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RESEARCH ARTICLE

Nash-type equlibria for systems of partially potential nonlinear equations

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Summary

In this paper we study the existence of a Nash-type equlibrium for a non-potential nonlinear system by combining variational methods with the monotonicity approach. The advance over existing research is that we can consider systems of Dirichlet problems in which the operator is not necessarily linear.

KEYWORDS:
non-potential system, Nash type equlibrium, Browder-Minty Theorem, Dirichlet problem

MSC CLASSIFICATION:
47J30, 47J05, 58E30

1 | INTRODUCTION

In this work we are concerned with solvability of a system of non-potential nonlinear equations

$$\begin{cases} \partial_{u_1}\mathcal{E}_1(u) = 0, \\ \vdots \\ \partial_{u_m}\mathcal{E}_m(u) = 0 \end{cases} \tag{1}$$

which is connected to a family of functionals $\{\mathcal{E}_k\}_{k=1}^m$, where $\mathcal{E}_k : \mathcal{H} = \prod_{k=1}^m \mathcal{H}_k \longrightarrow \mathbb{R}$ and $\mathcal{H}_k$ is a real Hilbert space for $k = 1, \dots, m$. Since system (1) arises from calculating partial (Gâteaux) derivatives of functionals $\{\mathcal{E}_k\}_{k=1}^m$, it has a type of partial potentiality imbedded in it, meaning that it is potential with respect to each variable separately with the other held fixed. It must be noted that (1) does not correspond to critical points of any Euler type action functional. Such a formulation from the very beginning does not permit the usage of the classical variational methods. Nevertheless the type of componentwise minimization of family $\{\mathcal{E}_k\}_{k=1}^m$ is still possible and will be investigated in this work. As in[1], we say that that an element $w \in \mathcal{H}$ is a *Nash-type equilibrium* for the system of functionals $\{\mathcal{E}_k\}_{k=1}^m$ if

$$\mathcal{E}_k(w) = \min_{u\in\mathcal{H}_k} \mathcal{E}_k(w_1, \dots, w_{k-1}, u, w_{k+1}, \dots, w_m)$$

for $k = 1, \dots, m$. There has been some research towards the existence of the Nash-type equilibria since it was started in[1] and further developed in a sequence of papers:[2,3,4], which investigate this concept from different point of view and with various approaches related to the use of the Perov type contractions. The comprehensive overview of results related to notion of the *Nash-type equilibrium* is containd in[5], Chapter 8. In order to get rid of the assumption of the Lipschitz continuity of the nonlinear term, in[6] the first two authors proposed the approach involving variational and monotonicity methods combined with concepts introduced in[1]. In a consequence, the Browder-Minty Theorem, in a form of the Strongly Montone Principle, is used instead of the Perov type contraction. The abstract results from[6] were meant for systems governed by densely defined operators and suitable nonlinearities. Such an abstract framework determined the application mainly to equations involving the (negative)

Laplacian without any perturbation. In order to include the perturbed Laplacian we decided to formulate the abstract setting with the use of the Gelfand triple. With such an approach we are able to get the most of the monotonicity methods and improve application allowing even in a classical setting for not necessarily self-adjoint operators. In all sources mentioned, only self-adjoint operators are considered In this work we also improve the methodology of the proof taken from[6] by simplifying some steps and also using somehow different approach towards main existence tool.

Paper is organized as follows: we start with some preliminaries necessary for the understanding of further concepts. Then we proceed with abstract existence result for a system of nonlinear equations based on the Browder-Minty Theorem. We further obtain the existence of the Nash-type equilibrium for system (1) under suitable assumptions. Applications are shown to systems of Dirichlet problem driven by the perturbed Laplacian.

2 | PRELIMINARIES AND AUXILIARY RESULTS

2.1 | M–matrices

In this section we will consider $m \times m$ real matrices. We say that a matrix $A = [a_{ij}]$ is *a nonsingular M-matrix* if A has the following representation

$$A = \lambda I - B,$$

where $B = [b_{ij}]$ is *nonnegative*, that is $b_{ij} \geq 0$ for all $i, j = 1, \dots, m$, and $\rho(B) < \lambda$. Here $\rho(B) = \max_{\xi \in \sigma(B)} |\xi|$ denotes the spectral radius of B and I is identity matrix. For the study of M-matrices we refer to[7] and[8]. We recall some useful criteria for being M-matrix using notation form[8]. The Euclidean inner product in $\mathbb{R}^m$ will be denoted by $\langle \cdot | \cdot \rangle$.

Theorem 1 ([8])**.** Assume that $A = [a_{ij}]$ has a representation $A = \lambda I - B$ for some non-negative matrix $B = [b_{ij}]$. Then the following conditions are equivalent:

1. A is a non-singular M-matrix;

F_{15}. A is invertible and A^{-1} is non-negative;

I_{25}. there exists a positive diagonal matrix[1] $D = [d_{ij}]$ such that is for all $x \in \mathbb{R}^m \setminus \{0\}$ there is

$$\langle DAx | x \rangle > 0;$$

N_{39}. A has a positive diagonal and there exists a positive diagonal matrix D such that for every $i = 1, \dots, m$ we have

$$a_{ii} d_{ii} > \sum_{j \neq i} |a_{ij}| d_{jj}. \tag{2}$$

The above given condition I_{25} differs from the one originally given in[8]. However, it is easy to show that it is equivalent, see[6] for details.

Remark 1. Using condition N_{39} in Theorem 1 we can show that for every non-singular M-matrix A and for any non-zero diagonal matrix $E = [e_{ij}]$ (non-necessary positive) there exists $\varepsilon^* > 0$ such that $A - \varepsilon E$ is also a non-singular M-matrix whenever $\varepsilon \in (0, \varepsilon^*)$. Indeed, if A a is non-singular M-matrix, then (2) holds for some positive diagonal matrix D. Taking

$$\varepsilon < \frac{\min\limits_{i=1,\dots,m} \left\{ a_{ii} d_{ii} - \sum\limits_{j \neq i} |a_{ij}| d_{jj} \right\}}{2 \max\limits_{i=1,\dots,m} |e_{ii}| \max\limits_{i=1,\dots,m} d_{ii}} =: \varepsilon^*$$

[1] That is $d_{ii} > 0$ for $i = 1, \dots m$ and $d_{ij} = 0$ for distinct i and j.

we obtain that that for every $k = 1, \dots, m$ there is

$$\begin{aligned}(a_{kk} - \varepsilon e_{kk})d_{kk} - \sum_{j\neq k} |a_{kj} - \varepsilon e_{kj}| d_{jj} =&(a_{kk} - \varepsilon e_{kk})d_{kk} - \sum_{j\neq k} |a_{kj}| d_{jj} \\ \geq& a_{kk}d_{kk} - \sum_{j\neq k} |a_{ij}| d_{jj} \\ &- \tfrac{1}{2} \min_{i=1,\dots,m} \left\{ a_{ii}d_{ii} - \sum_{j\neq i} |a_{ij}| d_{jj} \right\} > 0.\end{aligned}$$

Hence $A - \epsilon E$ has a positive diagonal. Therefore, by Theorem 1, $A - \epsilon E$ is a non-singular M-matrix.

M-matrices are closely connected to convergent matrices. Some comparison, given in a context of a nonlinear equation is given in[6]. A brief summary will be indicated in Example 2.

2.2 | On a Gelfand triple

In this section, following[9], [10], we provide some necessary background on a Gelfand triple coined to the case of the Hilbert space setting. Let $\mathcal{V}$ and $\mathcal{H}$ be Hilbert spaces. A triple $(\mathcal{V}; \mathcal{H}; \mathcal{V}^*)$ is said to be *an evolution triple* or *a Gelfand triple*, if $\mathcal{V} \subset \mathcal{H} \cong \mathcal{H}^* \subset \mathcal{V}^*$, the embedding of $\mathcal{V}$ into $\mathcal{H}$ is continuous and $\mathcal{V}$ is dense in $\mathcal{H}$. Note that it is assumed that we identify $\mathcal{H}^*$ with $\mathcal{H}$ via the Riesz Theorem.

Example 1. Putting $\mathcal{V} = H_0^1(0,1)$ and $\mathcal{H} = L^2(0,1)$ we have the most common example of a Gelfand triple.

We will denote by $\|\cdot\|_{\mathcal{V}}$, $\|\cdot\|_{\mathcal{V}^*}$ the norm in $\mathcal{V}$, $\mathcal{V}^*$ resp. and by $\langle \cdot, \cdot \rangle$ the relevant duality pairing. Here $\|\cdot\|_{\mathcal{H}}$ denotes the norm in $\mathcal{H}$ and $\langle \cdot | \cdot \rangle_{\mathcal{H}}$ – the associated scalar product. We identify elements from $\mathcal{H}$ with some elements from $\mathcal{V}^*$. Precisely speaking for any $y \in \mathcal{H}$ there is some $f_y \in \mathcal{V}^*$ such that

$$\left\langle f_y, x \right\rangle = \langle y | x \rangle_{\mathcal{H}}$$

for all $x \in \mathcal{V}$. Since the embedding $\imath : \mathcal{V} \longrightarrow \mathcal{H}$ is continuous, there is a constant $\gamma > 0$ such that

$$\|x\|_{\mathcal{H}} \leq \gamma \|x\|_{\mathcal{V}}$$

for any $x \in \mathcal{V}$. Then $\imath^* : \mathcal{H}^* \longrightarrow \mathcal{V}^*$ is the embedding from $\mathcal{H}^*$ into $\mathcal{V}^*$, here $\imath^*$ is the adjoint of $\imath$. Since we identify $\mathcal{H}$ with $\mathcal{H}^*$, then $\imath^*$ is continuous and $\mathcal{H}^*$ is dense in $\mathcal{V}^*$. When we assume that i is compact, which is very common for the applications, see Example 1, then so is $\imath^*$.

2.3 | On monotonicity methods

We describe monotonicity results which are required in the sequel and which are given in $\mathcal{H}$ after[10] and[9]. We recall that by $\langle \cdot, \cdot \rangle$ we denote an action on a linear and continuous functional on elements of $\mathcal{H}$. A functional $F : \mathcal{H} \to \mathbb{R}$ is said to be *Gâteaux differentiable* at $x_0 \in \mathcal{H}$ if there exists a continuous linear functional $f'(x_0) : \mathcal{H} \to \mathbb{R}$ such that for every $h \in \mathcal{H}$

$$\lim_{t\to 0} \frac{f(x_0 + th) - f(x_0)}{t} = \left\langle f'(x_0), h \right\rangle .$$

Operator $A : \mathcal{H} \to \mathcal{H}^*$ is called:

- *monotone*, if for all $u, v \in \mathcal{H}$ it holds
$$\langle A(u) - A(v), u - v \rangle \geq 0;$$
- *strictly monotone*, if for all $u, v \in \mathcal{H}$, $u \neq v$ it holds
$$\langle A(u) - A(v), u - v \rangle > 0;$$
- *m–strongly monotone*, if there exists a constant $m > 0$ such that for all $u, v \in \mathcal{H}$ it holds
$$\langle A(u) - A(v), u - v \rangle \geq m \|u - v\|_{\mathcal{H}}^2 ;$$
- *radially continuous*, if for all $u, v \in \mathcal{H}$ function
$$s \to \langle A(u + sv), v \rangle$$

is continuous on $[0,1]$;

- *demicontinuous* if $u_n \to u_0$ in $\mathcal{H}$ implies that $A\left(u_n\right) \rightharpoonup A\left(u_0\right)$ in $\mathcal{H}^*$;
- *Lipschitz continuous*, if there exists a constant $L > 0$ such that
$$\|A(u) - A(v)\|_{\mathcal{H}^*} \leq L\,\|u - v\|_{\mathcal{H}}$$
for all $u, v \in \mathcal{H}$
- *potential* if there exists a functional $F : \mathcal{H} \longrightarrow \mathbb{R}$, differentiable in the sense of Gâteaux on $\mathcal{H}$, and such that $F' = A$. Functional F is called the *potential* of A;
- *coercive* if
$$\lim_{\|u\|\to+\infty} \frac{\langle A(u), u\rangle}{\|u\|_{\mathcal{H}}} = \infty$$
or else if there exists a function
$$\gamma : \ [0,+\infty) \to \mathbb{R}, \quad \lim_{t\to+\infty} \gamma(t) = \infty,$$
such that
$$\langle A(u), u\rangle \geq \gamma\left(\|u\|_{\mathcal{H}}\right)\|u\|_{\mathcal{H}}$$
for all $u \in \mathcal{H}$.

Note that we do not necessarily identify $\mathcal{H}$ with its dual $\mathcal{H}^*$ in the above. We recall after[9] that a monotone and potential operator is necessarily demicontinuous and thus radially continuous. A monotone and radially continuous operator is necessarily demicontinuous. Thus for monotone operator we may impose radially–, hemi– or demicontinuity notion. A strongly monotone operator is strictly monotone and hence monotone. It is also coercive. Note also that the sum of two operators of various types of monotony forms an operator of the stronger type of monotony. The following lemma helps us with checking the monotonicity and extends a bit some known results:

Lemma 1. Assume that $\mathcal{H}$ and $\mathcal{Y}$ are Banach spaces. Let $\Lambda : \mathcal{Y} \longrightarrow \mathcal{H}$ be such a linear operator that for $u \in \mathcal{Y}$ it holds
$$\|u\|_{\mathcal{Y}} \leq \|\Lambda u\|_{\mathcal{H}}.$$
Assume that $A : \mathcal{H} \longrightarrow \mathcal{H}$ has any monotonicity property (namely, A is monotone or strictly monotone or strongly monotone). Then operator $T : \mathcal{Y} \longrightarrow \mathcal{Y}^*$ defined as follows
$$T = \Lambda^* A \Lambda$$
shares the monotonicity property of A.

Proof. Note that for every $u, v \in \mathcal{Y}$ we have
$$\begin{aligned}\langle T(u) - T(v), u - v\rangle &= \langle \Lambda^*(A(\Lambda u) - A(\Lambda v)), u - v\rangle \\ &= \langle A(\Lambda u) - A(\Lambda v), \Lambda u - \Lambda v\rangle.\end{aligned}$$
Assuming the m-strong monotonicity of A we get
$$\langle T(u) - T(v), u - v\rangle = \langle A(\Lambda u) - A(\Lambda v), \Lambda u - \Lambda v\rangle \geq m\|\Lambda(u-v)\|^2_{\mathcal{H}} \geq m\|u-v\|^2_{\mathcal{Y}}.$$
The remaining assertions are proved likewise. □

The existence result which we need in the sequel and which follows by investigating the convergence of a type of a Galerkin scheme is as follows:

Theorem 2 (Browder-Minty)**.** Assume that $A : \mathcal{H} \longrightarrow \mathcal{H}^*$ is radially continuous, coercive and strictly monotone. Then for any $f \in \mathcal{H}^*$ there is a unique solution to $A(u) = f$.

3 | ABSTRACT EXISTENCE AND EQUILIBRIUM RESULTS

In this section we introduce some abstract existence results for a system on m nonlinear equations which we next apply to the existence of a Nash-type equilibrium.

3.1 Problem setting

We provide a general assumptions for this section

Assumption 1. Let $(\mathcal{V}_k, \mathcal{H}_k, \mathcal{V}_k^*)$ for $k = 1, \dots, m$, $m \geq 2$, be a family of Gelfand triples such that $\mathcal{V}_k$ and $\mathcal{H}_k$ are Hilbert spaces. Moreover let $\mathcal{Y}_k$ be another Hilbert space.

Assumption 2. Linear operators $\Lambda_k : \mathcal{V}_k \longrightarrow \mathcal{Y}_k$, $k = 1, \dots, m$, are such that

$$\|u\|_{\mathcal{V}_k} = \|\Lambda_k u\|_{\mathcal{Y}_k}$$

for every $u \in \mathcal{V}_k$.

Assumption 3. For every $k = 1, \dots, m$ there exist $\beta_k > 0$ such that

$$\beta_k \|u\|_{\mathcal{H}_k} \leq \|u\|_{\mathcal{V}_k}$$

for each $u \in \mathcal{V}_k$.

The above assumptions have rather technical manner, namely Assumption 2 pertains the classical definition of a norm in $H_0^1(\Omega)$ and Assumption 3 to the Poincaré inequality. Now we provide a crucial assumptions for our investigation:

Assumption 4. For every $k = 1, \dots, m$ operator $A_k : \mathcal{Y}_k \longrightarrow \mathcal{Y}_k^*$ coercive and potential. Let $\gamma_k \geq 0$ be such that

$$\langle A_k(u) - A_k(v), u - v \rangle \geq \gamma_k \|u - v\|^2_{\mathcal{Y}_k}$$

for all $u, v \in \mathcal{Y}_k$.

Since we do not assume $\gamma_k > 0$ in Assumption 4, operators A_k, $k = 1, \dots, m$, are monotone, but they may not be strongly monotone. Moreover, by Lemma 5.4 in[11], operators A_k are demicontiuous.

Assumption 5. For every $k = 1, \dots, m$ operator $N_k : \mathcal{V} \longrightarrow \mathcal{H}_k^*$ is continuous in the following way: for every $u, v \in \mathcal{V}$ the function

$$[0,1] \ni t \mapsto \langle N_k(u + tv), v_k \rangle \in \mathbb{R}$$

is continuous.

Under Assumptions 1-5 we introduce following notions. Put $A = (A_1, \dots, A_m)$. By $\mathcal{A}_k$ we denote the potential of A_k for every $k = 1, \dots, m$. We define

$$G = \begin{bmatrix} \gamma_1 & 0 & \cdots & 0 \\ 0 & \gamma_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \gamma_m \end{bmatrix} \quad \text{and} \quad B = \begin{bmatrix} \beta_1 & 0 & \cdots & 0 \\ 0 & \beta_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \beta_m \end{bmatrix}. \tag{3}$$

We set $\mathcal{H} := \prod_{k=1}^m \mathcal{H}_k$ and equip $\mathcal{H}$ with the standard inner product

$$\langle u | v \rangle_{\mathcal{H}} = \sum_{k=1}^m \langle u_k | v_k \rangle_{\mathcal{H}_k}.$$

We also define $\mathcal{V} := \prod_{k=1}^m \mathcal{V}_k$, $\mathcal{Y} := \prod_{k=1}^m \mathcal{Y}_k$. Then $\mathcal{V}^* = \prod_{k=1}^m \mathcal{V}_k^*$ and $\mathcal{Y}^* = \prod_{k=1}^m \mathcal{Y}_k^*$. We will also denote $v = \left(v_1, v_2, \dots, v_m\right) \in \mathcal{V}$ and

$$\Lambda = \begin{bmatrix} \Lambda_1 & 0 & \cdots & 0 \\ 0 & \Lambda_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Lambda_m \end{bmatrix}.$$

Then

$$\Lambda^* = \begin{bmatrix} \Lambda_1^* & 0 & \cdots & 0 \\ 0 & \Lambda_2^* & \ddots & \vdots \\ \vdots & \ddots & \ddots & 0 \\ 0 & \cdots & 0 & \Lambda_m^* \end{bmatrix}.$$

As with standard calculations connected to matrices we mean that

$$\Lambda v = \left(\Lambda_1 v_1, \Lambda_2 v_2, \dots, \Lambda_m v_m\right)$$

for $\left(v_1, v_2, \dots, v_m\right) \in \mathcal{V}$. We also denote $N = (N_1, \dots, N_m)$. Notice that Assumption 5 means that N is radially continuous.

3.2 | Main abstract existence result

We recall that $u \in \mathcal{V}$ is a *weak solution* to the problem

$$\Lambda^* A\left(\Lambda u\right) = N(u) \tag{4}$$

if

$$\langle A\left(\Lambda u\right), \Lambda v\rangle = \langle N(u), v\rangle$$

for all $v \in \mathcal{V}$. This is equivalent to saying that u satisfies

$$\begin{cases} \langle A_1\left(\Lambda_1 u_1\right), \Lambda_1 v_1\rangle = \langle N_1(u), v_1\rangle, \\ \vdots \\ \langle A_m\left(\Lambda_m u_m\right), \Lambda_m v_m\rangle = \langle N_m(u), v_m\rangle, \end{cases}$$

for all $v \in \mathcal{V}$. The following theorem extends Theorem 8 obtained in[6].

Theorem 3. Let Assumptions 1-5 hold. Assume that there exists a matrix $C = [c_{ij}] \in M_{m\times m}(\mathbb{R})$ with non-negative off-diagonal elements such that for all $k = 1, \dots, m$ and every $u, v \in \mathcal{V}$ we have

$$\langle N_k(u) - N_k(v), u_k - v_k\rangle \leq \sum_{i=1}^{m} c_{ki}\|u_i - v_i\|_{\mathcal{H}_i}\|u_k - v_k\|_{\mathcal{H}_k}. \tag{5}$$

If $GB - C$ is a non-singular M-matrix, then the problem (4) has the unique weak solution.

Proof. Firstly observe that, by Remark 1, there exists $\tau \in (0, 1)$ such that $\tau GB - C$ is a non-singular M-matrix. By Theorem 1 we obtain that there exists a positive diagonal matrix D such that for any $x \in \mathbb{R}^m \setminus \{0\}$ we have

$$\langle D(\tau GB - C)x, x\rangle > 0. \tag{6}$$

We take $T : \mathcal{V} \longrightarrow \mathcal{V}^*$ given by

$$\langle T(u), v\rangle = \sum_{k=1}^{m} d_{kk}\langle A_k(\Lambda_k u_k), \Lambda_k v_k\rangle - \sum_{k=1}^{m} d_{kk}\langle N_k(u), v_k\rangle \tag{7}$$

for all $u, v \in \mathcal{V}$. It is clear that zeroes of T coincides with weak solutions to (4). Therefore in order to get the assertions it is enough to show that T satisfies assumptions of Theorem 2. The radial continuity of T is clear. To show that T is strictly monotone and coercive, we consider a decomposition $T = T_1 + T_2$, where

$$\langle T_1(u), v\rangle = \sum_{k=1}^{m} d_{kk}\langle A_k(\Lambda_k u_k), \Lambda_k v_k\rangle - \tau\sum_{k=1}^{m} d_{kk}\gamma_k\beta_k\langle u_k|v_k\rangle_{\mathcal{H}_k}$$

$$\langle T_2(u), v\rangle = \tau\sum_{k=1}^{m} d_{kk}\gamma_k\beta_k\langle u_k|v_k\rangle_{\mathcal{H}_k} - \sum_{k=1}^{m} d_{kk}\langle N_k(u), v_k\rangle$$

for all $u, v \in \mathcal{V}$. When $\gamma_k = 0$ for all $k = 1, \dots, m$, T_1 is monotone and coercive by Assumption 4. Otherwise we take $k \in \{1, \dots, m\}$ such that $\gamma_k > 0$. Using Assumptions 3 and 4 we obtain

$$\begin{aligned} \langle A_k(\Lambda_k u_k) - A_k(\Lambda_k v_k), \Lambda_k v_k - \Lambda_k v_k\rangle - \tau\gamma_k\beta_k\|u_k - v_k\|^2_{\mathcal{H}_k} & \\ \geq \gamma_k\|\Lambda_k u_k - \Lambda_k v_k\|^2_{\mathcal{Y}_k} - \tau\gamma_k\|u_k - v_k\|^2_{\mathcal{V}_k} & \\ \geq (1-\tau)\gamma_k\|u_k - v_k\|^2_{\mathcal{V}_k}. & \end{aligned}$$

Hence k-th component of T_1 is strongly monotone and hence clearly strictly monotone and coercive, all with respect to $\mathcal{V}_k$. Therefore it is immediate that T_1 is monotone and coercive. Moreover, by Cauchy-Schwartz inequality, (5) and (6), we have

$$\langle T_2(u) - T_2(v), u - v\rangle \geq \tau\sum_{k=1}^{m} d_{kk}\gamma_k\beta_k\|u_k - v_k\|^2_{\mathcal{H}_k} - \sum_{k=1}^{m}\sum_{i=1}^{m} c_{ki}\|u_i - v_i\|_{\mathcal{H}_i}\|u_k - v_k\|_{\mathcal{H}_k} > 0$$

for all $u, v \in \mathcal{V}$. Therefore both, T_1 and T_2, are monotone operators, while T_1 is additionally coercive and T_2 – strictly monotone. Since adding a monotone operator to the coercive operator yields still a coercive operator, we see using the above monotonicity and continuity relations that T is strictly monotone, coercive and radially continuous. Applying Theorem 2 we get the assertion. $\square$

The following example shows that the Theorem 3 is an extension of Perov Contraction Principle in the Hilbert spaces setting, see [12] for some details.

Example 2. Consider an algebraic equation of the form

$$\begin{cases} x = \frac{1}{2}\cos(x) + \frac{3}{4}y, \\ y = \frac{1}{4}x + \frac{1}{2}\cos(y). \end{cases} \tag{8}$$

Solutions to (8) clearly coincide with fixed points of $g : \mathbb{R}^2 \longrightarrow \mathbb{R}^2$ given by

$$g(x,y) = \left(\frac{1}{2}\cos(x) + \frac{3}{4}y, \frac{1}{4}x + \frac{1}{2}\cos(y)\right).$$

However, g is not a contraction, which follows by a standard characterization of optimal Lipschitz constant L_g, that is

$$L_g = \sup_{u\in\mathbb{R}^2} \max_{\|v\|_2=1} \left\|g'(u)v\right\|_2,$$

where $\|u\|_2$ stands for the Euclidean norm. However we can use the approach considered for instance in [12]. It is based on calculating Lipschitz constants for each coordinate separately. We get

$$|g_1(u) - g_1(v)| \le \frac{1}{2}|u_1 - v_1| + \frac{3}{4}|u_2 - v_2|,$$
$$|g_2(u) - g_2(v)| \le \frac{1}{4}|u_1 - v_1| + \frac{1}{2}|u_2 - v_2|$$

for every $u, v \in \mathbb{R}^2$. Since

$$A = \begin{bmatrix} \frac{1}{2} & \frac{3}{4} \\ \frac{1}{4} & \frac{1}{2} \end{bmatrix}$$

is convergent matrix, we can use a Perov Contraction Principle to obtain a unique solvability to (8). However, such an argumentation is not possible if we consider equation with non-Lipschitz right hand like for instance

$$\begin{cases} x = \frac{1}{2}\cos(x) + \frac{3}{4}y - x^3, \\ y = \frac{1}{4}x + \frac{1}{2}\cos(y) - y^3. \end{cases} \tag{9}$$

Nevertheless Theorem 3. Taking $\mathcal{H}_k = \mathcal{Y}_k = \mathcal{V}_k = \mathbb{R}$ for $k = 1, 2$, $N = g$ and A as an identity, we have

$$\left(g_1(u) - g_1(v)\right)(u_1 - v_1) \le \frac{1}{2}|u_1 - v_1|^2 + \frac{3}{4}|u_1 - v_1||u_2 - v_2|,$$
$$\left(g_2(u) - g_2(v)\right)(u_2 - v_2) \le \frac{1}{4}|u_1 - v_1||u_2 - v_2| + \frac{1}{2}|u_2 - v_2|^2$$

for each $u, v \in \mathbb{R}^2$. It is clear that $\gamma_k = \beta_k = 1$ for $k = 1, 2$. Moreover $I - A$ is M-matrix, since $\rho(A) < 1$. Therefore (9) has a unique solution.

Remark 2. Relations between above obtained results and Perov Contraction Principle has been already well described in [6]. Notice however that Theorem 3 is significant extension of Theorem 8 in [6]. It will be visualized by Example 3 in the sequel.

Now we answer the following important question: when *every* solution obtained by Theorem 3 is a Nash–type equilibrium?

Theorem 4. Let Assumptions 1-5 hold. Assume additionally that (5) holds, $GB - C$ is non-singular M-matrix and that functionals $\mathcal{N}_k : \mathcal{V} \longrightarrow \mathbb{R}$, $k = 1, \dots, m$, given by

$$\mathcal{N}_k(u) = \int_0^1 \langle N_k(tu), u_k\rangle dt$$

satisfy $\partial_{u_k}\mathcal{N}_k = N_k$ for $k = 1, \dots, m$, where $\partial_{u_k}\mathcal{N}_k$ denotes a Gâteaux derivative of $\mathcal{N}_k$ with respect to k-th variable. Then the unique weak solution to problem (4) (guaranteed by Theorem 3) is a Nash-type equilibrium for the system $\{\mathcal{E}_k\}_{k=1}^m$, where

$$\mathcal{E}_k(u) = \mathcal{A}_k\left(\Lambda_k u_k\right) - \mathcal{N}_k(u)$$

for $k = 1, \dots, m$.

Proof. Since for each $k = 1, \dots, m$ functional $\mathcal{N}_k$ has the k-th partial Gâteaux derivative and since each $\mathcal{A}_k$ is C^1 and Λ_k is linear and continuous we observe that every $\mathcal{E}_k$ has the k–th partial Gâteaux derivative. Let w be the weak solution to problem (4). Define functionals $\mathcal{J}_k : \mathcal{V}_k \longrightarrow \mathbb{R}$, $k = 1, \dots, m$, by

$$\mathcal{J}_k(u) = \mathcal{E}_k(w_1, \dots, w_{k-1}, u, w_{k+1}, \dots, w_m) \text{ for all } u \in \mathcal{V}_k.$$

Note that functionals $\mathcal{J}_k$, $k = 1, \dots, m$, are Gâteaux differentiable and

$$\mathcal{J}_k'(u) = \partial_u \mathcal{E}_k(w_1, \dots, w_{k-1}, u, w_{k+1}, \dots, w_m) \text{ for all } u \in \mathcal{V}_k.$$

Taking T given by (7) we see, by the above, that

$$\langle T(u), v\rangle = \sum_{k=1}^{m} d_{kk} \langle \partial_{u_k} \mathcal{E}_k(u), v_k\rangle.$$

Therefore, since T is monotone we obtain that for all $k = 1, \dots, m$ and every $u, v \in \mathcal{V}_k$

$$\langle \mathcal{J}_k'(u) - \mathcal{J}_k'(v), u - v\rangle \geq 0.$$

Hence $\mathcal{J}_k$, for $k = 1, \dots, m$, are convex. Since a critical point of a convex functional is necessarily an argument of a global minimum we obtain that

$$\mathcal{J}_k(w) = \min_{u \in \mathcal{V}_k} \mathcal{J}_k(u).$$

Therefore w in a Nash–type equilibrium for $\{\mathcal{E}_k\}_{k=1}^m$. □

4 | APPLICATIONS TO NONLINEAR PROBLEMS

We formulate the problem which serves as an example showing the advance over[6] and other sources mentioned:[2,3,5,1,4]. For simplicity we consider a system of two nonlinear elliptic equations of the form

$$\begin{cases} -\operatorname{div}\left(\varphi_{11}\left(x, \left|\frac{\partial u}{\partial x_1}(x)\right|\right)\frac{\partial u}{\partial x_1}(x), \dots, \varphi_{1l}\left(x, \left|\frac{\partial u}{\partial x_l}(x)\right|\right)\frac{\partial u}{\partial x_l}(x)\right) = f_1(x, u(x), v(x)), \\ -\operatorname{div}\left(\varphi_{21}\left(x, \left|\frac{\partial u}{\partial x_1}(x)\right|\right)\frac{\partial u}{\partial x_1}(x), \dots, \varphi_{2l}\left(x, \left|\frac{\partial u}{\partial x_l}(x)\right|\right)\frac{\partial u}{\partial x_l}(x)\right) = f_2(x, u(x), v(x)), \\ u|_{\partial\Omega} = v|_{\partial\Omega} = 0. \end{cases} \tag{10}$$

Now we give assumptions which allows us to use an abstract framework introduced in Section 3. To consider the boundary conditions in the sense of traces we impose

Assumption 6. Let $\Omega \subset \mathbb{R}^l$, $l \in \mathbb{N}$, be an open, bounded and connected set with a Lipschitz boundary.

Following[13] we denote $H^{-1}(\Omega) := H_0^1(\Omega)^*$. We take $L : H_0^1(\Omega) \longrightarrow L^2(\Omega; \mathbb{R}^l)$ given by the formula

$$Lu(x) = \nabla u(x) = \left(\frac{\partial u}{\partial x_1}(x), \dots, \frac{\partial u}{\partial x_m}(x)\right). \tag{11}$$

Taking

$$\mathcal{H}_k = L^2(\Omega), \quad \mathcal{Y}_k = L^2(\Omega; \mathbb{R}^l), \quad \mathcal{V}_k = H_0^1(\Omega) \tag{12}$$

we see that Assumption 1 holds. Moreover $\Lambda_1 = \Lambda_2 = L$ satisfy Assumption 2. Note that for every $y^* \in L^2(\Omega; \mathbb{R}^l)^*$ there exists a unique $y \in L^2(\Omega; \mathbb{R}^l)$ satisfying

$$\langle y^*, w\rangle = \sum_{i=1}^{l} \int_\Omega y_i(x) w_i(x) dx$$

for all $w \in L^2(\Omega; \mathbb{R}^l)$. Using the above notation we see that $\Lambda^* : L^2(\Omega; \mathbb{R}^l)^* \longrightarrow H^{-1}(\Omega)$ is defined by

$$\langle \Lambda^* y^*, v\rangle = \sum_{i=1}^{l} \int_\Omega y_i(x) \frac{\partial v}{\partial x_i}(x) dx$$

for all $v \in H_0^1(\Omega)$. We denote by λ_Ω the Poincaré constant, that is

$$\lambda_\Omega := \inf_{u \in H_0^1(\Omega), u \neq 0} \frac{\int_\Omega |\nabla u(x)|^2 dx}{\int_\Omega |u(x)|^2 dx}.$$

We see that Assumption 3 holds with $\beta_1 = \beta_2 = \lambda_\Omega$.

Assumption 7. For each $k = 1, 2$, $\varphi_k : \Omega \times [0, \infty) \longrightarrow \mathbb{R}$ is a Carathéodory function. Moreover

1. there exist positive numbers $r < R$ such that
$$r \leq \varphi_k(x, u) \leq R$$
for all $u \geq 0$, a.e. $x \in \Omega$ and $k = 1, 2$;
2. for each $k = 1, 2$ there exist $\delta_1, \delta_2 \geq 0$ such that
$$\varphi_k(x, u)u - \varphi_k(x, v)v \geq \delta_k(u - v)$$
for $k = 1, 2$, all $0 \leq u \leq v$ and a.e. $x \in \Omega$.

As in Assumption 4 we do not require $\delta_k > 0$ in Assumption 7. Under Assumption 7 we define operators $P_1, P_2 : L^2(\Omega; \mathbb{R}^l) \longrightarrow L^2(\Omega; \mathbb{R}^l)^*$ by the formula

$$\langle P_k(y), w \rangle = \sum_{i=1}^{l} \int_\Omega \varphi_{ki}(x, |y_i(x)|) y_i(x) w_i(x) dx, \tag{13}$$

for $x, y \in L^2(\Omega; \mathbb{R}^l)$ and $k = 1, 2$. Following[9] or[11] we can verify that P_1 and P_2 are continuous, monotone, coercive and potential operators. The proof relies on the Krasnosel'skii Theorem on the continuity of the Niemytskij operator and standard direct calculations. Moreover, if $\delta_k > 0$, operator P_k is δ_k-strongly monotone. Potential $\mathcal{P}_k$ of operator P_k reads

$$\mathcal{P}_k(u) = \sum_{i=1}^{l} \int_\Omega \int_0^{\left|\frac{\partial u}{\partial x_i}(x)\right|} \varphi_{ki}(x, t) t dt dx \tag{14}$$

for $u \in L^2(0, 1)$ and $k = 1, 2$. Therefore Assumption 4 is satisfied for $A_k = P_k$ with $\gamma_k = \delta_k$, $k = 1, 2$. Finally let us consider

Assumption 8. Let $f_1, f_2 : \Omega \times \mathbb{R}^2 \longrightarrow \mathbb{R}$ be Carathéodory functions satisfying:

if $l = 1$: there exist $h_1, h_2 \in L^2(\Omega; [0, \infty))$ functions $g_1, g_2 : \mathbb{R} \longrightarrow [0, \infty)$ such that
$$|f_k(t, u)| \leq h_k(t) g_k(u)$$
for a.e. $x \in \Omega$, all $u, v \in \mathbb{R}$ and $k = 1, 2$;

if $l \geq 2$: there exist constants $a_1, a_2 \geq 0$, $b_1, b_2 \in L^2(\Omega; [0, \infty))$ and a finite positive number $p \leq \frac{l}{l-2}$ (for $l = 2$, p can be any positive constant) such that
$$|f_k(x, u, v)| \leq a_k(|u|^p + |v|^p) + b_k(x)$$
for a.e. $x \in \Omega$, all $u, v \in \mathbb{R}$ and $k = 1, 2$.

We denote by $F_1, F_2 : H_0^1(\Omega)^2 \longrightarrow L^2(\Omega)^*$ the Nemytski operators associated with f_1 and f_2, respectively. Namely

$$\langle F_k(u_1, u_2), v \rangle = \int_\Omega f_k(x, u_1(x), u_2(x)) v(x) dx \tag{15}$$

and all $u_1, u_2 \in H_0^1(\Omega)$ and $v \in L^2(\Omega)$. Both operators are well defined by the Gagliardo-Nirenberg-Sobolev inequality. Hence Assumption 5 is satisfied with $N_k = F_k$ for $k = 1, 2$. Moreover taking

$$\mathcal{F}_1(u_1, u_2) = \int_\Omega \int_0^{u_1(x)} f_k(x, v, u_2(x)) dv dx,$$

$$\mathcal{F}_2(u_1, u_2) = \int_\Omega \int_0^{u_2(x)} f_k(x, u_1(x), v) dv dx$$

we get

$$\langle \partial_{u_k} \mathcal{F}_k(u_1, u_2), v\rangle = \int_\Omega f_k(t, u_1(x), u_2(x)) v(x) dx,$$

for $k = 1, 2$. Here $\partial_{u_k} \mathcal{F}_k(u_1, u_2)$ denotes a Gâteaux derivative of $\mathcal{F}_k$ at point (u_1, u_2) with respect to k-th variable. To study the existence of a Nash-type equilibrium for system (10) we consider $\mathcal{I}_k : H_0^1(\Omega)^2 \longrightarrow \mathbb{R}$ defined by the formula

$$\mathcal{I}_k(u_1, u_2) = \mathcal{P}_k(Lu_k) - \mathcal{F}_k(u_1, u_2) \tag{16}$$

for $u_1, u_2 \in H_0^1(\Omega)$ and $k = 1, 2$. Here L, $\mathcal{P}_k$ and $\mathcal{F}_k$ are given by (11), (14) and (15), respectively. Applying Theorem 4 to the above setting we get

Theorem 5. Let Assumptions 6-8 hold. Assume that there exist constants $c_{11} < \delta_1$, $c_{22} < \delta_2$ and $c_{12}, c_{21} \geq 0$ such that

$$\begin{aligned} \left(f_1(x, u_1, u_2) - f_1(x, v_1, v_2)\right)(u_1 - v_1) &\leq c_{11}|u_1 - v_1|^2 + c_{12}|u_1 - v_1||u_2 - v_2| \\ \left(f_2(x, u_1, u_2) - f_2(x, v_1, v_2)\right)(u_2 - v_2) &\leq c_{21}|u_1 - v_1||u_2 - v_2| + c_{22}|u_2 - v_2|^2 \end{aligned} \tag{17}$$

hold for all $u, v \in \mathbb{R}$, a.e. $x \in \Omega$. If

$$(\lambda_\Omega \delta_1 - c_{11})(\lambda_\Omega \delta_2 - c_{22}) > c_{12} c_{21}, \tag{18}$$

then system (10) has a unique solution, which is a Nash–type equilibrium for $\{\mathcal{I}_1, \mathcal{I}_2\}$ given by (16).

Proof. In the view of the above considerations it is clear that Assumptions 1-5 are satisfied if we consider a space setting (12) and take $\Lambda_1 = \Lambda_2 = L$, $A_k = P_k$ and $N_k = F_k$ for $k = 1, 2$. Operators L, P_k and F_k are given by (11), (13) and (15), respectively. Assumption (17) provides (5) with the same constants, while $GB - C$ is non-singular M-matrix by condition (18) (see condition F_{15} in Theorem 1). Therefore Theorem 3 provides existence and uniqueness of weak solution to the system (10). Such a solution is a Nash–type equilibrium due to Theorem 4. Indeed, it suffice to take $\mathcal{E}_k = \mathcal{I}_k$ given by (16) for $k = 1, 2$. □

Example 3. Let us consider function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ given by the formula

$$\varphi(u) := \begin{cases} 1 & \text{if } u \in [0, 1), \\ \frac{1}{u} & \text{if } u \in [1, 2), \\ 1 - \frac{1}{u} & \text{if } u \in [2, \infty). \end{cases}$$

Using Theorem 5, with $\varphi_1(x, u) = \varphi(u)$ and $\varphi_2 \equiv 1$, we can show that system

$$\begin{cases} -\frac{d}{dt}\left(\varphi\left(\left|\frac{du}{dt}\right|\right)\frac{du}{dt}\right) = v + \sin(v) - 2u, \\ -\frac{d^2 v}{dt^2} = \frac{u}{2} - v - v^3, \\ u(0) = u(1) = v(0) = v(1) = 0, \end{cases}$$

has a unique weak solution, which is a Nash-type equilibrium for a following system of functionals

$$\mathcal{E}_1(u, v) = \int_0^1 \int_0^{\left|\frac{du}{dt}(t)\right|} \varphi(s)\, s ds dt - \int_0^1 (v(t) + \sin(v(t)))\, u(t) dt + \int_0^1 |u(t)|^2 dt,$$

$$\mathcal{E}_2(u, v) = \frac{1}{2}\int_0^1 \left|\frac{dv}{dt}(t)\right|^2 dt - \frac{1}{2}\int_0^1 u(t)v(t) dt + \frac{1}{2}\int_0^1 |v(t)|^2 dt + \frac{1}{4}\int_0^1 |v(t)|^4 dt.$$

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