A new fractional derivative operator and its application to diffusion equation

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Abstract. In this paper, we introduced a new fractional derivative operator based on Lonezo-Hartely function, which is called G-function.With the help of the operator, we solved a fractional diffusion equations.Some applications related to the operator is also discussed as form of corollaries.

1. Introduction

If we study the fractional calculus and its generalization of derivative operator $D^v[f(x)]$ to non integer values of v, we approach the theory of differential calculus. That field is a growing field and day by day new research comes in our notice. It has been observe that almost in every field fractional calculus have a significant role. Due to it's nature of applicability it become the favourite field of researcher. In lituratuer We can see the areas which have deep connection with fractional calculus like mathematical modeling, viscoelasticity, chaos fractals, electrical, electronics, heat transfer, physics, chemistry, biology and many more see([6]– [9]).

In the field of research and technology mathematical modeling is quite important. And in recent time it has been observe that fractional differential equation plays an important role to describing the physical phenomena. Fractional differential operator is a generalization of ordinary differential operator of arbitrary order ([1], [16], [18]). Many operators are defined in literature time to time, which shows their importance in different fields. Out of these operator, it has been observe that effective memory function of fractional order derivative plays a significant role to describing various scientific phenomena.

Special roles in the applications of fractional calculus operators are played by the transcendental functions like Mittag-Leffler function, Miller-Ross function and their generalization,Rabotnov function,Lorenzo-Hartley function and many more. These function are used to propose the new non-singular derivative operator.The general fractional derivatives via the special kernels have the

properties, such as memory and integro-differential representations with the non-singular behaviors, were developed in [10]. As we study the Yang's paper [11] based on generalised operator and having the application of relaxation phenomenon. It has been observe that operators of Fractional calculus are used to describe mechanics, intermediate processes, phenomena in physics and many more different fields (see [13], [22]– [24]). The Lorenzo–Hartley's function is used to model a anomalous relaxation in dielectrics [3],of fractional order. Y. Feng and J. Liu, [12] describe the diffusion equation using a new fractional derivative within the Miller-Ross kernel. Yang in 2019 [4] examined the anomalous heat transfer model having fractional-order derivative with Rabotnov fractional-exponential kernel.
The main idea of this paper is to propose a fractional diffusion equation ([14]– [15]) within general fractional derivative via the Lorenzo-Hartley kernel. Analytical solution of proposed diffusion equation of fractional order, within the Lorenzo-Hartley kernel, under the consideration of all the parameters, is concluded in this paper and so further, we can also use Miller Ross function kernel, the Robotnov and Hartley's functions kernel and Erdelyi's function kernel by putting the particular values of the parameters to find the respective solution for the diffusion equation.

2. Mathematical Preliminaries and Definitions

Definition 1. *[19] Let $v, \mu \in \mathbb{R}$,$a \in \mathbb{C}$, $n \in \mathbb{N} \cup \{0\}$, The R-Function is given by*

$$R_{v,\mu}[a, c, \tau] = \sum_{n=0}^{\infty} \frac{(a)^n (\tau - c)^{(n+1)v-1-\mu}}{\Gamma((n+1)v - \mu)}, \quad \Re(v) > 0. \tag{2.1}$$

Definition 2. *[20] The Lorenzo-Hartley's function is introduced by Lorenzo and Hartley. It is a simple generalization of R-function.*

$$G_{v,\mu,d}(a, c, \tau) = \sum_{k=0}^{\infty} \frac{(d)_k a^k (\tau - c)^{(k+d)v-\mu-1}}{k!\Gamma((k + d)v - \mu)}, \qquad \Re(vd - \mu) > 0, \tag{2.2}$$

where $(d)_k = \left\{ \begin{matrix} 1, & k = 0 \\ d(d + 1)(d + 2)...(d + k - 1), d \neq 0 & , k \in \mathbb{N} \end{matrix} \right\}.$

As we study the liturature we found that the Lorenzo-Hartley function is also related to well-known special functions like Mittag-Leffler functions, the Robotnov and Hartley's functions, Agarwal function, the R-function and Erdelyi's function. Recently, Chaurasia and Pandey [5], generalized some fractional kinetic equations in computable forms by using the Lorenzo-Hartley's G-function.

Definition 3. *[19] Hartley and Lorenzo introduced the Rabotnov function. That function is studied by Robotnov for the application of solid mechanics.*

$$R_v[a, \tau] = \sum_{n=0}^{\infty} \frac{a^n \tau^{((n+1)\nu-1)}}{\Gamma((n + 1)\nu)}, \quad \Re(v) > 0. \tag{2.3}$$

The relationship with G-function is given by,

$$G_{v,0,1}(-b, 0, \tau) = \sum_{n=0}^{\infty} \frac{(1)_n (-b)^n \tau^{(n+1)v-1}}{n!\Gamma((n + 1)v)} = R_v[-b, \tau]. \tag{2.4}$$

Definition 4. *[10] We have the Mittag-Leffler function*

$$E_v[\tau] = \sum_{n=0}^{\infty} \frac{\tau^n}{\Gamma(nv + 1)}, \qquad \Re(v) > 0.$$

Mittag-Leffler function with the argument $(-a\tau^u)$ is given by,

$$E_v[-a\tau^v] = \sum_{n=0}^{\infty} \frac{(-a)^n \tau^{vn}}{\Gamma(nv+1)}, \qquad \Re(v) > 0. \tag{2.5}$$

The relationship with the G-function is given by

$$G_{v,v-1,1}(-a,0,\tau) = G_{v,v-1,1}(-a,0,\tau) = \sum_{n=0}^{\infty} \frac{(1)_n(-1)^n a^n (\tau)^{(n+1)v-(v-1)-1}}{n!\Gamma((n+1)v-(v-1))} = E_v[-a\tau^v]. \tag{2.6}$$

Definition 5. *[21] We have the definition of Generalized Mittag-Leffler Function,*

$$E^{\delta}_{v,w}(\tau) = \sum_{k=0}^{\infty} \frac{(\delta)_k}{\Gamma(kv+w)} \frac{\tau^k}{k!}, \quad \Re(v) > 0, \ \Re(w) > 0. \tag{2.7}$$

The relation with the G-function is given by,

$$G_{v,-\mu,\delta}(-a,\tau-\theta) = (\tau-\theta)^{\delta v+\mu-1} \sum_{k=0}^{\infty} \frac{(\delta)_k(-a)^k(\tau-\theta)^{kv}}{k!\Gamma((k+\delta)v+\mu)} = E^{\delta}_{v,v\delta+\mu}(-a(\tau-\theta)^v), \tag{2.8}$$

where, $\Re(v) > 0, \ \Re(\mu) > 0, \Re(\delta) > 0.$

Remark 1.

(1) *[19] The Mittag-Leffler function is generalized by Agarwal, known as Agarwal function, as follows*

$$E_{u,v}[\tau] = \sum_{n=0}^{\infty} \frac{\tau^{n+\frac{v-1}{u}}}{\Gamma(nu+v)}, \quad \Re(v) > 0. \tag{2.9}$$

The relationship with the G-function is given by

$$G_{u,u-v,1}(1,0,\tau^{1/u}) = \sum_{n=0}^{\infty} \frac{(1)_n 1^n \tau^{\left(n+\frac{v-1}{u}\right)}}{n!\Gamma(nu+v)} = \sum_{n=0}^{\infty} \frac{\tau^{\left(n+\frac{v-1}{u}\right)}}{\Gamma(nu+v)} \ , \Re(v) > 0, \tag{2.10}$$

$$E_{u,v}[\tau] = G_{u,u-v,1}(1,0,\tau^{1/u}). \tag{2.11}$$

(2) *[19] Erdelyi function is the generalization of the Mittag-Leffler function where the powers of t are integer*

$$E_{u,v}[\tau] = \sum_{n=0}^{\infty} \frac{\tau^n}{\Gamma(nu+v)} \ , \ \Re(v), \Re(u) > 0. \tag{2.12}$$

The relationship with the G-function is given by,

$$\begin{aligned} G_{u,u-v,1}(1,0,\tau) &= \sum_{n=0}^{\infty} \frac{(1)_n 1^n \tau^{(nu+v-1)}}{n!\Gamma(nu+v)} = \tau^{(v-1)} \sum_{n=0}^{\infty} \frac{\tau^{(nu)}}{\Gamma(nu+v)}, \\ &= \tau^{(v-1)} E_{u,v}[\tau^u]. \end{aligned} \tag{2.13}$$

(3) *[17] Miller and Ross function is given by*

$$M_v[\tau,a] = \sum_{n=0}^{\infty} \frac{a^n \tau^{(n+v)}}{\Gamma(n+v+1)}, \quad \Re(v) > 0. \tag{2.14}$$

The relation with the G-function is,

$$G_{1,-\nu,1}(a,0,\tau)=\sum_{n=0}^{\infty}\frac{(1)_n(a)^n\tau^{(n+\nu)}}{n!\Gamma(n+\nu+1)}=M_v[\tau,a],\quad \Re(v)>0. \tag{2.15}$$

Remark 2.

(1) *The fractional integral of order α, $\alpha \geqslant 0$ of Riemann-Lioville type of continuous real function f in $[a,b]$ is defined as*

$$ {}_aI_\tau{}^\alpha f(\tau) = \frac{1}{\Gamma(\alpha)} \int_a^\tau (\tau - s)^{\alpha-1} f(s) ds, \quad \tau \in [a,b]. \tag{2.16}$$

(2) *The fractional derivative of order α, $\alpha \geqslant 0$ of Riemann-Lioville type of continuous real function f in $[a,b]$ is defined as*

$$ {}_aD_\tau^\alpha f(\tau) = D^m {}_aI_\tau^{(-m+\alpha)} f(\tau), \qquad \tau \in [a,b] \ \text{ and } \ m \in \mathbb{N}. \tag{2.17}$$

(3) *We have the Laplace transform of fractional derivative*

$$ L[D_\tau^\alpha f(x,\tau); s] = s^\alpha F(x,s) - \sum_{r=0}^{m-1} s^{\alpha-r-1} f^{(r)}(x,0), \tag{2.18}$$

where function $f(x,\tau)$ is continuous function and $\alpha \in R^+$.

3. New General fractional operator using Lorenzo Hartley kernel

Let $v \in \mathbb{R}$, $0 < v < 1$ and $a \in \mathbb{C}$. Define the function X_v (with c=0 in Lorenzo Hartley function) as follows

$$ X_v(-a(\tau-\theta)^v) = G_{v,-\mu,\delta}(-a, \tau-\theta) = \sum_{k=0}^\infty \frac{(\delta)_k (-a)^k (\tau-\theta)^{(k+\delta)v+\mu-1}}{k!\Gamma((k+\delta)v+\mu)}, \qquad \Re(v\delta+\mu) > 0. \tag{3.1}$$

The definition of the general fractional derivative (GFD) and general fractional integral (GFI) within X_v are as follows:

(1) Let $v \in \mathbb{R}$, $0 < v < 1$, $-\infty < r, l < \infty$, and $a \in \mathbb{C}$. The left-sided GFI within X_v function in the Riemann-Liouville sense is defined by,

$$ {}_{LH}I_{l+}^{a,v}\Omega(\tau) = \int_l^\tau X_v(-a(\tau-\theta)^v)\Omega(\theta) d\theta, \tag{3.2}$$

and the right sided GFI within the X_v function in Riemann-Liouville seance is defined by,

$$ {}_{LH}I_{r-}^{a,v}\Omega(\tau) = \int_\tau^r X_v(-a(\tau-\theta)^v)\Omega(\theta) d\theta. \tag{3.3}$$

GFI within the X_v function in the Riemann-Liouville sense is defined by,

$$ {}_{LH}I_{0+}^{a,v}\Omega(\tau) = \int_0^\tau X_v(-a(\tau-\theta)^v)\Omega(\theta) d\theta. \tag{3.4}$$

(2) Let $v \in \mathbb{R}$, $0 < v < 1$, $-\infty < r,\ l < \infty$, and $a \in \mathbb{C}$. The left-sided GFD within the X_v function of Riemann-Liouville sense is defined by,

$$ {}_{LH}^{RL}D_{l+}^{a,v}\Omega(\tau) = \frac{d}{d\tau} \int_l^\tau X_v(-a(\tau-\theta)^v)\Omega(\theta) d\theta = \frac{d}{d\tau} {}_{LH}I_{l+}^{a,v}\Omega(\tau). \tag{3.5}$$

For n, the left-sided GFD within the X_v function of Riemann-Liouville sense is defined by,

$$ {}_{LH}^{RL}D_{l+}^{a,n,v}\Omega(\tau) = \frac{d^n}{d\tau^n}\int_l^\tau X_v(-a(\tau-\theta)^v)\Omega(\theta)d\theta = \frac{d^n}{d\tau^n}{}_{LH}I_{l+}^{a,v}\Omega(\tau). \tag{3.6} $$

The Right-sided GFD within the X_v function of Riemann-Liouville sense is defined by,

$$ {}_{LH}^{RL}D_{r-}^{a,v}\Omega(\tau) = -\frac{d}{d\tau}\int_\tau^r X_v(-a(\tau-\theta)^v)\Omega(\tau)d\tau = -\frac{d}{d\tau}{}_{LH}I_{r-}^{a,v}\Omega(\tau). \tag{3.7} $$

For n, the right-sided GFD within the X_v function of Riemann-Liouville sense is defined by,

$$ {}_{LH}^{RL}D_{r-}^{a,n,v}\Omega(\tau) = -\frac{d^n}{d\tau^n}\int_\tau^r X_v(-a(\tau-\theta)^v)\Omega(\theta)d\theta = -\frac{d^n}{d\tau^n}{}_{LH}I_{r-}^{a,v}\Omega(\tau). \tag{3.8} $$

GFD within the X_v function of Riemann-Liouville sense is defined by,

$$ {}_{LH}^{RL}D_{0+}^{a,n,v}\Omega(\tau) = \frac{d^n}{d\tau^n}\int_0^t X_v(-a(\tau-\theta)^v)\Omega(\theta)d\theta = \frac{d^n}{d\tau^n}{}_{LH}I_{0+}^{a,v}\Omega(\tau). \tag{3.9} $$

Remark 3.

(1) *GFI within the Rabotnov function of Riemann-Liouville sense is defined by,*

$$ {}_{Rb}I_{0+}^{a,v}\Omega(\tau) = \int_0^\tau R_v(-a(\tau-\theta)^v)\Omega(\theta)d\theta. \tag{3.10} $$

GFD within the Rabotnov function of Riemann-Liouville sense is defined by,

$$ {}_{Rb}^{RL}D_{0+}^{a,n,v}\Omega(\tau) = \frac{d^n}{d\tau^n}\int_0^\tau R_v(-a(\tau-\theta)^v)\Omega(\theta)d\theta = \frac{d^n}{d\tau^n}{}_{Rb}I_{0+}^{a,v}\Omega(\tau). \tag{3.11} $$

(2) *GFI within the Miller-Ross function of Riemann-Liouville sense is defined by*

$$ {}_{M}I_{0+}^{a,v}\Omega(\tau) = \int_0^\tau M_v(-a(\tau-\theta)^v)\Omega(\theta)d\theta. \tag{3.12} $$

GFD within the Miller Ross function of Riemann-Liouville sense is defined by,

$$ {}_{M}^{RL}D_{0+}^{a,n,v}\Omega(\tau) = \frac{d^n}{d\tau^n}\int_0^\tau M_v(-a(\tau-\theta)^v)\Omega(\theta)d\theta = \frac{d^n}{d\tau^n}{}_{LH}I_{0+}^{a,v}\Omega(\tau). \tag{3.13} $$

(3) *GFI with in the Erdelyi's function of Riemann-Liouville sense is defined by,*

$$ {}_{Er}I_{0+}^{1,v}\Omega(\tau) = \int_0^\tau E_{u,v}((\tau-\theta)^v)\Omega(\tau)d\tau. \tag{3.14} $$

GFD within the Erdelyi's function of Riemann-Liouville sense is defined by,

$$ {}_{Er}^{RL}D_{0+}^{1,n,v}\Omega(\tau) = \frac{d^n}{d\tau^n}\int_0^\tau E_{u,v}((\tau-\theta)^v)\Omega(\theta)d\theta = \frac{d^n}{d\tau^n}{}_{Er}I_{0+}^{1,v}\Omega(\tau). \tag{3.15} $$

4. Some Integral transform of New PRR LC Fractional Operator

In this section, we defined few theorems based on the new PRR LC fractional operator as follows:

Theorem 1. *Let $v \in \mathbb{R}$, $0 < v < 1$ and $a \in \mathbb{C}$. Then the Laplace transform of ${}_{LH}I^{a,v}_{0+}\Omega(\tau)$ is given by*

$$L\left[{}_{LH}I^{a,v}_{0+}\Omega(\tau)\right] = \frac{s^{-\mu}}{(s^v + a)^\delta}\Omega(s). \tag{4.1}$$

Proof. To find the Laplace transform of ${}_{LH}I^{a,v}_{0+}\Omega(\tau)$, we have

$${}_{LH}I^{a,v}_{0+}\Omega(\tau) = \int\limits_0^\tau X_v(-a(\tau-\theta)^v)\Omega(\theta)d\theta, \tag{4.2}$$

$$X_v(-a(\tau-\theta)^v) = \sum_{k=0}^{\infty} \frac{(\delta)_k(-a)^k(\tau-\theta)^{(k+\delta)v+\mu-1}}{k!\Gamma((k+\delta)v+\mu)}, \tag{4.3}$$

$$L\left[{}_{LH}I^{a,v}_{0+}\Omega(\tau)\right] = L\left[\int\limits_0^\tau X_v(-a(\tau-\theta)^v)\Omega(\theta)d\theta\right]. \tag{4.4}$$

Using Convolution theorem for Laplace transform is given by

$$L\left(\int\limits_0^\tau X_v(-a(\tau-\theta)^v).\Omega(\theta)d\theta\right) = L\left(X_v(-a\tau^v) * L\left(\Omega(\tau)\right)\right), \tag{4.5}$$

$$L\left[X_v(-a\tau)^v)\right] = \sum_{k=0}^{\infty} \frac{(\delta)_k(-a)^k L(\tau)^{(k+\delta)v+\mu-1}}{k!\Gamma((k+\delta)v+\mu)}, \tag{4.6}$$

$$L\left[X_v(-a\tau)^v)\right] = \frac{s^{-\mu}}{(s^v + a)^\delta}, \tag{4.7}$$

$$L\left[{}_{LH}I^{a,v}_{0+}\Omega(\tau)\right] = \frac{s^{-\mu}}{(s^v + a)^\delta}\Omega(s). \tag{4.8}$$

∎

Theorem 2. *Let $v \in \mathbb{R}$, $0 < v < 1$ and $a \in \mathbb{C}$. Then the Sumudu transform of ${}_{LH}I^{a,v}_{0+}\Omega(\tau)$ is given by,*

$$S\left[{}_{LH}I^{a,v}_{0+}\Omega(\tau)\right] = \frac{u^{\mu+v\delta}}{(1 + au^v)^\delta}\Omega(u). \tag{4.9}$$

Proof. Let $v \in \mathbb{R}$, $0 < v < 1$ and $a \in \mathbb{C}$. Then the Sumudu transform of ${}_{LH}I^{a,v}_{0+}\Omega(\tau)$ is given by,

$$S\left[{}_{LH}I^{a,v}_{0+}\Omega(\tau)\right] = \frac{u^{\mu+v\delta+1}}{(1 + au^v)^\delta}\Omega(1/u). \tag{4.10}$$

The Sumudu transform of (4.10) can be calculated as follows.
We have

$${}_{LH}I^{a,v}_{0+}\Omega(\tau) = \int\limits_0^\tau X_v(-a(\tau-\theta)^v)\Omega(\theta)d\theta, \tag{4.11}$$

$$X_v(-a(\tau-\theta)^v) = \sum_{k=0}^{\infty} \frac{(\delta)_k (-a)^k (\tau-\theta)^{(k+\delta)v+\mu-1}}{k!\Gamma((k+\delta)v+\mu)}. \tag{4.12}$$

Hence

$$S\left[{}_{LH}I_{0+}^{a,v}\Omega(\tau)\right] = S\left[\int_0^\tau X_v(-a(\tau-\theta)^v)\Omega(\theta)d\theta\right]. \tag{4.13}$$

Using Convolution theorem for Sumudu transform is given by,

$$S\left(\int_0^t X_v(-a(\tau-\theta)^v).\Omega(\theta)d\theta\right) = uS\left(X_v(-a\tau^v) * S\left(\Omega(\tau)\right)\right), \tag{4.14}$$

$$S\left[X_v(-a\tau)^v)\right] = \sum_{k=0}^{\infty} \frac{(\delta)_k (-a)^k S(\tau)^{(k+\delta)v+\mu-1}}{k!\Gamma((k+\delta)v+\mu)}, \tag{4.15}$$

$$S\left[X_v(-a\tau)^v)\right] = \frac{u^{\mu+v\delta-1}}{(1+au^v)^\delta}. \tag{4.16}$$

Hence

$$S\left[{}_{LH}I_{0+}^{a,v}\Omega(\tau)\right] = \frac{u^{\mu+v\delta}}{(1+au^v)^\delta}\Omega(u). \tag{4.17}$$

■

Theorem 3. *Let $v \in \mathbb{R},\ 0 < v < 1$ and $a \in \mathbb{C}$. Then the Laplace transform of ${}_{LH}^{RL}D_{0+}^{a,n,v}g(\tau)$, is given by,*

$$L\left({}^{RL}D_{0+}^{a,n,v}g(\tau)\right) = \frac{s^{n-\mu}}{(s^v+a)^\delta} - \sum_{k=0}^{n-k-1} D^k\left(\int_0^t X_v.g(\tau)d\tau\right)\Bigg|_{\tau=0}, \tag{4.18}$$

where k=0,1,2,3,...

Proof. The Laplace transform of fractional derivative is given by (2.18) where $g(\tau)$ is continuous function and $\alpha \in R^+$.
For the new operator,

$$L\left({}^{RL}D_{0+}^{a,n,v}g(\tau)\right) = s^n L\left(\int_0^\tau X_v.g(\tau)d\tau\right) - \sum_{k=0}^{n-k-1} D^k\left(\int_0^\tau X_v.g(\tau)d\tau\right)\Bigg|_{\tau=0}, \tag{4.19}$$

where k=0,1,2,3,...

$$L\left({}^{RL}D_{0+}^{a,n,v}g(\tau)\right) = \frac{s^{n-\mu}}{(s^v+a)^\delta} - \sum_{k=0}^{n-k-1} D^k\left(\int_0^t X_v.g(\tau)d\tau\right)\Bigg|_{\tau=0}. \tag{4.20}$$

■

5. Application of the new Generalized fractional derivative to Fractional diffusion equation

Theorem 4. *Let the following time fractional diffusion equation,*

$$ {}^{RL}_{LH}D^{a,v}_{0+}Y(x,\tau) = \frac{\partial^2}{\partial x^2}Y(x,\tau), \tag{5.1}$$

associated with the initial and boundary conditions given by,

$$\begin{aligned} &Y(x,0) = 0,\\ &Y(0,\tau) = \delta(\theta),\\ &Y(x,\tau) \to 0 \;\; as \;\; x \to \infty, \;\; \tau > 0, \end{aligned} \tag{5.2}$$

where δ is Kronecker delta. Then the solution of the diffusion equation (5.1) is

$$ Y(x,\tau) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} G_{v,(1-\mu)n/2,\delta n/2}(-a,0,\tau). \tag{5.3}$$

Proof. On taking Laplace transform,

$$ L\left({}^{RL}_{LH}D^{a,v}_{0+}Y(x,\tau)\right) = \frac{\partial^2}{\partial x^2}Y(x,s). \tag{5.4}$$

By equation (4.20),

$$ s.\frac{s^{-\mu}}{(s^v+a)^{\delta}}Y(x,s) - 0 = \frac{\partial^2}{\partial x^2}Y(x,s), \tag{5.5}$$

$$ \Rightarrow \left(\frac{\partial^2}{\partial x^2} - \frac{s^{1-\mu}}{(s^v+a)^{\delta}}\right)Y(x,s) = 0. \tag{5.6}$$

On solving this we get,

$$ Y(x,s) = Ae^{x\sqrt{s^{1-\mu}(s^v+a)^{-\delta}}} + Be^{-x\sqrt{s^{1-\mu}(s^v+a)^{-\delta}}}, \tag{5.7}$$

and since $Y(x,s) \to 0 \;\; as \;\; x \to \infty, \;\; s > 0$,

$$\begin{aligned} &\Rightarrow Ae^{x\sqrt{s^{1-\mu}(s^v+a)^{-\delta}}} = 0,\\ &\Rightarrow A = 0, B = 1,\\ &\Rightarrow Y(x,s) = e^{-x\sqrt{s^{1-\mu}(s^v+a)^{-\delta}}}. \end{aligned} \tag{5.8}$$

$$ \Rightarrow Y(x,s) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\left(\sqrt{\frac{s^{1-\mu}}{(s^v+a)^{\delta}}}\right)^n = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\left(\frac{s^{1-\mu}}{(s^v+a)^{\delta}}\right)^{n/2}, \tag{5.9}$$

$$ L^{-1}\left(Y(x,s)\right) = L^{-1}\left(\sum_{n=0}^{\infty} \frac{(-x)^n}{n!}\left(\frac{s^{1-\mu}}{(s^v+a)^{\delta}}\right)^{n/2}\right). \tag{5.10}$$

Now consider the following G-function,

$$\begin{aligned}
G_{v,1-\mu,\delta}(-a,c,\tau) &= \sum_{k=0}^{\infty}\frac{(\delta)_k(-a)^k(\tau-c)^{(k+\delta)v+\mu-1-1}}{k!\Gamma((k+\delta)v+\mu-1)},\\
L\left[G_{v,1-\mu,\delta}(-a,c,\tau)\right] &= L\left[\sum_{k=0}^{\infty}\frac{(\delta)_k(-a)^k\tau-c)^{(k+\delta)v+\mu-1-1}}{k!\Gamma((k+\delta)v+\mu-1)}\right],\\
&= e^{-cs}\sum_{k=0}^{\infty}\frac{(\delta)_k(-a)^kL(\tau)^{(k+\delta)v+\mu-1-1}}{k!\Gamma((k+\delta)v+\mu-1)},\\
&= e^{-cs}s^{1-\mu-v\delta}\frac{s^{v\delta}}{(s^v+a)^\delta}.
\end{aligned}\tag{5.11}$$

$$L\left[G_{v,1-\mu,\delta}(-a,0,\tau)\right] = \frac{s^{1-\mu}}{(s^v+a)^\delta},\quad c=0,\tag{5.12}$$

$$G_{v,(1-\mu)n/2,\delta n/2}(-a,0,\tau) = L^{-1}\left(\frac{s^{1-\mu}}{(s^v+a)^\delta}\right)^{n/2}.\tag{5.13}$$

The solution of the diffusion equation is given by

$$Y(x,\tau) = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}G_{v,(1-\mu)n/2,\delta n/2}(-a,0,\tau).\tag{5.14}$$

Also, with $\Re(v)>0$, $\Re(\mu)>0$, $\Re(\delta)>0$. We can write the solution in terms of Mittag-Leffler Function as,

$$Y(x,\tau) = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}(\tau)^{\frac{(v\delta-1+\mu)n}{2}-1}E^{\frac{\delta n}{2}}_{v,\frac{(v\delta-1+\mu)n}{2}}(-a\tau^v),\quad \Re(v\delta-\mu)>0,\tag{5.15}$$

where,

$$\begin{aligned}
&G_{v,-\mu,\delta}(-a,\tau) = \tau)^{\delta v+\mu-1}E^{\delta}_{v,v\delta+\mu}(-a\tau^v),\\
&\Rightarrow G_{v,(1-\mu)n/2,\delta n/2}(-a,\tau) = (\tau)^{\delta v+\mu-1}E^{\delta n/2}_{v,(v\delta+1-\mu)n/2}(-a\tau^v).
\end{aligned}\tag{5.16}$$

■

Corollary 1. *For the values, $\mu=0$ and $\delta=1$, the solution of diffusion equation (5.1) for GFD (3.11) within the Rabotnov kernel is given by,*

$$Y(x,\tau) = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}G_{v,n/2,n/2}(-a,0,\tau).\tag{5.17}$$

In terms of Mittag-Leffler function solution is given by

$$Y(x,\tau) = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}(\tau)^{\frac{(v-1)n}{2}-1}E^{\frac{n}{2}}_{v,\frac{(v-1)n}{2}}(-a\tau^v),\quad \Re(v)>0.\tag{5.18}$$

Corollary 2. *For the values, v=1, $\mu=v$ and $\delta=1$ the solution of the diffusion equation (5.1) for GFD (3.13) within Miller-Ross kernel is given by*

$$Y(x,\tau) = \sum_{n=0}^{\infty}\frac{(-x)^n}{n!}G_{1,-vn/2,n/2}(-a,0,\tau)].\tag{5.19}$$

And in terms of Mittag-leffler function,

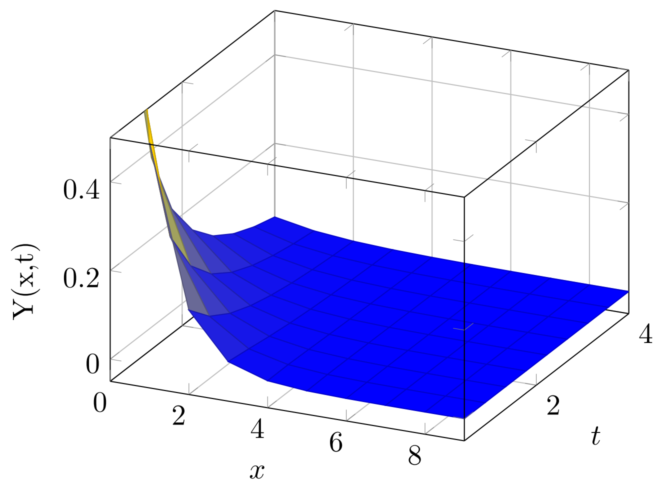
$$Y(x,\tau) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} (\tau)^{\frac{n(1-v)}{2}-1} E^{n/2}_{1,(1-v)n/2}(-a\tau^v). \tag{5.20}$$

Corollary 3. *For the values, v=u, $\mu = v - u$, $a = -1$ and $\delta = 1$ the solution of the diffusion equation (5.1) for GFD (3.15) within the Erdelyi's kernel given by,*

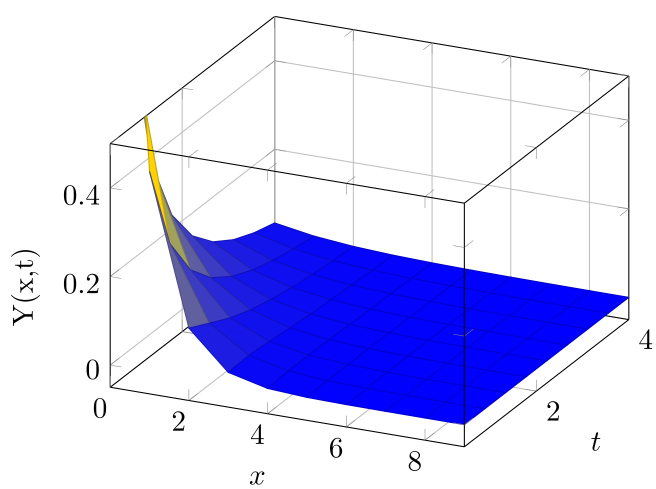
$$Y(x,\tau) = \sum_{n=0}^{\infty} \frac{(-x)^n}{n!} \tau^{1-v} G_{u,(u-v)n/2,n/2}(1,0,\tau)]. \tag{5.21}$$

6. Graphical Representation

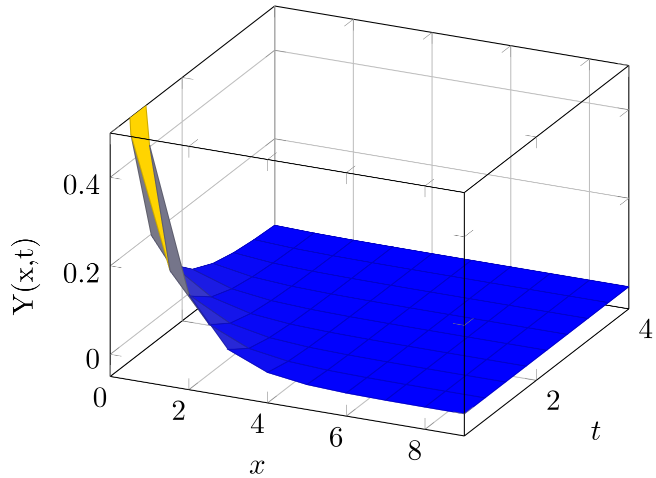
The followings are the graphical representation of solution of diffusion equation (5.1) with different values of μ, (fixing $\delta = 1$, v=1)
For, $\mu = 0.2$



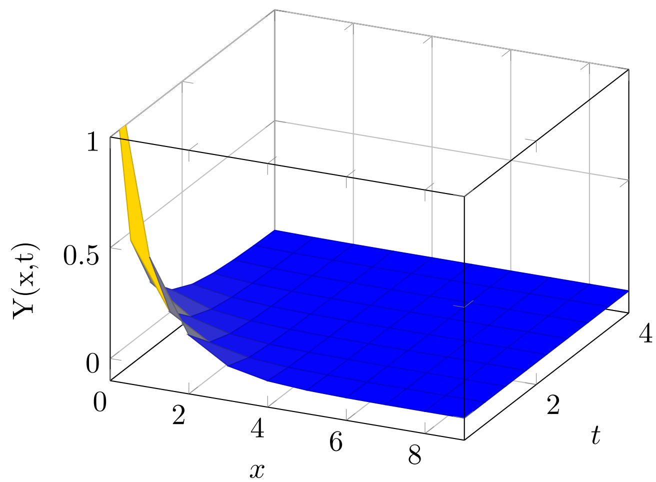
For, $\mu = 0.5$



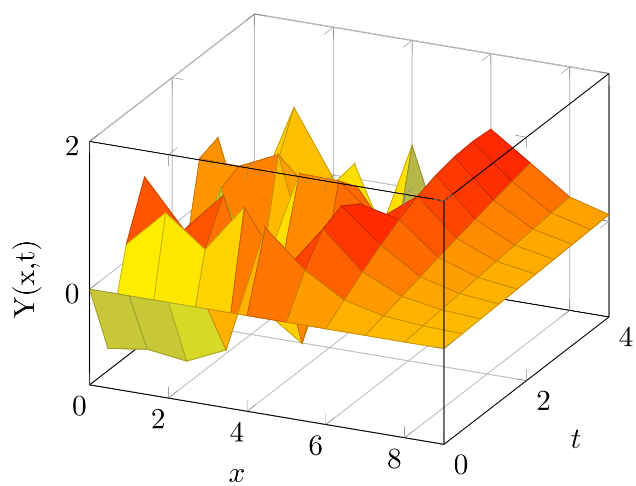
For, $\mu = 0.8$



For, $\mu = 0.1$



Now , the graph for analytical solution of diffusion equation with the ordinary derivative operator in calculus is shown below



The graphs are effectively showing the difference in solution of diffusion equation via fractional calculus and ordinary calculus.

7. Conclusion

In this paper, a new definition of general fractional operator is proposed. We also discuss the solution of diffusion equation, with the help Laplace transform. We found the solution for the diffusion equation in terms of transcedental functions which is effectively shown in graphs and comparison done with respect to the ordinary derivative operator.

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