Synchronization for Fractional FitzHugh-Nagumo Equations with Fractional Brownian Motion*

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Abstract: This paper is devoted to the study of Caputo-type fractional FitzHugh-Nagumo equations driven by fractional Brownian motion (fBm). We establish the existence and uniqueness of mild solution under some conditions on the coefficients. The exponential synchronization and finite-time synchronization for the stochastic FitzHugh-Nagumo equations are provided. The analysis of synchronization phenomenon for time-fractional FitzHugh-Nagumo equations perturbed by fBm are provided.

Key Words: FitzHugh-Nagumo Equations, Fractional Brownian motion, Caputo-type fractional derivative, Synchronization.

Mathematical Subject Classification: 60H15, 34D06.

1 Introduction

Although fractional calculus is an classical mathematical topic, a focal point of renewed interest on its applications to physics and engineering is dramatically increased during the past decades (see, e.g., [1, 3, 7, 17] and the references therein). It was found that many systems in interdisciplinary fields can be elegantly described with the help of fractional derivatives. Many systems are known to display fractional-order dynamics, such as electrode-electrolyte polarization [9], quantum evolution of complex systems [10] and so on. It is well known that chaos cannot occur in autonomous continuous-time systems of integer-order less than three according to the Poincare-Bendixon theorem [18]. However, in autonomous fractional-order systems, it is not the case (see, e.g. [11, 28] and the references therein). On the other hand, the fractional Brownian motion (fBm) appears naturally in the modeling of many complex phenomena in applications when the systems are subject to "rough" external forcing, as a centered Gaussian process, which differs significantly from the standard Brownian motion and semi-martingales [2]. Recently, Zeng and Yang [25] investigated

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the dynamics of stochastic Lorentz systems with fractional noise, and [22] proved the ergodicity of stochastic chaotic Robinovich system with fBm, we refer it to [22, 25] and the reference therein.

In the current paper, we investigate the Caputo-type fractional stochastic FitzHugh-Nagumo (FHN) equations driven by fBm of the form

$$\begin{cases} D_t^\beta u = [u - \frac{u^3}{3} - v + I]dt + dB_1^H(t), \\ D_t^\beta v = [r(u + a - bv)]dt + dB_2^H(t), \\ u(0) = u_0, \quad v(0) = v_0, \end{cases} \tag{1.1}$$

where u is the activity of the membrane potential and v is the recovery current that restores the resting state of the model. The parameter I is a constant bias current which can be considered as the effective external input current. $r > 0$ denotes a small singular perturbation parameter, a and b are parameters. The operator D_t^β denotes the Caputo-type fractional derivative, which can be used to denote the memory effects on past responses. $B_1^H(t)$ and $B_2^H(t)$ are independent fBms with Hurst parameter $H \in (0, 1)$. For the absence of stochastic terms and $\beta = 1$ in (1.1), the systems reduces to the classical FHN model, which was firstly introduced by FitzHugh [6] to study the nonlinear oscillations of the neuron by using phase diagrams. The FHN model is mathematically simple and produces a rich dynamical behavior that makes it possible to visualize the solution and to explain in geometric terms important phenomena, which related to the excitability and action potential generation mechanisms observed in biological neurons. Zhang et al. [26] investigated the type-II excitability if the injected current I is regarded as the control parameter when $r = 1/13, a = 0.7$ and $b = 0.8$, the neuron undergoes subcritical Adronov-Hopf bifurcation at $I = 0.3297$ where the state of the neuron changes from quiescence into periodic spiking. Moreover, when the input current $I = 1.4203$, then the neuron goes through subcritical Adronov-Hopf bifurcation again.

For the deterministic case, Liu ad Xie [13] proved that, there exists the Hope bifurcation point for the Caputo-type fractional FHN model, they also investigated the synchronization rate of fractional FHN neurous model, which is greater than that of the integer-order counterpart. Liu et al. [14] proved that there exists the Hopf bifurcation point, where the state of the model neuron changes from the quiescence into periodic spiking, they proved that the range of periodic spiking of the fractional-order model neuron is clearly smaller than that of the corresponding integer-order model neuron. For the stability analysis and synchronization analysis, we refer the readers to [15] and [16] for details. However, noise is ubiquitous in neural systems and it may arise from many different sources. In the stochastic model, Yamakou et al. [23] proved that there exists a global random attractor for the stochastic FHN system. Uda [20] discussed the ergodicity and spike rate for stochastic FitzHugh-Nagumo neural model. Li et al. [12] used Fourier coefficient and coherence resonance coefficient to measure the behavior of stochastic resonance and coherence resonance, respectively, and analyzed the effects of additive noise and multiplicative noise.

Recently, the influence of noise on the synchronization in stochastic model with Gaussian noise has been studied. Chen et al. [4] established the sufficient criteria for both complete synchronization and generalized synchronization of a class of chaotic systems, Wang et al. [21] studied the finite-time anti-synchronization control of memristive neural networks with stochastic perturbations, Yang and Cao [24] studied the finite-time stochastic synchronization problem for complex networks with stochastic noise perturbations.

As we know, there is a little of papers focus on the fractional FHN model driven by fBm. The novelty of this paper is to establish the regularity for the stochastic convolution of fBm for the Hurst parameter $H \in (\frac{1}{4}, 1)$, we also obtain the regularity for the stochastic convolution for Gaussian white noise. Based on the Banach fixed point theorem, we prove the global existence of the solution to stochastic systems (1.1), which is dependent on the order of fractional derivative and Hurst parameter H. The numerical simulations of Caputo-type fractional FHN system perturbed by the Gaussian noise and fBm are provided, respectively.

The rest of the paper is organized as follows. In Section 2, some basic concepts, the function setting and the definition of mild solution to the system (1.1) are presented. The existence and uniqueness of mild solutions are established in Section 3. In Section 4, the exponential synchronization and finite-time synchronization for the regular stochastic FHN model are provided. Numerical analysis of synchronization on time-fractional stochastic FHN model are given in Section 5. Conclusion is presented in section 6.

2 Preliminaries

In this section, the definitions of Caputo-type fractional derivatives and fBm are introduced, which can refer to [2] and [17].

Definition 2.1. *The Riemann-Liouville fractional integral of order $\beta > 0$ of function $f \in L^1([0,T]; X)$ is defined by*

$$I_t^\beta f(t) = \frac{1}{\Gamma(\beta)} \int_0^t (t-s)^{\beta-1} f(s) ds.$$

Definition 2.2. *The Caputo fractional derivative of order $\beta \in (0,1)$ of function $f \in C([0,T]; X)$ is defined by*

$$D_t^\beta f(t) := \frac{d}{dt}[I_t^{1-\beta}(f(t) - f(0))] = \frac{1}{\Gamma(1-\beta)} \int_0^t (t-s)^{-\beta} f'(s) ds.$$

Definition 2.3. *Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space, and $\{B^H(t)\}_{t\geq 0}$ be a continuous centered Gaussian stochastic process. If the covariance function of $\beta^H(t)$ satisfies*

$$R_H(t,s) = E[\beta^H(t)\beta^H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} + |t-s|^{2H}) = \int_0^{t\wedge s} K_H(t,r)K_H(s,r)dr, \ t, s \in \mathbb{R},$$

then, $B^H(t)$ is called a two-sided one-dimensional fBm with Hurst parameter H, where $K_H(t,s)$ is the square integrable kernel by

$$K_H(t,s) = C_H(t-s)^{H-1/2} + C_H(\frac{1}{2} - H) \int_s^t (r-s)^{H-3/2}[1 - (\frac{s}{r})^{1/2-H}]dr,$$

and

$$C_H = \sqrt{\frac{2H\Gamma(3/2-H)}{\Gamma(1/2+H)\Gamma(2-2H)}}$$

and

$$\frac{\partial K^H}{\partial t}(t,s) = C_H\left(H-\frac{1}{2}\right)(t-s)^{H-3/2}\left(\frac{s}{t}\right)^{\frac{1}{2}-H}.$$

We define the adjoint operator K_T^* on a possible subset of $L^2([0,T])$ by

$$(K_T^*\varphi)(s) = K_H(T,s)\varphi(s) + \int_s^T (\varphi(r)-\varphi(s))\frac{\partial K_H(r,s)}{\partial r}dr.$$

Therefore, the Wiener integral about $B^H(t)$ can be defined by

$$\int_0^t \varphi(s)dB^H(s) = \int_0^t (K_T^*\varphi)(s)dW(s), \qquad \forall\, t\in[0,T].$$

Now, we introduce some notations of functional spaces given as follows

$$M(T) = \Big\{(u(t),v(t))\Big|u(t),v(t)\in C[0,T]\Big\}$$

with the norm $\|(u,v)\| = \max_{t\in[0,T]}\big(|u(t)|+|v(t)|\big)$.

Definition 2.4. *A stochastic process* $(u(t),v(t))$ *is called a mild solution to* (1.1) *with initial value* (u_0,v_0) *if the following equation is satisfied*

$$\begin{cases} u(t) = u_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\big(u(\tau)-\frac{u(\tau)^3}{3}-v(\tau)+I\big)d\tau + \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}dB_1^H(\tau), \\ v(t) = v_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\big(u(\tau)+a-bv(\tau)\big)d\tau + \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}dB_2^H(\tau). \end{cases}$$

3 Well-posedness of stochastic FHN model

In this section, we will study the existence and uniqueness of mild solution to the time-fractional stochastic FHN model. Firstly, we will establish the regularity of the stochastic convolution

$$Z(t) = \int_0^t (t-\tau)^{\beta-1}dW(\tau)$$

for the Brownian motion ($H=\frac{1}{2}$), then prove the existence of mild solution for equations (1.1) with Brownian motion.

Lemma 3.1 ([8])**.** *If* $\beta>\frac{3}{4}$, *then the stochastic convolution* $\{Z(t)\}_{t\in[0,T]}$ *is continuous, and there exists* $C>0$ *such that*

$$\mathbb{E}|Z(t)|^2 \leqslant CT^{2\beta-1} < \infty.$$

Next, we are going to establish the basic properties of the following stochastic integrals

$$Z(t) = \int_0^t (t-\tau)^{\beta-1}dB^H(\tau)$$

with respect to fBm.

Lemma 3.2 ([8]). *If $t \in [0,T]$, the following results hold*

(1) For any $H \in (\frac{1}{2}, 1)$, assume that $2\beta + H - 2 > 0$ holds. Then the stochastic convolution $\{Z^H(t)\}_{t\in[0,T]}$ is continuous, and there exists $C > 0$ such that

$$\mathbb{E}|Z(t)|^2 \leqslant CT^{2\beta+2H-2} < \infty.$$

(2) For any $H \in (\frac{1}{4}, \frac{1}{2})$, assume that $1 - H < \beta < H + \frac{1}{2}$ and $3\beta + H - \frac{5}{2} > 0$ hold. Then the stochastic convolution $\{Z^H(t)\}_{t\in[0,T]}$ is continuous, and there exists $C > 0$ such that

$$\mathbb{E}|Z(t)|^2 \leqslant CT^{2\beta+2H-2} < \infty.$$

Theorem 3.1. *We set*

$$M(T) = \left\{(u(t), v(t))\Big|u(t), v(t) \in C[0,T], \max_{t\in[0,T]} \Big\{|u(t)|, |v(t)|\Big\} \leq P\right\},$$

the following results hold

(1) For any $H \in (\frac{1}{2}, 1)$. Assume that $2\beta + H - 2 > 0$ holds. Then the fractional stochastic equations (1.1) *has a local solution for T small enough in $M(T)$;*

(2) For any $H \in (\frac{1}{4}, \frac{1}{2})$. Assume that $1 - H < \beta < H + \frac{1}{2}$ and $3\beta + H - \frac{5}{2} > 0$ hold. Then the fractional stochastic equations (1.1) *has a local solution for T small enough in $M(T)$;*

(3) For any $H = \frac{1}{2}$ and if $\beta > \frac{3}{4}$. Then the fractional stochastic equations (1.1) *has a local mild solution for T small enough in $M(T)$.*

Proof. Define the map $F : M(T) \to M(T)$ by

$$\begin{cases} Fu(t) = u_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\big(u(\tau) - \frac{u(\tau)^3}{3} - v(\tau) + I\big)d\tau + \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} dB_1^H(\tau), \\ Fv(t) = v_0 + \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}\big(u(\tau) + a - bv(\tau)\big)d\tau + \frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1} dB_2^H(\tau), \end{cases}$$

for any $(u(t), v(t)) \in M(T)$. By Lemmas 3.1 and 3.2, it is easy to verify that F maps $M(T)$ into itself. Next, we will prove that equation (1.1) posses a local solution. For any $(u_1, v_1) \in M(T)$, $(u_2, v_2) \in M(T)$, and let $P = \max\limits_{t\in[0,T]}\{|u_1|, |u_2|\}$, then we have

$$\begin{aligned} &|Fu_1(t) - Fu_2(t)| \\ =&\frac{1}{\Gamma(\beta)}|\int_0^t (t-\tau)^{\beta-1}[(u_1(\tau) - u_2(\tau)) - \frac{u_1^3(\tau) - u_2^3(\tau)}{3} - (v_1(\tau) - v_2(\tau))]d\tau| \\ \leq&\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}(|u_1(\tau) - u_2(\tau)| + P^2|u_1(\tau) - u_2(\tau)| + |v_1(\tau) - v_2(\tau)|)d\tau \\ \leq&\frac{1}{\beta\Gamma(\beta)}(1 + P^2)T^\beta\|(u_1(t), v_1(t)) - (u_2(t), v_2(t))\|, \end{aligned}$$

and

$$\begin{aligned}|Fv_1(t)-Fv_2(t)| =&\frac{1}{\Gamma(\beta)}|\int_0^t (t-\tau)^{\beta-1}[(u_1(\tau)-u_2(\tau))-b(v_1(\tau)-v_2(\tau))]d\tau| \\ \leq&\frac{1}{\Gamma(\beta)}\int_0^t (t-\tau)^{\beta-1}(|u_1(\tau)-u_2(\tau)|+|b||v_1(\tau)-v_2(\tau)|)d\tau \\ \leqslant&\frac{1}{\beta\Gamma(\beta)}(1+|b|)T^\beta\|(u_1(t),v_1(t))-(u_2(t),v_2(t))\|.\end{aligned}$$

Thus

$$\|F(u_1(t),v_1(t))-F(u_2(t),v_2(t))\|\leqslant \frac{1}{\beta\Gamma(\beta)}(2+|b|+P^2)T^\beta\|(u_1(t),v_1(t))-(u_2(t),v_2(t))\|.$$

By choosing sufficient small T_1 such that $\frac{1}{\beta\Gamma(\beta)}(2+|b|+P^2)T^\beta<1$. Banach fixed point theorem implies that F has one unique fixed point in $M(T_1)$. The proof of Theorem 3.1 is complete. □

Theorem 3.2. *The following results hold*

(1) For any $H\in(\frac{1}{2},1)$. Assume that $2\beta+H-2>0$ holds. Then the fractional stochastic equations (1.1) *has a global mild solution;*

(2) For any $H\in(\frac{1}{4},\frac{1}{2})$. Assume that $1-H<\beta<H+\frac{1}{2}$ and $3\beta+H-\frac{5}{2}>0$ hold. Then the fractional stochastic equations (1.1) *has a global mild solution;*

(3) For any $H=\frac{1}{2}$. If $\beta>\frac{3}{4}$, then the fractional stochastic equations (1.1) *has a global mild solution.*

Proof. Consider the following equations

$$\begin{cases}D_t^\beta u=(u-\frac{u^3}{3}-v+I)\mathrm{d}t+\mathrm{d}B_1^H(t),\\ D_t^\beta v=r(u+a-bv)\mathrm{d}t+\mathrm{d}B_2^H(t),\\ u(\delta)=\varphi_u(\delta),v(\delta)=\psi_v(\delta).\end{cases}\tag{3.1}$$

It can be deduced similarly that there exists some $\delta_1>0$, such that equations (3.1) has one unique solution on $[\delta,\delta+\delta_1]$. Repeating the above arguments, we can deduce the equations (1.1) has a global mild solution. The proof of Theorem is complete. □

4 The synchronization of stochastic FHN model

In this section, we investigate the finite time synchronization and exponential synchronization for the following FHN model

$$\begin{cases}du=[u-\frac{u^3}{3}-v+I]dt,\\ dv=[r(u+a-bv)]dt.\end{cases}\tag{4.1}$$

We introduce the control terms Γ_1, Γ_2 and additive fractional noise into the FHN model, then the following response control system given by

$$\begin{cases} d\hat{u} = [\hat{u} - \frac{\hat{u}^3}{3} - \hat{v} + I + \Gamma_1]dt + dB_1^H(t), \\ d\hat{v} = [r(\hat{u} + a - b\hat{v}) + \Gamma_2]dt + dB_2^H(t), \end{cases} \tag{4.2}$$

where

$$\Gamma_1 = \frac{1}{3}(\hat{u}^2 u - \hat{u}u^2) - \frac{1}{2(\hat{u}) - u} - |\hat{u} - u|^\alpha, \quad \Gamma_2 = -\frac{1}{2(\hat{v} - v)} - |\hat{v} - v|^\alpha, \quad 0 < \alpha < 1.$$

Let $e_1 = u - \hat{u}$, $e_2 = v - \hat{v}$, then

$$\begin{cases} de_1 = [e_1 - (\frac{\hat{u}^3}{3} - u^3) - e_2]dt + \Gamma_1 dt + dB_1^H(t), \\ de_2 = [r(e_1 - be_2)]dt + \Gamma_2 dt + dB_2^H(t). \end{cases} \tag{4.3}$$

We denote that

$$A = \begin{pmatrix} 1 & -1 \\ r & -rb \end{pmatrix},$$

if $(rb+1)^2 \geq 4r$ for some positive real number r and b. Then, the direct calculations show that there exists a positive real number $\lambda > 0$ such that $\langle Ax, x\rangle \leq -\lambda |x|^2$. For example, $r = \frac{1}{13}, b = 0.8, a = 0.7$ and $I = 0.3297$, matrix A posses a positive eigenvalue, and the FHN model undergoes subcritical Adronov-Hopf bifurcation, see [26] for details.

Lemma 4.1 ([5])**.** *Assume that a continuous, positive-definite function $V(t)$ satisfies the following differential inequality:*

$$\dot{V}(t) \leq -\eta V^\alpha(t), \quad \forall\, t \geq t_0, \quad V(t_0) \geq 0,$$

where $\eta > 0$, $0 < \alpha < 1$ are two constants. Then, for any given t_0, $V(t)$ satisfies the following inequality

$$V^{1-\alpha}(t) \leq V^{1-\alpha}(t_0)) - \eta(1-\alpha)(t - t_)), \quad t_0 \leq t \leq t_1,$$

and

$$V(t) \equiv 0, \quad \forall\, t \geq t_1,$$

with t_1 given by

$$t_1 = t_0 + \frac{V^{1-\alpha}(t_0)}{\eta(1-\alpha)}.$$

Theorem 4.1. *If $(rb+1)^2 \geq 4r$ for some positive real number r and b. Then the error dynamics in* (4.3) *will converge to zero in finite time and the finite-time synchronization can be achieved.*

Proof. The application of Itô's formula to Lyapunov function

$$V(e(t)) = \frac{1}{2}\Big(|e_1(t)|^2 + |e_2(t)|^2\Big),$$

where $e(t) := (e_1(t), e_2(t))$, we deduce that

$$\begin{aligned} \mathbb{E}[\mathrm{d}V(e(t))] \leq & -\lambda |e(t)|^2 - \frac{1}{3}\mathbb{E}\langle \hat{u}^3 - u^3, \hat{u} - u\rangle + \mathbb{E}\langle \Gamma_1, \hat{u} - u\rangle + \frac{1}{2}\mathbb{E}\langle \Gamma_2, \hat{v} - v\rangle + 1 \\ \leq & -\mathbb{E}\Big(\lambda |e(t)|^2 + |e_1(t)|^{\alpha+1} + |e_2(t)|^{\alpha+1}\Big)\mathrm{d}t, \end{aligned}$$

which implies

$$\mathbb{E}[\mathrm{d}V(e(t))] \le -\mathbb{E}\big[|e_1(t)|^{\alpha+1} + |e_2(t)|^{\alpha+1}\big]\mathrm{d}t.$$

According to $0 < \alpha < 1$ and Jesen's inequality, we get

$$\Big(|e_1(t)|^{\alpha+1} + |e_2(t)|^{\alpha+1}\Big)^{\frac{1}{\alpha+1}} \ge \Big(|e_1(t)|^2 + |e_2(t)|^2\Big)^{\frac{1}{2}},$$

then

$$|e_1(t)|^{\alpha+1} + |e_2(t)|^{\alpha+1} \ge 2^{\frac{\alpha+1}{2}}\big(V(e(t))\big)^{\frac{\alpha+1}{2}}.$$

Hence, it follows that

$$\mathbb{E}[\mathrm{d}V(e(t))] \le -2^{\frac{\alpha+1}{2}}\mathbb{E}\Big[\big(V(e(t))\big)^{\frac{\alpha+1}{2}}\Big]\mathrm{d}t.$$

And

$$\mathbb{E}\Big[\big(V(e(0))\big)^{\frac{\alpha+1}{2}}\Big] = \Big(\mathbb{E}\big[V(e(0))\big]\Big)^{\frac{\alpha+1}{2}}.$$

By Lemma 4.1, $V(e(t))$ stochastically converges to zero in finite time, whose upper bound is

$$t_1 = \frac{\big(V(e(0))\big)^{\frac{1-\alpha}{2}}}{2^{\frac{\alpha+1}{2}}(1-\alpha)/2} = \frac{2^{\frac{\alpha-1}{2}}\big(|e_1(0)|^2 + |e_2(0)|^2\big)^{\frac{1-\alpha}{2}}}{2^{\frac{\alpha+1}{2}}(1-\alpha)/2} = \frac{\big(|e_1(0)|^2 + |e_2(0)|^2\big)^{\frac{1-\alpha}{2}}}{(1-\alpha)}.$$

The required assertion follows. □

In the sequel, we will study the exponential synchronization of FHN model (4.1). To this end, we consider the following response control systems

$$\begin{cases} d\tilde{u} = \Big[\tilde{u} - \frac{\tilde{u}^3}{3} - \tilde{v} + I + \Gamma_3(\tilde{u}, \tilde{v}, u, v)\Big]dt + (\tilde{u} - u)dB_1^H(t), \\ d\tilde{v} = \Big[r(\tilde{u} + a - b\tilde{v}) + \Gamma_4(\tilde{u}, \tilde{v}, u, v)\Big]dt + (\tilde{v} - v)dB_2^H(t), \end{cases} \tag{4.4}$$

where

$$\Gamma_3 = \frac{1}{3}(\tilde{u}^3 - u^3) - \tilde{u} - u, \quad \Gamma_4 = -(\tilde{v} - v).$$

Denote $e_3(t) := \tilde{u} - u$, $e_4(t) := \tilde{v} - v$ and $\tilde{e} := (e_3, e_4)$. Then FHN model (4.2) can be rewrite into the following system

$$\mathrm{d}\tilde{e} = [A\tilde{e} - \tilde{e}]\mathrm{d}t + \tilde{e}\mathrm{d}B^H(t). \tag{4.5}$$

Theorem 4.2. *If $(rb+1)^2 \ge 4r$ for some positive real number r and b. Then the error dynamics in (4.5) will converge to zero and the synchronization can be achieved.*

Proof. Choosing Lyapunov function $V(\tilde{e}(t)) = \frac{1}{2}\big[(e_3(t))^2 + (e_4(t))^2\big]$, and applying Itô's formula to $V(\tilde{e}(t))$, we derive

$$\begin{aligned}\mathbb{E}[\mathrm{d}V(\tilde{e}(t))] =& \mathbb{E}\Big(\langle A\tilde{e}(t), \tilde{e}(t)\rangle - \langle \tilde{e}(t), \tilde{e}(t)\rangle + \frac{1}{2}\big[|e_3(t)|^2 + |e_4(t)|^2\big]\Big)\mathrm{d}t \\ \le& -(\lambda + \frac{1}{2})\mathbb{E}|\tilde{e}(t)|^2\mathrm{d}t = -(2\lambda+1)\mathbb{E}\big[V(\tilde{e}(t))\big]\mathrm{d}t.\end{aligned}$$

Gronwall inequality guarantees that

$$\mathbb{E}|\tilde{e}(t)|^2 \le \mathbb{E}|\tilde{e}(0)|^2 \exp\big(-(2\lambda+1)t\big),$$

which implies that $\limsup_{t\to\infty}\frac{1}{t}\log(\mathbb{E}|\tilde{e}(t)|^2) \le -(2\lambda+1) < 0$. The required assertion follows. □

5 Stochastic synchronization of time-fractional FHN model

We consider the following stochastic differential equations

$$\begin{cases} D_t^\beta u_1 = \big[(u_1 - \frac{u_1^3}{3} - v_1 + I_1) + r_1(u_2 - u_1)\big]\mathrm{d}t + \sigma_1 \mathrm{d}B_1^H(t), \\ D_t^\beta v_1 = \big[r(u_1 + a - bv_1)\big]\mathrm{d}t + \sigma_2 \mathrm{d}B_2^H(t), \end{cases} \tag{5.1}$$

and

$$\begin{cases} D_t^\beta u_2 = \big[(u_2 - \frac{u_2^3}{3} - v_2 + I_2) + r_2(u_1 - u_2)\big]\mathrm{d}t + \sigma_1 \mathrm{d}B_1^H(t), \\ D_t^\beta v_2 = \big[r(u_2 + a - bv_2)\big]\mathrm{d}t + \sigma_2 \mathrm{d}B_2^H(t), \end{cases} \tag{5.2}$$

where a, b and r are parameters, I_1 and I_2 denotes the injected current, r_1 and r_2 are coupling intensity, σ_1 and σ_2 are disturbance intensity, $\beta \in (0,1)$ is the order of fractional derivative.

Let $e_1 = u_1 - u_2$ and $e_2 = v_1 - v_2$. If $t \to \infty$, we can obtain that $\|e\| = \|e_1\| + \|e_2\| \to 0$ a.s., and the two coupled neurons achieve complete synchronization. If $I_1 = I_2$, $r_1 = r_2$, When the initial values are different,the two neurons can achieve complete synchronization by adjusting the coupling strength. Choosing the parameters $u_1(0) = 0,\ v_1(0) = 0,\ u_2(0) = 1,\ v_2(0) = 1$, and

$$a = 0.7, \quad b = 0.8, \quad I_1 = I_2 = 0.4, \quad r_1 = r_2 = 0.1.$$

and we specify the MATLAB code for simulating $\|e\|$.

```
clear all;
I1=0.4; r=0.1;  a=0.7; b=0.8; I2=0.4; r1=0.1; r2=0.1; orders=0.5; H=1/2; Tstep=0.01;
TSim=200; Y0=[0 0 1 1]; israndom=1; cgm_1=0; cgm_2=0; cgm_3=0; cgm_4=0;
[e, T, Y]=fun_y(I1,r1,r,a,b,r2,I2,orders,H,Tstep,TSim,Y0,...
    israndom,cgm_1,cgm_2,cgm_3,cgm_4);

function [e, T, Y]=fun_y(I1,r1,r,a,b,r2,I2,orders,H,Tstep,TSim,Y0,...
             israndom,cgm_1,cgm_2,cgm_3,cgm_4)
n=round(TSim/Tstep); q1=orders; w1 =israndom*wfbm(H,n); w2 =israndom*wfbm(H,n);
for i=1:n-1
    dw1(i+1)=Tstep^(H)*(w1(i+1)-w1(i));
    dw2(i+1)=Tstep^(H)*(w2(i+1)-w2(i));
end
cp1=1;
for j=1:n
    c1(j)=(1-(1+q1)/j)*cp1;
    cp1=c1(j);
end
x1=zeros(n,1);x2=zeros(n,1); y1=x1; y2=x2;
x1(1)=Y0(1); y1(1)=Y0(2); x2(1)=Y0(3); y2(1)=Y0(4);
for i=2:n+1
    x1(i)=(x1(i-1)-x1(i-1)^3/3-y1(i-1)+I1+r1*(x2(i-1)-x1(i-1)))*Tstep^q1...
           -memo(x1, c1, i)+cgm_1*dw1(i-1);
    y1(i)=(r*(x1(i-1)-b*y1(i-1)+a))*Tstep^q1 - memo(y1, c1, i)+cgm_2*dw2(i-1);
    x2(i)=(x2(i-1)-x2(i-1)^3/3-y2(i-1)+I2+r2*(x1(i-1)-x2(i-1)))*Tstep^q1...
           -memo(x2, c1, i)+cgm_3*dw1(i-1);
    y2(i)=(r*(x2(i-1)-b*y2(i-1)+a))*Tstep^q1 - memo(y2, c1, i)+cgm_4*dw2(i-1);
end
Y(:,1)=x1; Y(:,2)=y1; Y(:,3)=x2; Y(:,4)=y2;
e=abs(x1-x2)+abs(y1-y2);
T=0:Tstep:TSim;
```

Figure 1, Figure 3 and Figure 8 show that the curves of error $\|e\|$ vary with time. It can be seen that the two neurons can achieve complete synchronization.

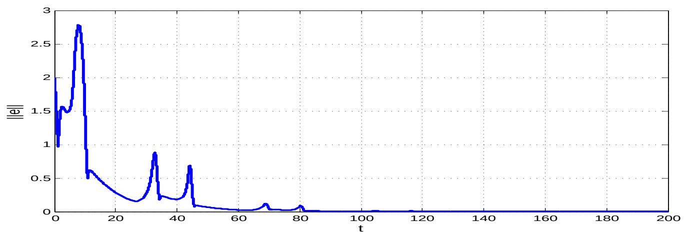

Figure 1: Time varying curve of synchronization error $\|e\|$ of integer-order deterministic FHN model, where $\beta = 1$ and $\sigma_1 = \sigma_2 = 0$.

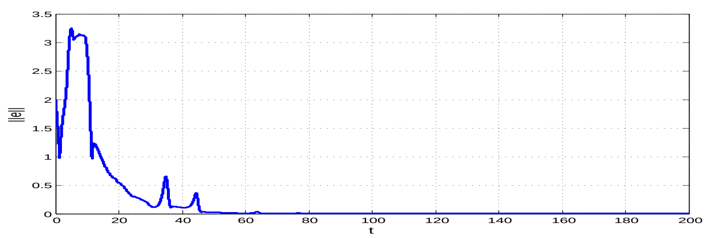

Figure 2: Time varying curve of synchronization error $\|e\|$ of fractional-order deterministic FHN model, where $\beta = 1$, $H = 0.5$ and $\sigma_1 = \sigma_2 = 0.25$.

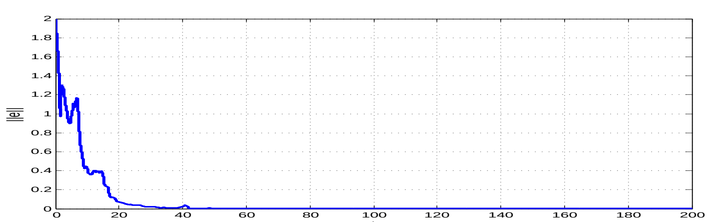

Figure 3: Time varying curve of synchronization error $\|e\|$ of integer-order deterministic FHN model, where $\beta = 1$, $H = 0.5$ and $\sigma_1 = \sigma_2 = 1$.

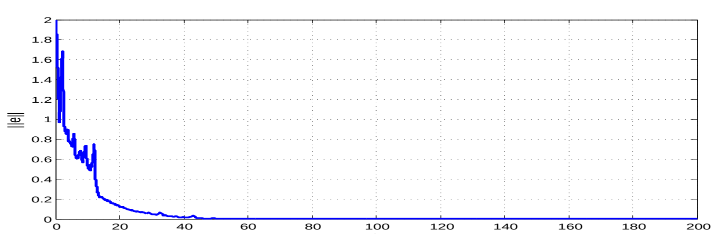

Figure 4: Time varying curve of synchronization error $\|e\|$ of fractional-order deterministic FHN model, where $\beta = 1$, $H = 0.5$ and $\sigma_1 = \sigma_2 = 4$.

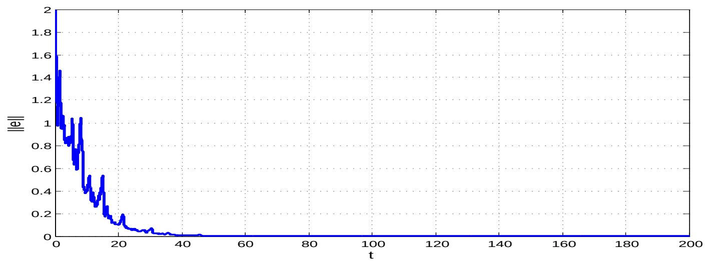

Figure 5: Time varying curve of synchronization error $\|e\|$ of fractional-order deterministic FHN model, where $\beta = 1$, $H = 0.375$ and $\sigma_1 = \sigma_2 = 4$.

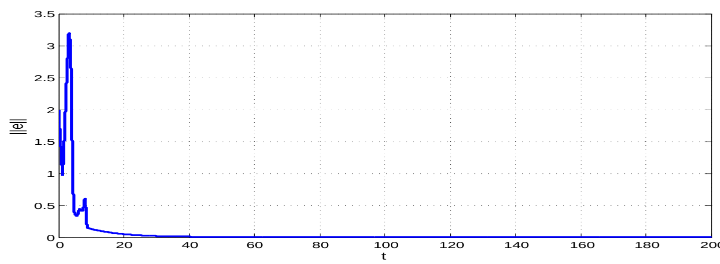

Figure 6: Time varying curve of synchronization error $\|e\|$ of fractional-order deterministic FHN model, where $\beta = 1$, $H = 0.8$ and $\sigma_1 = \sigma_2 = 4$.

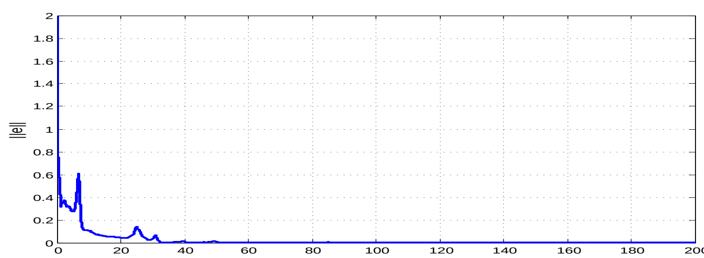

Figure 7: Time varying curve of synchronization error $\|e\|$ of fractional-order deterministic FHN model, where $\beta = 0.5$, $H = 0.8$ and $\sigma_1 = \sigma_2 = 4$.

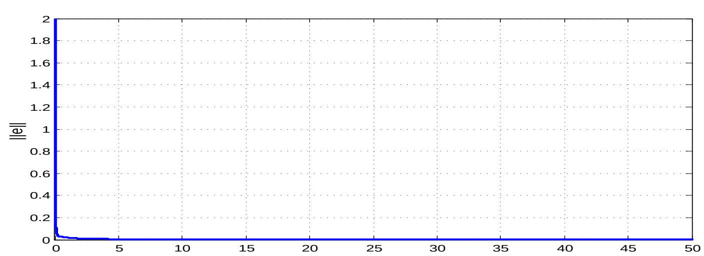

Figure 8: Time varying curve of synchronization error $\|e\|$ of fractional-order deterministic FHN model, where $\beta = 0.3$, $H = 0.8$ and $\sigma_1 = \sigma_2 = 4$.

It can be seen from the above figures that the rate of complete synchronization of neurons gradually increases with the decrease of the order of FHN model. Meanwhile, the discharge frequency of the neurons gradually increases with the decrease of the order of the FHN model. Therefore, it may be that the increase of discharge frequency makes the rate of complete synchronization of neurons gradually increase with the low order of the FHN model.

5.1 Hurst parameter H effect on the synchronization

In this subsection, we analysis the stochastic effect on the synchronization of stochastic fractional systems (5.1) and (5.2). Figure 9, Figure 10 and Figure 11 respectively show the synchronization diagram of two neurons when the order is $\beta = 0.5$ and $\sigma_1 = \sigma_2 = 1$ and fBm $H = 0.375, 0.5, 0.8$.

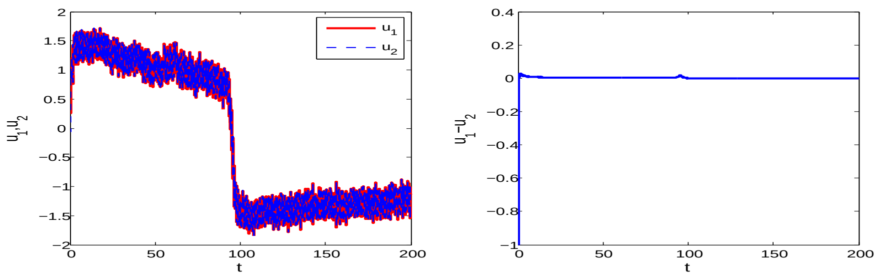

Figure 9: $\beta = 0.5$, $H = 0.375$

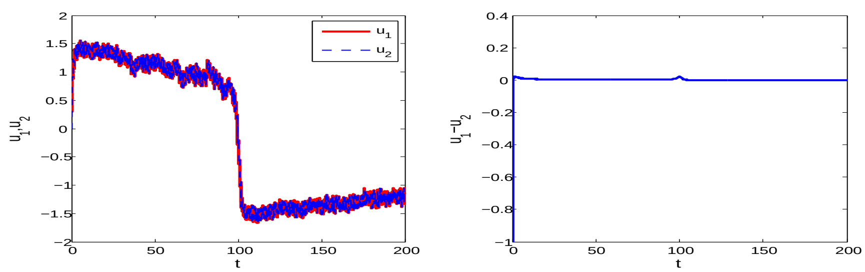

Figure 10: $\beta = 0.5$, $H = 0.5$

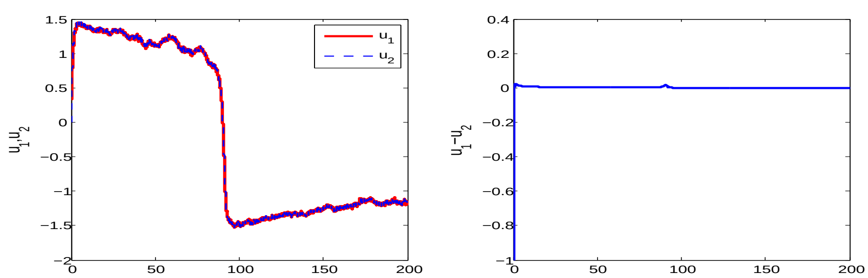

Figure 11: $\beta = 0.5$, $H = 0.8$

Figure 12, Figure 13 and Figure 14 show that the synchronization diagram of two neurons when $\beta = 0.9$ and $\sigma_1 = \sigma_2 = 1$, order $H = 0.375, 0.5, 0.8$.

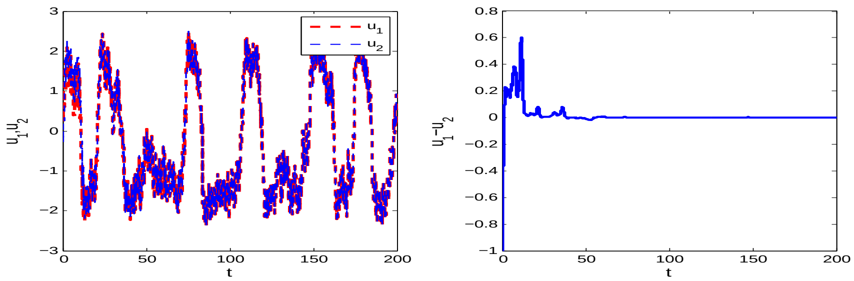

Figure 12: $\beta = 0.9$, $H = 0.375$

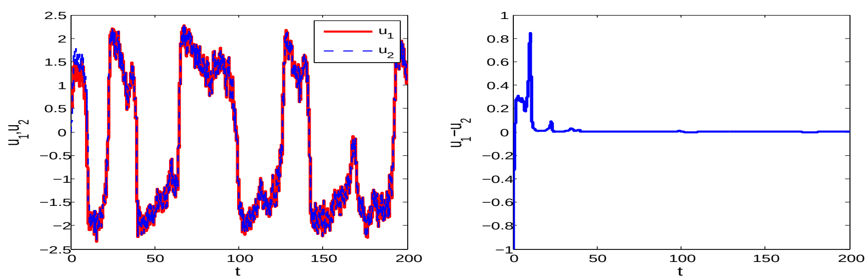

Figure 13: $\beta = 0.9$, $H = 0.5$

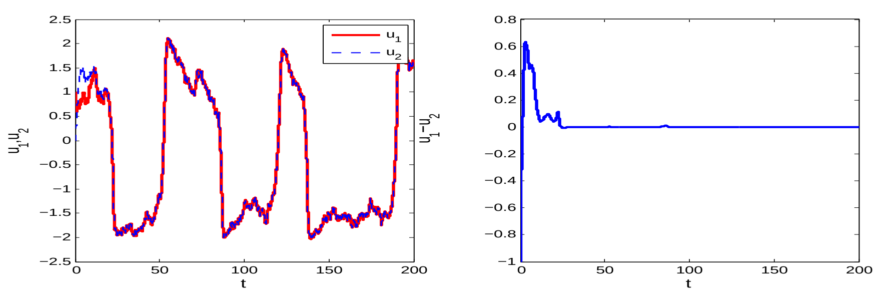

Figure 14: $\beta = 0.9$, $H = 0.8$

5.2 Fractional order effect on the synchronization

Figure 15, Figure 16 and Figure 17 show that the synchronization diagram of two neurons when $H = 0.5$ and $\sigma_1 = \sigma_2 = 1$, order $\beta = 0.5, 0.8, 1$.

Figure 18, Figure 19 and Figure 20 show that the synchronization diagram of two neurons when $H = 0.375$ and $\sigma_1 = \sigma_2 = 1$, order $\beta = 0.5, 0.8, 1$.

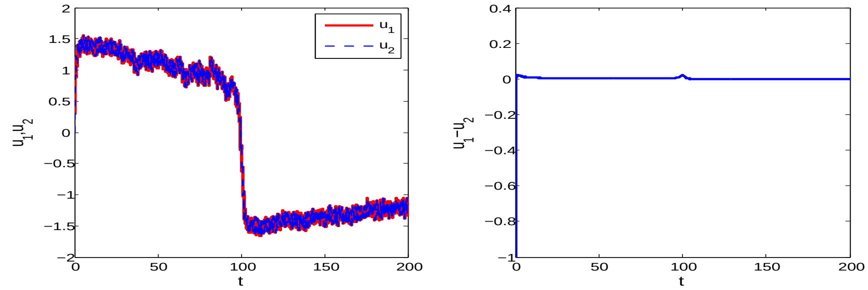

Figure 15: $H = 0.5$, $\beta = 0.5$

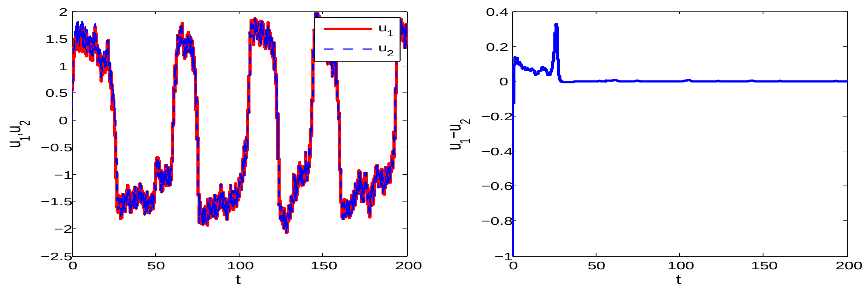

Figure 16: $H = 0.5$, $\beta = 0.8$

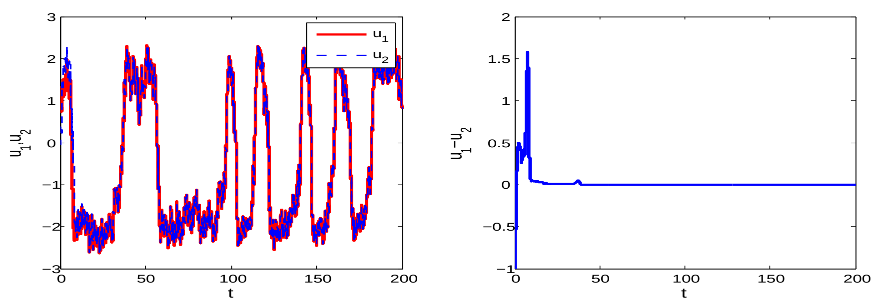

Figure 17: $H = 0.5$, $\beta = 1$

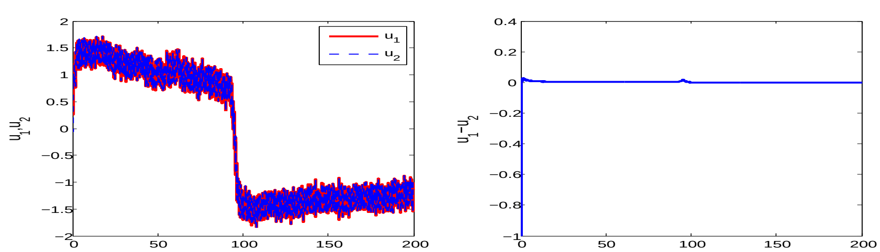

Figure 18: $H = 0.375$, $\beta = 0.5$

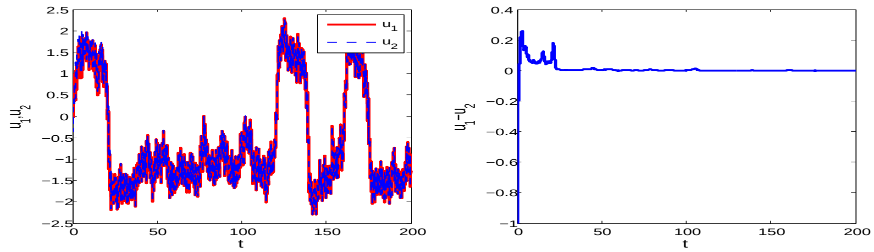

Figure 19: $H = 0.375$, $\beta = 0.8$

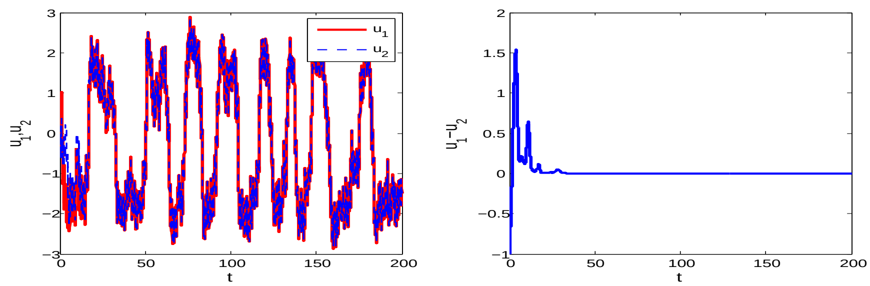

Figure 20: $H = 0.375$, $\beta = 1$

6 Conclusion

In this paper, we study the exponential synchronization and finite-time synchronization of Caputo-type fractional FitzHugh-Nagumo equations driven by fBm. By using stochastic analysis, we establish the sufficient conditions for the existence and uniqueness of the mild solution. Then, we obtain the exponential stability and finite-time stability results by constructing suitable Lyapunov functions. Numerical experiments verify the effectiveness of the theoretical analysis. Finally, we numerically investigated the effects on synchronization of the Hurst parameter in fBm, and the effects of beat parameter on the synchronization phenomenon of time-fractional FitzHugh-Nagumo equations.

The simulation show that the rate of complete synchronization of neurons gradually increases with the decrease of the order of FHN model. Meanwhile, the discharge frequency of the neurons gradually increases with the decrease of the order of the FHN model. Therefore, it may be that the increase of discharge frequency makes the rate of complete synchronization of neurons gradually increase with the low order of the FHN model.

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