NOTE ON GEOMETRIC ALGEBRAS AND CONTROL PROBLEMS WITH $SO(3)$–SYMMETRIES

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Abstract. We study the role of symmetries in control systems by means of geometric algebra approach. We discuss two specific control problems on Carnot group of step 2 invariant with respect to the action of $SO(3)$. We understand geodesics as curves in suitable geometric algebras which allows us to asses an efficient algorithm for local control.

1. Introduction

Geometric control theory uses geometric methods to control various mechanical systems, [12, 5]. We use methods of sub–Riemannian geometry and Hamiltonian concept, [2, 1]. As a reasonable starting point, we consider mechanisms moving in the plane, typically wheeled mechanisms like cars, cars with trailers, robotic snakes, etc., see e.g. [7, 10]. The movement of planar mechanisms is always invariant with respect to the action of the Euclidean group $SE(2)$. As prototypes of planar mechanisms we choose those consisting of a body in the shape of a triangle and three legs connected to the vertices of the body by joints which can be of various types and combinations. Although such mechanisms have almost the same shape, the configuration spaces may be very different. In particular, possible motions of the mechanism induce a specific filtration in the configuration space. We present two examples that carry the filtration $(3, 6)$ and $(4, 7)$, respectively, [10, 9].

To control mechanisms locally we consider the nilpotent approximations of the original control systems, [4]. Although the appropriate configuration spaces have the same filtration, they are endowed with more symmetries in general, [14]. We always have symmetries generated by Lie algebra of right–invariant vector fields and some additional symmetries that act non–trivially on the distribution. In nilpotent

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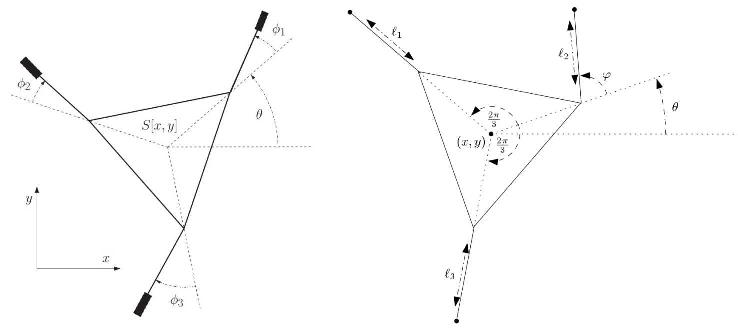

FIGURE 1. Generalized trident snakes

approximations of our control problems, there is a subgroup of symmetries which is isomorphic to the Lie group $SO(3)$, [11, 9]. This leads us to the idea of local control in geometric algebra approach.

We reformulate the control problems in geometric algebras $\mathbb{G}_3$ and $\mathbb{G}_4$, [16, 17, 19]. We use the natural $SO(3)$ invariant operation in geometric algebra to reduce the set of geodesics to a simpler set of curves in geometric algebra, []. Namely, each geodesic is a linear combination of orthogonal vectors, and $SO(3)$ acts on geodesics by the action on the appropriate orthonormal system of vectors. So it is sufficient to study geodesics for one fixed orthonormal basis, i.e. we can study just geodesics in the moduli space over the action of $SO(3)$.

We present local control algorithm for finding geodesics through the origin and arbitrary point in its neighbourhood. The algorithm is based on a problem how to use rotors in order to compare two orthogonal bases. We provide an efficient method to such comparison using geometric algebras. We demonstrate our algorithm on two specific examples.

2. Nilpotent control problems

We focus on two control problems such that their symmetry groups contain $SO(3)$ as subgroups. The first system has the growth vector $(3,6)$ and the other one has the growth vector $(4,7)$, [14, 9].

2.1. **Control problems on a Carnot group of step 2.** By nilpotent control problems we mean invariant control problems on Carnot groups, particularly we consider a Carnot group G of step 2 with filtration (m,n), [1, Section 13] or [15, 13]. If we denote local coordinates by $(x,z) \in \mathbb{R}^m \oplus \mathbb{R}^{n-m}$, we can model the corresponding Lie algebra $\mathfrak{c}$ of vector fields

$$\begin{aligned} X_i &= \partial_{x_i} - \frac{1}{2}\sum_{l=1}^{n-m}\sum_{j=1}^{m} c^l_{ij} x_j \partial_{z_l}, \quad && j = 1,\dots,m \\ X_{m+j} &= \partial_{z_j} && j = 1,\dots,m-n, \end{aligned} \tag{1}$$

where c^k_{jl} are the structure constants of Lie algebra $\mathfrak{g}$ and the symbol ∂ stands for partial derivative. We discuss the related optimal control problem

$$\dot{q}(t) = u_1 X_1 + \cdots + u_m X_m \tag{2}$$

for $t > 0$ and q in G and the control $u = (u_1(t), \dots, u_m(t)) \in \mathbb{R}^m$ with the boundary condition $q(0) = q_1$, $q(T) = q_2$ for fixed points $q_1, q_2 \in G$, where we minimize the cost functional $\frac{1}{2}\int_0^T (u_1^2 + \cdots + u_m^2)\mathrm{d}t$. Solutions $q(t)$ then correspond to sub–Riemannian geodesics, i.e. admissible curves parametrized by constant speed whose sufficiently small arcs are length minimizers.

We use Hamiltonian approach to this control problem and we follow [1, Sections 7 and 13]. Let us note that there are no strict abnormal extremals for step 2 Carnot groups, [1, Section 13]. Left–invariant vector fields X_i, $i = 1, \dots, m$ form a basis of TG and determine left–invariant coordinates on G. Then we define the corresponding left–invariant coordinates h_i, $i = 1, \dots, m$ and w_i, $i = 1, \dots, n-m$ on the fibres of T^*G by $h_i(\lambda) = \lambda(X_i)$ and $w_i(\lambda) = \lambda(X_{m+i})$, for an arbitrary 1–form λ on G. Thus we can use (x_i, w_i) as global coordinates on T^*G.

It turns out that the geodesics are exactly the projections of normal Pontryagin extremals, i.e. integral curves of left–invariant normal Hamiltonian

$$H = \frac{1}{2}(h_1^2 + h_2^2 + \cdots + h_m^2), \tag{3}$$

on G. Assume that $\lambda(t) = (x_i(t), z_i(t), h_i(t), w_i(t))$ in T^*G is a normal extremal. Then the controls u_j to the system (2) satisfy $u_j(t) = h_j(\lambda(t))$ and the system on the base space is of the form

$$\begin{aligned} \dot{x}_i &= h_i, \quad i = 1, \dots, m \\ \dot{z}_j &= -\frac{1}{2}\sum_{i=1}^{m} c_{ik}^j h_i x_k, \quad j = 1, \dots, n-m \end{aligned} \tag{4}$$

for $q = (x_i, z_i)$. Using $u_j(t) = h_j(\lambda(t))$ and the equation $\dot{\lambda}(t) = \vec{H}(\lambda(t))$ for normal extremals, we can write the system on fibres as

$$\begin{aligned} \dot{h}_i &= -\sum_{l=1}^{m-n}\sum_{j=1}^{m} c_{ij}^l h_j w_l, \quad & i = 1, \dots, m, \\ \dot{w}_j &= 0, & j = 1, \dots, n-m, \end{aligned} \tag{5}$$

where c_{ij}^l are the structure constants of the Lie algebra $\mathfrak{g}$ for the basis X_i. We see immediately that the solutions w_i, $i = 1, \dots, n-m$ are constants which we denote by

$$w_1 = K_1, \ \dots, w_{n-m} = K_{n-m}. \tag{6}$$

If $K_1 = \cdots = K_{n-m} = 0$ then $h(t) = h(0)$ is constant and the geodesic $(x_i(t), z_i(t))$ is a line in G such that $z_i(t) = 0$. If at least one of K_i is non–zero, the first part of the fibre system (5) forms a homogeneous system of ODEs $\dot{h} = -\Omega h$ with constant coefficients for $h := (h_1, \dots, h_m)^T$ and the system matrix Ω. Its solution is given by $h(t) = e^{-t\Omega}h(0)$, where $h(0)$ is the initial value of vector h at the origin.

2.2. **Left–invariant control problem with the growth vector** $(3, 6)$**.** In compliance with the notation of (1), we consider three vector fields given on $\mathbb{R}^6$ with local coordinates $(x_1, x_2, x_3, z_1, z_2, z_3)$ in the form

$$
\begin{aligned}
X_1 &= \partial_{x_1} + \frac{x_3}{2}\partial_{z_2} - \frac{x_2}{2}\partial_{z_3},\\
X_2 &= \partial_{x_2} + \frac{x_1}{2}\partial_{z_3} - \frac{x_3}{2}\partial_{z_1},\\
X_3 &= \partial_{x_3} + \frac{x_2}{2}\partial_{z_1} - \frac{x_1}{2}\partial_{z_2}.
\end{aligned} \tag{7}
$$

The only non–trivial Lie brackets are

$$
X_4 = [X_1, X_2] = \partial_{z_3}, \quad X_5 = [X_1, X_3] = -\partial_{z_2}, \quad X_6 = [X_2, X_3] = \partial_{z_1}. \tag{8}
$$

Together, these six fields determine a step 2 nilpotent Lie algebra $\mathfrak{m}$ with multiplication given by Table 1.

	X_1	X_2	X_3	X_4	X_5	X_6
X_1	0	X_4	X_5	0	0	0
X_2	$-X_4$	0	X_6	0	0	0
X_3	$-X_5$	$-X_6$	0	0	0	0
X_4	0	0	0	0	0	0
X_5	0	0	0	0	0	0
X_6	0	0	0	0	0	0

TABLE 1. Lie algebra $\mathfrak{m}$

Then there is a Carnot group M such that the fields X_i, $i = 1\dots, 6$ are left–invariant for the corresponding group structure. When identified with $\mathbb{R}^6 = \mathbb{R}^3 \oplus \mathbb{R}^3$, the group structure on M reads that

$$
(x, z) \cdot (x', z') = (x + x', z + z' + \frac{1}{2} x \times x') \tag{9}
$$

for $x = (x_1, x_2, x_3)$ and $z = (z_1, z_2, z_3)$, where $\times$ stands for the vector product on $\mathbb{R}^3$. In particular, $\mathcal{M} = \langle X_1, X_2, X_3 \rangle$ forms a 3–dimensional left–invariant distribution on M. We define the left–invariant sub–Riemannian metric g_M on $\mathcal{M}$ by declaring X_1, X_2, X_3 orthonormal.

The geodesics of the control problem are solutions to the control system (4),(5), with $(m, n) = (3, 6)$ and the structure constants can be read off Table 1. Hence, the fibre system is given by $w_1 = K_1, w_2 = K_2, w_3 = K_3$ and $\dot{h} = -\Omega h$, where K_1, K_2, K_3 are constants, $h := (h_1, h_2, h_3)^T$ and

$$
\Omega = \begin{pmatrix} 0 & K_1 & K_2 \\ -K_1 & 0 & K_3 \\ -K_2 & -K_3 & 0 \end{pmatrix}. \tag{10}
$$

Its solution is given by the exponential $h(t) = e^{-t\Omega} h(0)$, where $h(0)$ is the initial value of vector h at the origin. We write an explicit formula for the general solution in terms of eigenvectors of (10). If at least one of constants K_i is non–zero, the kernel of Ω_w, i.e. zero eigenspace, is one–dimensional, generated by vector $(K_3, K_2, K_1)^T$. Its orthogonal complement corresponds to the sum of eigenspaces appropriate to the eigenvalues $\pm iK$, where we denote $K := \sqrt{K_1^2 + K_2^2 + K_3^2}$, and is generated by vectors $(-K_1K_3, -K_1K_2, K_2^2 + K_3^2) \pm i(K_2, -K_3, 0)$. Thus solution to the fibre system can be written as

$$
h(t) = (C_1 \cos(Kt) - C_2 \sin(Kt))v_1 + (C_1 \sin(Kt) + C_2 \cos(Kt))v_2 + C_3 v_3, \tag{11}
$$

where v_1, v_2, v_3 is an eigenspace–adapted real orthonormal basis given by

$$v_1 = \frac{1}{K\sqrt{K_2^2+K_3^2}} \begin{pmatrix} -K_1K_3 \\ -K_1K_2 \\ K_2^2+K_3^2 \end{pmatrix}, v_2 = \frac{1}{\sqrt{K_2^2+K_3^2}} \begin{pmatrix} K_2 \\ -K_3 \\ 0 \end{pmatrix}, v_3 = \frac{1}{K} \begin{pmatrix} K_3 \\ K_2 \\ K_1 \end{pmatrix}$$

and C_1, C_2, C_3 are constants that satisfy the level set condition $H = 1/2$, i.e. $\|h(t)\| = 1$, that reads $C_1^2 + C_2^2 + C_3^2 = 1$. Let us note that the choice $C_1 = C_2 = 0$ leads to constant solutions that are irrelevant as control functions.

Let us emphasize that the base system (4) can be written in terms of vector product as follows

$$\begin{aligned} \dot{x} &= h, \\ \dot{z} &= \frac{1}{2} x \times h \end{aligned} \tag{12}$$

for vectors $x = (x_1, x_2, x_3)^T$ and $z = (z_1, z_2, z_3)^T$. Its general solution is obtained by substituting (11) for h and by consequent direct integration. We are interested in solutions passing through the origin, i.e. we impose the initial condition

$$x_i(0) = 0, z_i(0) = 0, \quad i = 1, 2, 3. \tag{13}$$

However, finding a geodesic towards a given point, leads to a problem of solving non–trivial algebraic equations. Here we exploit intrinsic symmetries of the system and geometric algebra approach, see Section 4.

2.3. **Left–invariant control problem with growth vector $(4, 7)$.** Let us consider four vector fields given on $\mathbb{R}^7$ with local coordinates $(x, \ell_1, \ell_2, \ell_3, y_1, y_2, y_3)$ in the form

$$\begin{aligned} &Y_0 = \partial_x - \frac{\ell_1}{2}\partial_{y_1} - \frac{\ell_2}{2}\partial_{y_2} - \frac{\ell_3}{2}\partial_{y_3}, \\ &Y_1 = \partial_{\ell_1} + \frac{x}{2}\partial_{y_1}, \quad Y_2 = \partial_{\ell_2} + \frac{x}{2}\partial_{y_2}, \quad Y_3 = \partial_{\ell_3} + \frac{x}{2}\partial_{y_3}, \end{aligned} \tag{14}$$

The only non–trivial Lie brackets are

$$Y_4 = [Y_0, Y_1] = \partial_{y_1}, \quad Y_5 = [Y_0, Y_2] = \partial_{y_2}, \quad Y_6 = [Y_0, Y_3] = \partial_{y_3}. \tag{15}$$

These fields then determine a step 2 nilpotent Lie algebra $\mathfrak{n}$ with multiplication given by Table 2.

	Y_0	Y_1	Y_2	Y_3	Y_4	Y_5	Y_6
Y_0	0	Y_4	Y_5	Y_6	0	0	0
Y_1	$-Y_4$	0	0	0	0	0	0
Y_2	$-Y_5$	0	0	0	0	0	0
Y_3	$-Y_6$	0	0	0	0	0	0
Y_4	0	0	0	0	0	0	0
Y_5	0	0	0	0	0	0	0
Y_6	0	0	0	0	0	0	0

Table 2. Lie algebra $\mathfrak{n}$

There is a Carnot group N such that the fields Y_i, $i = 1, \dots, 7$ are left–invariant for the corresponding group structure. The group structure on N, when identified with $\mathbb{R}^7 = \mathbb{R} \oplus \mathbb{R}^3 \oplus \mathbb{R}^3$, yields

$$(x, \ell, y) \cdot (x', \ell', y') = (x + x', \ell + \ell', y + y' + \frac{1}{2}(\ell \times \ell')). \tag{16}$$

In particular, $\mathcal{N} = \langle Y_0, Y_1, Y_2, Y_3 \rangle$ forms a 4–dimensional left–invariant distribution on N. Moreover, there is a natural decomposition

$$\mathcal{N} = \langle Y_0 \rangle \oplus \langle Y_1, Y_2, Y_3 \rangle \tag{17}$$

of $\mathcal{N}$ into 1–dimensional distribution and 3–dimensional involutive distribution, both left–invariant. We define the left–invariant sub–Riemannian metric g_N on $\mathcal{N}$ by declaring Y_0, Y_1, Y_2, Y_3 orthonormal.

We apply the same method as in previous section to discuss the control problem. The geodesics of the control problem are solutions to the control system (4),(5), with $(m, n) = (4, 7)$ and the structure constants can be read off Table 2.3. Hence, the first part of the fibre system (5) is given by $w_1 = K_1, w_2 = K_2, w_3 = K_3$, where K_1, K_2, K_3 are constants. Second part of the fibre system is in this case of the explicit matrix form $\dot{h} = -\Omega h$, where $h := (h_0, h_1, h_2, h_3)^T$ and

$$\Omega = \begin{pmatrix} 0 & K_1 & K_2 & K_3 \\ -K_1 & 0 & 0 & 0 \\ -K_2 & 0 & 0 & 0 \\ -K_3 & 0 & 0 & 0 \end{pmatrix}. \tag{18}$$

Its solution is given by $h(t) = e^{-t\Omega}h(0)$, where $h(0)$ is the initial value of vector h at the origin and we write its explicit form in terms of the eigenvectors of (18). If $K_1 = K_2 = K_3 = 0$ then $h(t) = h(0)$ is constant and the geodesic $(x(t), \ell_i(t), y_i(t))$ is a line in N such that $y_i = 0$. If at least one of the constants K_i is non–zero, the kernel of Ω_w that corresponds to the eigenspace for the eigenvalue 0 is two–dimensional and is generated by vectors $(0, -K_3, 0, K_1)^T$ and $(0, -K_2, K_1, 0)^T$. Its orthogonal complement corresponds to the sum of eigenspaces appropriate to the eigenvalues $\pm iK$, where we denote $K := \sqrt{K_1^2 + K_2^2 + K_3^2}$, and is generated by the eigenvectors $(0, K_1, K_2, K_3)^T \pm i(1, 0, 0, 0)^T$. Thus the solution to the vertical system for non–zero K is in this case of the form

$$\begin{aligned} h_0 &= C_1 \cos(Kt) - C_2 \sin(Kt) \\ \bar{h} &= (C_1 \sin(Kt) + C_2 \cos(Kt))r_1 + Cr_2 \end{aligned} \tag{19}$$

where $\bar{h} = (h_1, h_2, h_3)^T$ and r_1, r_2 are orthogonal unit vectors given by

$$r_1 = \frac{1}{K}\begin{pmatrix} K_1 \\ K_2 \\ K_3 \end{pmatrix}, \quad r_2 = \frac{1}{C}\left(C_3 \begin{pmatrix} -K_3 \\ 0 \\ K_1 \end{pmatrix} + C_4 \begin{pmatrix} -K_2 \\ K_1 \\ 0 \end{pmatrix}\right)$$

with C_1, C_2, C_3, C_4 being real constants and with the normalization factor $C = \sqrt{(C_3K_3 + C_4K_2)^2 + K_1^2(C_3^2 + C_4^2)}$. The level set condition $\|h(t)\| = 1$ reads $C_1^2 + C_2^2 + C_3^2 = 1$. Let us note that the choice $C_1 = C_2 = 0$ leads to constant solutions that are irrelevant as control functions. Thus we assume that at least one of the constants C_1, C_2 is non–zero. Let us emphasize that vectors r_1, r_2 are orthonormal. The base system (4) takes the explicit form of

$$\begin{aligned} \dot{x} &= h_0, \\ \dot{\ell} &= \bar{h}, \\ \dot{y} &= \frac{1}{2}(x\bar{h} - h_0\ell). \end{aligned} \tag{20}$$

As discussed above, we are interested in solutions passing through the origin, i.e. we impose the initial condition

$$x(0) = 0, \ell_i(0) = 0, y_i(0) = 0, \quad i = 1, 2, 3. \tag{21}$$

By substitution of (19), the system (20) can be directly integrated. In section 4 we show how to use symmetries of the system and geometric algebra approach to find a geodesic towards a given point.

2.4. **Symmetries of the control systems.** Symmetries of the control systems (2) coincide with symmetries of the corresponding left–invariant sub–Riemannian structure $(M, \mathcal{M}, g_M)$ and $(N, \mathcal{N}, g_N)$, respectively. These are precisely automorphisms on groups preserving distributions and sub–Riemannian metrics. The group $SO(3)$ acts on $\mathbb{R}^3$ and preserves vector product which implies the following statement.

Proposition 1. *For each $R \in SO(3)$, the map*

$$(x, z) \mapsto (Rx, Rz) \tag{22}$$

maps geodesics of the system from Section 2.2 starting at the origin to geodesics starting at the origin. For each $R \in SO(3)$ the map

$$(x, \ell, y) \mapsto (x, R\ell, Ry) \tag{23}$$

maps geodesics of the system from Section 2.3 starting at the origin to geodesics starting at the origin.

Proof. Follows from invariance of (12) and (20) with respect to the action of $R \in SO(3)$. □

3. Geometric algebras $\mathbb{G}_m$

Construction of the universal real geometric algebra is well-known, for details see e.g. books [16, 17, 19] or paper [8]. We provide only a brief description in a special case which we are going to use later. In general, geometric algebras are based on symmetric bilinear form of arbitrary signature. Here, we deal with real vector space $\mathbb{R}^m$ endowed with a positive definite symmetric bilinear form B only.

3.1. **Geometric product.** For an associated orthonormal basis $(e_1, \dots, e_m)$ of $\mathbb{R}^n$ we use

$$B(e_i, e_j) = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases} \text{ where } 1 \leq i, j \leq m.$$

Let us recall that the Grassmann algebra is an associative algebra with the anti-symmetric outer product $\wedge$ defined by the rule

$$e_i \wedge e_j + e_j \wedge e_i = 0 \ \text{ for } 1 \leq i, j \leq m.$$

The Grassmann blade of grade r is $e_A = e_{i_1} \wedge \cdots \wedge e_{i_r}$, where the multi-index A is a set of indices ordered in the natural way $1 \leq i_1 \leq \cdots \leq i_r \leq m$, and we put $e_\emptyset = 1$. Blades of grades $0 \leq r \leq m$ form the basis of the graded Grassmann algebra $\Lambda(\mathbb{R}^m)$. Next, we introduce the inner product

$$e_i \cdot e_j = B(e_i, e_j), \ \ 1 \leq i, j \leq m,$$

leading to the so-called geometric product in the Clifford algebra

$$e_i e_j = e_i \cdot e_j + e_i \wedge e_j, \;\; 1 \leq i, j \leq m.$$

The respective definitions of inner, outer and geometric products are then extended to blades of the grade r as follows. For inner product we put

$$e_j \cdot e_A = e_j \cdot (e_{i_1} \wedge \cdots \wedge e_{i_r}) = \sum_{k=1}^{r} (-1)^k B(e_j, e_{i_k}) e_{A \setminus \{i_k\}},$$

where $e_{A \setminus \{i_k\}}$ is the blade of grade $r-1$ created by deleting e_{i_k} from e_A. This product is also called the left contraction in literature. For outer product we have

$$e_j \wedge e_A = \begin{cases} e_j \wedge e_{i_1} \wedge \cdots \wedge e_{i_r} & \text{if } j \notin A \\ 0 & \text{if } j \in A \end{cases}$$

and for geometric product we define

$$e_j e_A = e_j \cdot e_A + e_j \wedge e_A.$$

Finally, these definitions are linearly extended to the whole vector space $\Lambda(\mathbb{R}^m)$. Thus we get an associative algebra over this vector space, the so-called real Clifford algebra, denoted by $\mathbb{G}_m$. Note that this algebra is naturally graded; the grade zero and grade one elements are identified with $\mathbb{R}$ and $\mathbb{R}^m$, respectively.

3.2. **Geometric algebra $\mathbb{G}_m$.** Euclidean geometric algebra $\mathbb{G}_m$ is generated by m Euclidean basis vectors $e_1, e_2, \ldots, e_m$ by

$$e_i^2 = B(e_i, e_i) = 1 \text{ for all } i = 1, \ldots, m.$$

The vectors in $\mathbb{R}^m$ with coordinates $(x_1, \ldots, x_n)$ are given by $x = x_1 e_1 + \cdots + x_n e_n$ and the square with respect to geometric product $x^2 = x_1^2 + \cdots + x_n^2 \in \mathbb{R}$ coincides with the square of the Euclidean norm of x. Vector x represents a one-dimensional subspace (line) p in $\mathbb{R}^m$ defined by scalar multiples of x which in $\mathbb{G}_m$ is expressed by formula $u \in p \Leftrightarrow u \wedge x = 0$. In the same way, a plane π generated by two vectors x and y is represented by $x \wedge y$ in the sense $u \in \pi \Leftrightarrow u \wedge x \wedge y = 0$. In general, any r-dimensional subspace $V_r \subseteq \mathbb{R}^m$ is represented by a blade A_r of grade r such that

$$V_r = NO(A_r) = \{x \in \mathbb{R}^m : x \wedge A_r = 0\} \tag{24}$$

holds. Such a representation is called the outer product null space (OPNS) representation in literature. In particular, the whole space $\mathbb{R}^m$ is represented by a blade of maximal grade, so called pseudoscalar. Similarly one defines the inner product null space (IPNS) representation A^*_{m-r} of V_r as a blade of grade $m-r$ such that $x \cdot A^*_{m-r} = 0$ if and only if $x \in V_r$. The OPNS and IPNS representations are mutually dual with respect to duality on $\mathbb{G}_m$ defined by multiplication by pseudoscalar, namely

$$A^* = AI,$$

where A is a blade and I is a pseudoscalar. Indeed, one can show that $(x \wedge A)I = x \cdot (AI)$ for each vector $x \in \mathbb{R}^m$, in particular $x \wedge A = 0$ if and only if $x \cdot A^* = 0$.

Remark 3.3. *OPNS representations of $\mathbb{G}_3$ are summarized in Table 3. For example in $\mathbb{G}_3$, a plane generated by vectors u, v has OPNS representation $u \wedge v$. Its IPNS representation $(u \wedge v)^*$ is vector perpendicular to the plane. More specifically* *for pseudoscalar* $I = e_1 \wedge e_2 \wedge e_3 = e_1e_2e_3$ *we receive the usual vector product in geometric algebra form as*

$$u \times v = -(u \wedge v)I. \tag{25}$$

grade	name	blades	dimension	objects
0	scalars	1	1	numbers
1	vectors	$e_1,\ e_2,\ e_3$	3	lines
2	bivectors	$e_1 \wedge e_2,\ e_1 \wedge e_3,\ e_2 \wedge e_3$	3	planes
3	pseudoscalars	$e_1 \wedge e_2 \wedge e_3$	1	volume forms

Table 3. Blades of geometric algebra $\mathbb{G}_3$

3.4. **Transformations in $\mathbb{G}_m$.** Let us have two vectors $x, n \in \wedge^1\mathbb{R}^m \subset \mathbb{G}_m$, such that $n \cdot n = n^2 = 1$. The action of n on x by conjugation nxn is a reflection with respect to the plane orthogonal to n, which can be proved by straightforward computation

$$nx^{\perp}n = (n \wedge x^{\perp})n = -(x^{\perp} \wedge n)n = -x^{\perp}nn = -x^{\perp},$$

where $x = x^{||} + x^{\perp}$ is the orthogonal decomposition with respect to n and with the fact that $nx^{||}n = x^{||}$. Conjugation preserves grades of blades and is an outermorphism $n(u_1 \wedge \cdots \wedge u_l)n = (nu_1n) \wedge \cdots \wedge (nu_ln)$ for any vectors $u_1, \ldots, u_n$. Composition of two reflections is a rotation, so in $\mathbb{G}_m$ a rotation is represented by conjugation with respect to geometric multiplication of two vectors. To find a rotor between vectors x and y we have a nice formula at hand.

Lemma 1. *Let x and y be unit vectors in $\mathbb{G}_m$, i.e. $x, y \in \wedge^1\mathbb{G}_m$, then the formula*

$$R_{xy} = \overline{1 + yx}, \tag{26}$$

where the bar symbol stands for normalization $\bar{u} = u/\sqrt{u \cdot u}$, defines rotation in the plane $x \wedge y$ which maps vector x to y and acts trivially on $(x \wedge y)^*$.

Proof. Multiplication of two vectors $x, y \in \wedge^1\mathbb{R}^m \subset \mathbb{G}_m$, such that $x^2 = y^2 = 1$, defines a multivector

$$xy = \cos(\theta) - \sin(\theta)(x \wedge y),$$

where θ is the angle between x and y. It is easy to see that $(xy)(xy)^* = xyyx = 1$. The classical property of this geometric algebra is that the conjugation by such multivector xy represents rotation in the plane $x \wedge y$ with respect to angle 2θ, see Lemma 4.2 in [17] for more details.

Using standard trigonometric formulas we can verify this by straightforward calculation

$$\begin{aligned} R_{xy} &= \overline{1 + yx} = \frac{1 + \cos(\theta) + (x \wedge y)\sin(\theta)}{\sqrt{(1 + \cos(\theta))^2 + \sin^2(\theta)}} = \frac{1 + \cos(\theta) + (x \wedge y)\sin(\theta)}{\sqrt{2 + 2\cos(\theta)}} \\ &= \sqrt{\frac{1 + \cos(\theta)}{2}} + (x \wedge y)\sqrt{\frac{1 - \cos^2(\theta)}{2(1 + \cos(\theta))}} = \sqrt{\frac{1 + \cos(\theta)}{2}} + (x \wedge y)\sqrt{\frac{1 - \cos(\theta)}{2}} \\ &= \cos(\frac{\theta}{2}) + (x \wedge y)\sin(\frac{\theta}{2}). \end{aligned}$$

Finally $(x \wedge y)^* \cdot x = x \wedge y \wedge x = 0$ and $(x \wedge y)^* \cdot y = 0$ hold, so we compute

$$xy(x \wedge y)^* yx = xyyx(x \wedge y)^* = (x \wedge y)^*$$

which proves the statement. □

3.5. **Rotor construction.** Let $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ be a pair of bases such that the scalar products $x_i \cdot x_j = y_i \cdot y_j$ are the same for all $i, j = 1, \dots, n$ and these bases have the same orientation, i.e. $x_1 \wedge \cdots \wedge x_n = y_1 \wedge \cdots \wedge y_n$. In the sequel, we show how to find the explicit rotation R such that $Rx_iR^* = y_i$ for all $i = 1, \dots n$.

First, we define a complete flag $\{V\}$ as an increasing sequence of subspaces of vector space $\mathbb{R}^n$:

$$\{0\} \subset V_1 \subset V_2 \subset \cdots \subset V_k = \mathbb{R}^n,$$

such that $\dim(V_i) = i$. We consider a complete flags $\{V\}$ and $\{W\}$ by setting $V_i = \langle x_1, \dots, x_i \rangle = NO(x_1 \wedge \cdots \wedge x_i)$ and $W_i = \langle y_1, \dots, y_i \rangle = NO(y_1 \wedge \cdots \wedge y_i)$, respectively.

Consequently, we map the complete flag $\{V\}$ to the complete flag $\{W\}$ in n steps. In particular step j, we suppose $V_i = W_i$ for $i > j$ and we find rotations R_i such that $R_i V_i R_i^* = W_i$ for $i > j - 1$. The algorithm ends after n steps which is formulated in the following lemma.

Lemma 2. *Let $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ be a pair of bases such that $x_i \cdot x_j = y_i \cdot y_j$ for all $i, j = 1, \dots, n$ and $x_1 \wedge \cdots \wedge x_n = y_1 \wedge \cdots \wedge y_n$. If $\{V\}$ and $\{W\}$ are the corresponding complete flags, respectively, and if $V_i = W_i$ for all $i = 1, \dots, n$, then $x_i = y_i$ for all $i = 1, \dots, n$.*

Proof. If $V_1 = W_1$ and $v_1 \cdot v_1 = w_1 \cdot w_1$ then $v_1 = w_1$. If $v_2 \in W_2$ and $w_1 = v_1$ and $v_2 \cdot w_1 = w_2 \cdot w_1$ and the orientation is the same (rotations preserve orientation) then $w_2 = v_2$ etc. □

Each particular step is correctly defined because of the following lemma.

Lemma 3. *Let us have two complete flags $\{V\}$ and $\{W\}$, such that $V_j = W_j$, for $j > i$. The rotor R_i between the hyperplanes $V_i \oplus V_{i+1}^{\perp}$ and $W_i \oplus W_{i+1}^{\perp}$ constructed by equation (26) maps V_i to W_i.*

Proof. By definition $V_i \subset V_{i+1}$, so $V_{i+1}^{\perp} \subset V_i^{\perp}$ and $V_i \oplus V_{i+1}^{\perp}$ is an orthogonal decomposition of the hyperplane. Any rotation preserves orthogonal decomposition, so it maps V_i to W_i because it acts as an identity on $V_{i+1} = W_{i+1}$ and thus $V_{i+1}^{\perp} = W_{i+1}^{\perp}$. □

All these properties can be used to formulate the following theorem with the constructive proof.

Theorem 3.6. *Let $(x_1, \dots, x_n)$ and $(y_1, \dots, y_n)$ be a pair of bases such that $x_i \cdot x_j = y_i \cdot y_j$ for all $i, j = 1, \dots, n$ and $x_1 \wedge \cdots \wedge x_n = y_1 \wedge \cdots \wedge y_n$. Then we can construct a rotor R such that $Rx_iR^* = y_i$ for all $j, i = 1, \dots n$.*

Proof. Let $\{V\}$ and $\{W\}$ be a pair of corresponding complete flags, respectively, so $V_i = NO(x_1 \wedge \cdots \wedge x_i)$ and $W_i = NO(y_1 \wedge \cdots \wedge y_i)$. We construct a rotor $R = R_1 \cdots R_n$ which maps the complete flag $\{V\}$ to complete flag $\{W\}$ so that $V_i = W_i$ for all $i = 1, \dots n$ and the result follows by Lemma 2.

We define R_n as identity. As the next step, we find the rotation R_{n-1} between hyperplanes $V_{n-1} \wedge V_n^{\perp} \cong V_{n-1}$ and $W_{n-1} \wedge W_n^{\perp} \cong W_{n-1}$ which maps complete flag $\{V\}$ to complete flag $\{RVR^*\}$ in such a way that $W_{n-1} = RV_{n-1}R$, where $R = R_{n-1}R_n$.

As an induction step, we consider the rotor $R = R_j \cdots R_n$ such that $RV_iR^* = W_i$ for all indices $i \geq j$. According to Lemma 3 the rotation R_{j-1} between hyperplanes $(RV_{j-1}R^*) \wedge (RV_jR^*)^\perp$ and $W_{j-1} \wedge W_j^\perp$ maps complete flag $\{RVR^*\}$ to complete flag $\{R_jRVR^*R_j^*\}$ in such a way that $W_i = R_jRV_iR^*R_j^*$ for all $i \geq j-1$.

After n steps the rotor $R = R_1 \cdots R_n$ maps the complete flag $\{V\}$ to the complete flag $\{W\}$ in such a way that $V_i = W_i$ for all $i = 1, \dots, n$ and so $Rx_iR^* = y_i$ for all $j, i = 1, \dots, n$ because of Lemma 2. □

The explicit construction in the proof of the theorem gives us the following algorithm.

Algorithm 1. *Calculate rotor $R = R_1 \dots R_n$*

Require: $x_i \cdot x_j = y_i \cdot y_j$ and $x_1 \wedge \cdots \wedge x_n = y_1 \wedge \cdots \wedge y_n$
Ensure: $y_i = Rx_iR^*$
$V_i \leftarrow x_1 \wedge \cdots \wedge x_i$
$W_i \leftarrow y_1 \wedge \cdots \wedge y_i$
$R \leftarrow Id$
for $n > i > 0$ **do**
  $V_i \leftarrow RV_iR^*$
  $H_V \leftarrow V_i \wedge W_{i+1}^*$
  $H_W \leftarrow W_i \wedge W_{i+1}^*$
  $R_i \leftarrow \overline{1 + \bar{H}_V^* \bar{H}_W^*}$
  $R \leftarrow R_iR$
end for

end

4. Nilpotent control problems in GA approach

In this section, we are using symmetries from $SO(3)$ to define an equivalence relation on the set of geodesics passing through the origin, see Proposition 1. We find a convenient representative of any equivalence class and describe the moduli space in the language of GA.

4.1. Geodesics of $(3, 6)$.

Since the vector product $x \times h$ coincides with the dual of wedge product $x \wedge h$ according to (25), the horizontal system (12) can be written in the form

$$\begin{aligned} \dot{x} &= h, \\ \dot{z} &= \frac{1}{2} x \wedge h \end{aligned} \tag{27}$$

where $x \in \wedge^1\mathbb{R}^3$ represents a line and $z \in \wedge^2\mathbb{R}^3$ represents a plane in $\mathbb{R}^3$. In this way we see geodesics as curves in geometric algebra $\mathbb{G}_3$.

Proposition 2. *Each sub–Riemannian geodesic satisfying the initial condition $x_i(0) = 0, z_i(0) = 0$, $i = 1, 2, 3$ is equivalent to a curve in $M \cong \wedge^1\mathbb{R}^3 \oplus \wedge^2\mathbb{R}^3 \subset \mathbb{G}_3$*

and, up to the action of suitable $R \in SO(3)$, it is of the form

$$
\begin{aligned}
q(t) = x(t) + z(t) = {} & D(1-\cos(Kt))e_1 + D\sin(Kt)e_2 + C_3Kte_3 \\
& + D^2(Kt - \sin(Kt))e_1 \wedge e_2 - C_3D(Kt - 2\sin(Kt) + Kt\cos(Kt))e_1 \wedge e_3 \\
& - C_3D(2 - Kt\sin(Kt) - 2\cos(Kt))e_2 \wedge e_3,
\end{aligned} \tag{28}
$$

where $K > 0$ and D, C_3 satisfy the level set equation $D^2 + C_3^2 = 1$.

Proof. The solution to the vertical system (11) can be rewritten as

$$h(t) = D\sin(Kt)\bar{v}_1 + D\cos(Kt)\bar{v}_2 + C_3v_3, \tag{29}$$

where we denote $D = \sqrt{C_1^2 + C_2^2}$, and $\bar{v}_1, \bar{v}_2$ form an orthonormal basis of the kernel complement obtained by rotation of orthonormal vectors v_1, v_2, namely

$$\bar{v}_1 = \frac{1}{\sqrt{C_1^2 + C_2^2}}(-C_1v_1 + C_2v_2),\ \bar{v}_2 = \frac{1}{\sqrt{C_1^2 + C_2^2}}(C_2v_1 + C_1v_2).$$

The vectors $\bar{v}_1, \bar{v}_2, v_3$ are orthonormal with respect to the Euclidean metric on $\mathbb{R}^3$ by definition. So, there is an orthogonal matrix $R \in SO(3)$ that aligns vectors $\bar{v}_1, \bar{v}_2, v_3$ with the standard basis of $\mathbb{R}^3$. Thus we get

$$\bar{v}_1 = Re_1, \quad \bar{v}_2 = Re_2, \quad v_3 = Re_3,$$

where e_1, e_2 and e_3 are elements of standard Euclidean basis of $\mathbb{R}^3$. According to (22), rotor R defines a representative of geodesic class $(R^Tx(t), R^Tz(t))$ which is a solution to (27) for $h(t) = D\sin(Kt)e_1 + D\cos(Kt)e_2 + C_3e_3$. The solution (28) then follows by direct integration when the initial condition is applied. Equation for the level set follows from the definition of D. □

The action of $SO(3)$ on $M \cong \mathbb{R}^6$ given by equation (22) defines a moduli space $M/SO(3)$. We see M as a subset of $\mathbb{G}_3$ and the group $SO(3)$ is represented by rotors instead of matrices, which act on M by conjugation. The action preserves vector and bivector parts, scalar product, norm and dualisation. We can see the elements of M as pairs consisting of lines and planes. The natural invariants are the norms of lines' directional vectors, norms of the planes' normal vectors and angles between these pairs of vectors. Square norm of the normal vector of the plane $z^* \cdot z^*$ is $-z \cdot z$. Scalar product between the directional vector of the line x and normal vector of the plane z can be rewritten as $(x \wedge z)^*$ because $(x \cdot z^*)^* = x \wedge z$ and $x \cdot z^* = (x \wedge z)^*$. Altogether, we consider three invariants

- square of the the norm of the vector x, i.e. $x \cdot x$,
- square of the the norm of the bivector z, i.e. $z \cdot z$,
- element $(x \wedge z)^*$,

where $\cdot$ coincides with the inner product on $\mathbb{G}_3$. In particular, these invariant elements form a coordinate system on the moduli space $M/SO(3)$.

Proposition 3. *Each geodesic starting at the origin defines a curve in the moduli space $M/SO(3)$, which is determined by invariants in the following way*

$$
\begin{aligned}
x \cdot x = {} & -2D^2(\cos(Kt) - 1) + C_3^2K^2t^2 \\
z \cdot z = {} & -D^2(-D^2(\cos(Kt))^2 + (2K^2t^2 - 8)C_3^2\cos(Kt) \\
& - 2Kt(4C_3^2 + D^2)\sin(Kt) + (2K^2t^2 + 8)C_3^2 + D^2(K^2t^2 + 1)) \\
(x \wedge z)^* = {} & D^2C_3(K^2t^2 + K\sin(Kt)t - 4 + 4\cos(Kt))
\end{aligned} \tag{30}
$$

Proof. Follows directly from (28). □

4.2. **Geodesics of** $(4,7)$**.** The base system (20) can be seen as a system in geometric algebra $\mathbb{G}_4$

$$\begin{aligned} \dot{x} + \dot{\ell} &= h_0 + \bar{h}, \\ \dot{y} &= x \wedge \bar{h} + \ell \wedge h_0, \end{aligned} \tag{31}$$

where we assume that x and h_0 are collinear with e_1 and $\ell, \bar{h}$ lie in the subspace generated by e_2, e_3, e_4. The form of the second equation implies that y is given by the wedge product of e_1 and a vector from this subspace. Hence the solution $y(t)$ can be viewed as a curve of planes in $\mathbb{G}_4$.

Proposition 4. *Each sub–Riemannian geodesic on Carnot group N satisfying the initial condition* (13) *is equivalent to a curve in $N \cong \wedge^1\mathbb{R}^4 \oplus \wedge^2\mathbb{R}^4 \subset \mathbb{G}_4$ and, up to the action of suitable $R \in SO(3)$, it is of the form*

$$\begin{aligned} q(t) = x(t) + \ell(t) + y(t) &= (C_1\cos(Kt) + C_2\sin(Kt) - C_1)e_1 \\ &+ (C_1\sin(Kt) - C_2\cos(Kt) + C_2)e_2 + Ce_3 \\ &- (C_1^2 + C_2^2)(tK - \sin(Kt))e_1 \wedge e_2 - \frac{C}{K}((2C_1 - C_2Kt)\sin(Kt) \\ &- (C_1Kt + 2C_2)\cos(Kt) + 2C_2 - tC_1K)e_1 \wedge e_3, \end{aligned} \tag{32}$$

where $K > 0$ and constants C_1, C_2, C satisfying the level condition $C_1^2 + C_2^2 + C^2 = 1$.

Proof. According to the vertical system (19), the vector $\bar{h}(t)$ lies in the subspace generated by vectors r_1, r_2 for any t. Since the vectors r_1 and r_2 are orthonormal, there is an orthogonal matrix $R \in SO(3)$ that aligns these vectors with the second and third vector of the standard basis of $\mathbb{R}^3$, i.e

$$r_1 = Re_2, \quad r_2 = Re_3.$$

Due to the symmetry of this system, see (23), this rotor defines a representative of the geodesic class $(x(t), R^T\ell(t), R^Ty(t))$ which is the solution to the horizontal system (20) for

$$\bar{h}(t) = (C_1\sin(Kt) + C_2\cos(Kt))e_1 + Ce_2$$

or, equivalently, a curve in $\mathbb{R}^4 \oplus \Lambda^2\mathbb{R}^4 \in \mathbb{G}_4$ given by the solution of (31). By direct integration of this equation and by imposing the initial conditions, we get the formula (32) for the solution. □

The action of $SO(3)$ on $N \cong \mathbb{R}^7$ given by (23), defines a moduli space $N/SO(3)$. We see N as a subset of $\mathbb{G}_4$ and the group $SO(3)$ is represented by rotors instead of matrices, which act on N by conjugation. The action preserves vector and bivector part, the split $x+\ell$, scalar product, norm and dualisation. The orbits of this actions are determined by natural invariants. For the same reason as in the case of $(3,6)$ and due to the invariant split, we have three invariants as follows

- value of the coordinate x,
- square of the norm of the vector ℓ, i.e. $\ell \cdot \ell$,
- square of the norm of the bivector y, i.e. $y \cdot y$.

We need one more invariant for dimensional reasons but the element $(\ell \wedge y)^*$ is not scalar but vector. On the other hand, $\ell \wedge y$ is a multiple of blade $e_1e_2e_3$, so the value of $(\ell \wedge y)e_1e_2e_3$ is scalar. As the last invariant we consider

- value of $(\ell \wedge y)e_1e_2e_3$.

These form the coordinate system on the moduli space $N/SO(3)$.

Proposition 5. *Each geodesic starting at the origin defines a curve in the moduli space $N/SO(3)$, which is determined by invariants in the following way*

$$
\begin{aligned}
x &= C_1(\cos(Kt)-1) + C_2\sin(Kt),\\
\ell\cdot\ell &= C^2t^2 - {C_1}^2(\cos(Kt))^2 - 2C_1C_2\sin(Kt)\cos(Kt) + {C_2}^2(\cos(Kt))^2\\
&\quad + 2C_2C_1\sin(Kt) - 2{C_2}^2\cos(Kt) + {C_1}^2 + {C_2}^2\\
y\cdot y &= \frac{1}{K^2}((-C^2t^2{C_2}^2 - ({C_1}^2+{C_2}^2)^2)K^2 + 4C^2C_1C_2Kt - 4C^2{C_1}^2)(\sin(Kt))^2\\
&\quad + (-2C^2(C_1Kt+2C_2)(KtC_2-2C_1)\cos(Kt)\\
&\quad - 2t(-({C_1}^2+{C_2}^2)^2K^3 + C^2K^2tC_1C_2 - 2C^2(t{C_2}^2+{C_1}^2)K + 4C^2C_1C_2))\sin(Kt)\\
&\quad - C^2(C_1Kt+2C_2)^2(\cos(Kt))^2 - 2C^2t(KC_1-2C_2)(C_1Kt+2C_2)\cos(Kt)\\
&\quad - t^2(({C_1}^2+{C_2}^2)^2K^4 + C^2K^2{C_1}^2 - 4C^2KC_1C_2 + 4C^2{C_2}^2),\\
&(\ell\wedge y)^* = \frac{1}{K}((-2C_2KtC_1 + 2{C_1}^2 - 2{C_2}^2)(\cos(Kt))^2 +\\
&((tK{C_1}^2 - tK{C_2}^2 + 4C_1C_2)\sin(Kt) + 2{C_2}^2(t+1))\cos(Kt) - 2C_1C_2(t+1)\sin(Kt)\\
&+ (t^2K^2-2t){C_2}^2 + 2C_2KtC_1 + (t^2K^2-2){C_1}^2)Ce4
\end{aligned}
\tag{33}
$$

Proof. Follows directly from (32). ☐

5. Examples

In the sequel, we present two examples of controls based on symmetries in geometric algebra approach. We have the following scheme based on Algorithm 1.

(1) For the target point q_t compute invariants of the chosen particular control system (2).
(2) Solve the system of non–linear equations (30) or (33) in the moduli space.
(3) Find the family of curves (28) or (32) going from the origin to the same point q_f that belongs to the same $SO(3)$ orbit of q_t.
(4) Find $R \in SO(3)$, such that $R(q_f) = q_t$.
(5) Apply R on the set of curves (28) or (32) to get a family of curves going from the origin to the target point q_t.

The explicit calculations were acquired using a CAS system Maple similarly to the paper [6].

5.1. **Example in** $(3,6)$**.** Our goal is to find the geodesic going from the origin to the target point

$$q_t = (x_t, z_t) = -e_1 + e_2 - 2e_3 - e_1\wedge e_2 - 6e_1\wedge e_3 + 4e_1\wedge e_3$$

using invariants (30) in the target point. We have

$$x\cdot x = 6,\ \ z\cdot z = -53,\ \ (x\wedge z)^* = -16$$

and together with the level set condition we get the system with invariants at q_t. We solve the system numerically in Maple and present the solution with rounding up to four decimal digits

$$C_3 = 0.3646, D = 1.1205, K = 0.8487 \tag{34}$$

$$t = 5.0410 \tag{35}$$

Using the constants (34), we get the geodesic in the moduli space from the origin to the point q_f in the form

$$\begin{aligned} q = (x,z) &= -1.120(\cos(0.8487t) - 1)e_1 + 1.120 \sin(0.8487t)e_2 + 0.3094t\, e_3 \\ &- 1.066\,(-t + 1.178 \sin(0.8487t))e_1 \wedge e_2 \\ &+ 0.3467\,(-t + 2.357 \sin(0.8487) - t \cos(0.8487t))e_1 \wedge e_3 \\ &+ 0.3467\,(t \sin(0.8487t) + 2.357 \cos(0.8487t) - 0.8170)e_2 \wedge e_3 \end{aligned}$$

and at the time $t = 5.0410$ we reach the point

$$\begin{aligned} q_f = (x_f, z_f) &= 1.560e_3 - 1.017e_2 + 1.592e_1 \\ &+ 6.5102e_1 \wedge e_2 - 1.7536e_1 \wedge e_3 - 2.7461e_2 \wedge e_3. \end{aligned} \tag{36}$$

We are looking for the rotor which maps the multivector q_t on multivector q_f. We consider complete flags

$$\{0\} \subset NO(x_t) \subset NO(x_t \wedge z_t^*) \subset NO(z_t \wedge z_t^*) \cong \mathbb{R}^3,$$

$$\{0\} \subset NO(x_f) \subset NO(x_f \wedge z_f^*) \subset NO(z_f \wedge z_f^*) \cong \mathbb{R}^3.$$

We set $R_n = R_3 = \mathrm{id}$ and map the plane $x_t \wedge z_t^*$ to the plane $x_f \wedge z_f^*$ by rotor $R_{n-1} = R_2$ according to the formula (26). Explicitly,

$$R_2 := 0.7402 + 0.54863e_1 \wedge e_2 + 0.20912e_1 \wedge e_3 + 0.3276e_2 \wedge e_3$$

and we can map the multivector q_t on multivector $q_s = (x_s, z_s) = R_2 q_t {R_2}^*$ in such a way that x_s and z_s lie in the plane $x_f \wedge z_f^*$. Explicitly,

$$q_s = -0.9736e_1 + 0.6217e_2 - 2.1599e_3 - 4.7859e_1 \wedge e_2 + 2.9524e_1 \wedge e_3 + 4.6235e_2 \wedge e_3.$$

Finally, we map the plane $x_s \wedge (x_s \wedge z_s)^*$ to the plane $x_f \wedge (x_f \wedge z_f)^*$ by rotor

$$R_{n-2} = R_1 = 0.1934 - 0.8268e_1 \wedge e_3 + 0.52803e_2 \wedge e_3.$$

Altogether, we found rotor $R = R_1 R_2 R_3$ and, when applied on (28), we got a geodesic going from origin to the point q_t in the form

$$\begin{aligned} q = (x,z) &= (1.075 \cos(0.8487\,t) + 0.07650\,t - 1.075 - 0.1554 \sin(0.8487\,t))e_1 \\ &+ (0.5879 \sin(0.8487\,t) + 0.2598\,t + 0.1575 - 0.1575 \cos(0.8487\,t))e_2 \\ &+ (-0.1496\,t + 0.2758 \cos(0.8487\,t) + 0.9411 \sin(0.8487\,t) - 0.2758)e_3 \\ &+ (-0.2011 \cos(0.8487\,t) + 0.2912 \cos(0.8487\,t)t - 0.08533 \sin(0.8487\,t)t \\ &+ 0.2011 - 0.2242\,t - 0.07897 \sin(0.8487\,t))e_1 \wedge e_2 \\ &+ (-0.1149 \cos(0.8487\,t) - 1.077\,t - 0.04875 \sin(0.8487\,t)t + 0.1149 \\ &+ 1.483 \sin(0.8487\,t) - 0.1819 \cos(0.8487\,t)t)e_1 \wedge e_3 \\ &- 0.04807\,((1.0\,t + 16.30) \cos(0.8487\,t) + (4.101 + 6.917\,t) \sin(0.8487\,t) \\ &- 4.481\,t - 16.30)e_2 \wedge e_3. \end{aligned}$$

In Figure 2 we present trajectories (x_1, x_2, x_3) and (z_1, z_2, z_3), respectively.

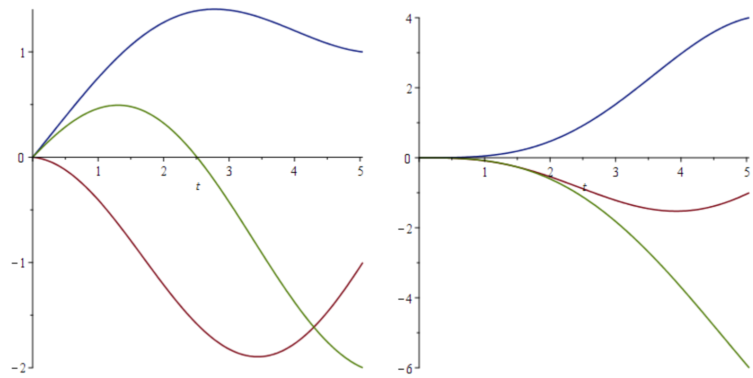

FIGURE 2. Trajectories (x_1, x_2, x_3) and (z_1, z_2, z_3)

5.2. **Example in** $(4, 7)$**.** Our goal is to find the geodesic going from the origin to the target point

$$q_t = (x_t, \ell_t, y_t) = 2e_1 - 3e_2 + e_3 + 3e_4 + 4e_1 \wedge e_2 + 2e_1 \wedge e_3 + 2e_1 \wedge e_4$$

using invariants (33) at the target point. We have

$$x = 2, \;\; \ell \cdot \ell = 19, \;\; y \cdot y = -24, \;\; (\ell \wedge y)^* \cdot (\ell \wedge y)^* = 440$$

and together with the level set condition we get the system with invariants at q_t. We solve the system numerically in Maple and we present the solution with constants rounded up to four decimal digits as follows

$$C = 0.7370, C_1 = -1.063, C_2 = -0.0937, K = 0.6334, \tag{37}$$

$$t = 5.887. \tag{38}$$

Using the constants (37) we get a geodesic in the moduli space from the origin to the point q_f in the form

(39)

$$\begin{aligned} q = (x, \ell, y) &= 2e_1 + (-1.063\,\sin(0.6334t) + 0.09371\cos(0.6334t) - 0.0937)e_2 \\ &+0.737e_3t + (-0.7835\cos(0.6334t)t - 0.2181\cos(0.6334t) - 0.0690\sin(0.6334t)t \\ &+2.4737\sin(0.6334t) - 0.5654t)e_1 \wedge e_3 + (1.139\sin(0.6334t) - 0.7214t)e_1 \wedge e_2 \end{aligned}$$

and at the time $t = 5.887$ we reach the point

$$q_f = (x_f, \ell_f, y_f) = 2e_1 + 4.339e_3 + 0.4173e_2 - 0.4527e_1 \wedge e_3 - 4.878e_1 \wedge e_2. \tag{40}$$

We are looking for the rotor which maps the multivector q_t on multivector q_f. We consider complete flags

$$\{0\} \subset NO(\ell_t) \subset NO(\ell_t \wedge (\ell_t \wedge y_t)^*) \subset NO(\ell_t \wedge y_t^*) \subset NO(y_t \wedge y_t^*) \cong \mathbb{R}^4,$$

$$\{0\} \subset NO(\ell_f) \subset NO(\ell_f \wedge (\ell_f \wedge y_f)^*) \subset NO(\ell_f \wedge y_f^*) \subset NO(y_f \wedge y_f^*) \cong \mathbb{R}^4.$$

First, we map the hyperplane $\ell_t \wedge y_t^*$ to the hyperplane $\ell_f \wedge y_f^*$, but these to coincide with hyperplane orthogonal to e_1, so $R_1 = \mathrm{id}$. The next step is to map hyperplane $(\ell_t \wedge (\ell_t \wedge y_t)^*)) \wedge e_1$ to the hyperplane $(\ell_f \wedge (\ell_f \wedge y_f)^*)) \wedge e_1$ by rotor R_2 according to the formula (26). Explicitly,

$$R_2 := 0.4167 + 0.6291e_2 \wedge e_3 + 0.6532e_2 \wedge e_4 - 0.06282e_3 \wedge e_4.$$

As the last step, we map the hyperplane perpendicular to ℓ_t to hyperplane perpendicular to ℓ_f by the rotor

$$R_3 = 0.2179 - 0.5337 e_2 \wedge e_3 + 0.4483 e_2 \wedge e_4 - 0.6831 e_3 \wedge e_4$$

Altogether, we found the rotor $R = R_3 R_2 R_1$ and, when applied on (39), we have got a geodesic going from origin to the point q_t in the form

$$\begin{aligned} q = (x, \ell, y) &= -0.9(-1.181 + 1.181\cos(0.6334t) + 0.1041\sin(0.6334t))e_1 \\ &+ (0.07147 + 0.8107\sin(0.6334t) - 0.07147\cos(0.6334t) - 0.4555t)e_2 \\ &+ (0.4627\sin(0.6334t) + 0.04079 - 0.04079\cos(0.6334t) + 0.2007t)e_3 \\ &- 0.1000(-5.086\sin(0.6334t) - 5.435t + 0.4484\cos(0.6334t) - 0.4484)e_4 \\ &+ 0.1(0.4268\sin(0.6334t)t + 4.843\cos(0.6334t)t + 1.348\cos(0.6334t) \\ &+ 8.996t - 23.98\sin(0.6334t))e_1 \wedge e_2. \end{aligned}$$

In Figure 3 we present trajectories $(x, \ell_2, \ell_3, \ell_4)$ and (y_1, y_2, y_3), respectively.

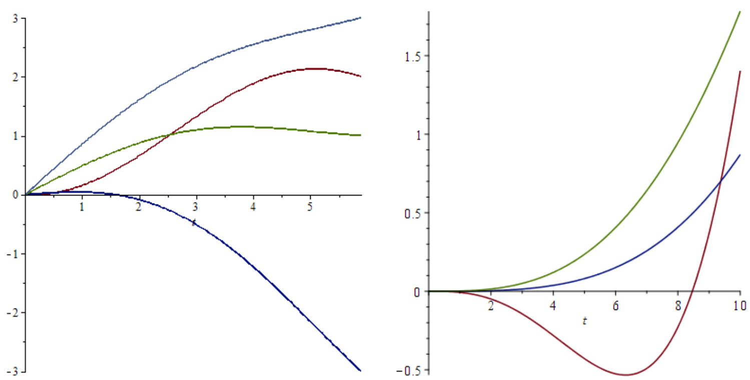

Figure 3. Trajectories $(x, \ell_2, \ell_3, \ell_4)$ and (y_1, y_2, y_3)

6. Conclusion

We demonstrated the use of geometric algebra for controlling systems invariant with respect to orthogonal transformations. The main contribution of GA lies in a construction of a rotor between two bases of a vector space. We assessed an algorithm and demonstrated its use on two particular examples with filtration $(3, 6)$ corresponding to a trident snake robot control, and $(4, 7)$ corresponding to a trident snake with flexible leg. All calculations were acquired using a Maple packages Clifford, [18] and DifferentialGeometry, [3].

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