTS

In this section, we show that $U_\delta(t)$ converges pointwise and converges in the mean to $U(t)$ on bounded intervals $[t_0, T)$ when $\delta \to 0$. In the forthcoming discussions, we assume that there exists a Lipschitz constant $\gamma \geqslant 0$ such that

$$|f(t,y_1,y_2,y_3) - f(t,z_1,z_2,z_3)| \leqslant \gamma\left(|y_1-z_1| + |y_2-z_2| + |y_3-z_3|\right). \tag{24}$$

We also require the differential form of the Gronwall's inequality:

Lemma 1. Let $B>0$ and f be a continuous function defined on $[a,b]$. If f is differentiable in (a,b) and $\frac{\mathrm{d}}{\mathrm{d}t}f(t) \leqslant A + Bf(t)$ for $t \in (a,b)$, then $f(t) \leqslant \frac{A}{B}\left(e^{B(t-a)}-1\right)$ for all $t \in [a,b]$.

Theorem 2. Suppose that the NDDE (13) with a piecewise continuous initial function $\phi(t)$ has a unique solution $U(t)$. Then, with the aforementioned assumptions on $\phi_\delta(t)$, $U_\delta(t)$ converges pointwise to $U(t)$ for $t \in \bigcup_{k=1}^{K}[\zeta_{k-1}, \zeta_k)$ as $\delta \to 0$.

Proof. For all $t \in [t_{-1}, t_0)$, given the pointwise convergence of $\phi_\delta(t)$ and $\frac{\mathrm{d}}{\mathrm{d}t}\phi_\delta(t)$, there exists a corresponding $\widetilde{\delta}(t)$ such that

$$\left|\phi_{\delta_1}(t) - \phi_{\delta_2}(t)\right| \leqslant \varepsilon, \qquad \left|\frac{\mathrm{d}}{\mathrm{d}t}\phi_{\delta_1}(t) - \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\delta_2}(t)\right| \leqslant \varepsilon, \quad \text{when } \delta_1,\delta_2 \in (0, \widetilde{\delta}(t)]. \tag{25}$$

Suppose $t \in [t_0, \zeta_1)$ and $\delta_1, \delta_2 \in (0, \widetilde{\delta}(t)]$, then from (13) we have

$$\frac{\mathrm{d}}{\mathrm{d}t}U_{\delta_1}(t) = f\left(t, U_{\delta_1}(t), \phi_{\delta_1}(t-\tau_1(t)), \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\delta_1}(t-\tau_2(t))\right),$$

$$\frac{\mathrm{d}}{\mathrm{d}t}U_{\delta_2}(t) = f\left(t, U_{\delta_2}(t), \phi_{\delta_2}(t-\tau_1(t)), \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\delta_2}(t-\tau_2(t))\right).$$

By subtracting these equations for $t \in [t_0, \zeta_1)$ and utilizing the Lipschitz condition (24), we obtain

$$\begin{aligned}\left|\frac{\mathrm{d}}{\mathrm{d}t}U_{\delta_1}(t) - \frac{\mathrm{d}}{\mathrm{d}t}U_{\delta_2}(t)\right| &\leqslant \gamma\left|U_{\delta_1}(t) - U_{\delta_2}(t)\right| + \gamma\left|\phi_{\delta_1}(t-\tau_1(t)) - \phi_{\delta_2}(t-\tau_1(t))\right| \\ &\quad + \gamma\left|\frac{\mathrm{d}}{\mathrm{d}t}\phi_{\delta_1}(t-\tau_2(t)) - \frac{\mathrm{d}}{\mathrm{d}t}\phi_{\delta_2}(t-\tau_2(t))\right|.\end{aligned} \tag{26}$$

Then, (25) implies

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}U_{\delta_1}(t) - \frac{\mathrm{d}}{\mathrm{d}t}U_{\delta_2}(t)\right| \leqslant 2\gamma\varepsilon + \gamma\left|U_{\delta_1}(t) - U_{\delta_2}(t)\right|. \tag{27}$$

Applying Lemma 1 to (27), results

$$\left|U_{\delta_1}(t) - U_{\delta_2}(t)\right| \leqslant \Lambda_{1,1}\varepsilon \quad \text{for } t \in [t_0, \zeta_1) \text{ whenever } \delta_1,\delta_2 \in (0, \widetilde{\delta}(t)], \tag{28}$$

where $\Lambda_{1,1} = 2(e^{\gamma h_1} - 1) > 0$. In addition, we can deduce from (27) and (28) that

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}U_{\delta_1}(t) - \frac{\mathrm{d}}{\mathrm{d}t}U_{\delta_2}(t)\right| \leqslant \Lambda_{1,2}\varepsilon \quad \text{for } t \in [t_0, \zeta_1) \text{ whenever } \delta_1, \delta_2 \in (0, \widetilde{\delta}(t)], \tag{29}$$

where $\Lambda_{1,2} = 2\gamma e^{\gamma h_1} > 0$. Thus, with $\Lambda_1 = \max\{\Lambda_{1,1}, \Lambda_{1,2}\}$, we have

$$\left|U_{\delta_1}(t) - U_{\delta_2}(t)\right| \leqslant \Lambda_1\varepsilon, \quad \text{for } t \in [t_0, \zeta_1) \text{ whenever } \delta_1, \delta_2 \in (0, \widetilde{\delta}(t)],$$

and

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}U_{\delta_1}(t) - \frac{\mathrm{d}}{\mathrm{d}t}U_{\delta_2}(t)\right| \leqslant \Lambda_1\varepsilon,$$

and the conditions that apply on $[t_{-1}, t_0)$ apply on $[t_0, \zeta_1)$ provided that ε is replaced by $\Lambda_1\varepsilon$. It follows that the preceding arguments can be repeated on the subinterval $[\zeta_1, \zeta_2)$ to establish (with the same method of proof) that

$$\left|U_{\delta_1}(t) - U_{\delta_2}(t)\right| \leqslant \Lambda_2\varepsilon, \quad \text{for } t \in [\zeta_1, \zeta_2) \text{ whenever } \delta_1, \delta_2 \in (0, \widetilde{\delta}(t)],$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}U_{\delta_1}(t) - \frac{\mathrm{d}}{\mathrm{d}t}U_{\delta_2}(t)\right| \leqslant \Lambda_2\varepsilon.$$

By induction, for each subinterval $[\zeta_{k-1}, \zeta_k)$ contained in $[t_0, T)$

$$\left|U_{\delta_1}(t) - U_{\delta_2}(t)\right| \leqslant \Lambda_k\varepsilon, \quad \text{for } t \in [\zeta_{k-1}, \zeta_k) \text{ whenever } \delta_1, \delta_2 \in (0, \widetilde{\delta}(t)],$$

$$\left|\frac{\mathrm{d}}{\mathrm{d}t}U_{\delta_1}(t) - \frac{\mathrm{d}}{\mathrm{d}t}U_{\delta_2}(t)\right| \leqslant \Lambda_k\varepsilon.$$

It follows that there exists a value $\Lambda_*(t) \in (0, \infty)$ such that, for all $t \in [t_0, T)$,

$$\left|U_{\delta_1}(t) - U_{\delta_2}(t)\right| \leqslant \Lambda_*(t) \sup_{s\in[t_{-1},t_0]} \left|\phi_{\delta_1}(s) - \phi_{\delta_2}(s)\right|. \tag{30}$$

This discussion implies the existence of $U_*(t)$ such that $U_\delta(t)$ converges pointwise to $U_*(t)$ for $t \in [t_0, T)$ as $\delta \to 0$. In view of the uniqueness of the solution $U(t)$ of (13), $U_*(t) = U(t)$ and we have established the pointwise convergence of $U_\delta(t)$ to $U(t)$ for all $t \in [t_0, T)$. □

Theorem 3. Suppose that the NDDE (13) with a piecewise continuous initial function $\phi(t)$ has a unique solution $U(t)$. Then, with the aforementioned assumptions on $\phi_\delta(t)$, $U_\delta(t)$ converges in the mean to $U(t)$ as $\delta \to 0$.

Proof. Consider the mesh $\{\eta_{\delta,i}\}_{i=-3}^{3K}$ with the mesh points

$$t_{-1} < t_0 - \sigma - \delta < t_0 - \sigma < t_0 < t_1 - \sigma - \delta < t_1 - \sigma < t_1 < \cdots < T.$$

By our assumptions, $\phi(t) = \phi_\delta(t)$ for $t \in [t_{-1}, t_0 - \sigma - \delta]$ and also $\phi(t_0) = \phi_\delta(t_0)$; therefore, the uniqueness of the solution implies that $U(t) = U_\delta(t)$ on the subinterval $[\eta_{\delta,0}, \eta_{\delta,1})$.

Let $t \in [\eta_{\delta,1}, \eta_{\delta,2})$. From (13) we obtain

$$U(t) = U(\eta_{\delta,1}) + \int_{\eta_{\delta,1}}^{t} f\Big(s, U(s), \phi(s - \tau_1(s)), \frac{\mathrm{d}}{\mathrm{d}s}\phi(s - \tau_2(s))\Big)\,\mathrm{d}s,$$

$$U_\delta(t) = U_\delta(\eta_{\delta,1}) + \int_{\eta_{\delta,1}}^{t} f\Big(s, U_\delta(s), \phi_\delta(s - \tau_1(s)), \frac{\mathrm{d}}{\mathrm{d}s}\phi_\delta(s - \tau_2(s))\Big)\,\mathrm{d}s.$$

Note that $U(\eta_{\delta,1}) = U_\delta(\eta_{\delta,1})$. By subtracting these equations and utilizing the Lipschitz condition (24), we get

$$\begin{aligned}\left|U(t) - U_\delta(t)\right| \leqslant \gamma\Bigg(&\int_{\eta_{\delta,1}}^{t} \left|U(s) - U_\delta(s)\right| \mathrm{d}s + \int_{\eta_{\delta,1}}^{t} \left|\phi(s - \tau_1(s)) - \phi_\delta(s - \tau_1(s))\right| \mathrm{d}s \\ &+ \int_{\eta_{\delta,1}}^{t} \left|\frac{\mathrm{d}}{\mathrm{d}s}\phi(s - \tau_2(s)) - \frac{\mathrm{d}}{\mathrm{d}s}\phi_\delta(s - \tau_2(s))\right| \mathrm{d}s\Bigg).\end{aligned} \tag{31}$$

We also have $\lim_{\delta\to 0}\left\|\phi_\delta(t)-\phi(t)\right\|_{L^1[t_{-1},t_0]}=0$ and $\lim_{\delta\to 0}\left\|\frac{\mathrm{d}}{\mathrm{d}t}\phi_\delta(t)-\frac{\mathrm{d}}{\mathrm{d}t}\phi(t)\right\|_{L^1[t_{-1},t_0]}=J_0$. Thus, for a given constant $\varepsilon>0$, there exists a value $\delta_*>0$ such that, when $\delta\in(0,\delta_*]$ and $t\in[\eta_{\delta,1},\eta_{\delta,2})$,

$$\int\limits_{\eta_{\delta,1}}^{t}\left|\phi(s-\tau_1(s))-\phi_\delta(s-\tau_1(s))\right|\,\mathrm{d}s\leqslant\varepsilon,\quad \int\limits_{\eta_{\delta,1}}^{t}\left|\frac{\mathrm{d}}{\mathrm{d}s}\phi(s-\tau_2(s))-\frac{\mathrm{d}}{\mathrm{d}s}\phi_\delta(s-\tau_2(s))\right|\,\mathrm{d}s\leqslant J_0+\varepsilon.$$

Therefore, when $\delta\in(0,\delta_*]$ inequality (31) can be rewritten as

$$\left|U(t)-U_\delta(t)\right|\leqslant\gamma(2\varepsilon+J_0)+\gamma\int\limits_{\eta_{\delta,1}}^{t}\left|U(s)-U_\delta(s)\right|\,\mathrm{d}s. \tag{32}$$

Introducing $A=\gamma\left(2\varepsilon+J_0\right)$, $B=\gamma$ and $f(t)=\int_{\eta_{\delta,1}}^{t}\left|U(s)-U_\delta(s)\right|\,\mathrm{d}s$ in Lemma 1, and with $t=\eta_{\delta,2}$, we can conclude that

$$\int\limits_{\eta_{\delta,1}}^{\eta_{\delta,2}}\left|U(t)-U_\delta(t)\right|\,\mathrm{d}t\leqslant\left(2\varepsilon+J_0\right)\left(e^{\gamma\delta}-1\right).$$

Consequently, there exists a value $\delta^*>0$ such that for $\delta\in(0,\delta^*]$,

$$\int\limits_{\eta_{\delta,1}}^{\eta_{\delta,2}}\left|U(t)-U_\delta(t)\right|\,\mathrm{d}t\leqslant\Delta_1\varepsilon,$$

where $\Delta_1=2\varepsilon+J_0$.

This shows that $U_\delta(t)$ converges in the mean to $U(t)$ in the subinterval $[\eta_{\delta,1},\eta_{\delta,2})$ as $\delta\to 0$. Moreover, considering the pointwise convergence of $U_\delta(t)$ to $U(t)$, there exist constants $\Lambda_1,\overline{\delta}>0$ such that for $\delta\in(0,\overline{\delta}]$,

$$\left|U(\eta_{\delta,2})-U_\delta(\eta_{\delta,2})\right|\leqslant\Lambda_1\varepsilon. \tag{33}$$

Next, let $t\in[\eta_{\delta,2},\eta_{\delta,3})$. With the same method of proof and with the aid of (33), we can establish that, if $\delta\in(0,\overline{\delta}]$ then

$$\int\limits_{\eta_{\delta,2}}^{\eta_{\delta,3}}\left|U(t)-U_\delta(t)\right|\,\mathrm{d}t\leqslant\frac{1}{\gamma}\left(e^{\gamma\sigma}-1\right)\left|U(\eta_{\delta,2})-U_\delta(\eta_{\delta,2})\right|\leqslant\Delta_2\varepsilon,$$

where $\Delta_2=\frac{\Lambda_1}{\gamma}\left(e^{\gamma\sigma}-1\right)$ and, moreover,

$$\left|U(\eta_{\delta,3})-U_\delta(\eta_{\delta,3})\right|\leqslant\Lambda_1\varepsilon. \tag{34}$$

It follows that the preceding arguments can be repeated to establish that for $\delta\in(0,\overline{\delta}]$,

$$\int\limits_{\eta_{\delta,3}}^{\eta_{\delta,4}}\left|U(t)-U_\delta(t)\right|\,\mathrm{d}t\leqslant\Delta_3\varepsilon,$$

where $\Delta_3=\frac{\Lambda_1}{\gamma}\left(e^{\gamma(h_1-\sigma-\delta)}-1\right)$, and there exists a constant $\Lambda_2>0$ such that

$$\left|U(\eta_{\delta,4})-U_\delta(\eta_{\delta,4})\right|\leqslant\Lambda_2\varepsilon. \tag{35}$$

Now, whenever $t\in[\eta_{\delta,4},\eta_{\delta,5})$ we also require an upper bound for $\int_{\eta_{\delta,1}}^{\eta_{\delta,2}}\left|\frac{\mathrm{d}}{\mathrm{d}t}U(t)-\frac{\mathrm{d}}{\mathrm{d}t}U_\delta(t)\right|\,\mathrm{d}t$. To this end, we can verify from (13) and the Lipschitz condition (24) that

$$\int\limits_{\eta_{\delta,1}}^{\eta_{\delta,2}}\left|\frac{\mathrm{d}}{\mathrm{d}t}U(t)-\frac{\mathrm{d}}{\mathrm{d}t}U_\delta(t)\right|\,\mathrm{d}t\leqslant\gamma\Delta_1(1+\varepsilon).$$

This inequality together with the preceding arguments, for $\delta\in(0,\overline{\delta}]$ results

$$\int\limits_{\eta_{\delta,4}}^{\eta_{\delta,5}}\left|U(t)-U_\delta(t)\right|\,\mathrm{d}t\leqslant\Delta_4\varepsilon,$$

where $\Delta_4 = \left(\Delta_1 + \gamma\Delta_1 + \frac{\Delta_2}{\gamma}\right)\varepsilon + \gamma\Lambda_1$. In addition, we have

$$\left|U(\eta_{\delta,5}) - U_\delta(\eta_{\delta,5})\right| \leqslant \Lambda_2\varepsilon. \tag{36}$$

By induction, for each subinterval $[\eta_{\delta,i}, \eta_{\delta,i+1})$, whenever $\delta \in (0, \overline{\delta}]$, there exists a positive value Δ_i such that

$$\int_{\eta_{\delta,i}}^{\eta_{\delta,i+1}} \left|U(t) - U_\delta(t)\right| \,\mathrm{d}t \leqslant \Delta_i\, \varepsilon.$$

This discussion together with the uniqueness of the solution of (13) implies that $U_\delta(t)$ converges in the mean to $U(t)$ as $\delta \to 0$.

□

5 | PERTURBATION–COLLOCATION METHOD FOR APPROXIMATING DISCONTINUOUS SOLUTIONS

Consider the subintervals $I^{(k)}$, $k = 1, 2, \ldots, K$ described in Subsection 3.1. On each $I^{(k)}$ the local solution $U_k(t)$ of the NDDE (13) is smooth but the global solution $U(t)$ may have jump discontinuities at some of the points ζ_k, which are caused by discontinuity in the initial function. Therefore, the numerical scheme presented in Subsection 3.1 is no longer valid. The reason is that the continuity condition at the interface of mesh intervals is generally not satisfied. Moreover, the size of the jumps at points where the solution is discontinuous are not governed by the scheme (16). Nevertheless, the solution on the mesh interval $I^{(k)}$ is defined by a differential equation in terms of the solutions obtained in preceding mesh intervals. Hence, a modification of the scheme (16) can be derived over a successive new mesh intervals provided that the size of the jumps can be governed properly.

To this end, we first consider a perturbed initial function $\phi_\delta(t)$ as discussed in Subsection 3.2. Then, we utilize the perturbing scheme proposed in [11] and combine it with the multistep LGR collocation method of Subsection 3.1 to compute a continuous numerical approximation $U_\delta^N(t)$ to $U_\delta(t)$. Since $U_\delta(t)$ has large derivatives in the vicinity of points where $U(t)$ has jump discontinuities, a robust numerical procedure for computing approximations to $U_\delta(t)$ should take appropriately a small step-size in the neighbourhood of discontinuity points. To meet this requirement in the explained multistep LGR collocation method, we consider the set of mesh points $M = \{\eta_{\delta,i}\}_{i=-3}^{3K}$ described in Theorem 3. We then follow the numerical scheme (16).

Note that, when $\sigma = 0$ it is desirable that $\phi_\delta(t)$ satisfies a sewing condition, which provides two-sided derivative for $U_\delta(t)$ at all breaking points and more accurate numerical results would be obtained. In numerical experiments, we will show that the multistep LGR collocation method interacts well with the proposed perturbation scheme even for small values of δ.

Remark 1. It follows from Theorems 2 and 3 that, when the initial function is discontinuous, the NDDE (13) *could be relatively ill-condition with respect to the perturbation in the initial function. Indeed, due to accumulation of pointwise errors in a step by step calculation, very small values of δ may be required to retrieve the accuracy of the solution on very large intervals that could be numerically challenging.*

6 | NUMERICAL RESULTS

Consider the following nonlinear explicit NDDE,

$$\frac{\mathrm{d}}{\mathrm{d}t}U(t) = U(t-1)\frac{\mathrm{d}}{\mathrm{d}t}U(t-\tau), \tag{37}$$

with discontinuous initial functions

$$\phi(t) = \begin{cases} t, & t < 0 \\ 1, & t = 0 \end{cases} \qquad \frac{\mathrm{d}}{\mathrm{d}t}\phi(t) = \begin{cases} 1, & t < 0 \\ -1, & t = 0. \end{cases} \tag{38}$$

The condition $\frac{\mathrm{d}}{\mathrm{d}t}\phi(0) = -1$ is imposed to fulfill the sewing condition. This problem attracts attention in [11,37] because it can only be rewritten in Hale's form when $\tau = 1$. We solve this problem for $\tau = 1$ and $\tau = 2$ by smoothing the initial functions as described in Section 3.

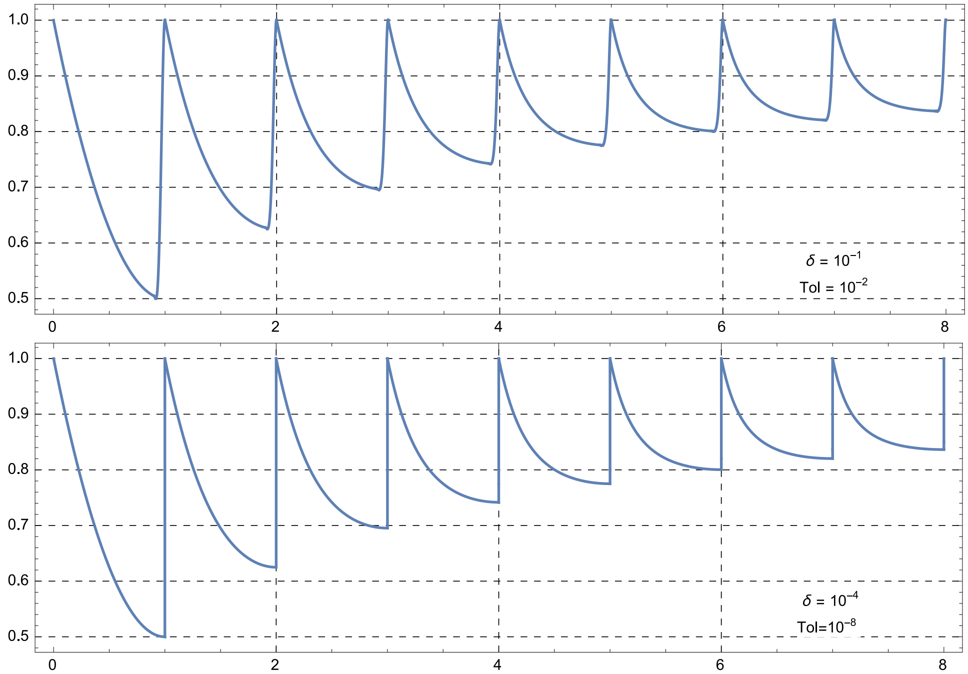

FIGURE 1 Numerical solutions of the perturbed problem (39)-(41) for $\tau = 1$.

TABLE 1 The numerical errors at $t = 10$ for $\tau = 1$

δ	10^{-1}	10^{-2}	10^{-3}	10^{-4}
$N = 5$	5.02e − 02	3.10e − 02	2.92e − 02	2.90e − 02
$N = 25$	2.71e − 09	3.06e − 10	1.50e − 10	9.92e − 11

Derivatives must be interpreted as right-hand derivatives at the points $\{n\tau\}$ for $n \in \mathbb{Z}^+$. Figs. 1 –2 illustrate numerical approximations to the solution of the perturbed problem

$$\frac{\mathrm{d}}{\mathrm{d}t}U_\delta(t) = U_\delta(t-1)\frac{\mathrm{d}}{\mathrm{d}t}U_\delta(t-\tau), \quad t \geqslant 0, \tag{39}$$

$$\phi_\delta(t) = \begin{cases} t, & -\tau \leqslant t \leqslant -\delta \\ \frac{-2(1+\delta)}{\delta^3}t^3 - \frac{3+4\delta}{\delta^2}t^2 - t + 1, & -\delta < t \leqslant 0 \end{cases}, \tag{40}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}U_\delta(t) = \frac{\mathrm{d}}{\mathrm{d}t}\phi_\delta(t), \quad t < 0. \tag{41}$$

The qualitative behaviour of the approximate solution is clearly sensitive to the time lag value τ. Moreover, it is observed that the size of the jumps are governed properly in our method. Table 1 shows the numerical errors at $t = 10$ for $\tau = 1$ and various values of δ. The exact value of this case is $U(10) = 1$. It is seen that the numerical errors decrease exponentially as N increases. Moreover, they decrease linearly as δ decreases. This demonstrates the efficiency and accuracy of the method explained in Section 5.

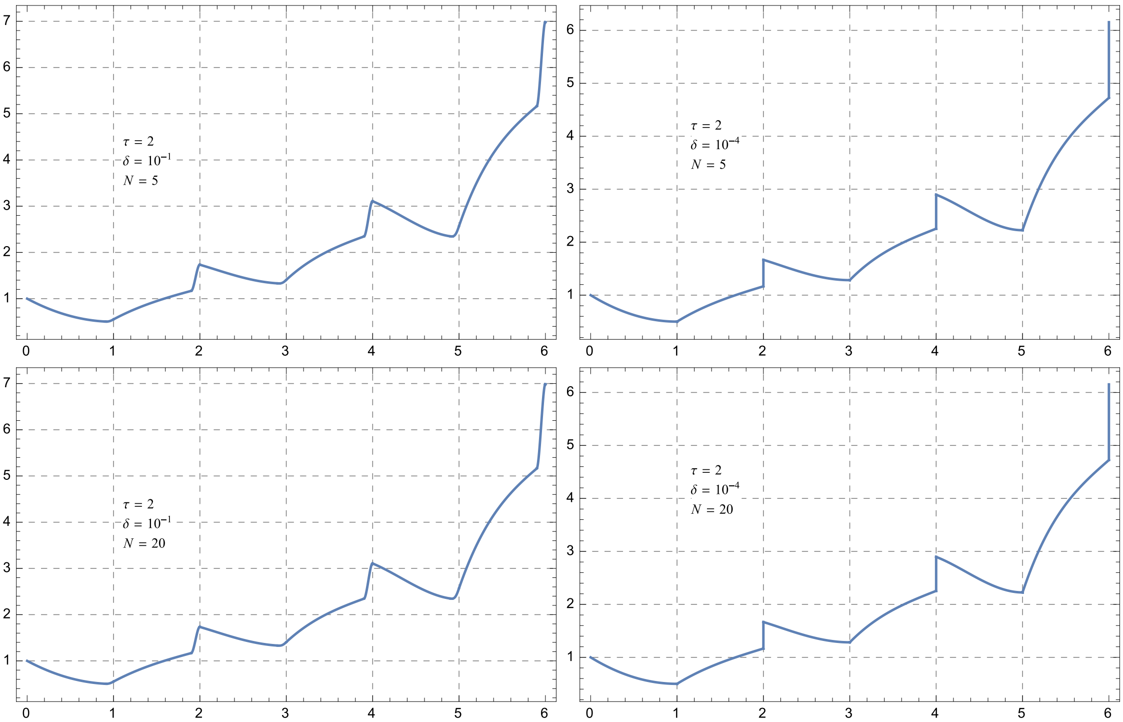

FIGURE 2 Numerical solutions of the perturbed problem (39)-(41) for $\tau = 2$.

7 | CONCLUSIONS

We have developed an efficient approach based on the hybrid of perturbation scheme and the adaptive LGR collocation method for approximating discontinuous solutions of explicit NDDEs with constant or time dependent delays. The limiting behavior of the solution $U_\delta(t)$ as $\delta \to 0$ was also discussed. The pointwise and mean convergence of the continuous solution of the perturbed NDDE to the discontinuous solution of the original NDDE has been proved. There is scope for further work on the treatment of NDDEs with vanishing and state-dependent delays.

DATA SHARING

Data sharing not applicable to this article as no datasets were generated or analyzed during the current study.

FUNDING

The authors did not receive support from any organization for the submitted work.

CONFLICT OF INTEREST

The authors have no conflict of interest to declare.

References

1. A. Bellen and M. Zennaro, Numerical Methods for Delay Differential Equations, Oxford University Press, Oxford, ISBN 0–19–850654–6, 2003.

2. L.E. El'sgol'ts and S.B. Norkin, Introduction to the Theory and Application of Differential Equations with Deviating Arguments, Academic Press, New York, 1973.

3. Y. Kuang, Delay differential equations with applications in population dynamics, Mathematics in Science and Engineering, vol. 191, Academic Press, Boston, 1993.

4. M.J. Crawley, Natural Enemies: The Population Biology of Predators, Parasites and Disease, Blackwell Scientific Publications, Oxford, 1992.

5. H. Kocak and A. Yildirim, Series solution for a delay differential equation arising in electrodynamics, Int. J. Numer. Method. Biomed. Engin. 25(11) (2009) pp. 1084–1096.

6. T. Candan,Existence of positive periodic solutions of first-order neutral differential equations, Mathematical Methods in the Applied Sciences. 40(1) (2017) pp. 205-209.

7. E. Biçer, On the periodic solutions of thirdorder neutral differential equation, Mathematical Methods in the Applied Sciences. 44(2) (2021) pp. 2013-2020.

8. R.D. Driver, Existence and continuous dependence of solutions of a neutral functional–differential equation, Rational, Mech. Anal. 19(2) (1965) pp. 149–186.

9. L.J. Grimm, Existence and continuous dependence for a class of nonlinear neutral differential equations, Proceedings of the American Mathematical Society. 29(3) (1971) pp. 467–473.

10. Z. Jackiewicz, Existence and uniqueness of solutions of neutral delay-differential equations with state dependent delays, Funkcialaj Ekvacioj. 30 (1987) pp. 9–17.

11. C.T.H. Baker and C.A.H. Paul, Discontinuous solutions of neutral delay differential equations, Appl. Numer. Math. 56 (2006) pp. 284–304.

12. M.V. Barbarossa, K.P. Hadeler and C. Kuttler, State–dependent neutral delay equations from population dynamics, J. Math. Bio. 69(4) (2014) pp. 1027–1056.

13. M. Mohr, M.V. Barbarossa and C. Kuttler, Predator–prey interactions, age structures and delay equations, Math. Model. Nat. Phenom. 9(1) (2014) pp. 92–107.

14. M.V. Barbarossa, C. Kuttler and J. Zinsl, Delay equations modeling the effects of phase–specific drugs and immunotherapy on proliferating tumor cells, Math. Biosc. Eng. 9(2) (2012) pp. 241–257.

15. H. Hayashi, Numerical solution of retarded and neutral delay differential equations using continuous Runge-Kutta methods, Ph.D. thesis, Department of Computer Science, University of Toronto, ON, Canada, 1996.

16. W.H. Enright and H. Hayashi, Convergence analysis of the solution of retarded and neutral delay differential equations by continuous numerical methods, SIAM J. Numer. Anal. 35(2) (1998) pp. 572–585.

17. W.H. Enright, Software for ordinary and delay differential equations: Accurate discrete approximate solutions are not enough, Appl. Numer. Math. 56(3-4) (2006) pp. 459–471.

18. H. ZivariPiran and H. Enright, An efficient unified approach for the numerical solution of delay differential equations, Numer. Algor. 53(2-3) (2010) pp. 397–417.

19. U.M. Ascher and L.R. Petzold, The numerical solution of delay-differential algebraic equations of retarded and neutral type, SIAM J. Numer. Anal. 32 (1995) pp. 1635–1657.

20. Y. Liu, Numerical solution of implicit neutral functional–differential equations, SIAM J. Numer. Anal. 36 (1999) pp. 516–528.

21. N. Guglielmi and E. Hairer, Implementing Radau IIA methods for stiff delay differential equations, Computing. 67 (2001) pp. 1–12.

22. Z. Jackiewicz and E. Lo, Numerical solution of neutral functional differential equations by Adams methods in divided difference form, J. Comput. Appl. Math. 189 (2006) pp. 592–605.

23. J.C. Jimenez, L.M. Pedroso, F. Carbonell and V. Hernandez, Local linearization method for numerical integration of delay differential equations, SIAM J. Numer. Anal. 44(6) (2006) pp. 2584–2609.

24. A. Bellen, Preservation of superconvergence in the numerical integration of delay differential equations with proportional delay, IMA J. Numer. Anal. 22(4) (2002) pp. 529-536.

25. N. Guglielmi and E. Hairer, Computing breaking points in implicit delay differential equations, Adv. Comput. Math. 29 (2008) pp. 229–247.

26. C. Canuto, M.Y. Hussaini, A. Quarteroni and T.A. Zang, Spectral Methods: Fundamentals in Single Domains, Springer–Verlag, Berlin, 2006.

27. Z.-q Wang and B.-y Guo, Legendre–Gauss–Radau collocation method for solving initial value problems of first order ordinary differential equations, J. Sci. Comput. 52(1) (2012) pp. 226-225.

28. K. Ito, H.T. Tran and A. Manitius, A fully–discrete spectral method for delay–differential equations, SIAM J. Numer. Anal. 28 (1991) pp. 1121–1140.

29. M. Maleki, A. Davari, Analysis of an adaptive collocation solution for retarded and neutral delay systems, Numer. Algor. 88 (2021) pp. 67–91.

30. M. Maleki, I. Hashim and S. Abbasbandy, Solution of time–varying delay systems using an adaptive collocation method, Appl. Math. Comput. 219 (2012) pp. 1434–1448.

31. M. Maleki and I. Hashim, Adaptive pseudospectral methods for solving constrained linear and nonlinear time–delay optimal control problems, J. Franklin. Inst. 351 (2014) pp. 811–839.

32. I. Ali, H. Brunner and T. Tang, A spectral method for pantograph–type delay differential equations and its convergence analysis, J. Comput. Math. 27 (2009) pp. 254–265.

33. Z.Q. Wang and L.L. Wang, A Legendre–Gauss collocation method for nonlinear delay differential equations, Discrete Contin. Dyn. Syst. Ser. B 13 (2010) pp. 685–708.

34. H. Brunner, H. Xie and R. Zhang, Analysis of collocation solutions for a class of functional equations with vanishing delays, IMA J. Numer. Anal. 31(2) (2011) pp. 698–718.

35. T. Meng, Z. Wang and L. Yi, An h-p version of the Chebyshev spectral collocation method for nonlinear delay differential equations, Numer. Methods Partial Differential Eq. 35(2) (2019) pp. 1–17.

36. A. Bellen and N. Guglielmi, Solving neutral delay differential equations with state-dependent delays, J. Comput. Appl. Math. 229 (2009) pp. 350–362.

37. V.B. Kolmanovski and A. Myshkis, Introduction to the theory and applications of functional–differential equations, Kluwer Academic, Dordrecht, ISBN 0–7923–5504–0, 1999.