Two linearized finite difference schemes for time fractional nonlinear diffusion-wave equations with fourth order derivative

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Abstract

In this paper, we present a finite difference and a compact finite difference schemes for the time fractional nonlinear diffusion-wave equations (TFNDWEs) with the space fourth order derivative. To reduce the smoothness requirement in time, the considered TFNDWEs are equivalently transformed into their partial integro-differential forms with the classical first order integrals and the Caputo derivative. The finite difference scheme is constructed by using Crank-Nicolson method combined with the midpoint formula, the weighted and shifted Grünwald difference formula and the second order convolution quadrature formula to deal with the temporal discretizations. Meanwhile, the classical central difference formula and fourth order Stephenson scheme are used in spacial direction. Then, the compact finite difference scheme is developed by using the fourth order compact difference formula for the spatial direction. The stability and convergence of the proposed schemes are strictly proved by using the discrete energy method. Finally, some numerical experiments are presented to support our theoretical results.

Keywords Fractional nonlinear diffusion-wave equations. Linearized schemes. Fourth order derivative. Stability. Convergence.

1 Introduction

Fractional partial differential equations (FPDEs) have attracted considerable attention in various fields. Though research shows that many phenomena can be described by FPDEs such as physics [1, 2], engineering [3, 4], and other sciences [5, 6, 7]. However, finding the exact solutions of FPDEs by using current analytical methods such as, Laplace transform, Green's function, and Fourier-Laplace transform (see [8, 9, 10, 11] for examples) are difficult to achieve if not

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impossible [12, 9]. Thus, proposing new methods to find numerical solutions of these equations has practical importance. Due to this fact, in recent years several numerical methods were proposed for solving FPDEs, for instances see [13, 14, 15, 16, 17, 18, 19, 20] and the references therein.

In this paper, the following nonlinear time fractional diffusion-wave equation with fourth order derivative in space and homogeneous initial boundary conditions will be considered

$$\frac{\partial^2 u(x,t)}{\partial t^2} + {}_0^C D_t^\alpha u(x,t) + K_c \frac{\partial^4 u(x,t)}{\partial x^4} = \frac{\partial^2 u(x,t)}{\partial x^2} + g(u) + f(x,t), \qquad (1.1)$$

where $1 < \alpha < 2$, $f(x,t)$ is a known function, $g(u)$ is a nonlinear function of u with $g(0) = 0$ and satisfies the Lipschitz condition, and ${}_0^C D_t^\alpha u(x,t)$ denotes the temporal Caputo derivative with order α defined as

$${}_0^C D_t^\alpha u(x,t) = \frac{1}{\Gamma(2-\alpha)} \int_0^t (t-s)^{1-\alpha} \frac{\partial^2 u(x,s)}{\partial s^2} ds.$$

Recently, there exist many works on numerical methods for time fractional diffusion-wave equations (TFDWEs), see [21, 22, 23, 24, 25, 26, 27, 28, 29, 30] and the references therein. Chen et al. [23] proposed the method of separation of variables with constructing the implicit difference scheme for fractional diffusion-wave equation with damping. Heydari et al. [26] have proposed Legendre wavelets (LWs) for solving TFDWEs where fractional operational matrix of integration for LWs was derived. Bhrawy et al. [21] have proposed Jacobi tau spectral procedure combined with the Jacobi operational matrix for solving TFDWEs. Ebadian et al. [24] have proposed triangular function (TFs) methods for solving a class of nonlinear TFDWEs where fractional operational matrix of integration for the TFs was derived. Mohammed et al. [29] have proposed shifted Legengre collocation scheme and sinc function for solving TFDWEs with variable coefficients. Zhou et al. [30] have applied Chebyshev wavelets collocation for solving a class of TFDWEs where fractional integral formula of a single Chebyshev wavelets in the Riemann-Liouville sense was derived. Khalid et al. [28] have proposed the third degree modified extended B-spline functions for solving TFDWEs with reaction and damping terms. Some other numerical methods are presented for solving time fractional diffusion equations, see [31, 32, 33, 34] and the references therein.

To the best of our knowledge, there is no existing numerical method which can be used to solve Eq. (1.1) neither directly nor by transferring Eq. (1.1) into equivalent integro-differential equation. Thus, the aim of this study is devoted to constructing the high order numerical schemes to solve Eq. (1.1), and carrying out the corresponding numerical analysis for the proposed schemes. Herein, we firstly transform Eq. (1.1) into the equivalent partial integro-differential equations by using the integral operator. Secondly, the Crank-Nicolson technique is applied to deal with the temporal direction. Then, we use the midpoint formula to discretize the first order derivative, use the weighted and shifted Grüunwald difference formula to discretize the Caputo derivative, and apply the second order convolution quadrature formula to approximate the first order integral. The classical central difference formula, the fourth order Stephenson scheme, and the fourth order compact difference formula are applied for spatial approximations.

The rest of this paper is organized as follows. In Section 2, some preparations and useful lemmas are provided and discussed. In Section 3 the finite difference scheme is constructed and analyzed. In Section 4, the compact finite difference scheme is deduced, and the convergence and the unconditional stability are strictly proved. Numerical experiments are provided to support the theoretical results in Section 5. Finally, some concluding remarks are given.

2 Preliminaries

Lemma 2.1. *(see [35]) Eq. (1.1) is equivalent to the following partial integro-differential equation,*

$$\frac{\partial u(x,t)}{\partial t} + {}_0^C D_t^{\alpha-1} u(x,t) + K_c \cdot {}_0J_t \frac{\partial^4 u(x,t)}{\partial x^4} = {}_0J_t \frac{\partial^2 u(x,t)}{\partial x^2} + {}_0J_t g(u) + F(x,t), \tag{2.1}$$

where $F(x,t) = {}_0J_t f(x,t)$ and ${}_0J_t$ is first order integral operator, i.e., ${}_0J_t u(\cdot,t) = \int_0^t u(\cdot,s)ds$.

To discretize Eq. (2.1), we introduce the temporal step size $\tau = T/N$ with a positive integer N, $t_n = n\tau$, and $t_{n+1/2} = (n+1/2)\tau$. Similarly, define the spatial step size $h = L/M$ with a positive integer M, and denote $x_i = ih$. Then, define a grid function space $\Theta_h = \{v_i^n | 0 \le n \le N, 0 \le i \le M, v_0^n = v_M^n = 0\}$, and introduce the following notations, inner product, and norm, i.e., for $u^n, v^n \in \Theta_h$, we define

$$\begin{aligned} \Delta_x u_i^n =& \frac{1}{2h}\left(u_{i+1}^n - u_{i-1}^n\right), \quad \delta_x^2 u_i^n = \frac{1}{h^2}\left(u_{i-1}^n - 2u_i^n + u_{i+1}^n\right), \\ \langle u^n, v^n \rangle =& h \sum_{i=1}^{M-1} u_i^n v_i^n, \quad ||u^n||^2 = \langle u^n, u^n \rangle, \end{aligned}$$

$$\mathcal{H} u_i^n = \begin{cases} \left(1 + \frac{h^2}{12}\delta_x^2\right) u_i^n = \frac{1}{12}\left(u_{i-1}^n + 10u_i^n + u_{i+1}^n\right), & 1 \le i \le M-1, \\ u_i^n, & i = 0 \text{ or } M. \end{cases}$$

Lemma 2.2. *(see [36, 37]) If $u(\cdot,t) \in C^2([0,T])$ and $0 < \gamma < 1$, then it holds*

$${}_0J_t u(\cdot, t_{n+1/2}) = \frac{1}{2}\left[{}_0J_t u(\cdot, t_{n+1}) + {}_0J_t u(\cdot, t_n)\right] + O(\tau^2).$$

Furthermore, if $u(\cdot,t) \in C^3([0,T])$, then we have

$$u_t(\cdot, t_{n+1/2}) = \frac{u(\cdot, t_{n+1}) - u(\cdot, t_n)}{\tau} + O(\tau^2) = \delta_t u(\cdot, t_{n+1/2}) + O(\tau^2),$$

and

$$ {}_0^C D_t^\gamma u(\cdot, t_{n+1/2}) = \frac{1}{2}\left({}_0^C D_t^\gamma u(\cdot, t_{n+1}) + {}_0^C D_t^\gamma u(\cdot, t_n)\right) + O(\tau^2). $$

Lemma 2.3. *(see [38, 39]) Let $\{\omega_k\}$ be the weights from generating function $\left(3/2 - 2z + z^2/2\right)^{-1}$, i.e., $\omega_k = 1 - 3^{-(k+1)}$. If $u(\cdot,t) \in C^2([0,T])$ and $u(\cdot,0) = u_t(\cdot,0) = 0$, then we have*

$$ {}_0J_{t_{n+1}} u(\cdot,t) - \tau \sum_{k=0}^{n+1} \omega_{n+1-k} u(\cdot, t_k) = O(\tau^2). $$

Lemma 2.4. *(see [40]) For $u(\cdot,t) \in L^1(\mathbb{R})$, ${}_{-\infty}^{RL} D_t^{\gamma+2} u(\cdot,t)$ and its Fourier transform belong to $L^1(\mathbb{R})$, if we use the weighted and shifted Grünwald difference operator to approximate the Riemann-Liouville derivative, then it holds*

$$ {}_0^{RL} D_0^\gamma u(\cdot, t_{k+1}) = \tau^{-\gamma} \sum_{j=0}^{k+1} \sigma_j^{(\gamma)} u(\cdot, t_{k+1-j}) + O(\tau^2), \quad 0 < \gamma < 1, $$

where

$$ \sigma_0^{(\gamma)} = \frac{2+\gamma}{2} c_0^{(\gamma)}, \ \sigma_j^{(\gamma)} = \frac{2+\gamma}{2} c_j^{(\gamma)} - \frac{\gamma}{2} c_{j-1}^{(\gamma)}, \quad j \geq 1, $$

and $c_j^{(\gamma)} = (-1)^j \begin{pmatrix} \gamma \\ j \end{pmatrix}$ for $j \geq 0$.

Lemma 2.5. *(see [41]) Suppose $u(x,\cdot) \in C^4([x_{i-1}, x_{i+1}])$, let $\zeta(s) = u^{(4)}(x_i + sh, \cdot) + u^{(4)}(x_i - sh, \cdot)$, then*

$$ \delta_x^2 u(x_i, \cdot) = \frac{u(x_{i-1},\cdot) - 2u(x_i,\cdot) + u(x_{i+1},\cdot)}{h^2} = u_{xx}(x_i,\cdot) + \frac{h^2}{24}\int_0^1 \zeta(s)(1-s)^3 ds. $$

Lemma 2.6. *(see [41]) Suppose $u(x,\cdot) \in C^6([x_{i-1}, x_{i+1}])$, $1 \leq i \leq M-1$, and $\zeta(s) = 5(1-s)^3 - 3(1-s)^5$. Then it holds that*

$$ \begin{aligned} &\frac{1}{12}\left[u_{xx}(x_{i-1},\cdot) + 10u_{xx}(x_i,\cdot) + u_{xx}(x_{i+1},\cdot)\right] - \frac{1}{h^2}\left[u(x_{i-1},\cdot) - 2u(x_i,\cdot) + u(x_{i+1},\cdot)\right] \\ &= \frac{h^4}{360}\int_0^1 \left[u^{(6)}(x_i - sh,\cdot) + u^{(6)}(x_i + sh,\cdot)\right]\zeta(s) ds. \end{aligned} $$

Lemma 2.7. *(see [37]) Assume that $u(\cdot,t) \in C^1([0,T]) \cap C^2((0,T])$, then the following approximation holds*

$$ u(\cdot, t_{n+1}) = 2u(\cdot, t_n) - u(\cdot, t_{n-1}) + O(\tau^2). $$

Lemma 2.8. *(see [42]) For any grid function $w^n \in \Theta_h$, we have*

$$ \frac{2}{3}\|w^n\|^2 \leq \langle \mathcal{H} w^n, w^n \rangle \leq \|w^n\|^2. $$

Lemma 2.9. *(see [43]) For any grid function $w^n, v^n \in \Theta_h$, it holds*

$$\langle \delta_x^2 w^n, v^n \rangle = -\langle \delta_x w^n, \delta_x v^n \rangle.$$

Lemma 2.10. *(see [44]) Let $\{\sigma_k^{(\alpha-1)}\}$ be the weighted coefficients defined in Lemma 2.4, then for any positive integer n and $w^n \in \Theta_h$, it holds that*

$$\sum_{m=0}^{n} \sum_{k=0}^{m} \sigma_k^{(\alpha-1)} \langle \mathcal{H} w^{m-k}, w^m \rangle \geq 0.$$

Lemma 2.11. *(see [19, 45]) Let $\{\omega_k\}$ and $\{\sigma_k^{(\alpha-1)}\}$ be the weights defined in Lemma 2.3 and 2.4, respectively. Then for any positive integer K and real vector $(V_1, V_2, \cdots, V_K)^T$, the inequalities*

$$\sum_{n=0}^{K-1} \left(\sum_{j=0}^{n} \omega_j V_{n+1-j} \right) V_{n+1} \geq 0,$$
$$\sum_{n=0}^{K-1} \left(\sum_{j=0}^{n} \sigma_j^{(\alpha-1)} V_{n+1-j} \right) V_{n+1} \geq 0$$

hold.

Lemma 2.12. *(see [46, 47]) Assume that $u(x, \cdot) \in C^8([0, L])$ with $u(0, \cdot) = u(L, \cdot) = u_x(0, \cdot) = u_x(L, \cdot) = 0$, and define the operator δ_x^4 by*

$$\delta_x^4 u_i^n = \frac{12}{h^2} \left(\Delta_x v_i^n - \delta_x^2 u_i^n \right),$$

where v_i^n is a compact approximation of $u_x(x_i, t_n)$, i.e.,

$$\frac{1}{6} v_{i-1}^n + \frac{2}{3} v_i^n + \frac{1}{6} v_{i+1}^n = \Delta_x u_i^n.$$

Then we have the following approximation

$$\delta_x^4 u_i^n = \frac{\partial^4 u(x_i, t_n)}{\partial x^4} + O(h^4).$$

Furthermore, let $u^n = \left(u_1^n, u_2^n, \cdots, u_{M-1}^n\right)^T$, then the matrix representation of the operator δ_x^4 is

$$\mathbf{S} u^n = \frac{6}{h^4} \left(3\mathbf{K}\mathbf{P}^{-1}\mathbf{K} + 2\mathbf{D} \right) u^n,$$

where

$$\mathbf{K} = \begin{pmatrix} 0 & 1 & & & \\ -1 & 0 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & -1 & 0 & 1 \\ & & & -1 & 0 \end{pmatrix}_{(M-1)\times(M-1)},$$

$$
\mathbf{P} = \begin{pmatrix} 4 & 1 & & & \\ 1 & 4 & 1 & & \\ & \ddots & \ddots & \ddots & \\ & & 1 & 4 & 1 \\ & & & 1 & 4 \end{pmatrix}_{(M-1)\times(M-1)},
$$

and $\mathbf{D} = 6\mathbf{I} - \mathbf{P}$ *with the identity matrix* $\mathbf{I}$.

Lemma 2.13. *(see [46]) The matrix* $\mathbf{S}$ *defined in Lemma 2.12 is symmetric positive definite.*

It follows from Lemma 2.13, there is an invertible matrix $\mathbf{B}$ such that, $\mathbf{S} = \mathbf{B}^T\mathbf{B}$. Then for $w^n, v^n \in \Theta_h$, we have

$$
\langle \mathbf{S}w^n, v^n \rangle = \langle \mathbf{B}^T\mathbf{B}w^n, v^n \rangle = \langle \mathbf{B}w^n, \mathbf{B}v^n \rangle. \tag{2.2}
$$

3 Derivation and analysis of the finite difference scheme

In this section, a finite difference scheme with the accuracy $O(\tau^2 + h^2)$ for nonlinear Problem (2.1) is constructed and analyzed.

Assume that $u(x,t) \in C^{8,3}_{x,t}([0,L] \times [0,T])$, and $u(\cdot,0) = u_t(\cdot,0) = 0$. Consider Eq. (2.1) at the point $u(x_i, t_{n+1/2})$, we have

$$
\begin{aligned}
\left.\frac{\partial u(x_i,t)}{\partial t}\right|_{t=t_{n+1/2}} = &-{}^C_0D^{\alpha-1}_{t_{n+1/2}}u(x_i,t) - K_c \cdot {}_0J_{t_{n+1/2}}\frac{\partial^4 u(x_i,t)}{\partial x^4} + {}_0J_{t_{n+1/2}}\frac{\partial^2 u(x_i,t)}{\partial x^2} \\
&+ {}_0J_{t_{n+1/2}}g(u(x_i,t)) + F(x_i,t_{n+1/2}).
\end{aligned}
$$

The Crank-Nicolson technique and Lemma 2.2 for the above equation yield

$$
\begin{aligned}
\frac{u(x_i,t_{n+1}) - u(x_i,t_n)}{\tau} = &-\frac{1}{2}\left[{}^C_0D^{\alpha-1}_{t_{n+1}}u(x_i,t) + {}^C_0D^{\alpha-1}_{t_n}u(x_i,t)\right] \\
&- \frac{K_c}{2}\left[{}_0J_{t_{n+1}}\frac{\partial^4 u(x_i,t)}{\partial x^4} + {}_0J_{t_n}\frac{\partial^4 u(x_i,t)}{\partial x^4}\right] \\
&+ \frac{1}{2}\left[{}_0J_{t_{n+1}}\frac{\partial^2 u(x_i,t)}{\partial x^2} + {}_0J_{t_n}\frac{\partial^2 u(x_i,t)}{\partial x^2}\right] \\
&+ \frac{1}{2}\left[{}_0J_{t_{n+1}}g(x_i,t) + {}_0J_{t_n}g(x_i,t)\right] \\
&+ F(x_i,t_{n+1/2}) + O(\tau^2).
\end{aligned} \tag{3.1}
$$

Let $u(x_i,t_n) = u_i^n$. Since the initial values are 0, thus the Riemann−liouville derivative is equivalent to Caputo derivative. We apply Lemmas 2.3 and 2.4 to discretize the first order integral operator and Caputo derivative in Eq. (3.1) respectively, apply Lemma 2.12 to discretize $\frac{\partial^4 u(x_i,t)}{\partial x^4}$, and Lemma 2.5 to discretize

$\frac{\partial^2 u(x_i,t)}{\partial x^2}$, then we get

$$\begin{aligned}\frac{u_i^{n+1}-u_i^n}{\tau} =& -\frac{\tau^{1-\alpha}}{2}\left[\sum_{k=0}^{n+1}\sigma_k^{(\alpha-1)}u_i^{n+1-k}+\sum_{k=0}^{n}\sigma_k^{(\alpha-1)}u_i^{n-k}\right]\\ &-\frac{K_c\tau}{2}\left[\sum_{k=0}^{n+1}\omega_k\delta_x^4u_i^{n+1-k}+\sum_{k=0}^{n}\omega_k\delta_x^4u_i^{n-k}\right]\\ &+\frac{\tau}{2}\left[\sum_{k=0}^{n+1}\omega_k\delta_x^2u_i^{n+1-k}+\sum_{k=0}^{n}\omega_k\delta_x^2u_i^{n-k}\right]\\ &+\frac{\tau}{2}\left[\sum_{k=0}^{n+1}\omega_k g(u_i^{n+1-k})+\sum_{k=0}^{n}\omega_k g(u_i^{n-k})\right]\\ &+F_i^{n+\frac{1}{2}}+(R_1)_i^{n+1}, \end{aligned} \tag{3.2}$$

where

$$(R_1)_i^{n+1}=O(\tau^2+h^2+h^4)=O(\tau^2+h^2).$$

It is clear that Eq. (3.2) is a nonlinear system with respect to the unknown u_i^{n+1}. To linearly solve Eq. (3.2), we use $u_i^1=u_i^0+\tau(u_t)_i^0+O(\tau^2)$ and Lemma 2.7 to linearize Eq. (3.2) for $n=0$ and $1\le n\le N-1$, respectively, and then multiply Eq. (3.2) by τ, i.e.,

$$\begin{aligned}u_i^1-u_i^0 =& -\frac{\tau^{2-\alpha}}{2}\left[\sum_{k=0}^{1}\sigma_k^{(\alpha-1)}u_i^{1-k}+\sigma_0^{(\alpha-1)}u_i^0\right]\\ &-\frac{K_c\tau^2}{2}\left[\sum_{k=0}^{1}\omega_k\delta_x^4u_i^{1-k}+\omega_0\delta_x^4u_i^0\right]\\ &+\frac{\tau^2}{2}\left[\sum_{k=0}^{1}\omega_k\delta_x^2u_i^{1-k}+\omega_0\delta_x^2u_i^0\right]\\ &+\frac{\tau^2}{2}\left[\omega_0 g(u_i^0+\tau(u_t)_i^0)+\omega_1 g(u_i^0)+\omega_0 g(u_i^0)\right]\\ &+\tau F_i^{n+\frac{1}{2}}+O(\tau^3+\tau h^2), \end{aligned} \tag{3.3}$$

and

$$\begin{aligned}u_i^{n+1}-u_i^n =& -\frac{\tau^{2-\alpha}}{2}\left[\sum_{k=0}^{n+1}\sigma_k^{(\alpha-1)}u_i^{n+1-k}+\sum_{k=0}^{n}\sigma_k^{(\alpha-1)}u_i^{n-k}\right]\\ &-\frac{K_c\tau^2}{2}\left[\sum_{k=0}^{n+1}\omega_k\delta_x^4u_i^{n+1-k}+\sum_{k=0}^{n}\omega_k\delta_x^4u_i^{n-k}\right]\\ &+\frac{\tau^2}{2}\left[\sum_{k=0}^{n+1}\omega_k\delta_x^2u_i^{n+1-k}+\sum_{k=0}^{n}\omega_k\delta_x^2u_i^{n-k}\right]\end{aligned}$$

$$
+\frac{\tau^2}{2}\left[\sum_{k=1}^{n+1}\omega_k g(u_i^{n+1-k})+\sum_{k=0}^{n}\omega_k g(u_i^{n-k})\right]
+\frac{\tau^2\omega_0}{2}g(2u_i^n-u_i^{n-1})+\tau F_i^{n+\frac{1}{2}}+O(\tau^3+\tau h^2),\quad \text{for } 1\le n\le N-1. \tag{3.4}
$$

Noting $(u_t)_i^0=0$, neglecting the truncation error term $O(\tau^3+\tau h^2)$ in both above equations, and replacing the u_i^n with its numerical solution U_i^n, we deduce the following finite difference scheme for Problem (2.1)

$$
\begin{aligned}
U_i^1-U_i^0=&-\frac{\tau^{2-\alpha}}{2}\left[\sum_{k=0}^{1}\sigma_k^{(\alpha-1)}U_i^{1-k}+\sigma_0^{(\alpha-1)}U_i^0\right]\\
&-\frac{K_c\tau^2}{2}\left[\sum_{k=0}^{1}\omega_k\delta_x^4U_i^{1-k}+\omega_0\delta_x^4U_i^0\right]\\
&+\frac{\tau^2}{2}\left[\sum_{k=0}^{1}\omega_k\delta_x^2U_i^{1-k}+\omega_0\delta_x^2U_i^0\right]\\
&+\frac{\tau^2}{2}\left[\omega_0 g(U_i^0)+\omega_1 g(U_i^0)+\omega_0 g(U_i^0)\right]\\
&+\tau F_i^{n+\frac{1}{2}},
\end{aligned}\tag{3.5}
$$

and

$$
\begin{aligned}
U_i^{n+1}-U_i^n=&-\frac{\tau^{2-\alpha}}{2}\left[\sum_{k=0}^{n+1}\sigma_k^{(\alpha-1)}U_i^{n+1-k}+\sum_{k=0}^{n}\sigma_k^{(\alpha-1)}U_i^{n-k}\right]\\
&-\frac{K_c\tau^2}{2}\left[\sum_{k=0}^{n+1}\omega_k\delta_x^4U_i^{n+1-k}+\sum_{k=0}^{n}\omega_k\delta_x^4U_i^{n-k}\right]\\
&+\frac{\tau^2}{2}\left[\sum_{k=0}^{n+1}\omega_k\delta_x^2U_i^{n+1-k}+\sum_{k=0}^{n}\omega_k\delta_x^2U_i^{n-k}\right]\\
&+\frac{\tau^2}{2}\left[\sum_{k=1}^{n+1}\omega_k g(U_i^{n+1-k})+\sum_{k=0}^{n}\omega_k g(U_i^{n-k})\right]\\
&+\frac{\tau^2\omega_0}{2}g(2U_i^n-U_i^{n-1})+\tau F_i^{n+\frac{1}{2}},\quad \text{for } 1\le n\le N-1.
\end{aligned}\tag{3.6}
$$

Now, let us analyze the convergence and the unconditional stability of the Scheme (3.5) and (3.6).

Theorem 3.1. *Assume $u(x,t)\in C_{x,t}^{8,3}([0,L]\times[0,T])$ and $u(\cdot,0)=u_t(\cdot,0)=0$, and let $u(x,t)$ be the exact solution of Eq. (2.1) and $\{U_i^n|0\le i\le M, 1\le n\le N\}$ be the numerical solution for Scheme (3.7) and (3.8). Then, for $1\le n\le N$, it holds that*

$$
\|u^n-U^n\|\le C(\tau^2+h_x^2+h_y^2).
$$

Proof. Let us start by analyzing the error of (3.6). Subtracting Eq. (3.6) from Eq. (3.4), we have

$$
\begin{aligned}
e_i^{n+1} - e_i^n = & -\frac{\tau^{2-\alpha}}{2}\left[\sum_{k=0}^{n+1}\sigma_k^{(\alpha-1)}e_i^{n+1-k} + \sum_{k=0}^{n}\sigma_k^{(\alpha-1)}e_i^{n-k}\right] \\
& - \frac{K_c\tau^2}{2}\left[\sum_{k=0}^{n+1}\omega_k\delta_x^4 e_i^{n+1-k} + \sum_{k=0}^{n}\omega_k\delta_x^4 e_i^{n-k}\right] \\
& + \frac{\tau^2}{2}\left[\sum_{k=0}^{n+1}\omega_k\delta_x^2 e_i^{n+1-k} + \sum_{k=0}^{n}\omega_k\delta_x^2 e_i^{n-k}\right] \\
& + \frac{\tau^2}{2}\sum_{k=0}^{n}(\omega_{k+1}+\omega_k)\left[g(u_i^{n-k}) - g(U_i^{n-k})\right] \\
& + \frac{\tau^2\omega_0}{2}\left[g(2u_i^n - u_i^{n-1}) - g(2U_i^n - U_i^{n-1})\right] \\
& + O(\tau^3 + \tau h^2),
\end{aligned}
$$

where $e_i^n = u_i^n - U_i^n$. Since $e_i^0 = 0$, the above equation becomes

$$
\begin{aligned}
e_i^{n+1} - e_i^n = & -\frac{\tau^{2-\alpha}}{2}\left[\sum_{k=0}^{n}\sigma_k^{(\alpha-1)}(e_i^{n+1-k} + e_i^{n-k})\right] \\
& - \frac{K_c\tau^2}{2}\left[\sum_{k=0}^{n}\omega_k\delta_x^4\left(e_i^{n+1-k} + e_i^{n-k}\right)\right] \\
& + \frac{\tau^2}{2}\left[\sum_{k=0}^{n}\omega_k\delta_x^2\left(e_i^{n+1-k} + e_i^{n-k}\right)\right] \\
& + \frac{\tau^2}{2}\sum_{k=0}^{n}(\omega_{k+1}+\omega_k)\left[g(u_i^{n-k}) - g(U_i^{n-k})\right] \\
& + \frac{\tau^2\omega_0}{2}\left[g(2u_i^n - u_i^{n-1}) - g(2U_i^n - U_i^{n-1})\right] \\
& + O(\tau^3 + \tau h^2).
\end{aligned}
$$

Multiplying the both sides of the above equation by $h(e_i^{n+1} + e_i^n)$ and summing over $1 \le i \le M-1$. Then using Lemmas 2.9, 2.12, and Eq. (2.2), we have

$$
\begin{aligned}
\|e^{n+1}\|^2 - \|e^n\|^2 = & -\frac{\tau^{2-\alpha}}{2}\sum_{k=0}^{n}\sigma_k^{(\alpha-1)}\langle e^{n+1-k} + e^{n-k}, e^{n+1} + e^n\rangle \\
& - \frac{K_c\tau^2}{2}\sum_{k=0}^{n}\omega_k\langle B(e^{n+1-k} + e^{n-k}), B(e^{n+1} + e^n)\rangle \\
& - \frac{\tau^2}{2}\sum_{k=0}^{n}\omega_k\langle \delta_x(e^{n+1-k} + e^{n-k}), \delta_x(e^{n+1} + e^n)\rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\tau^2}{2}\sum_{k=0}^{n}(\omega_{k+1}+\omega_k)\langle g(u^{n-k})-g(U^{n-k}),e^{n+1}+e^n\rangle\\
&+\frac{\tau^2\omega_0}{2}\langle g(2u^n-u^{n-1})-g(2U^n-U^{n-1}),e^{n+1}+e^n\rangle\\
&+\langle O(\tau^3+\tau h^2),e^{n+1}+e^n\rangle.
\end{aligned}
$$

Summing the above equation over n from 1 to $J-1$ leads to

$$
\begin{aligned}
\|e^J\|^2-\|e^1\|^2=&-\frac{\tau^{2-\alpha}}{2}\sum_{n=1}^{J-1}\sum_{k=0}^{n}\sigma_k^{(\alpha-1)}\langle e^{n+1-k}+e^{n-k},e^{n+1}+e^n\rangle\\
&-\frac{K_c\tau^2}{2}\sum_{n=1}^{J-1}\sum_{k=0}^{n}\omega_k\langle B(e^{n+1-k}+e^{n-k}),B(e^{n+1}+e^n)\rangle\\
&-\frac{\tau^2}{2}\sum_{n=1}^{J-1}\sum_{k=0}^{n}\omega_k\langle \delta_x(e^{n+1-k}+e^{n-k}),\delta_x(e^{n+1}+e^n)\rangle\\
&+\frac{\tau^2}{2}\sum_{n=1}^{J-1}\sum_{k=0}^{n}(\omega_{k+1}+\omega_k)\langle g(u^{n-k})-g(U^{n-k}),e^{n+1}+e^n\rangle\\
&+\frac{\tau^2\omega_0}{2}\sum_{n=1}^{J-1}\langle g(2u^n-u^{n-1})-g(2U^n-U^{n-1}),e^{n+1}+e^n\rangle\\
&+\sum_{n=1}^{J-1}\langle O(\tau^3+\tau h^2),e^{n+1}+e^n\rangle. \qquad (3.7)
\end{aligned}
$$

Now, we turn to analyze $\|e^1\|$. Subtracting Eqs. (3.5) from Eq. (3.3), and by the similar deductions as above, we can derive that

$$
\begin{aligned}
\|e^1\|^2=&-\frac{\tau^{2-\alpha}}{2}\sigma_0^{(\alpha-1)}\langle e^1+e^0,e^1+e^0\rangle\\
&-\frac{K_c\tau^2}{2}\omega_0\langle B(e^1+e^0),B(e^1+e^0)\rangle\\
&-\frac{\tau^2}{2}\omega_0\langle \delta_x(e^1+e^0),\delta_x(e^1+e^0)\rangle\\
&+\tau^2\omega_0\langle g(u^0)-g(U^0),e^1+e^0\rangle\\
&+\frac{\tau^2\omega_1}{2}\langle g(u^0)-g(U^0),e^1+e^0\rangle\\
&+\langle O(\tau^3+\tau h^2),e^1+e^0\rangle. \qquad (3.8)
\end{aligned}
$$

Sum up Eq. (3.7) and Eq. (3.8), and apply Lemma 2.11, it deduces that

$$
\begin{aligned}
\|e^J\|^2 \leq & \frac{\tau^2}{2}\sum_{n=1}^{J-1}\sum_{k=0}^{n}(\omega_{k+1}+\omega_k)\langle g(u^{n-k})-g(U^{n-k}), e^{n+1}+e^n\rangle \\
&+\frac{\tau^2\omega_0}{2}\sum_{n=1}^{J-1}\langle g(2u^n-u^{n-1})-g(2U^n-U^{n-1}), e^{n+1}+e^n\rangle \\
&+\tau^2\omega_0\langle g(u^0)-g(U^0), e^1+e^0\rangle+\frac{\tau^2\omega_1}{2}\langle g(u^0)-g(U^0), e^1+e^0\rangle \\
&+C\sum_{n=1}^{J-1}\langle O(\tau^3+\tau h^2), e^{n+1}+e^n\rangle. \qquad (3.9)
\end{aligned}
$$

Using the Lipschitz condition of g and exchanging the order of two summations in the above inequality, we have

$$
\begin{aligned}
\|e^J\|^2 \leq & C\tau^2\sum_{k=0}^{J-1}\sum_{n=k}^{J-1}(\omega_{n+1-k}+\omega_{n-k})\|e^k\|\|e^{n+1}+e^n\| \\
&+C\tau^2\sum_{n=1}^{J-1}\|e^n\|\|e^{n+1}+e^n\|+C\sum_{n=1}^{J-1}(\tau^3+\tau h^2)\|e^{n+1}+e^n\|. \qquad (3.10)
\end{aligned}
$$

Assuming $\|e^P\|=\max_{0\leq p\leq N}\|e^p\|$. Since $\tau\sum_{n=k}^{N}(\omega_{n+1-k}+\omega_{n-k})$ is bounded (see [39]), then the above inequality yields

$$
\|e^P\|\leq C\tau\sum_{k=0}^{P-1}\|e^k\|+C(\tau^2+h^2). \qquad (3.11)
$$

Once the discrete Gronwall inequality has been applied to Inequality (3.11), we arrive at the estimate

$$
\|e^P\|\leq C(\tau^2+h^2),
$$

thus finishing the proof. □

Theorem 3.2. *Let $\{U_i^n|0\leq i\leq M, 0\leq n\leq N\}$ be the numerical solution of Scheme (3.5) and (3.6) for Problem (2.1). Then for $1\leq K\leq N$, it holds*

$$
\|U^K\|\leq C\left(\max_{0\leq n\leq N}\|g(U^n)\|+\max_{0\leq n\leq N-1}\|F^{n+\frac{1}{2}}\|\right). \qquad (3.12)
$$

Proof. Multiplying (3.6) by $h(U_i^{n+1}+U_i^n)$ and summing up for i from 1 to $M-1$.

We have

$$
\begin{aligned}
\|U^{n+1}\|^2-\|U^n\|^2=&-\frac{\tau^{2-\alpha}}{2}\sum_{k=0}^{n}\sigma_k^{(\alpha-1)}\langle U^{n+1-k}+U^{n-k},U^{n+1}+U^n\rangle\\
&-\frac{K_c\tau^2}{2}\sum_{k=0}^{n}\omega_k\langle\delta_x^4(U^{n+1-k}+U^{n-k}),U^{n+1}+U^n\rangle\\
&+\frac{\tau^2}{2}\sum_{k=0}^{n}\omega_k\langle\delta_x^2(U^{n+1-k}+U^{n-k}),U^{n+1}+U^n\rangle\\
&+\frac{\tau^2}{2}\sum_{k=0}^{n}(\omega_{k+1}+\omega_k)\langle g(U^{n-k}),U^{n+1}+U^n\rangle\\
&+\frac{\tau^2\omega_0}{2}\langle g(2U^n-U^{n-1}),U^{n+1}+U^n\rangle\\
&-\frac{K_c\tau^2}{2}\omega_{n+1}\langle\delta_x^4U^0,U^{n+1}+U^n\rangle-\frac{\tau^{2-\alpha}}{2}\sigma_{n+1}^{(\alpha-1)}\langle U^0,U^{n+1}+U^n\rangle\\
&+\frac{\tau^2}{2}\omega_{n+1}\langle\delta_x^2U^0,U^{n+1}+U^n\rangle+\tau\langle F^{n+\frac{1}{2}},U^{n+1}+U^n\rangle.
\end{aligned}
$$

Note that Eq. (1.1) is equipped with the homogeneous initial conditions, thus it deduces

$$
\begin{aligned}
\|U^{n+1}\|^2-\|U^n\|^2=&-\frac{\tau^{2-\alpha}}{2}\sum_{k=0}^{n}\sigma_k^{(\alpha-1)}\langle U^{n+1-k}+U^{n-k},U^{n+1}+U^n\rangle\\
&-\frac{K_c\tau^2}{2}\sum_{k=0}^{n}\omega_k\langle\delta_x^4(U^{n+1-k}+U^{n-k}),U^{n+1}+U^n\rangle\\
&+\frac{\tau^2}{2}\sum_{k=0}^{n}\omega_k\langle\delta_x^2(U^{n+1-k}+U^{n-k}),U^{n+1}+U^n\rangle\\
&+\frac{\tau^2}{2}\sum_{k=0}^{n}(\omega_{k+1}+\omega_k)\langle g(U^{n-k}),U^{n+1}+U^n\rangle\\
&+\frac{\tau^2\omega_0}{2}\langle g(2U^n-U^{n-1}),U^{n+1}+U^n\rangle\\
&+\tau\langle F^{n+\frac{1}{2}},U^{n+1}+U^n\rangle.
\end{aligned}
$$

Applying the similar deductions to get Eq. (3.9), it achieves that

$$
\begin{aligned}
\|U^J\|^2\le&C\tau\sum_{k=0}^{J-1}\|g(U^k)\|\left(\|U^{n+1}\|+\|U^n\|\right)\\
&+\frac{\tau^2}{2}\omega_0\sum_{n=1}^{J-1}\|g(2U^n-U^{n-1})\|\left(\|U^{n+1}\|+\|U^n\|\right)\\
&+C\tau\sum_{n=1}^{J-1}\|F^{n+\frac{1}{2}}\|\left(\|U^{n+1}\|+\|U^n\|\right).
\end{aligned}
\tag{3.13}
$$

One can estimate $\|g(2U^n - U^{n-1})\|$ as the following

$$\begin{aligned}\|g(2U^n - U^{n-1})\| =&\|g(2U^n - U^{n-1}) - g(U^n) + g(U^n)\|,\\ \leq&\|g(2U^n - U^{n-1}) - g(U^n)\| + \|g(U^n)\|,\\ \leq&C(\|U^n\| + \|U^{n-1}\|) + \|g(U^n)\|. \end{aligned} \tag{3.14}$$

Substituting Eq. (3.14) into Eq. (3.13) and using Young's inequality, then we have

$$\|U^J\|^2 \leq C\tau\sum_{n=0}^{J-1}\|U^n\|^2 + C\max_{0\leq n\leq N}\|g(U^n)\|^2 + C\max_{0\leq n\leq N-1}\|F^{n+\frac{1}{2}}\|^2. \tag{3.15}$$

By applying the Gronwall inequality to (3.15), it becomes

$$\|U^J\|^2 \leq C\left(\max_{0\leq n\leq N}\|g(U^n)\|^2 + \max_{0\leq n\leq N-1}\|F^{n+\frac{1}{2}}\|^2\right),$$

and this completes the proof. □

4 Derivation and analysis of the compact finite difference scheme

In this section, a compact finite difference scheme with accuracy $O(\tau^2 + h^4)$ for nonlinear Problem (2.1) is presented and analyzed.

Now let us act on both sides of Eq. (3.1) with the compact operator $\mathcal{H}$. Then, by using Lemma 2.6, we obtain

$$\begin{aligned}\mathcal{H}\left[\frac{u(x_i,t_{n+1}) - u(x_i,t_n)}{\tau}\right] =& -\frac{1}{2}\mathcal{H}\left[{}_0^CD_{t_{n+1}}^{\alpha-1}u(x_i,t) + {}_0^CD_{t_n}^{\alpha-1}u(x_i,t)\right]\\ &-\frac{K_c}{2}\mathcal{H}\left[{}_0J_{t_{n+1}}\frac{\partial^4u(x_i,t)}{\partial x^4} + {}_0J_{t_n}\frac{\partial^4u(x_i,t)}{\partial x^4}\right]\\ &+\frac{1}{2}\left[{}_0J_{t_{n+1}}\delta_x^2u(x_i,t) + {}_0J_{t_n}\delta_x^2u(x_i,t)\right]\\ &+\frac{1}{2}\mathcal{H}\left[{}_0J_{t_{n+1}}g(x_i,t) + {}_0J_{t_n}g(x_i,t)\right]\\ &+\mathcal{H}F_i^{n+\frac{1}{2}} + O(\tau^2 + h^4). \end{aligned} \tag{4.1}$$

Apply the similar deductions to get Eqs. (3.3) and (3.4), it achieves

$$\begin{aligned}\mathcal{H}\left[u_i^1 - u_i^0\right] =& -\frac{\tau^{2-\alpha}}{2}\mathcal{H}\left[\sum_{k=0}^{1}\sigma_k^{(\alpha-1)}u_i^{1-k} + \sigma_0^{(\alpha-1)}u_i^0\right]\\ &-\frac{K_c\tau^2}{2}\mathcal{H}\left[\sum_{k=0}^{1}\omega_k\delta_x^4u_i^{1-k} + \omega_0\delta_x^4u_i^0\right]\end{aligned}$$

$$
+\frac{\tau^2}{2}\left[\sum_{k=0}^{1}\omega_k\delta_x^2 u_i^{1-k}+\omega_0\delta_x^2 u_i^0\right]
+\frac{\tau^2}{2}\mathcal{H}\left[\omega_0 g(u_i^0)+\omega_1 g(u_i^0)+\omega_0 g(u_i^0)\right]+\tau\mathcal{H}F_i^{n+\frac{1}{2}}
+O(\tau^3+\tau h^4), \tag{4.2}
$$

and

$$
\begin{aligned}
\mathcal{H}\left[u_i^{n+1}-u_i^n\right]=&-\frac{\tau^{2-\alpha}}{2}\mathcal{H}\left[\sum_{k=0}^{n+1}\sigma_k^{(\alpha-1)}u_i^{n+1-k}+\sum_{k=0}^{n}\sigma_k^{(\alpha-1)}u_i^{n-k}\right]\\
&-\frac{K_c\tau^2}{2}\mathcal{H}\left[\sum_{k=0}^{n+1}\omega_k\delta_x^4 u_i^{n+1-k}+\sum_{k=0}^{n}\omega_k\delta_x^4 u_i^{n-k}\right]\\
&+\frac{\tau^2}{2}\left[\sum_{k=0}^{n+1}\omega_k\delta_x^2 u_i^{n+1-k}+\sum_{k=0}^{n}\omega_k\delta_x^2 u_i^{n-k}\right]\\
&+\frac{\tau^2}{2}\mathcal{H}\left[\sum_{k=1}^{n+1}\omega_k g(u_i^{n+1-k})+\sum_{k=0}^{n}\omega_k g(u_i^{n-k})\right]\\
&+\frac{\tau^2\omega_0}{2}\mathcal{H}g(2u_i^n-u_i^{n-1})+\tau\mathcal{H}F_i^{n+\frac{1}{2}}\\
&+O(\tau^3+\tau h^4),\quad \text{for } 1\le n\le N-1.
\end{aligned}\tag{4.3}
$$

Neglecting the truncation error term $O(\tau^3+\tau h^4)$ in both above equations, and replacing the u_i^n with its numerical solution U_i^n, we deduce the following compact finite difference scheme for Problem (2.1)

$$
\begin{aligned}
\mathcal{H}\left[U_i^1-U_i^0\right]=&-\frac{\tau^{2-\alpha}}{2}\mathcal{H}\left[\sum_{k=0}^{1}\sigma_k^{(\alpha-1)}U_i^{1-k}+\sigma_0^{(\alpha-1)}U_i^0\right]\\
&-\frac{K_c\tau^2}{2}\mathcal{H}\left[\sum_{k=0}^{1}\omega_k\delta_x^4 U_i^{1-k}+\omega_0\delta_x^4 U_i^0\right]\\
&+\frac{\tau^2}{2}\left[\sum_{k=0}^{1}\omega_k\delta_x^2 U_i^{1-k}+\omega_0\delta_x^2 U_i^0\right]\\
&+\frac{\tau^2}{2}\mathcal{H}\left[\omega_0 g(U_i^0)+\omega_1 g(U_i^0)+\omega_0 g(U_i^0)\right]\\
&+\tau\mathcal{H}F_i^{n+\frac{1}{2}},
\end{aligned}\tag{4.4}
$$

and

$$
\begin{aligned}
\mathcal{H}\left[U_i^{n+1}-U_i^n\right] =& -\frac{\tau^{2-\alpha}}{2}\mathcal{H}\left[\sum_{k=0}^{n+1}\sigma_k^{(\alpha-1)}U_i^{n+1-k}+\sum_{k=0}^{n}\sigma_k^{(\alpha-1)}U_i^{n-k}\right]\\
&-\frac{K_c\tau^2}{2}\mathcal{H}\left[\sum_{k=0}^{n+1}\omega_k\delta_x^4U_i^{n+1-k}+\sum_{k=0}^{n}\omega_k\delta_x^4U_i^{n-k}\right]\\
&+\frac{\tau^2}{2}\left[\sum_{k=0}^{n+1}\omega_k\delta_x^2U_i^{n+1-k}+\sum_{k=0}^{n}\omega_k\delta_x^2U_i^{n-k}\right]\\
&+\frac{\tau^2}{2}\mathcal{H}\left[\sum_{k=1}^{n+1}\omega_k g(U_i^{n+1-k})+\sum_{k=0}^{n}\omega_k g(U_i^{n-k})\right]\\
&+\frac{\tau^2\omega_0}{2}\mathcal{H}g(2U_i^n-U_i^{n-1})+\tau\mathcal{H}F_i^{n+\frac{1}{2}}, \quad \text{for } 1\le n\le N-1.
\end{aligned}
\tag{4.5}
$$

Theorem 4.1. *Assume* $u(x,t)\in C_{x,t}^{8,3}([0,L]\times[0,T])$ *and* $u(\cdot,0)=u_t(\cdot,0)=0$, *and let* $u(x,t)$ *be the exact solution of Eq. (2.1) and* $\{U_i^n|0\le i\le M, 1\le n\le N\}$ *be the numerical solution for Scheme (4.4) and (4.5). Then, for* $1\le n\le N$, *it holds that*

$$\|u^n-U^n\|\le C(\tau^2+h^4).$$

Proof. Let us start by analyzing the error of (4.5). Subtracting Eq. (3.5) from Eq. (4.3), we have

$$
\begin{aligned}
\mathcal{H}\left[e_i^{n+1}-e_i^n\right] =& -\frac{\tau^{2-\alpha}}{2}\mathcal{H}\left[\sum_{k=0}^{n+1}\sigma_k^{(\alpha-1)}e_i^{n+1-k}+\sum_{k=0}^{n}\sigma_k^{(\alpha-1)}e_i^{n-k}\right]\\
&-\frac{K_c\tau^2}{2}\mathcal{H}\left[\sum_{k=0}^{n+1}\omega_k\delta_x^4e_i^{n+1-k}+\sum_{k=0}^{n}\omega_k\delta_x^4e_i^{n-k}\right]\\
&+\frac{\tau^2}{2}\left[\sum_{k=0}^{n+1}\omega_k\delta_x^2e_i^{n+1-k}+\sum_{k=0}^{n}\omega_k\delta_x^2e_i^{n-k}\right]\\
&+\frac{\tau^2}{2}\mathcal{H}\sum_{k=0}^{n}(\omega_{k+1}+\omega_k)\left[g(u_i^{n-k})-g(U_i^{n-k})\right]\\
&+\frac{\tau^2\omega_0}{2}\mathcal{H}\left[g(2u_i^n-u_i^{n-1})-g(2U_i^n-U_i^{n-1})\right]\\
&+O(\tau^3+\tau h^4),
\end{aligned}
$$

where $e_i^n = u_i^n - U_i^n$. Since $e_i^0 = 0$, the above equation becomes

$$
\begin{aligned}
\mathcal{H}\left[e_i^{n+1} - e_i^n\right] = &- \frac{\tau^{2-\alpha}}{2}\left[\sum_{k=0}^{n} \sigma_k^{(\alpha-1)} \mathcal{H}(e_i^{n+1-k} + e_i^{n-k})\right] \\
&- \frac{K_c \tau^2}{2}\left[\sum_{k=0}^{n} \omega_k \mathcal{H} \delta_x^4 \left(e_i^{n+1-k} + e_i^{n-k}\right)\right] \\
&+ \frac{\tau^2}{2}\left[\sum_{k=0}^{n} \omega_k \delta_x^2 \left(e_i^{n+1-k} + e_i^{n-k}\right)\right] \\
&+ \frac{\tau^2}{2} \sum_{k=0}^{n} (\omega_{k+1} + \omega_k) \mathcal{H}\left[g(u_i^{n-k}) - g(U_i^{n-k})\right] \\
&+ \frac{\tau^2 \omega_0}{2} \mathcal{H}\left[g(2u_i^n - u_i^{n-1}) - g(2U_i^n - U_i^{n-1})\right] \\
&+ O(\tau^3 + \tau h^4).
\end{aligned}
$$

Multiplying the both sides of the above equation by $h(e_i^{n+1} + e_i^n)$ and summing over $1 \le i \le M-1$. Then using Lemmas 2.8, 2.9, and Eq. (2.2), we have

$$
\begin{aligned}
\|e^{n+1}\|^2 - \|e^n\|^2 \le &- \frac{\tau^{2-\alpha}}{2} \sum_{k=0}^{n} \sigma_k^{(\alpha-1)} \langle \mathcal{H}(e^{n+1-k} + e^{n-k}), e^{n+1} + e^n \rangle \\
&- \frac{K_c \tau^2}{2} \sum_{k=0}^{n} \omega_k \langle \mathcal{H} B(e^{n+1-k} + e^{n-k}), B(e^{n+1} + e^n) \rangle \\
&- \frac{\tau^2}{2} \sum_{k=0}^{n} \omega_k \langle \delta_x(e^{n+1-k} + e^{n-k}), \delta_x(e^{n+1} + e^n) \rangle \\
&+ \frac{\tau^2}{2} \sum_{k=0}^{n} (\omega_{k+1} + \omega_k) \langle \mathcal{H}\left(g(u^{n-k}) - g(U^{n-k})\right), e^{n+1} + e^n \rangle \\
&+ \frac{\tau^2 \omega_0}{2} \langle \mathcal{H}\left(g(2u^n - u^{n-1}) - g(2U^n - U^{n-1})\right), e^{n+1} + e^n \rangle \\
&+ \langle O(\tau^3 + \tau h^4), e^{n+1} + e^n \rangle.
\end{aligned}
$$

Summing the above inequality over n from 1 to $J-1$ leads to

$$
\begin{aligned}
\|e^J\|^2 - \|e^1\|^2 \le &- \frac{\tau^{2-\alpha}}{2} \sum_{n=1}^{J-1} \sum_{k=0}^{n} \sigma_k^{(\alpha-1)} \langle \mathcal{H}(e^{n+1-k} + e^{n-k}), e^{n+1} + e^n \rangle \\
&- \frac{K_c \tau^2}{2} \sum_{n=1}^{J-1} \sum_{k=0}^{n} \omega_k \langle \mathcal{H} B(e^{n+1-k} + e^{n-k}), B(e^{n+1} + e^n) \rangle \\
&- \frac{\tau^2}{2} \sum_{n=1}^{J-1} \sum_{k=0}^{n} \omega_k \langle \delta_x(e^{n+1-k} + e^{n-k}), \delta_x(e^{n+1} + e^n) \rangle \\
&+ \frac{\tau^2}{2} \sum_{n=1}^{J-1} \sum_{k=0}^{n} (\omega_{k+1} + \omega_k) \langle \mathcal{H}\left(g(u^{n-k}) - g(U^{n-k})\right), e^{n+1} + e^n \rangle
\end{aligned}
$$

$$
\begin{aligned}
&+\frac{\tau^2\omega_0}{2}\sum_{n=1}^{J-1}\langle\mathcal{H}\left(g(2u^n-u^{n-1})-g(2U^n-U^{n-1})\right), e^{n+1}+e^n\rangle\\
&+\sum_{n=1}^{J-1}\langle O(\tau^3+\tau h^4), e^{n+1}+e^n\rangle.
\end{aligned}\tag{4.6}
$$

Now, we turn to analyze $\|e^1\|$. From Eqs. (4.4), (4.2), and by the similar deductions as above, we can derive that

$$
\begin{aligned}
\|e^1\|^2\le&-\frac{\tau^{2-\alpha}}{2}\sigma_0^{(\alpha-1)}\langle\mathcal{H}(e^1+e^0), e^1+e^0\rangle\\
&-\frac{K_c\tau^2}{2}\omega_0\langle\mathcal{H}B(e^1+e^0), B(e^1+e^0)\rangle\\
&-\frac{\tau^2}{2}\omega_0\langle\delta_x(e^1+e^0), \delta_x(e^1+e^0)\rangle\\
&+\frac{\tau^2}{2}(\omega_1+\omega_0)\langle\mathcal{H}\left(g(u^0)-g(U^0)\right), e^1+e^0\rangle\\
&+\frac{\tau^2\omega_0}{2}\langle\mathcal{H}\left(g(u^0)-g(U^0)\right), e^1+e^0\rangle\\
&+\langle O(\tau^3+\tau h^4), e^1+e^0\rangle.
\end{aligned}\tag{4.7}
$$

Sum up Eq. (4.6) and Eq. (4.7), and apply Lemmas 2.11 and 2.12, it deduces that

$$
\begin{aligned}
\|e^J\|^2\le&\frac{\tau^2}{2}\sum_{n=1}^{J-1}\sum_{k=0}^{n}(\omega_{k+1}+\omega_k)\langle\mathcal{H}\left(g(u^{n-k})-g(U^{n-k})\right), e^{n+1}+e^n\rangle\\
&+\frac{\tau^2\omega_0}{2}\sum_{n=1}^{J-1}\langle\mathcal{H}\left(g(2u^n-u^{n-1})-g(2U^n-U^{n-1})\right), e^{n+1}+e^n\rangle\\
&+\frac{\tau^2}{2}(\omega_1+\omega_0)\langle\mathcal{H}\left(g(u^0)-g(U^0)\right), e^{n+1}+e^n\rangle\\
&+\frac{\tau^2\omega_0}{2}\langle\mathcal{H}\left(g(u^0)-g(U^0)\right), e^{n+1}+e^n\rangle\\
&+C\sum_{n=1}^{J-1}\langle O(\tau^3+\tau h^4), e^{n+1}+e^n\rangle.
\end{aligned}
$$

According to the same technique as for dealing with (3.9), we can achieve

$$
\|e^P\|\le C(\tau^2+h^4),
$$

thus finishing the proof. □

Theorem 4.2. *Let $\{U_i^n|0\le i\le M, 0\le n\le N\}$ be the numerical solution of Scheme (4.4) and (4.5) for problem (2.1). Then for $1\le K\le N$, it holds*

$$
\|U^K\|\le C\left(\max_{0\le n\le N}\|g(U^n)\|+\max_{0\le n\le N-1}\|F^{n+\frac{1}{2}}\|\right).
$$

5 Numerical experiments

Consider the following problem with exact solution $u(x,t) = t^{2+\alpha}\sin^2(\pi x)$

$$\frac{\partial^2 u(x,t)}{\partial t^2} + {}_0^C D_t^\alpha u(x,t) + \frac{\partial^4 u(x,t)}{\partial x^4} = \frac{\partial^2 u(x,t)}{\partial x^2} + f(x,t) + g(u),$$

where $T = 1$, $0 < x < 1$, $0 < t \leq T$, and $1 < \alpha < 2$. The nonlinear function $g(u) = u^2$ and $f(x,t)$ is

$$\begin{aligned} f(x,t) = {} & (2+\alpha)(1+\alpha)t^\alpha \sin^2(\pi x) + \frac{\Gamma(3+\alpha)}{2}t^2 \sin^2(\pi x) - 8\pi^4 t^{2+\alpha}\cos(2\pi x) \\ & -2\pi^2 t^{2+\alpha}\cos(2\pi x) - t^{2(2+\alpha)}\sin^4(\pi x). \end{aligned}$$

It is clear that $u(x,t)$ satisfies all smoothness conditions required by Theorems 3.1 and 4.1, so that both of our schemes can be applied in this example. In Figures 1 and 2, we compare the exact solution with the numerical solution of finite difference Scheme (3.5) and (3.6) and compact finite difference Scheme (4.4) and (4.5). We easily see that the exact solution can be well approximated by the numerical solutions of our schemes.

First, we in Table 1 show the computational results of finite difference Scheme (3.5) and (3.6). We set $\alpha = 1.25$, 1.5 and 1.75, respectively. Obviously, these settings meet the smoothness assumption of Theorem 3.1. The temporal and spatial convergence approach to 2.

In Tables 2 and 3 we test the numerical convergence order of compact finite difference Scheme (4.4) and (4.5). It is clear that all of the settings of α in Tables 2 and 3 satisfy the conditions of Theorem 4.1. And Tables 2 and 3 show the numerical convergence order in time and space approach to 2 and 4, respectively.

Table 1: Errors and numerical convergence orders of Scheme (3.5) and (3.6) for different α.

$\tau = h$	$\alpha = 1.25$		$\alpha = 1.5$		$\alpha = 1.75$	
	error	order	error	order	error	order
1/5	6.6627×10^{-2}		7.8031×10^{-2}		8.9815×10^{-2}	
1/10	1.8412×10^{-2}	1.8555	2.1839×10^{-2}	1.8371	2.5456×10^{-2}	1.8190
1/20	4.8132×10^{-3}	1.9355	5.7273×10^{-3}	1.9310	6.6917×10^{-3}	1.9275
1/40	1.2137×10^{-3}	1.9876	1.4621×10^{-3}	1.9698	1.7210×10^{-3}	1.9591

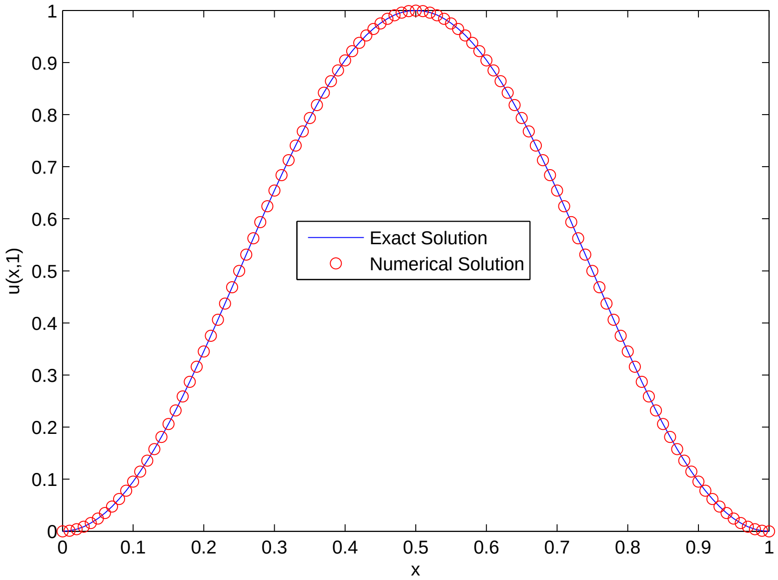

Figure 1: The comparison of numerical solution of Scheme (3.5) and (3.6) with the exact solution for $\tau = h = 0.01$ and $\alpha = 1.6$.

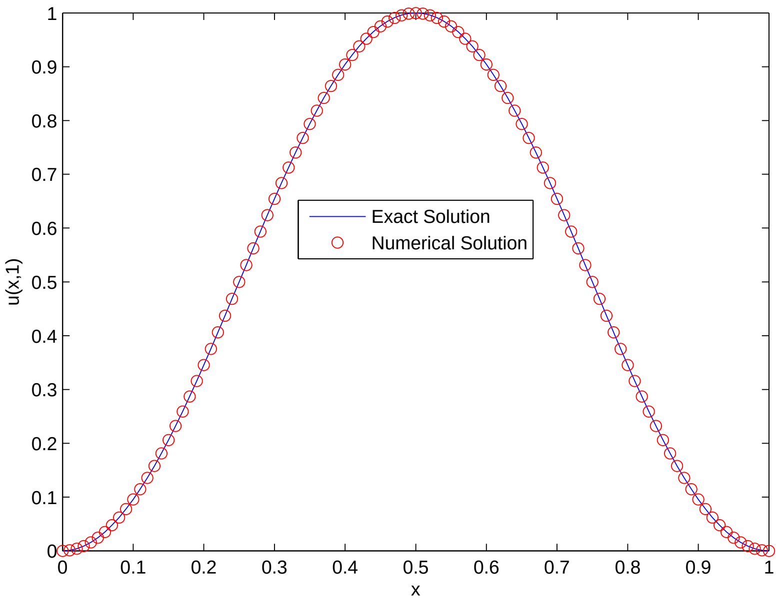

Figure 2: The comparison of numerical solution of the compact finite difference Scheme (4.4) and (4.5) with the exact solution for $\tau = h = 0.01$ and $\alpha = 1.6$.

Table 2: Errors and temporal numerical convergence orders of Scheme (4.4) and (4.5) for $h = 0.001$ and different α.

τ	$\alpha = 1.25$		$\alpha = 1.5$		$\alpha = 1.75$	
	error	order	error	order	error	order
1/5	7.0844×10^{-2}		8.2129×10^{-2}		9.3783×10^{-2}	
1/10	1.9012×10^{-2}	1.8978	2.2432×10^{-2}	1.8724	2.6040×10^{-2}	1.8486
1/20	4.9407×10^{-3}	1.9441	5.8538×10^{-3}	1.9381	6.8169×10^{-3}	1.9335
1/40	1.2436×10^{-3}	1.9901	1.4919×10^{-3}	1.9723	1.7506×10^{-3}	1.9612

Table 3: Errors and spatial numerical convergence orders of Scheme (4.4) and (4.5) for $\tau = 0.0005$ and different α.

h	$\alpha = 1.25$		$\alpha = 1.5$		$\alpha = 1.75$	
	error	order	error	order	error	order
1/5	3.8110×10^{-3}		3.7871×10^{-3}		3.7555×10^{-3}	
1/10	2.5308×10^{-4}	3.9125	2.5141×10^{-4}	3.9130	2.4922×10^{-4}	3.9135
1/20	2.2087×10^{-5}	3.5183	2.1851×10^{-5}	3.5243	2.1557×10^{-5}	3.5312
1/40	1.8261×10^{-6}	3.5964	1.7163×10^{-6}	3.6703	1.5904×10^{-6}	3.7607

6 Concluding Remarks

We in this paper constructed two linearized finite difference schemes for modified time fractional nonlinear diffusion-wave equations with the space fourth-order derivative. The equations are reduced to equivalent partial integro-differential equations, the Crank-Nicolson technique and the midpoint formula, the weighted and shifted Grünwald difference formula and the second order convolution formula based on the generating function $(3/2 - 2z + z^2/2)^{-1}$, the classical central difference formula, the fourth-order approximation, and the compact difference approach. The finite difference Scheme (3.5) and (3.6) has the accuracy $O(\tau^2 + h^2)$. The compact finite difference Scheme (4.4) and (4.5) has the accuracy $O(\tau^2 + h^4)$. It should be mentioned that our schemes require the exact solution $u(\cdot, t) \in C^3([0, T])$, while it requires $u(\cdot, t) \in C^4([0, T])$ if one discretize Eq. (1.1) directly to get the second order accuracy in time. Theoretically, the convergence and the unconditional stability of the two proposed schemes are proved and discussed. All of the numerical experiments can support our theoretical results.

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