ON NONLINEAR FRACTIONAL SCHRÖDINGER EQUATION WITH INDEFINITE POTENTIAL AND HARDY POTENTIAL

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Abstract. This paper is concerned with a class of fractional Schrödinger equation with Hardy potential

$$(-\Delta)^s u + V(x)u - \frac{\kappa}{|x|^{2s}}u = f(x,u),\ x \in \mathbb{R}^N,$$

where $s \in (0,1)$ and $\kappa \geq 0$ is a parameter. Under some suitable conditions on the potential V and the nonlinearity f, we prove the existence of ground state solutions when the parameter κ lies in a given range by using the non-Nehari manifold method. Moreover, we investigate the continuous dependence of ground state energy about κ. Finally, we are able to explore the asymptotic behaviors of ground state solutions as κ tends to 0.

1. Introduction and main results

We consider the following nonlinear fractional Schrödinger equation with Hardy potential

$$(-\Delta)^s u + V(x)u - \frac{\kappa}{|x|^{2s}}u = f(x,u),\ x \in \mathbb{R}^N, \tag{1.1}$$

where $s \in (0,1)$, $\kappa \geq 0$, $N > 2s$, V is external potential, f is nonlinear function with subcritical growth, $(-\Delta)^s$ is the usual fractional Laplacian operator, defined by

$$(-\Delta)^s u = C_{N,s}\text{P.V.}\int_{\mathbb{R}^N} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dy,\ \forall x \in \mathbb{R}^N, \tag{1.2}$$

here P.V. stands for the Cauchy principal value, and

$$C_{N,s} := \frac{2^{2s}\Gamma(\frac{N+2s}{2})}{2\pi^{\frac{N}{2}}|\Gamma(-s)|} > 0$$

is a normalized constant and Γ is the usual Gamma function.

As well known, problem (1.1) arises when one considers standing wave solutions of the following time-dependent fractional Schrödinger equation

$$i\hbar\frac{\partial\Psi}{\partial t} = \frac{\hbar^{2s}}{2m}(-\Delta)^s\Psi + \left(V(x) - \frac{\kappa}{|x|^{2s}} + E\right)\Psi - g(x,|\Psi|),\ (x,t) \in \mathbb{R}^N \times \mathbb{R},$$

where Ψ represents the wave function, V is an external potential, $\frac{\kappa}{|x|^{2s}}$ is Hardy potential, m is the mass of free particle and the nonlinear coupling g describes a self-interaction among many particles. We note that fractional Schrödinger equation was first introduced by Laskin [20], and comes from an expansion of the Feynman path integral from Brownian-like to Lévy-like quantum mechanical paths. In Laskin's studies, the Feynman path integral leads to the classical Schrödinger equation and the path integral over Lévy trajectories leads to the fractional Schrödinger equation. More in general, the study of nonlinear elliptic equations involving nonlocal and fractional operators has gained tremendous popularity during the last decade, because of intriguing structure of these operators and their application in many areas of research such as optimization, finance, phase transition phenomena, minimals surfaces, game theory, and population dynamics.

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On the other hand, the other feature of problem (1.1) is that the equation includes the singular Hardy potential $\frac{\kappa}{|x|^{2s}}$. Physically, the Hardy potential describes the motion and the interactions (attractive and repulsive) between two charged particles, which plays a crucial role in quantum mechanics. Besides, it also arises in many other areas such as nuclear physics, molecular physics and quantum cosmology, see [14] for more background and applications. We note that the main reason of interest in Hardy potential relies in their criticality: indeed it has the same homogeneity as the operator $(-\Delta)^s + V$ and does not belong to the Kato's class, hence it cannot be regarded as a lower order perturbation term, This feature causes some new difficulties for overcoming the lack of compactness, and this is one of the main motivations why we investigate problem (1.1).

It is known, but not completely trivial, that $(-\Delta)^s$ reduces to the classical Laplacian $-\Delta$ as $s \to 1$. To be more precise, when $s = 1$, the classical nonlinear Schrödinger equation with Hardy potential

$$-\Delta u + V(x)u - \frac{\kappa}{|x|^2}u = f(x,u), \ x \in \mathbb{R}^N,$$

has received extensive attention in recent years by many researchers. Applying the topological and variational arguments, some authors studied the existence of positive solutions, sign-changing solutions, multiple solutions, ground state solutions and some related properties of solutions under some suitable conditions, respectively. We refer the readers to see the papers [4, 5, 7, 16, 17, 19, 28] and the references therein. In addition, for other related results about the coupled Schrodinger system with Hardy potential, see for instance [35, 36, 39]. These works also motivate us to study the fractional Schrödinger equations with Hardy potential in the present work.

Concerning the nonlocal framework, from the mathematical point of view, the main difficulty of the fractional problem lies in that the fractional Laplacian $(-\Delta)^s$ has nonlocal characteristic. Accordingly, some arguments used to deal with the local case do not work in nonlocal case, and some nontrivial additional technical difficulties also arise. The seminal work initiated by Caffarelli and Silvestre [8] in which the authors made greatest achievement in overcoming this difficulty by the extension method. Under this framework of extension, the nonlocal problem can be transformed into the local problem. Recently, for the case $\kappa = 0$, there have been many works focused on the study of fractional Schrödinger equation (1.1) by using variational method and the extension method. For instance, the papers [3, 6, 9, 13, 18, 22, 26, 34] studied the existence, multiplicity and regularity results of solutions under different assumptions on the potential and nonlinearity. We also refer to the monograph by Molica Bisci, Rădulescu and Servadei [23] for a very nice introduction for the nonlocal fractional variational problems.

Regarding the study of fractional nonlocal equations with Hardy potential we would like to mention the recent papers [1, 2, 11, 12]. More precisely, Bieganowski [1] studied the existence and asymptotic behaviors of ground state solutions to problem (1.1) with sign-changing nonlinearities by using the mountain pass argument and Nehari manifold method. It should be pointed that, in [1], the author supposed that the potential V is positive and satisfies the asymptotically periodic condition. After that, Bieganowski and co-authors [2] generalized these results to semirelativistic Choquard equations. Fall and Felli [11, 12] also proved some properties of relativistic Schrödinger opeartor with Hardy potential, such as the unique continuation properties and sharp essential self-adjointness, and carefully analyzed the asymptotics of solutions at the singularity.

Inspired by the papers [1], in this paper we are interested in problem (1.1) with general periodic indefinite potential. In order to better understand our purpose, we would like to introduce the recent paper by Fang and Ji [13]. Indeed, under the condition of periodic and sign-changing for the potential, Fang and Ji [13] proved the fractional Schrödinger operator $(-\Delta)^s + V$ has purely continuous spectrum which is bounded below and consists of closed disjoint intervals, see [13, Theorem 1.1]. So, in this framework, we can know that such problem has the strongly indefinite variational structure. In the sense it is easy to see that zero is no longer a local minimum point of the energy functional, thus the usual mountain pass theorem and Nehari manifold method do not work. Naturally, we require more delicate approach to treat our problem. Therefore, the problem we considered is completely different from the problem studied by Bieganowski [1], This is the main motivation of the present paper and we will give an affirmative answer, which also complement and extend the results before. To the best of our knowledge, it seems that there is no work considered this problem in the literature up to now.

More specifically, the main ingredients of the present paper are three aspects as follows. Firstly, we prove the existence of ground state solutions for sufficiently small $\kappa \geq 0$ under periodic and asymptotically periodic conditions. Secondly, we investigate the continuous dependence of ground state energy about parameter κ. Finally, we analyze the asymptotic convergence of ground state solutions as $\kappa \to 0$.

Moreover, throughout the paper, we introduce the following hypotheses on the potential V:

(V) $V \in C(\mathbb{R}^N, \mathbb{R})$ is $\mathbb{Z}^N$-periodic in x and $0 \notin \sigma((-\Delta)^s + V)$ and $\sigma((-\Delta)^s + V) \cap (-\infty, 0) \neq \emptyset$, where $\sigma\left((-\Delta)^s + V\right)$ denotes the spectrum of Schrödinger operator $(-\Delta)^s + V$.

Before stating the results, we introduce the following notation. We use $\mathscr{W}$ to denote the class of functions $g \in C(\mathbb{R}^N, \mathbb{R}^+) \cap L^\infty(\mathbb{R}^N, \mathbb{R}^+)$ such that for every $\epsilon > 0$, the set $\left\{x \in \mathbb{R}^N : |g(x)| \geq \epsilon\right\}$ has finite Lebesgue measure. Meanwhile, we assume that f satisfies the following conditions:

(f_1) $f \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ and there exist $p \in (2, 2_s^*)$ and $c > 0$ such that
$$|f(x,s)| \leq c(1 + |s|^{p-1}) \text{ for all } (x,s),$$
where $2_s^* = \frac{2N}{N-2s}$ is the fractional Sobolev critical exponent;

(f_2) $f(x,s) = o(|s|)$ as $|s| \to 0$ uniformly in x;

(f_3) $\frac{F(x,s)}{s^2} \to \infty$ as $s \to \infty$ uniformly in x, where $F(x,s) = \int_0^s f(x,t)dt$;

(f_4) $f(x,s)$ is $\mathbb{Z}^N$-periodic in x;

(f_5) $\frac{f(x,s)}{|s|}$ is non-decreasing on $(-\infty, 0)$ and $(0, \infty)$;

(f_6) there exist constant $p_0 \in (2, 2_s^*)$ and function $a \in \mathscr{W}$, $\widehat{f} \in C(\mathbb{R}^N \times \mathbb{R}, \mathbb{R})$ is $\mathbb{Z}^N$-periodic in x such that
(i) $F(x,s) > \widehat{F}(x,s) = \int_0^s \widehat{f}(x,t)dt$ for all (x,s),
(ii) $|f(x,s) - \widehat{f}(x,s)| \leq a(x)(1 + |s|^{p_0 - 1})$ for all (x,s),
(iii) $\frac{\widehat{f}(x,s)}{|s|}$ is non-decreasing in s on $(0, +\infty)$ and $(0, \infty)$;

(f_7) there exist $c_0 > 0$ and $2 < q \leq p$ such that
$$\frac{1}{2} f(x,s)s - F(x,s) \geq \begin{cases} c_0|s|^2, & \text{for } |s| < 1, \\ c_0|s|^q, & \text{for } |s| \geq 1. \end{cases}$$

Let κ^* and ν_0 be two positive constants and m_κ denote the ground state energy of problem (1.1), where κ^*, ν_0 and m_κ will be given in Section 2. Now we introduce the main results of this paper. First, for the periodic case we have the following theorem.

Theorem 1.1. *Assume that* (V) *and* (f_1)*-*(f_5) *are satisfied and* $0 \leq \kappa < \kappa^* \nu_0^2$*. Then problem* (1.1) *has at least a ground state solutions.*

For the asymptotically periodic case we have the following result.

Theorem 1.2. *Suppose that* (V)*,* (f_1)*-*(f_3)*,* (f_5) *and* (f_6) *are satisfied and* $0 \leq \kappa < \kappa^* \nu_0^2$*. Then problem* (1.1) *has at least a ground state solutions.*

We also show the convergence property of the ground state energy as follows.

Theorem 1.3. *Assume that* (V)*,* (f_1)*-*(f_3)*,* (f_5) *and* (f_4) *(or* (f_6) *) hold. Then the ground state energy has the convergence property:* $\lim_{\kappa \to 0^+} m_\kappa = m_0$*.*

Evidently, the ground state solutions u_κ obtained in Theorems 1.1 and 1.2 is related to parameter κ. The following theorem shows the asymptotic behavior of u_κ as $\kappa \to 0$, which illustrate the relationship between $\kappa > 0$ and $\kappa = 0$ in problem (1.1).

Theorem 1.4. *Under the conditions of Theorem* 1.1 *or Theorem* 1.2*, let* u_κ *be a ground state solution of problem* (1.1)*. Then for any sequence* $\{\kappa_n\}$ *with* $\kappa_n \to 0$ *as* $n \to \infty$*, passing to a subsequence,* $u_{\kappa_n} \to u_0$ *as* $n \to \infty$ *in* $H^s(\mathbb{R}^N)$*, where* u_0 *is a ground state solution of the following problem*
$$(-\Delta)^s u + V(x)u = f(x,u), \ x \in \mathbb{R}^N.$$

To complete the proofs of main results, let us now outline the main strategies and approaches. Our strategies are based on variational arguments and refined analysis techniques. Firstly, we note that the conditions (f_5) and (f_6)-(iii) are not strictly increasing, so we do not use the generalized Nehari manifold method introduced by Szulkin and Weth [29] to find ground state solutions. To circumvent this obstacle created by the non-decreasing, we intend to adapt the non-Nehari manifold method developed by Tang [31] to handle the present problem. The main idea of this method is to construct a special Cerami sequence at some level outside the generalized Nehari manifold by combining the generalized linking theorem and the diagonal method, then show that the Cerami sequence is bounded. Secondly, we will make use of the technique of limit problem to analyze carefully the behavior of Cerami sequence, and establish two global compactness results for bounded Cerami sequences to overcome the lack of embedding compactness. Moreover, combining the global compactness results and the energy comparison argument, we can establish the existence of ground state solutions. Thirdly, using some analysis techniques, we prove the convergence property of the ground state energy and asymptotic behaviors of ground state solutions.

The structure of this paper is the following. In Section 2, we establish the variational framework to problem (1.1) and give some useful preliminary lemmas. In Section 3, we prove two global compactness results by analyzing the properties of Cerami sequence, and we give the completed proofs of Theorems 1.1 and 1.2. In Section 4, we prove the asymptotic behaviors of solutions and finish the proofs of Theorems 1.3 and 1.4.

2. Variational setting and preliminary results

Throughout the paper, we use $|\cdot|_q$ to denote the usual L^q-norm, and use $(\cdot,\cdot)_2$ to denote the usual L^2 inner product, c, c_i or C_i stand for different positive constants.

In the following we introduce the fractional Sobolev spaces [23] and some related conclusions. For any $s \in (0,1)$, we define $\mathcal{D}^{s,2}(\mathbb{R}^N)$ as the completion of $C_0^\infty(\mathbb{R}^N)$ with respect to

$$[u]^2_{\mathcal{D}^{s,2}} = \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dydx.$$

Let us introduce the fractional Sobolev space

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dydx < +\infty \right\}$$

endowed with the natural norm

$$\|u\|_{H^s} = \left(C_{N,s} \int_{\mathbb{R}^{2N}} \frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}} dydx + \int_{\mathbb{R}^N} |u|^2 dx \right)^{\frac{1}{2}}.$$

We note that $H^s(\mathbb{R}^N)$ can be also equivalently represented as

$$H^s(\mathbb{R}^N) = \left\{ u \in L^2(\mathbb{R}^N) : (-\Delta)^{\frac{s}{2}} u \in L^2(\mathbb{R}^N) \right\},$$

and the norm $\|u\|_{H^s}$ can be rewritten as

$$\|u\|_{H^s} = \left(\int_{\mathbb{R}^N} \left[|(-\Delta)^{\frac{s}{2}} u|^2 + |u|^2 \right] dx \right)^{\frac{1}{2}}.$$

Let $H = (-\Delta)^s + V$ denote the fractional Schrödinger operator. According to the continuity of V, we can see that V is bounded in $\mathbb{R}^N$. Moreover, under the condition (V), H is a self-adjoint operator, and it has purely continuous spectrum which is bounded below and consists of closed disjoint intervals due to [13, Theorem 1.1]. Then L^2 have the orthogonal decomposition

$$L^2 = L^- \oplus L^+, u = u^- \oplus u^+,$$

such that H is negative definite in L^- and positive definite in L^+. Let $|H|$ be the absolute value of H, $|H|^{\frac{1}{2}}$ be its square root, and let $E = \mathcal{D}(|H|^{\frac{1}{2}})$ be the Hilbert space equipped with the inner product

$$(u,v)=(|H|^{\frac{1}{2}}u,|H|^{\frac{1}{2}}v)_2=\int_{\mathbb{R}^N}|H|^{\frac{1}{2}}u|H|^{\frac{1}{2}}vdx,$$

then the induced norm $\|u\|=(u,u)^{\frac{1}{2}}$. From the boundedness of V, we know that the norm $\|u\|$ is equivalent to $\|u\|_{H^s}$, that is, there exist ν_0 and ν_1 such that

$$\nu_0\|u\|_{H^s}\le\|u\|\le\nu_1\|u\|_{H^s}. \tag{2.1}$$

Therefore $E=H^s(\mathbb{R}^N)$. Furthermore, according to the orthogonal decomposition of L^2, we have the decomposition of E:

$$E=E^-\oplus E^+,\text{where } E^-=E\cap L^-,\ E^+=E\cap L^+,$$

Clearly, E is orthogonal with respect to the two inner products $(\cdot,\cdot)$ and $(\cdot,\cdot)_2$. Using the polar decomposition of operator, we also have

$$Hu^-=-|H|u^-,Hu^+=|H|u^+,\ \text{for all } u=u^-+u^+\in E.$$

Define a bilinear map $B(u,v)$ as follows

$$B(u,v)=\int_{\mathbb{R}^N}\left[(-\Delta)^{\frac{s}{2}}u(-\Delta)^{\frac{s}{2}}v+V(x)uv\right]dx. \tag{2.2}$$

Then, according to the above decomposition, for each $u\in E$ we have

$$B(u,u)=B(u^+,u^+)+B(u^-,u^-)=(u^+,u^+)-(u^-,u^-)=\|u^+\|^2-\|u^-\|^2. \tag{2.3}$$

Let us recall the following embedding property for fractional Sobolev spaces, see the monograph by Molica Bisci-Rădulescu-Servadei [23] for more details.

Lemma 2.1. *The embedding $E\hookrightarrow L^q(\mathbb{R}^N)$ is continuous for any $2\le q\le 2_s^*$. Moreover, the embedding $E\hookrightarrow L^q(\mathbb{R}^N)$ is locally compact for any $2\le q<2_s^*$.*

From Lemma 2.1, we can see that there exists constant $\gamma_q>0$ such that

$$|u|_q\le\gamma_q\|u\|,\ \forall\ u\in E,\ q\in[2,2_s^*]. \tag{2.4}$$

We also recall the fractional Hardy inequality (see [15, Theorem 1.1]), which is very crucial to deal with Hardy potential.

Lemma 2.2. *There exists $\Lambda_{N,s}>0$ such that for every $u\in E$ and $N>2s$, then there holds*

$$\Lambda_{N,s}\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^{2s}}dx\le\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx, \tag{2.5}$$

where

$$\Lambda_{N,s}=2\pi^{\frac{N}{2}}\frac{\Gamma(\frac{N+2s}{4})^2|\Gamma(-s)|}{\Gamma(\frac{N-2s}{4})^2\Gamma(\frac{N+2s}{2})}.$$

We define

$$\kappa^*=\Lambda_{N,s}C_{N,s}=4^s\left[\frac{\Gamma(\frac{N+2s}{4})}{\Gamma(\frac{N-2s}{4})}\right]^2.$$

In particular, for the local case ($s=1$) we obtain

$$\kappa^*=4\left[\frac{\Gamma(\frac{N+2}{4})}{\Gamma(\frac{N-2}{4})}\right]^2=4(\frac{N}{4}-\frac{1}{2})^2=\frac{(N-2)^2}{4}.$$

From (2.1) and (2.5) we can deduce that

$$\begin{aligned}\kappa^*\int_{\mathbb{R}^N}\frac{|u(x)|^2}{|x|^{2s}}dx&\le C_{N,s}\int_{\mathbb{R}^{2N}}\frac{|u(x)-u(y)|^2}{|x-y|^{N+2s}}dydx\\&\le\|u\|_{H^s}^2\le\frac{1}{\nu_0^2}\|u\|^2.\end{aligned} \tag{2.6}$$

Now, on E we define the the energy functional corresponding to problem (1.1)

$$\Phi_\kappa(u) = \frac{1}{2}\int_{\mathbb{R}^N}\left[|(-\Delta)^{\frac{s}{2}}u|^2 + V(x)|u|^2\right]dx - \frac{\kappa}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2s}}dx - \int_{\mathbb{R}^N}F(x,u)dx.$$

In view of (2.2) and (2.3) we get

$$\begin{aligned}\Phi_\kappa(u) =& \frac{1}{2}B(u,u) - \frac{\kappa}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2s}}dx - \int_{\mathbb{R}^N}F(x,u)dx\\ =& \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \frac{\kappa}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2s}}dx - \int_{\mathbb{R}^N}F(x,u)dx\end{aligned}$$

for $u = u^+ + u^- \in E$. Since E^+ and E^- are infinite dimensional, then Φ_κ is strongly indefinite.

We deduce from the assumptions (f_1), (f_2) and (f_5) that for any $\epsilon > 0$, there exists positive constant C_ϵ such that

$$\left\{\begin{array}{l} f(x,s) \le \epsilon s + C_\epsilon |s|^{p-1} \\ F(x,s) \le \epsilon |s|^2 + C_\epsilon |s|^p \end{array}\right. \text{ for all } (x,s) \in \mathbb{R}^N \times \mathbb{R} \text{ and } p \in (2, 2_s^*) \tag{2.7}$$

and

$$\frac{1}{2}f(x,s)s \ge F(x,s) \ge 0, \text{ for all } (x,s) \in \mathbb{R}^N \times \mathbb{R}. \tag{2.8}$$

Therefore, according to (2.6) and (2.7) and using a standard argument, we can show that $\Phi_\kappa \in C^1(E,\mathbb{R})$. Clearly, the critical points of Φ_κ are solutions of problem (1.1), and

$$\langle \Phi_\kappa'(u), v\rangle = (u^+, v^+) - (u^-, v^-) - \kappa\int_{\mathbb{R}^N}\frac{uv}{|x|^{2s}}dx - \int_{\mathbb{R}^N}f(x,u)vdx, \text{ for } u, v \in E.$$

For more information about the strongly indefinite variational problems, we refer to the monographs by Ding [10] and Willem [33].

Below we recall that a functional $\Phi \in C^1(E,\mathbb{R})$ is said to be weakly sequentially lower semicontinuous

$$\text{“if for any } u_n \rightharpoonup u \text{ in } E \Rightarrow \ \Phi(u) \le \liminf_{n\to\infty}\Phi(u_n);\text{”}$$

and Φ' is said to be weakly sequentially continuous

$$\text{“if } \lim_{n\to\infty}\langle \Phi'(u_n), \varphi\rangle = \langle \Phi'(u), \varphi\rangle \text{ for each } \varphi \in E.\text{”}$$

We say that Φ satisfies that “Cerami condition”, if the following property holds:

“if $\{u_n\} \subseteq E$ is a Cerami sequence such that
$\{\Phi(u_n)\} \subseteq \mathbb{R}$ is bounded,
and $(1 + \|u_n\|)\Phi'(u_n) \to 0$ in E^* as $n \to \infty$,
then it has a strongly convergent subsequence.”

For the sake of simplicity, let

$$\Psi_\kappa(u) = \frac{\kappa}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2s}}dx + \int_{\mathbb{R}^N}F(x,u)dx.$$

Using Lemma 2.1, Lemma 2.2 and Fatou's lemma, we can easily check the following result.

Lemma 2.3. *Assume that (V), (f_1),(f_2) and (f_5) are satisfied and $\kappa \ge 0$. Then Ψ_κ is weakly sequentially lower semi-continuous, and Ψ_κ' is weakly sequentially continuous.*

In order to find the ground state solutions of the problem (1.1), we define the following Nehari-Pankov manifold $\mathscr{N}_\kappa$ (also called generalized Nehari manifold [29])

$$\mathscr{N}_\kappa = \{u \in E\backslash E^- : \langle \Phi_\kappa'(u), u\rangle = 0 \text{ and } \langle \Phi_\kappa'(u), v\rangle = 0, \ \forall v \in E^-\}.$$

and the ground state energy m_κ of Φ_κ defined by

$$m_\kappa = \inf_{u\in\mathscr{N}_\kappa}\Phi_\kappa(u).$$

Lemma 2.4. *Assume that (V), (f_1), (f_2), (f_3) and (f_5) are satisfied and $0 \le \kappa < \kappa^* \nu_0^2$, and let $u \in E$, $w \in E^-$ and $t \ge 0$. Then we have*

$$\Phi_\kappa(u) \ge \Phi_\kappa(tu + w) + \frac{1-t^2}{2}\langle \Phi'_\kappa(u), u\rangle - t\langle \Phi'_\kappa(u), w\rangle. \tag{2.9}$$

In particular, let $u \in \mathscr{N}_\kappa$, $w \in E^-$ and $t \ge 0$, there holds

$$\Phi_\kappa(u) \ge \Phi_\kappa(tu + w). \tag{2.10}$$

Proof. By the virtue of (f_5), due to [31, Lemma 2.3], for any $x \in \mathbb{R}^N$ we have

$$f(x,s) \le \frac{f(x,\tau)}{|\tau|}|s|, \ s \le \tau; \quad f(x,s) \ge \frac{f(x,\tau)}{|\tau|}|s|, \ s \ge \tau;$$

and

$$\left(\frac{1-t^2}{2}\tau^2 - t\tau\sigma\right)\frac{f(x,\tau)}{|\tau|} \ge \int_{t\tau+\sigma}^{\tau} f(x,s)ds, \ \tau \ge 0, \sigma \in \mathbb{R}. \tag{2.11}$$

By a direct computation we can obtain

$$\begin{aligned}
&\Phi_\kappa(u) - \Phi_\kappa(tu+w) - \frac{1-t^2}{2}\langle \Phi'_\kappa(u), u\rangle + t\langle \Phi'_\kappa(u), w\rangle \\
&= \int_{\mathbb{R}^N}\left[\frac{1-t^2}{2}f(x,u) - tf(x,u)w - \int_{tu+w}^{u} f(x,s)ds\right] + \frac{1}{2}\|w\|^2 + \frac{\kappa}{2}\int_{\mathbb{R}^N}\frac{|w|^2}{|x|^{2s}}dx.
\end{aligned}$$

Thus, taking advantage of (2.11) we can see easily that (2.9) holds.

Evidently, let $u \in \mathscr{N}_\kappa$ and $w \in E^-$, then

$$\langle \Phi'_\kappa(u), u\rangle = \langle \Phi'_\kappa(u), w\rangle = 0.$$

Consequently, (2.10) holds. This ends the proof of the lemma. □

Next we need the generalized linking theorem and show that Φ_κ has linking structure.

Lemma 2.5. *Let X be a real Hilbert space with $X = X^- \oplus X^+$, and let $\Phi \in C^1(X, \mathbb{R})$ be of the form*

$$\Phi(u) = \frac{1}{2}\left(\|u^+\|^2 - \|u^-\|^2\right) - \Psi(u), \quad u = u^- + u^+ \in X^- \oplus X^+.$$

Assume that the following conditions hold:

- (Ψ_1) *$\Psi \in C^1(X, \mathbb{R})$ is bounded from below and weakly sequentially lower semi-continuous;*
- (Ψ_2) *Ψ' is weakly sequentially continuous;*
- (Ψ_3) *there exist $R > \rho > 0$ and $e \in X^+$ with $\|e\| = 1$ such that*

$$\alpha := \inf \Phi(S_\rho^+) > \sup \Phi(\partial Q),$$

where

$$S_\rho^+ = \left\{u \in X^+ : \|u\| = \rho\right\}, \qquad Q = \left\{v + se : v \in X^-, \ s \ge 0, \ \|v + se\| \le R\right\}.$$

Then there exist a constant $c \in [\alpha, \sup \Phi(Q)]$ and a sequence $\{u_n\} \subset X$ satisfying

$$\Phi(u_n) \to c \text{ and } (1 + \|u_n\|)\|\Phi'(u_n)\| \to 0.$$

Lemma 2.6. *Assume that (V), (f_1), (f_2), (f_3) and (f_5) are satisfied and $0 \le \kappa < \kappa^* \nu_0^2$. Then*

(i) *there exists $\rho > 0$ such that*

$$m_\kappa \ge \alpha := \inf\left\{\Phi_\kappa(u) : u \in E^+, \|u\| = \rho\right\} > 0.$$

(ii) $\|u^+\|^2 \ge \max\left\{\|u^-\|^2, 2m_\kappa\right\}$ *for all $u \in \mathscr{N}_\kappa$.*

Proof. (i) Clearly, from Lemma 2.4 we can easily see that $m_\kappa \geq \alpha := \inf\{\Phi_\kappa(u) : u \in E^+, \|u\| = \rho\}$. So, we only need to show that $\alpha > 0$. Indeed, let $u \in E^+$ and $0 \leq \kappa < \kappa^*\nu_0^2$. Using (2.4), (2.6) and (2.7) we have

$$\begin{aligned}\Phi_\kappa(u) &= \frac{1}{2}\|u\|^2 - \frac{\kappa}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2s}}dx - \int_{\mathbb{R}^N}F(x,u)dx \\ &\geq \frac{1}{2}\left(1 - \frac{\kappa}{\kappa^*\nu_0^2}\right)\|u\|^2 - \epsilon\gamma_2^2\|u\|^2 - \gamma_p^p C_\epsilon\|u\|^p.\end{aligned}$$

Since $0 \leq \kappa < \kappa^*\nu_0^2$ and $p > 2$, we infer that there exists $\rho > 0$ small enough such that

$$\alpha := \inf\left\{\Phi_\kappa(u) : u \in E^+, \|u\| = \rho\right\} > 0.$$

(ii) Let $u \in \mathscr{N}_\kappa$, by (2.8), it is easy to see that

$$\begin{aligned}m_\kappa \leq& \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \frac{\kappa}{2}\int_{\mathbb{R}^N}\frac{|u|^2}{|x|^{2s}}dx - \int_{\mathbb{R}^N}F(x,u)dx \\ \leq& \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2),\end{aligned}$$

which implies that $\|u^+\|^2 \geq \max\left\{\|u^-\|^2, 2m_\kappa\right\}$. So, we finish the proof. $\square$

Lemma 2.7. *Assume that* (V), (f_1), (f_2), (f_3) *and* (f_5) *are satisfied and* $0 \leq \kappa < \kappa^*\nu_0^2$. *Then for any* $e \in E^+$, $\sup\Phi_\kappa(E^- \oplus \mathbb{R}^+e) < \infty$, *and there is* $R_e > 0$ *independent of* κ *such that*

$$\Phi_\kappa(u) < 0, \ \forall\ u \in E^- \oplus \mathbb{R}^+e, \ \|u\| \geq R_e.$$

In particular, there is a $R_0 > \rho$ *independent of* κ *such that* $\sup\Phi(\partial Q_R) \leq 0$ *for* $R \geq R_0$, *where*

$$Q_R = \left\{se + w : w \in E^-, \ s \geq 0, \ \|se + w\| \leq R\right\}. \tag{2.12}$$

Proof. Let $e \in E^+$, $t \geq 0$ and $u = te + u^- \in E^- \oplus \mathbb{R}^+e$. Note that $\kappa \geq 0$, then

$$\Phi_\kappa(u) \leq \Phi_0(u) = \frac{1}{2}(\|u^+\|^2 - \|u^-\|^2) - \int_{\mathbb{R}^N}F(x,u)dx.$$

Consequently, we only need to demonstrate the result holds for the functional Φ_0, and the proof of the functional Φ_0 is standard, see [10, 35]. So we omit the details. $\square$

From Lemmas 2.3, 2.5, 2.6 and 2.7, we can deduce easily that the following conclusion holds.

Lemma 2.8. *Assume that* (V), (f_1), (f_2), (f_3) *and* (f_5) *are satisfied and* $0 \leq \kappa < \kappa^*\nu_0^2$. *Then there exist a constant* $\tilde{c}_\kappa \in [\alpha, \sup\Phi_\kappa(Q)]$ *and a correspond sequence* $\{u_n\} \subset E$ *such that*

$$\Phi_\kappa(u_n) \to \tilde{c}_\kappa \ \textit{and}\ \|\Phi_\kappa'(u_n)\|(1 + \|u_n\|) \to 0.$$

In what follows we will take advantage of the non-Nehari method developed by Tang [31] to construct a special $(C)_{c_\kappa}$-sequence for some $c_\kappa \in [\alpha, m_\kappa]$, which is very crucial in our analysis.

Lemma 2.9. *Assume that* (V), (f_1), (f_2), (f_3) *and* (f_5) *are satisfied and* $0 \leq \kappa < \kappa^*\nu_0^2$. *Then there exist a constant* $c_\kappa \in [\alpha, m_\kappa]$ *and a correspond sequence* $\{u_n\} \subset E$ *such that*

$$\Phi_\kappa(u_n) \to c_\kappa \ \textit{and}\ \|\Phi_\kappa'(u_n)\|(1 + \|u_n\|) \to 0.$$

Proof. We follow the idea of [31] to complete the proof. Note that, according to the definition of m_κ we can choose $v_k \in \mathscr{N}_\kappa$ such that

$$m_\kappa \leq \Phi_\kappa(v_k) < m_\kappa + \frac{1}{k}, \ k \in \mathbb{N}. \tag{2.13}$$

From Lemma 2.6, we can see that $\|v_k^+\|^2 \geq 2m_\kappa > 0$. Denote $e_k = v_k^+/\|v_k^+\|$. Then $e_k \in E^+$ and $\|e_k\| = 1$. According to Lemma 2.7, we infer that there exists $R_k > \max\{\rho, \|v_k\|\}$ such that $\sup\Phi_\kappa(\partial Q_k) \leq 0$, where

$$Q_k = \{se_k + w : w \in E^-, \ s \geq 0, \ \|se_k + w\| \leq R_k\}, \ k \in \mathbb{N}. \tag{2.14}$$

Thus, using Lemma 2.8 we can get a constant $c_{\kappa,k} \in [\alpha, \sup \Phi_\kappa(Q_k)]$ and a correspond sequence $\{u_{k,n}\} \subset E$ such that

$$\Phi_\kappa(u_{k,n}) \to c_{\kappa,k} \text{ and } \|\Phi'_\kappa(u_{k,n})\|(1 + \|u_{k,n}\|) \to 0, \ k \in \mathbb{N}. \tag{2.15}$$

We observe that from Lemma 2.4 we can obtain

$$\Phi_\kappa(v_k) \geq \Phi_\kappa(tv_k + z), \ \forall\, t \geq 0, \ z \in E^-. \tag{2.16}$$

Since $v_k \in Q_k$, we deduce from (2.14) and (2.16) that $\sup \Phi_\kappa(Q_k) = \Phi_\kappa(v_k)$. Therefore, according to (2.13) and (2.15) we have

$$\Phi_\kappa(u_{k,n}) \to c_{\kappa,k} < m_\kappa + \frac{1}{k} \text{ and } \|\Phi'_\kappa(u_{k,n})\|(1 + \|u_{k,n}\|) \to 0, \ k \in \mathbb{N}.$$

Using the diagonal method, we can choose a sequence $\{n_k\} \subset \mathbb{N}$ such that

$$\Phi_\kappa(u_{k,n_k}) < m_\kappa + \frac{1}{k} \text{ and } \|\Phi'_\kappa(u_{k,n_k})\|(1 + \|u_{k,n_k}\|) < \frac{1}{k}, \ k \in \mathbb{N}.$$

Let $u_k = u_{k,n_k}, k \in \mathbb{N}$. Up to a subsequence, we see that

$$\Phi_\kappa(u_k) \to c_\kappa \in [\alpha, m_\kappa] \text{ and } \|\Phi'_\kappa(u_k)\|(1 + \|u_k\|) \to 0.$$

So, we complete the proof. $\square$

Similarly to the proof [29, Lemma 2.6], one can get the following important conclusion which will be very useful later.

Lemma 2.10. *Assume that (V), (f_1), (f_2), (f_3) and (f_5) are satisfied and $0 \leq \kappa < \kappa^* \nu_0^2$, and let $u \in E \setminus E^-$. Then $\mathscr{N}_\kappa \cap (E^- \oplus \mathbb{R}^+ u) \neq \emptyset$, i.e., there are $t > 0$ and $v \in E^-$ such that $tu + v \in \mathscr{N}_\kappa$.*

3. Ground state solutions

In this section, we are going to prove the existence of ground state solutions to problem (1.1). That is, we give the completed proofs of Theorem 1.1 and Theorem 1.2. We begin by analyzing the behaviors of $(C)_{c_\kappa}$-sequence which play a fundamental role in the study.

Lemma 3.1. *Assume that (V), (f_1), (f_2), (f_3) and (f_5) are satisfied and $0 \leq \kappa < \kappa^* \nu_0^2$. Then the $(C)_{c_\kappa}$-sequence $\{u_n\} \subset E$ obtained in Lemma 2.9 is bounded.*

Proof. Let $\{u_n\} \subset E$ be a $(C)_{c_\kappa}$-sequence satisfying

$$(1 + \|u_n\|)\Phi'_\kappa(u_n) \to 0 \text{ and } \Phi_\kappa(u_n) \to c_\kappa. \tag{3.1}$$

We argue by contradiction, assume that $\|u_n\| \to \infty$ as $n \to \infty$. Setting $v_n = u_n/\|u_n\|$, then $\|v_n\| = 1$. After passing to a subsequence, we may assume that $v_n \rightharpoonup v$ in E, $v_n \to v$ in $L^q_{loc}(\mathbb{R}^N)$ for $q \in [2, 2_s^*)$ and $v_n(x) \to v(x)$ a.e. $x \in \mathbb{R}^N$. Let

$$\delta := \lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |v_n^+|^2 dx.$$

If $\delta = 0$, using the Lions concentration compactness lemma (see [21]), we know that $v_n^+ \to 0$ in $L^q(\mathbb{R}^N)$ for any $q \in (2, 2_s^*)$. Employing (2.7), we can deduce that

$$\int_{\mathbb{R}^N} F(x, sv_n^+)dx \to 0 \text{ as } n \to \infty, \ \text{for any } s > 0. \tag{3.2}$$

Let $t_n = s/\|u_n\|$, from (2.6), (2.9), (3.1) and (3.2) we can infer that there exists $C > 0$,

$$
\begin{aligned}
C \geq \Phi_\kappa(u_n) &\geq \Phi_\kappa(t_n u_n + (-t_n u_n^-)) - \frac{t_n^2 - 1}{2}\langle \Phi'_\kappa(u_n), u_n\rangle + t_n^2 \langle \Phi'_\kappa(u_n), u_n^-\rangle \\
&= \Phi_\kappa(s v_n^+) + o(1) \\
&= \frac{s^2}{2}\left(\|v_n^+\|^2 - \kappa \int_{\mathbb{R}^N} \frac{|v_n^+|^2}{|x|^{2s}} dx\right) + o(1) \\
&\geq \frac{s^2}{2}\left(1 - \frac{\kappa}{\kappa^* \nu_0^2}\right)\|v_n^+\|^2 + o(1).
\end{aligned} \tag{3.3}
$$

On the other hand, from (2.8) we can deduce that

$$
\begin{aligned}
\Phi'_\kappa(u_n), u_n\rangle =& \|u_n^+\|^2 - \|u_n^-\|^2 - \int_{\mathbb{R}^N} \left[\frac{1}{2} f(x, u_n) u_n - F(x, u_n)\right] dx \\
\leq& \|u_n^+\|^2 - \|u_n^-\|^2,
\end{aligned}
$$

consequently, from Lemma 2.6-(ii), we can get

$$
2\|u_n^+\|^2 \geq \|u_n^+\|^2 + \|u_n^-\|^2 + \langle \Phi'_\kappa(u_n), u_n\rangle = \|u_n\|^2 + \langle \Phi'_\kappa(u_n), u_n\rangle.
$$

Evidently, we can see that $\|v_n^+\|^2 \geq c_3 > 0$. Therefore, we can get a contradiction in (3.3) if s is large enough.

From the above discussions, we know $\delta = 0$ does not occur, and we may assume that $\delta > 0$. Up to a subsequence, there exists $\{k_n\} \subset \mathbb{Z}^N$ such that

$$
\int_{B(k_n, 1+\sqrt{N})} |v_n^+|^2 dx > \frac{\delta}{2}.
$$

Let us define $\widehat{v}_n(x) = v_n(x + k_n)$, then we have

$$
\int_{B(0, 1+\sqrt{N})} |\widehat{v}_n^+|^2 dx > \frac{\delta}{2}.
$$

Passing to a subsequence, $\widehat{v}_n^+ \to \widehat{v}^+$ in $L^2_{loc}(\mathbb{R}^N)$ and $\widehat{v}^+ \neq 0$, which implies that $|u_n(x + k_n)| = |\widehat{v}_n(x)|\|u_n\| \to \infty$. According to (f_3) we obtain

$$
\frac{F(x + k_n, u_n(x + k_n))}{\|u_n\|^2} = \frac{F(x + k_n, u_n(x + k_n))}{|u_n(x + k_n)|^2}|\widehat{v}_n|^2 \to \infty \text{ for } x \in \{\widehat{v} \neq 0\}.
$$

Thus, it follows from Fatou's lemma that

$$
\begin{aligned}
0 &= \lim_{n\to\infty} \frac{\Phi_\kappa(u_n)}{\|u_n\|^2} \\
&\leq \lim_{n\to\infty} \left[\frac{1}{2}\left(\|v_n^+\|^2 - \|v_n^-\|^2 - \kappa \int_{\mathbb{R}^N} \frac{|v_n|^2}{|x|^{2s}} dx\right) - \int_{\{\widehat{v}\neq 0\}} \frac{F(x + k_n, u_n(x + k_n))}{\|u_n\|^2} dx\right] \\
&\leq \frac{1}{2} - \int_{\{\widehat{v}\neq 0\}} \lim_{n\to\infty} \inf \frac{F(x + k_n, u_n(x + k_n))}{\|u_n\|^2} dx \\
&\to -\infty,
\end{aligned}
$$

which yields a contradiction. The proof is completed. $\square$

We introduce the following result which is very useful to deal with the Hardy-type term and plays a very important role in the proof of the global compactness result.

Lemma 3.2. *For any $u \in E$, if $|x_n| \to \infty$, then we have the conclusion*

$$
\int_{\mathbb{R}^N} \frac{|u(x - x_n)|^2}{|x|^{2s}} dx \to 0 \ \textit{as}\ n \to \infty.
$$

Following the idea of [1] and combining Lemma 2.2 and a approximation argument, we can easily prove Lemma 3.2. More details can be found in [1, Lemma 2.5], we omit it here.

Next we will make use of Lemma 3.1 and Lemma 3.2 to establish a global compactness result, which plays an important role in dealing with the difficulty caused by the lack of compactness of the Sobolev embedding.

Lemma 3.3. *Assume that (V), (f_1), (f_2), (f_3) and (f_5) are satisfied and $0 \le \kappa < \kappa^* \nu_0^2$, and let $\{u_n\}$ be a bounded $(C)_{c_\kappa}$-sequences of Φ_κ. Then there exists $u_\kappa \in E$ such that $\Phi_\kappa'(u_\kappa) = 0$, moreover, we have either*
(i) *$u_n \to u_\kappa$ in E, or*
(ii) *there exist number $k \ge 1$, nontrivial critical points $u_1, \cdots, u_k$ of Φ_0 and k sequences of points $\{x_n^i\} \subset \mathbb{Z}^N$, $1 \le i \le k$, such that*

$$|x_n^i| \to +\infty,\ |x_n^i - x_n^j| \to +\infty,\ \text{if } i \neq j,\ i,j = 1,2,\cdots,k,$$

$$\left\| u_n - u_\kappa - \sum_{i=1}^{k} u_i(\cdot - x_n^i) \right\| \to 0 \ \text{and}\ c_\kappa = \Phi_\kappa(u_\kappa) + \sum_{i=1}^{k} \Phi_0(u_i).$$

Proof. Let $\{u_n\}$ be a bounded $(C)_{c_\kappa}$-sequences of Φ_κ. From Lemma 2.2 and (2.6) we know that $\{u_n\}$ is bounded in $L^2(\mathbb{R}^N, \frac{1}{|x|^{2s}})$. Then, after passing to a subsequence, we may assume that

$$\begin{aligned} &u_n \rightharpoonup u_\kappa \text{ in } E,\ u_n \rightharpoonup u_\kappa \text{ in } L^2\left(\mathbb{R}^N, \frac{1}{|x|^{2s}}\right) \\ &u_n \to u_\kappa \text{ in } L^2_{loc}(\mathbb{R}^N),\ u_n(x) \to u_\kappa(x) \text{ a.e. on } \mathbb{R}^N. \end{aligned}$$

It follows from Lemma 2.3 that $\Phi_\kappa'(u_\kappa) = 0$. Setting $v_n = u_n - u_\kappa$, then

$$v_n^+ \rightharpoonup 0 \text{ in } E^+,\ v_n^- \rightharpoonup 0 \text{ in } E^- \text{ and } v_n \rightharpoonup 0 \text{ in } L^2(\mathbb{R}^N, \frac{1}{|x|^{2s}}).$$

Computing directly, we have

$$\begin{aligned} \|v_n^+\|^2 &= \|u_n^+\|^2 - \|u_\kappa^+\|^2 + o(1), \\ \|v_n^-\|^2 &= \|u_n^-\|^2 - \|u_\kappa^-\|^2 + o(1), \\ \|v_n\|^2 &= \|u_n\|^2 - \|u_\kappa\|^2 + o(1). \end{aligned} \tag{3.4}$$

$$\int_{\mathbb{R}^N} \frac{|v_n|^2}{|x|^{2s}} dx = \int_{\mathbb{R}^N} \frac{|u_n|^2}{|x|^{2s}} dx - \int_{\mathbb{R}^N} \frac{|u_\kappa|^2}{|x|^{2s}} dx + o(1). \tag{3.5}$$

Using some standard arguments from [10] we can easily check that

$$\int_{\mathbb{R}^N} [F(x,u_n) - F(x,u_\kappa) - F(x,v_n)]\, dx = o(1), \tag{3.6}$$

$$\int_{\mathbb{R}^N} [f(x,u_n) - f(x,u_\kappa) - f(x,v_n)]\, \phi dx = o(1),\ \forall \phi \in E, \tag{3.7}$$

and

$$\int_{\mathbb{R}^N} \frac{u_n \phi}{|x|^{2s}} dx - \int_{\mathbb{R}^N} \frac{u_\kappa \phi}{|x|^{2s}} dx - \int_{\mathbb{R}^N} \frac{v_n \phi}{|x|^{2s}} dx = o(1),\ \forall \phi \in E. \tag{3.8}$$

Therefore, we deduce from (3.4), (3.5) and (3.6) that

$$\Phi_\kappa(u_n) - \Phi_\kappa(u_\kappa) - \Phi_\kappa(v_n) = o(1). \tag{3.9}$$

Similarly, according to (3.7) and (3.8) we also have

$$\langle \Phi_\kappa'(u_n), \phi \rangle - \langle \Phi_\kappa'(u_\kappa), \phi \rangle - \langle \Phi_\kappa'(v_n), \phi \rangle = o(1),\ \forall \phi \in E. \tag{3.10}$$

Next we discuss the following two cases: (a) $\{v_n\}$ is vanishing, and (b) $\{v_n\}$ is non-vanishing. For the case (a), if $\{v_n\}$ is vanishing, then

$$\lim_{n\to\infty}\sup_{y\in\mathbb{R}^N}\int_{B(y,1)}|v_n|^2dx=0.$$

Making use of the Lions concentration compactness lemma (see [21]), we can know that $v_n\to 0$ in $L^q(\mathbb{R}^N)$ for any $q\in(2,2_s^*)$. On the other hand, according to the facts that the orthogonal projection of E on E^+ and E^- is continuous in $L^q(\mathbb{R}^N)$, we have $v_n^+\to 0$ and $v_n^-\to 0$ in $L^q(\mathbb{R}^N)$ for any $q\in(2,2_s^*)$. Thus it follows from (2.7) that

$$\begin{aligned}&\int_{\mathbb{R}^N}\left[f(x,u_n)-f(x,u_\kappa)\right]v_n^+dx=o(1),\\&\int_{\mathbb{R}^N}\left[f(x,u_\kappa)-f(x,u_n)\right]v_n^-dx=o(1).\end{aligned}\tag{3.11}$$

Since $\{u_n\}$ is bounded $(C)_{c_\kappa}$-sequences, we can get that $\Phi_\kappa'(u_n)=o(1)$ and $\langle\Phi_\kappa'(u_n),v_n^\pm\rangle=o(1)$. Combining the above facts and $\Phi_\kappa'(u_\kappa)=0$, we deduce that

$$\begin{aligned}o(1)&=\langle\Phi_\kappa'(u_n),v_n^+\rangle\\&=(u_n,v_n^+)-\kappa\int_{\mathbb{R}^N}\frac{u_nv_n^+}{|x|^{2s}}dx-\int_{\mathbb{R}^N}f(x,u_n)v_n^+dx\\&=\|v_n^+\|^2-\kappa\int_{\mathbb{R}^N}\frac{v_nv_n^+}{|x|^{2s}}dx-\int_{\mathbb{R}^N}f(x,u_n)v_n^+dx+\langle\Phi_\kappa'(u_\kappa),v_n^+\rangle+\int_{\mathbb{R}^N}f(x,u_\kappa)v_n^+dx\\&=\|v_n^+\|^2-\kappa\int_{\mathbb{R}^N}\frac{|v_n^+|^2}{|x|^{2s}}dx-\kappa\int_{\mathbb{R}^N}\frac{v_n^-v_n^+}{|x|^{2s}}dx+\int_{\mathbb{R}^N}\left[f(x,u_\kappa)-f(x,u_n)\right]v_n^+dx.\end{aligned}$$

This, together with (2.6), implies that

$$\left[1-\frac{\kappa}{\kappa^*\nu_0^2}\right]\|v_n^+\|^2\le\kappa\int_{\mathbb{R}^N}\frac{v_n^-v_n^+}{|x|^{2s}}dx+\int_{\mathbb{R}^N}\left[f(x,u_n)-f(x,u_\kappa)\right]v_n^+dx+o(1).\tag{3.12}$$

Using the same arguments we also have

$$\begin{aligned}o(1)&=\langle\Phi_\kappa'(u_n),v_n^-\rangle\\&=-\|v_n^-\|^2-\kappa\int_{\mathbb{R}^N}\frac{|v_n^-|^2}{|x|^{2s}}dx-\kappa\int_{\mathbb{R}^N}\frac{v_n^-v_n^+}{|x|^{2s}}dx+\int_{\mathbb{R}^N}\left[f(x,u_\kappa)-f(x,u_n)\right]v_n^-dx,\end{aligned}$$

it follows that

$$\|v_n^-\|^2\le-\kappa\int_{\mathbb{R}^N}\frac{v_n^-v_n^+}{|x|^{2s}}dx+\int_{\mathbb{R}^N}\left[f(x,u_\kappa)-f(x,u_n)\right]v_n^-dx+o(1).\tag{3.13}$$

Therefore, from (3.11), (3.12) and (3.13) we conclude that $\|v_n\|\to 0$ in E, and $u_n\to u_\kappa$ in E. Consequently, we prove that the conclusion (i) holds.

For the case (b), if $\{v_n\}$ is non-vanishing, then there exist $\delta>0$, $\varrho>1$ and $\{y_n\}\subset\mathbb{Z}^N$ such that

$$\liminf_{n\to\infty}\int_{B(y_n,\varrho)}|v_n|^2dx\ge\delta.\tag{3.14}$$

Evidently, $\{y_n\}$ is unbounded. After passing to a subsequence, we may assume that $|y_n|\to\infty$. Let us define $\tilde v_n=v_n(x+y_n)$. According to (3.14), up to a subsequence, we can find $u_1\ne 0$ such that $\tilde v_n\rightharpoonup u_1$ in E, $\tilde v_n\to u_1$ in $L^q_{loc}(\mathbb{R}^N)$ for $q\in(2,2_s^*)$ and $\tilde v_n(x)\to u_1(x)$ a.e. $x\in\mathbb{R}^N$.

Applying the Hölder inequality and Lemma 3.2, it is easy to see that

$$\left|\int_{\mathbb{R}^N}\frac{1}{|x|^{2s}}v_n\varphi(x-y_n)dx\right|\le\left[\int_{\mathbb{R}^N}\frac{|v_n|^2}{|x|^{2s}}dx\right]^{\frac12}\left[\int_{\mathbb{R}^N}\frac{|\varphi(x-y_n)|^2}{|x|^{2s}}dx\right]^{\frac12}\to 0,\ \forall\varphi\in E.$$

Taking advantage of the above conclusion, we have

$$
\begin{aligned}
o(1) &= \langle \Phi_\kappa'(v_n), \varphi(x-y_n)\rangle \\
&= (v_n^+, \varphi^+(x-y_n)) - (v_n^-, \varphi^-(x-y_n)) - \kappa\int_{\mathbb{R}^N} \frac{v_n\varphi(x-y_n)}{|x|^{2s}}dx \\
&\quad - \int_{\mathbb{R}^N} f(x,v_n)\varphi(x-y_n)dx \\
&= (\tilde{v}_n^+, \varphi^+) - (\tilde{v}_n^-, \varphi^-) - \int_{\mathbb{R}^N} f(x,\tilde{v}_n)\varphi dx + o(1) \\
&= \langle \Phi_0'(\tilde{v}_n), \varphi\rangle + o(1),
\end{aligned} \tag{3.15}
$$

which implies that $\langle \Phi_0'(u_1), \varphi\rangle = 0$. Hence, u_1 is a nontrivial critical point of Φ_0.

Now denote $v_n^1 = u_n - u_\kappa - u_1(\cdot - y_n)$. Similarly to (3.4) and (3.5), we have

$$
\begin{aligned}
&\|u_n^+\|^2 - \|u_\kappa^+\|^2 - \|u_1^+\|^2 - \|v_n^{1+}\|^2 = o(1), \\
&\|u_n^-\|^2 - \|u_\kappa^-\|^2 - \|u_1^-\|^2 - \|v_n^{1-}\|^2 = o(1), \\
&\|u_n\|^2 - \|u_\kappa\|^2 - \|u_1\|^2 - \|v_n^1\|^2 = o(1),
\end{aligned} \tag{3.16}
$$

and

$$
\int_{\mathbb{R}^N} \frac{|u_n-u_\kappa|^2}{|x|^{2s}}dx - \int_{\mathbb{R}^N} \frac{|u_1(x-y_n)|^2}{|x|^{2s}}dx - \int_{\mathbb{R}^N} \frac{|v_n^1|^2}{|x|^{2s}}dx = o(1). \tag{3.17}
$$

Applying Lemma 3.2 and the Brezis-Lieb Lemma, from (3.17) we can get

$$
\int_{\mathbb{R}^N} \frac{|v_n^1|^2}{|x|^{2s}}dx = \int_{\mathbb{R}^N} \frac{|u_n|^2}{|x|^{2s}}dx - \int_{\mathbb{R}^N} \frac{|u_\kappa|^2}{|x|^{2s}}dx + o(1). \tag{3.18}
$$

Similarly to (3.6), there holds

$$
\int_{\mathbb{R}^N} \left[F(x,u_n) - F(x,u_\kappa) - F(x,u_1) - F(x,v_n^1)\right] dx = o(1). \tag{3.19}
$$

Therefore, using (3.16)-(3.19), we conclude that

$$
\Phi_\kappa(u_n) - \Phi_\kappa(u_\kappa) - \Phi_\kappa(v_n^1) - \Phi_0(u_1) = o(1) \tag{3.20}
$$

and we take $x_n^1 := y_n$. We replace v_n by v_n^1 and repeat the above arguments in vanishing case and non-vanishing case. If

$$
\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |v_n^1|^2 dx = 0.
$$

Following the proof of conclusion (i), we have $v_n^1 \to 0$ in E. Then we deduce from (3.16) and (3.20) that $k=1$.

Otherwise, arguing as the proof of non-vanishing case, we can may find $\{y_n\} \subset \mathbb{Z}^N$ such that (3.14) holds for $\{v_n^1\}$. Then passing to a subsequence $|y_n| \to \infty$ and $|y_n - x_n^1| \to \infty$ as $n\to\infty$. Adapting the above argument, let $\tilde{v}_n^1(x) = v_n(x+y_n)$, then we can find $u_2 \neq 0$ such that, up to a subsequence, $\tilde{v}_n^1 \rightharpoonup u_2$ in E, $\tilde{v}_n^1 \to u_2$ in $L^q_{loc}(\mathbb{R}^N)$ for $q \in (2, 2_s^*)$ and $\tilde{v}_n^1(x) \to u_2(x)$ a.e. $x \in \mathbb{R}^N$. Following the proof in (3.15), we see that u_2 is a nontrivial critical point of Φ_0. Denote $v_n^2 = u_n - u_\kappa - u_1(\cdot - x_n^1) - u_2(\cdot - y_n)$, and similarly to (3.16) and (3.20), we have

$$
\begin{aligned}
&\|u_n^+\|^2 - \|u_\kappa^+\|^2 - \|u_1^+\|^2 - \|u_2^+\|^2 - \|v_n^{2+}\|^2 = o(1), \\
&\|u_n^-\|^2 - \|u_\kappa^-\|^2 - \|u_1^-\|^2 - \|u_2^-\|^2 - \|v_n^{2-}\|^2 = o(1), \\
&\|u_n\|^2 - \|u_\kappa\|^2 - \|u_1\|^2 - \|u_2\|^2 - \|v_n^2\|^2 = o(1), \\
&\Phi_\kappa(u_n) - \Phi_\kappa(u_\kappa) - \Phi_\kappa(v_n^2) - \Phi_0(u_1) - \Phi_0(u_2) = o(1),
\end{aligned}
$$

and $x_n^2 := y_n$. Again repeating the above arguments, we claim that the iterations must stop after finite steps. Indeed, using $\langle \Phi_0'(u), u^{\pm}\rangle = 0$ and (2.7) we have

$$\|u^+\|^2 \leq \int_{\mathbb{R}^N} |f(x,u)u^+|dx \leq \epsilon\|u^+\|\|u\| + C_\epsilon\|u^+\|\|u\|^{p-1}$$

and

$$\|u^-\|^2 \leq \int_{\mathbb{R}^N} |f(x,u)u^-|dx \leq \epsilon\|u^-\|\|u\| + C_\epsilon\|u^-\|\|u\|^{p-1}.$$

Consequently,

$$\|u\|^2 \leq 2\epsilon\|u\|^2 + 2C_\epsilon\|u\|^p.$$

Obviously, there is a constant $\rho_0 > 0$ such that

$$\|u\| \geq \rho_0 \text{ for any } u \neq 0 \text{ with } \Phi_0'(u) = 0, \tag{3.21}$$

which implies the claim above is true. So we finish the proof of the lemma. □

Now we are in a position to finish the proofs of Theorem 1.1.

Proof of Theorem 1.1. From Lemma 2.9, we see that there exists a $(C)_{c_\kappa}$-sequence $\{u_n\} \subset E$ of Φ_κ such that

$$\Phi_\kappa(u_n) \to c_\kappa \leq m_\kappa \text{ and } (1 + \|u_n\|)\|\Phi_\kappa'(u_n)\| \to 0.$$

Lemma 3.1 shows that $\{u_n\}$ is bounded in E, then up to a subsequence, $u_n \rightharpoonup u_\kappa$ in E, $u_n(x) \to u_\kappa(x)$ a.e. $x \in \mathbb{R}^N$. Using Lemma 2.3 we have $\Phi_\kappa'(u_\kappa) = 0$. If $u_\kappa \neq 0$, we can see that $u_\kappa \in \mathscr{N}_\kappa$. On the other hand, from (2.8) and Fatou's lemma, we get

$$\begin{aligned} m_\kappa \geq c_\kappa &= \lim_{n\to\infty}\left[\Phi_\kappa(u_n) - \frac{1}{2}\langle \Phi_\kappa'(u_n), u_n\rangle\right] \\ &= \lim_{n\to\infty}\int_{\mathbb{R}^N}\left[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right]dx \\ &\geq \int_{\mathbb{R}^N}\lim_{n\to\infty}\left[\frac{1}{2}f(x,u_n)u_n - F(x,u_n)\right]dx \\ &= \Phi_\kappa(u_\kappa) - \frac{1}{2}\langle \Phi_\kappa'(u_\kappa), u_\kappa\rangle = \Phi_\kappa(u_\kappa), \end{aligned}$$

which implies that $\Phi_\kappa(u_\kappa) \leq m_\kappa$. Consequently, from the definition of m_κ, we have $\Phi_\kappa(u_\kappa) = m_\kappa$. So u_κ is a ground state solution of problem (1.1).

Next we show that $u_\kappa \neq 0$. Indeed, for the case $\kappa = 0$, the functional Φ_0 has the property of translation invariance under the condition (f_4). Then, using a standard variational argument and concentration compactness principle, we can get a nontrivial ground state solution $u_0 \in \mathscr{N}_0$ satisfying $\Phi_0(u_0) = m_0$. According to Lemma 2.10, we know that there exist $t_0 > 0$ and $v_0 \in E^-$ such that $t_0u_0 + v_0 \in \mathscr{N}_\kappa$. This combines with Lemma 2.4 we can get

$$m_0 = \Phi_0(u_0) \geq \Phi_0(t_0u_0 + v_0) > \Phi_\kappa(t_0u_0 + v_0) \geq m_\kappa \geq c_\kappa.$$

Consequently, by Lemma 3.3 we get $k = 0$, that is, $u_n \to u_\kappa$ in E. So, $u_\kappa \neq 0$. This finishes the proof of Theorem 1.1. □

In the following we study the asymptotically periodic case. Firstly, we need to introduce two useful results due to [30, 37].

Lemma 3.4. *Assume that (f_6) is satisfied, and let $\{u_n\} \subset E$ be a bounded sequence and $\varphi_n(x) = \varphi(x - x_n)$, where $\varphi \in E$ and $\{x_n\} \subset \mathbb{R}^N$. If $|x_n| \to \infty$, then we have*

$$\int_{\mathbb{R}^N} |\widehat{f}(x,u_n) - f(x,u_n)\varphi_n|dx \to 0.$$

Lemma 3.5. *Assume that $a \in \mathscr{W}$ and $\tau \in [2, 2_s^*)$, and let $\{u_n\} \subset E$ be a sequence such that $u_n \rightharpoonup u$ in E. Then*

$$\lim_{n\to\infty}\int_{\mathbb{R}^N} a(x)|u_n|^\tau dx = \int_{\mathbb{R}^N} a(x)|u|^\tau dx.$$

Moreover, from (f_6), (2.7) and Lemma 3.5, we can obtain that

$$\int_{\mathbb{R}^N}\left[\widehat{F}(x,u_n)-F(x,u_n)\right]dx\to 0.$$

To restore the lack of compactness, we need to use the techniques of the limit problem. And we consider the following limit problem of problem (1.1)

$$-(\Delta)^s u+V(x)u=\widehat{f}(x,u),\ x\in\mathbb{R}^N, \tag{3.22}$$

where $\widehat{f}$ is $\mathbb{Z}^N$-periodic in the x-variables and satisfies the conditions given in (f_6). We define the corresponding energy functional of problem (3.22)

$$\widehat{\Phi}_0(u)=\frac{1}{2}(\|u^+\|^2-\|u^-\|^2)-\int_{\mathbb{R}^N}\widehat{F}(x,u)dx,\ \text{for } u\in E.$$

From [13, Theorem 1.2], we can conclude that problem (3.22) has a ground state solution u with

$$\widehat{m}_0=\widehat{\Phi}_0(u)=\inf_{\widehat{\mathcal{N}_0}}\widehat{\Phi}_0>0,$$

where

$$\widehat{\mathcal{N}_0}:=\{u\in E\backslash E^-:\langle\widehat{\Phi}_0'(u),u\rangle=0 \text{ and } \langle\widehat{\Phi}_0'(u),v\rangle=0 \text{ for any } v\in E^-\}.$$

Applying Lemma 3.4 and Lemma 3.5 and following the analogous arguments as in the proofs of Lemma 3.3, we also can establish a global compactness result for bounded $(C)_{c_\kappa}$-sequences of Φ_κ under the asymptotically periodic condition. We now present the result as follow.

Lemma 3.6. *Assume that (V), (f_1), (f_2), (f_3), (f_5) and (f_6) are satisfied and $0\le\kappa<\kappa^*\nu_0^2$, and let $\{u_n\}$ be a bounded $(C)_{c_\kappa}$-sequences of Φ_κ at level $c_\kappa\ge 0$. Then there exist u_κ such that $\Phi_\kappa'(u_\kappa)=0$, moreover, we have either*
(i) *$u_n\to u_\kappa$ in E, or*
(ii) *there exist a number $l\ge 1$, nontrivial critical points $u_1,\cdots,u_l$ of $\widehat{\Phi}_0$ and l sequences of points $\{x_n^i\}\subset\mathbb{Z}^N$, $1\le i\le l$, such that*

$$|x_n^i|\to+\infty,\ |x_n^i-x_n^j|\to+\infty,\ \text{if } i\ne j,\ i,j=1,2,\cdots,l,$$

$$\left\|u_n-u_\kappa-\sum_{i=1}^l u_i(\cdot-x_n^i)\right\|\to 0 \text{ and } c_\kappa=\Phi_\kappa(u_\kappa)+\sum_{i=1}^l\widehat{\Phi}_0(u_i).$$

Proof of Theorem 1.2. We adapt the idea of the proof of Theorem 1.1, and we replace m_0, Φ_0 and $\mathcal{N}_0$ by $\widehat{m}_0$, $\widehat{\Phi}_0$ and $\widehat{\mathcal{N}_0}$ in the proof. The remaining proof is similar to the proof of Theorem 1.1 with suitable modification, here we omit the details. □

4. Asymptotic behaviors

In this section we study the continuous dependence of ground state energy about parameter κ. and the asymptotic convergence of ground state solutions when $\kappa\to 0$. Moreover, we complete the proofs of Theorem 1.3 and Theorem 1.4.

Proof of Theorem 1.3. Evidently, we observe that if $\kappa_1\ge\kappa_2$, then $\Phi_{\kappa_1}(u)\le\Phi_{\kappa_2}(u)$. Hence $m_{\kappa_1}\le m_{\kappa_2}$, this shows that m_κ is decreasing.

Next we prove that $m_\kappa\to m_0$ as $\kappa\to 0$. We first need to describe the relationship between m_κ and m_0. Let $u_\kappa\in\mathcal{N}_\kappa$ be a ground state of Φ_κ. Applying Lemma 2.10, we can see that there exist $t_\kappa>0$ and $v_\kappa\in E^-$ such that $t_\kappa u_\kappa+v_\kappa\in\mathcal{N}_0$. Then, in view of Lemma 2.4 we have

$$\begin{aligned}m_\kappa&=\Phi_\kappa(u_\kappa)\ge\Phi_\kappa(t_\kappa u_\kappa+v_\kappa)\\&=\Phi_0(t_\kappa u_\kappa+v_\kappa)-\frac{\kappa}{2}\int_{\mathbb{R}^N}\frac{|t_\kappa u_\kappa+v_\kappa|^2}{|x|^{2s}}dx\\&\ge m_0-\frac{\kappa}{2}\int_{\mathbb{R}^N}\frac{|t_\kappa u_\kappa+v_\kappa|^2}{|x|^{2s}}dx.\end{aligned}\tag{4.1}$$

Similarly, let $u_0 \in \mathscr{N}_0$ be a ground state solution of Φ_0 with $\Phi_0(u_0) = m_0$. Again using Lemma 2.10, there exist $t_0 > 0$ and $v_0 \in E^-$ such that $t_0u_0 + v_0 \in \mathscr{N}_\kappa$. According to Lemma 2.4, we have

$$\begin{aligned} m_0 &= \Phi_0(u_0) \geq \Phi_0(t_0u_0 + v_0) \\ &= \Phi_\kappa(t_0u_0 + v_0) + \frac{\kappa}{2}\int_{\mathbb{R}^N} \frac{|t_0u_0 + v_0|^2}{|x|^{2s}}dx \\ &\geq m_\kappa + \frac{\kappa}{2}\int_{\mathbb{R}^N} \frac{|t_0u_0 + v_0|^2}{|x|^{2s}}dx. \end{aligned} \tag{4.2}$$

We take a sequence $\kappa_n \to 0^+$. Let $u_{\kappa_n} \in \mathscr{N}_{\kappa_n}$, then we can see that

$$\Phi_{\kappa_n}(u_{\kappa_n}) = m_{\kappa_n} \leq m_0 \text{ and } \langle \Phi'_{\kappa_n}(u_{\kappa_n}), u_{\kappa_n}\rangle = 0. \tag{4.3}$$

To simplify the notation, we denote u_{κ_n} by u_n. From Lemma 3.1, we know that $\{u_n\}$ is bounded in E. If

$$\lim_{n\to\infty} \sup_{y\in\mathbb{R}^N} \int_{B(y,1)} |u_n^+|^2 dx = 0,$$

then the Lions concentration compactness lemma yields that $u_n^+ \to 0$ in $L^q(\mathbb{R}^N)$ for $q \in (2, 2_s^*)$. Therefore, from the fact that $u_n \in \mathscr{N}_{\kappa_n}$, we get

$$\|u_n^+\|^2 = \kappa_n \int_{\mathbb{R}^N} \frac{u_n u_n^+}{|x|^{2s}}dx + \int_{\mathbb{R}^N} f(x, u_n)u_n^+ dx \to 0.$$

Then, we can conclude that

$$\limsup_{n\to\infty} \Phi_{\kappa_n}(u_n) \leq 0,$$

which contradicts Lemma 2.6-(i). Hence, there exist $\delta > 0$, $\varrho > 1$ and $\{y_n\} \subset \mathbb{Z}^N$ such that

$$\liminf_{n\to\infty} \int_{B(y_n,\varrho)} |u_n^+|^2 dx \geq \delta.$$

After passing to a subsequence, we may find $u \in E$ such that $u_n^+(\cdot + y_n) \to u^+$ in $L^2_{loc}(\mathbb{R}^N)$ and $u^+ \neq 0$. Moreover, we may assume that $u_n(\cdot + y_n) \rightharpoonup u$ in E, $u_n(x + y_n) \to u(x)$, $u_n^+(x + y_n) \to u^+(x)$ a.e. $x \in \mathbb{R}^N$. From Lemma 2.10, we infer that there exist $t_n > 0$ and $v_n \in E^-$ such that $t_nu_n + v_n \in \mathscr{N}_0$. Then by (2.8) we have

$$\begin{aligned} \|u_n^+\|^2 &= \left\|u_n^- + \frac{v_n}{t_n}\right\|^2 + \frac{1}{t_n^2}\int_{\mathbb{R}^N} f(x, t_nu_n + v_n)(t_nu_n + v_n)dx \\ &\geq \left\|u_n^- + \frac{v_n}{t_n}\right\|^2 + 2\int_{\mathbb{R}^N} \frac{F(x, t_n(u_n + \frac{v_n}{t_n}))}{t_n^2}dx. \end{aligned} \tag{4.4}$$

According to (f_3) and (4.4), we can see that $\{u_n + \frac{v_n}{t_n}\}$ is bounded in E. Hence, passing to a subsequence, there exists $v \in E^-$ such that $(u_n^- + \frac{v_n}{t_n})(x) \to v(x)$ a.e. $x \in \mathbb{R}^N$.

Now we verify that $\{t_n\}$ is also bounded. We use a contradiction argument to show this fact. If not, $|t_nu_n(x) + v_n(x)| = t_n|u_n + \frac{v_n}{t_n}| \to \infty$ provided that $(u^+ + v)(x) \neq 0$. Moreover, from (f_3) and Fatou's lemma, we can easily check that

$$\int_{\mathbb{R}^N} \frac{F(x, t_n|u_n + \frac{v_n}{t_n}|)}{t_n^2}dx \to \infty.$$

This contradicts (4.4), and so $\{t_n\}$ is bounded. Then $\{t_nu_n^+\}$ and $\{t_nu_n^- + v_n\}$ are both bounded. Consequently, using (2.6) we get

$$\frac{\kappa_n}{2}\int_{\mathbb{R}^N} \frac{|t_nu_n + v_n|^2}{|x|^{2s}}dx \leq \frac{\kappa_n}{2\kappa^*\nu_0^2}\|t_nu_n + v_n\|^2 \to 0 \text{ as } \kappa_n \to 0^+. \tag{4.5}$$

Finally, from (4.1), (4.3) and (4.5) we see that $m_\kappa \to m_0$ as $\kappa \to 0$. The proof is now complete. □

Proof of Theorem 1.4. Let $\{\kappa_n\}$ be a sequence with $\kappa_n \to 0^+$ as $n \to \infty$ and $\{u_{\kappa_n}\}$ be a sequence of ground state solutions of problem (1.1) with $\kappa = \kappa_n$. For convenience of notation, we denote $u_n := u_{\kappa_n}$. According to Lemma 3.1, we know that $\{u_n\}$ is bounded in E. Then passing to a subsequence, we may assume that $u_n \rightharpoonup u_0$ in E, $u_n \to u_0$ in $L^q_{loc}(\mathbb{R}^N)$ for $q \in (2, 2^*_s)$ and $u_n(x) \to u_0(x)$ a.e. $x \in \mathbb{R}^N$. Note that for any $\varphi \in E$, using the Hölder inequality and (2.6) we get

$$\langle \Phi_0'(u_n), \varphi\rangle = \langle \Phi_{\kappa_n}'(u_n), \varphi\rangle + \kappa_n \int_{\mathbb{R}^N} \frac{u_n\varphi}{|x|^{2s}} dx \to 0.$$

This implies that $\Phi_0'(u_0) = 0$. Then u_0 is a nontrivial critical point of Φ_0.

Next we show that u_0 is a ground state solution of Φ_0. Setting

$$\widetilde{F}(x,u) := \frac{1}{2} f(x,u)u - F(x,u).$$

Applying Fatou's lemma and the conclusion of Theorem 1.3, we have

$$\begin{aligned} m_0 &= \lim_{n\to\infty} \Phi_{\kappa_n}(u_n) = \lim_{n\to\infty} \left[\Phi_{\kappa_n}(u_n) - \frac{1}{2}\langle \Phi_{\kappa_n}'(u_n), u_n\rangle\right] \\ &= \lim_{n\to\infty} \int_{\mathbb{R}^N} \widetilde{F}(x,u_n)dx \geq \int_{\mathbb{R}^N} \widetilde{F}(x,u_0)dx \\ &= \Phi_0(u_0) - \frac{1}{2}\langle \Phi_0'(u_0), u_0\rangle = \Phi_0(u_0) \geq m_0, \end{aligned}$$

So, we conclude that u_0 is a ground state solution of Φ_0. Moreover, we also have

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \widetilde{F}(x,u_n)dx = \int_{\mathbb{R}^N} \widetilde{F}(x,u_0)dx. \tag{4.6}$$

Finally, we claim that $u_n \to u_0$ in E. Using a standard argument we can prove that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \left[\widetilde{F}(x,u_n) - \widetilde{F}(x,u_n-u_0) - \widetilde{F}(x,u_0)\right] dx = 0.$$

This, together with (4.6), implies that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} \widetilde{F}(x,u_n-u_0)dx = 0. \tag{4.7}$$

Moreover, combing with (f_7) we have

$$\begin{aligned} \int_{\mathbb{R}^N} |u_n-u_0|^2 dx &\leq \int_{\Omega_1} |u_n-u_0|^2 dx + \int_{\Omega_2} |u_n-u_0|^q dx \\ &\leq \int_{\mathbb{R}^N} \widetilde{F}(x,u_n-u_0)dx, \end{aligned} \tag{4.8}$$

where

$$\Omega_1 = \{x \in \mathbb{R}^N : |u_n-u_0| < 1\} \text{ and } \Omega_2 = \{x \in \mathbb{R}^N : |u_n-u_0| \geq 1\}.$$

Clearly, from (4.8) we can deduce that $u_n \to u_0$ in $L^2(\mathbb{R}^N)$. Since $\{u_n\}$ is bounded in E, then $\{u_n\}$ is also bounded in $L^2(\mathbb{R}^N)$ and $L^{2^*_s}(\mathbb{R}^N)$. Employing the Hölder inequality we get

$$\int_{\mathbb{R}^N} |u_n-u_0|^\nu dx \leq \left[\int_{\mathbb{R}^N} |u_n-u_0|^2 dx\right]^{\frac{\mu}{2}} \left[\int_{\mathbb{R}^N} |u_n-u_0|^{2^*_s} dx\right]^{\frac{\nu-\mu}{2^*_s}} \to 0,$$

where $\mu = 2(2^*_s-\nu)/(2^*_s-2)$ and $2 < \nu < 2^*_s$. Consequently, using (2.6), the Hölder inequality and the continuity of orthogonal projection of E on $E^\pm$, we conclude that

$$\begin{aligned} \|u_n^+ - u_0^+\|^2 =& \langle \Phi_{\kappa_n}'(u_n), u_n^+ - u_0^+\rangle - (u_0^+, u_n^+ - u_0^+) \\ &+ \kappa_n \int_{\mathbb{R}^N} \frac{u_n(u_n^+ - u_0^+)}{|x|^{2s}} dx + \int_{\mathbb{R}^N} f(x,u_n)(u_n^+ - u_0^+)dx \\ &\to 0, \end{aligned}$$

which implies that $u_n^+ \to u_0^+$ in E. Using the same argument, we can easily show that $u_n^- \to u_0^-$ in E. So $u_n \to u_0$ in E. We complete the proof of Theorem 1.4. $\square$

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