The Stefan problem in a thermomechanical context with fracture and fluid flow[1]

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Summary
The classical Stefan problem, concerning mere heat-transfer during solid-liquid phase transition, is here enhanced towards mechanical effects. The Eulerian description at large displacements is used with convective and Zaremba-Jaumann corotational time derivatives, linearized by exploiting the additive Green-Naghdi's decomposition in (objective) rates. In particular, the liquid phase is a viscoelastic fluid while creep and rupture of the solid phase is considered in the Jeffreys viscoelastic rheology exploiting the phase-field model, exploiting a concept of slightly (so-called "semi") compressible materials. The L^1-theory for the heat equation is adopted for the Stefan problem relaxed by allowing for kinetic superheating/supercooling effects during the solid-liquid phase transition. A rigorous proof of existence of week solutions is provided for an incomplete melting, exploiting a time-discretisation approximation.

Keywords: Solid-liquid phase transition, creep, Jeffreys rheology, semi-compressible fluids, Eulerian formulation, Stefan problem, melting, solidification, enthalpy formulation, phase-field fracture, fully convective model.

AMS Classification: 35Q74, 35R37, 74A30, 74R20, 76A10, 80A22.

1 Introduction

The formulation of *solid-liquid phase transition* has a long history and had been widely scrutinized under the name of the Stefan problem [43], being a prominent free- or moving-boundary problem in the second half of the 20th century, cf. the survey [48]. The original motivation of Josef Stefan was to model melting of ice in polar caps of the Earth into merely static water (considering the heat-transfer problem in one dimension only). Actually, the *melting/freezing* (*solidification*) phenomena can be found in other occasions in our planet Earth: melting/solidification of Iron with Nickel (in solid inner core vs fluidic outer core) or, without a sharp phase-transition temperature, of rocks vs magma (volcanism in the crust and magma migration in the mantle). Moreover, this phenomena is important in other parts of our Solar system, in particular for icy moons of Jupiter (Europa and Ganymede) and of Saturn (Enceladus and Titan). Beside these geophysical applications, other motivations can certainly be found in engineering, e.g. recrystallization and fusing of steel and of other metals.

In contrast to the original models of the Stefan problem dealing with mere temperature (or enthalpy) evolution, there are some mechanical phenomena with which this thermal problem is ultimately coupled. In particular, the liquid phase is really fluidic and can easily flow. In turn, the solid phase may exhibit *creep*

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and damage or *fracture*, as often seen on the Earth's ice sheet [42] or in the lithospheric rocks (as so-called aseismic slip and tectonic earthquakes, respectively). Fracture can also be clearly seen on the surface of the mentioned Jupiter's and Saturn's moons. Involving creep by a permanent inelastic strain is important for a longer-time scales and for healing (reversible damage). This allows us to model *Maxwellian or Jeffreys visco-elastic rheologies* in the solid part.

Rather surprisingly, so far the Stefan problem was scrutinized in mechanical context only in very simplified situations, like solidification with the incompressible Navier-Stokes and completely rigid solid with only simplified thermomechanical coupling e.g. in [20, 23, 33, 50] or a Darcy flow independent of a possible phase transformation in [32]. Thus realistic models for e.g. floating and melting ice floats in oceans, which was the original motivation of Josef Stefan for his 1-dimensional heat-transfer problem, was not yet successfully formulated and analysed.

Such a thermo-mechanical model will be formulated here in Section 3, after preparatory calculations for casting first a merely visco-elasto-dynamical model in a *convective stress/velocity formulation* in Section 2 to explain how the corotational time derivative for strains in isotropic materials arises. A specialization of general model towards solid-fluid phase transformation and *phase-field fracture* in solid will then be stated in Section 4 and involvement of *superheating/cooling effects* into the model in Section 5. The model from Section 3 in the modification from Section 5 will be then analyzed by converging a time-discretisation combined with a certain regularization in Section 6, but only for an incomplete melting. Eventually, various comments and outlined enhancements of this model will be outlined in Section 7. For geophysical modelling of rock-magma phase transition see [38].

2 A preliminary model: visco-elastodynamics in Eulerian stress-velocity formulation

There are several aspects which should be reflected in the model. First, the so-called *fluid-structure interaction* is itself a difficult problem because fluids dictate Eulerian description while solids are (belived to be) better treated in Lagragian coordinates. We however devise the model in the *fully Eulerian description.* This needs a careful concept of the *objective time derivatives.* In fluidic part, it is also more natural to formulate the model in terms of stresses and velocities rather than in strains and displacements. It then suggests to use the stress/velocity formulation in the solid part, too.

As this "monolithic" Eulerian formulation is not standard, let us first introduce it for the mere visco-elastodynamics without considering any internal variable, which is perhaps of its own importance itself and which justifies the strain/velocity formulation used later. As to the mentioned objective time derivatives, for the transport of scalars, there is no controversy and the *convective* (material) *time derivative* is used when scalars represent an *intensive variable* (like pressure, damage phase-field variable, or temperature). For tensors (or vectors), there are however many options used in literature, like Oldroyd's or Truesdell's or Lie's derivatives. For stress rates (in contrast to general strain rates), the *Zaremba-Jaumann*'s [24, 51] *derivative* (which is the simplest variant of the so-called corotional objective derivative) is the proper one, as also derived by M. Biot [4, p.494] and advocated e.g. in [7, 17, 18, 49] and is widely used both in theoretical and numerical modelling in particular in geophysics, see e.g. [2, 8, 21, 28, 31]. Perhaps, in the context of the solid-liquid phase transformation, the simple convincing argument is that, when the stress tensor is just a pressure (i.e. in the form of the unit matrix multiplied by a scalar), then the Zaremba-Jaumann derivative degenerates to the convective derivative, which is indeed expected in fluids, cf. Remark 2.2 below. The other important attribute of this derivative is that symmetric (resp. trace-free) tensors stay symmetric (resp. trace-free) when transported.

We use the notation $\mathbb{R}^{d\times d}_{\rm sym} = \{A \in \mathbb{R}^{d\times d};\ A^\top = A\}$ and $\mathbb{R}^{d\times d}_{\rm dev} = \{A \in \mathbb{R}^{d\times d}_{\rm sym};\ {\rm tr}\,A = 0\}$ where ${\rm tr}\,A = \sum_{i=1}^d A_{ii}$ denotes the trace of the matrix A. We will use the decomposition of square matrices on symmetric and skew-symmetric parts:

$$\operatorname{sym} A := \frac{1}{2}A^\top + \frac{1}{2}A \quad \text{ and } \quad \operatorname{skew} A := \frac{1}{2}A^\top - \frac{1}{2}A = A - \operatorname{sym} A\,, \tag{2.1}$$

We will also use another decomposition to the spherical (volumetric) and the deviatoric parts, defined by

$$A = \operatorname{sph} A + \operatorname{dev} A \quad \text{ with } \quad \operatorname{sph} A := \frac{1}{d}(\operatorname{tr} A)\mathbb{I} \quad \text{ and } \quad \operatorname{dev} A := A - \frac{1}{d}(\operatorname{tr} A)\mathbb{I}\,, \tag{2.2}$$

where "tr" denotes for a trace of a matrix. Note that it is an orthogonal decomposition and that $\operatorname{dev} A$ is trace-free, i.e. $\operatorname{sph} A : \operatorname{dev} A = 0$ and $\operatorname{tr}(\operatorname{dev} A) = 0$. In this notation, $\operatorname{sym} A \in \mathbb{R}^{d\times d}_{\rm sym}$ and $\operatorname{dev}(\operatorname{sym} A) = \operatorname{sym}(\operatorname{dev} A) \in \mathbb{R}^{d\times d}_{\rm dev}$.

Also we will use the standard notation AB for the product of two matrices while $A : B$ will denote the scalar product of these matrices. We recall the algebra

$$A : (BC) = (B^\top A) : C = (AC^\top) : B \tag{2.3}$$

for any square matrices A, B, and C. Moreover, "$\cdot$" and "$\vdots$" will be used for the scalar products of the vectors and for the 3rd-order tensors, respectively.

We consider a fixed bounded domain $\varOmega \subset \mathbb{R}^d$ with a Lipschitz boundary $\varGamma$ and a times interval $I = [0, T]$. The basic variables and data for this preliminary model are summarized in the following table:

$\boldsymbol{v}$ velocity
$\boldsymbol{e}(\boldsymbol{v}) = \operatorname{sym}(\nabla\boldsymbol{v})$
ϱ mass density (considered constant)
$\boldsymbol{T}$ the Cauchy stress
$\boldsymbol{S}$ the (first) Piola-Kirchhoff stress tensor
$\boldsymbol{D}$ the dissipative part of the stress tensor

φ^* stored energy (in terms of the stress $\boldsymbol{S}$)
$\mathbb{C}$ elasticity tensor (hence $\mathbb{C}^{-1}$ the compliance tensor)
$\mathbb{D}$ Stokes viscosity tensor
$\boldsymbol{f}$ bulk force
$\boldsymbol{g}$ tangential traction force,

We will assume that $\mathbb{C}$ is fully symmetric in the sense $\mathbb{C}_{ijkl} = \mathbb{C}_{klij} = \mathbb{C}_{jikl}$ while $\mathbb{D}$ suffices to satisfy only a minor symmetry $\mathbb{D}_{ijkl} = \mathbb{D}_{jilk}$, both these tensors being positive definite, and $\mathbb{C}$ as well as the mass density $\varrho > 0$ independent of $x \in \varOmega$. These assumptions are important to ensure energetic consistency of the following system considered on $I{\times}\varOmega$:

$$\varrho\dot{\boldsymbol{v}} = \operatorname{div}\boldsymbol{T} - \frac{\varrho}{2}(\operatorname{div}\boldsymbol{v})\,\boldsymbol{v} + \boldsymbol{f}\,, \quad \text{where} \quad \dot{\boldsymbol{v}} = \frac{\partial\boldsymbol{v}}{\partial t} + (\boldsymbol{v}{\cdot}\nabla)\boldsymbol{v} \quad \text{and} \quad \boldsymbol{T} = \boldsymbol{S} + \varphi^*(\boldsymbol{S})\mathbb{I} + \boldsymbol{D}$$

$$\text{with} \quad \varphi^*(\boldsymbol{S}) = \frac{1}{2}\boldsymbol{S}{:}\mathbb{C}^{-1}\boldsymbol{S} \quad \text{and} \quad \boldsymbol{D} = \mathbb{D}\boldsymbol{e}(\boldsymbol{v})\,, \tag{2.4a}$$

$$\overset{\circ}{\boldsymbol{S}} = \mathbb{C}\boldsymbol{e}(\boldsymbol{v}) \quad \text{where} \quad \overset{\circ}{\boldsymbol{S}} = \underbrace{\dot{\boldsymbol{S}} - \frac{\nabla\boldsymbol{v}{-}\nabla\boldsymbol{v}^\top}{2}\boldsymbol{S} - \boldsymbol{S}\frac{\nabla\boldsymbol{v}^\top{-}\nabla\boldsymbol{v}}{2}}_{\text{Zaremba-Jaumann co-rotational derivative}} \quad \text{with} \quad \dot{\boldsymbol{S}} = \underbrace{\frac{\partial\boldsymbol{S}}{\partial t} + (\boldsymbol{v}{\cdot}\nabla)\boldsymbol{S}}_{\text{convective derivative}}\,. \tag{2.4b}$$

We can equally write $\overset{\circ}{\boldsymbol{S}} = \dot{\boldsymbol{S}} - \operatorname{skew}(\nabla\boldsymbol{v})\boldsymbol{S} + \boldsymbol{S}\operatorname{skew}(\nabla\boldsymbol{v})$. The penultimate term $\frac{\varrho}{2}(\operatorname{div}\boldsymbol{v})\,\boldsymbol{v}$ in the first equation (2.4a) is a force introduced by Temam [44] to balance the energy in flows of fluids which are so-called quasi-incompressible, cf. also [45] for a certain justification. The pressure contribution $\varphi^*(\boldsymbol{S})$ to the overall (Cauchy) stress is due to the concept of considering the stored energy per unit actual (not referential) volume and it matches the energy balance, cf. (2.13) below.

Of course, this system of nonlinear parabolic equations should be accompanied by some boundary conditions, e.g.

$$\boldsymbol{v}\cdot\boldsymbol{n} = 0 \quad \text{ and } \quad \big[\boldsymbol{T}\boldsymbol{n}\big]_{\rm T} = \boldsymbol{g} \tag{2.5}$$

with $\boldsymbol{n}$ denoting the unit outward normal to $\varGamma$, $[\cdot]_{\rm T}$ the tangential component of a vector, i.e. $[\boldsymbol{\sigma}]_{\rm T} = \boldsymbol{\sigma} - (\boldsymbol{\sigma}{\cdot}\boldsymbol{n})\boldsymbol{n}$ for a vector $\boldsymbol{\sigma}$. Naturally, $\boldsymbol{g}{\cdot}\boldsymbol{n} = 0$ is to be assumed. Further, we prescribe the initial conditions

$$\boldsymbol{v}|_{t=0} = \boldsymbol{v}_0 \quad \text{ and } \quad \boldsymbol{S}|_{t=0} = \boldsymbol{S}_0\,. \tag{2.6}$$

The energy balance behind this model is revealed by adding the momentum equation (2.4a) tested by $\boldsymbol{v}$ and the stress equation (2.4b) tested by $\mathbb{C}^{-1}\boldsymbol{S}$. We use several calculations exploiting integration over Ω and Green's formula. In particular:

Exploiting the Green formula, the calculus for the convective term

$$\begin{aligned}\int_\Omega \varrho(\widetilde{\boldsymbol{v}}\cdot\nabla)\boldsymbol{v}\cdot\boldsymbol{v}\,\mathrm{d}x &= \int_\Gamma \varrho|\boldsymbol{v}|^2(\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n})\,\mathrm{d}S - \int_\Omega \varrho\boldsymbol{v}\,\mathrm{div}(\widetilde{\boldsymbol{v}}\otimes\boldsymbol{v})\,\mathrm{d}x\\ &= \int_\Gamma \varrho|\boldsymbol{v}|^2(\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n})\,\mathrm{d}S - \int_\Omega \varrho|\boldsymbol{v}|^2\mathrm{div}\,\widetilde{\boldsymbol{v}} + \varrho\widetilde{\boldsymbol{v}}\cdot\nabla\boldsymbol{v}\cdot\boldsymbol{v} + |\boldsymbol{v}|^2(\nabla\varrho\cdot\widetilde{\boldsymbol{v}})\,\mathrm{d}x\\ &= \int_\Gamma \frac{\varrho}{2}|\boldsymbol{v}|^2(\widetilde{\boldsymbol{v}}\cdot\boldsymbol{n})\,\mathrm{d}S - \int_\Omega \frac{\varrho}{2}|\boldsymbol{v}|^2\mathrm{div}\,\widetilde{\boldsymbol{v}} + \frac12|\boldsymbol{v}|^2(\nabla\varrho\cdot\widetilde{\boldsymbol{v}})\,\mathrm{d}x \end{aligned}\tag{2.7}$$

is used for $\widetilde{\boldsymbol{v}} = \boldsymbol{v}$ and for ϱ constant so that $\nabla\varrho = 0$ and the last term in (2.7) vanishes; here the role of the bulk force $\frac12\varrho(\mathrm{div}\,\boldsymbol{v})\boldsymbol{v}$ in (2.4a) is revealed.

Further, by the assumed symmetry of $\mathbb{C}$ and $\mathbb{D}$,

$$\int_\Omega \mathrm{div}\,(\boldsymbol{S}+\varphi^*(\boldsymbol{S})\mathbb{I}+\mathbb{D}\boldsymbol{e}(\boldsymbol{v}))\cdot\boldsymbol{v}\,\mathrm{d}x = \int_\Gamma (\boldsymbol{S}+\varphi^*(\boldsymbol{S})\mathbb{I}+\mathbb{D}\boldsymbol{e}(\boldsymbol{v})):(\boldsymbol{v}\otimes\boldsymbol{n})\,\mathrm{d}S - \int_\Omega (\boldsymbol{S}+\varphi^*(\boldsymbol{S})\mathbb{I}+\mathbb{D}\boldsymbol{e}(\boldsymbol{v})):\boldsymbol{e}(\boldsymbol{v})\,\mathrm{d}x. \tag{2.8}$$

For the conservative-stress terms, we futher use

$$\begin{aligned}\int_\Omega (\boldsymbol{S}+\varphi^*(\boldsymbol{S})\mathbb{I}):\boldsymbol{e}(\boldsymbol{v})\,\mathrm{d}x &= \int_\Omega \boldsymbol{S}:\boldsymbol{e}(\boldsymbol{v}) + \varphi^*(\boldsymbol{S})\cdot\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x = \int_\Omega \boldsymbol{S}:\mathbb{C}^{-1}\overset{\circ}{\boldsymbol{S}} + \varphi^*(\boldsymbol{S})\cdot\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varphi^*(\boldsymbol{S})\,\mathrm{d}x + \int_\Omega \varphi^*(\boldsymbol{S})\cdot\mathrm{div}\,\boldsymbol{v} + \mathbb{C}^{-1}\boldsymbol{S}:\big((\boldsymbol{v}\cdot\nabla)\boldsymbol{S} - (\mathrm{skew}\nabla\boldsymbol{v})\boldsymbol{S} - (\mathrm{skew}\nabla\boldsymbol{v})^\top\big)\,\mathrm{d}x\\ &= \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varphi^*(\boldsymbol{S})\,\mathrm{d}x + \int_\Omega \mathrm{skew}\big(\boldsymbol{S}(\mathbb{C}^{-1}\boldsymbol{S})^\top - (\mathbb{C}^{-1}\boldsymbol{S})\boldsymbol{S}^\top\big):\nabla\boldsymbol{v}\,\mathrm{d}x = \frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \varphi^*(\boldsymbol{S})\,\mathrm{d}x\,, \end{aligned}\tag{2.9}$$

where we used also (2.4b). In particular, relying on that $\mathbb{C}$ is constant in space, we used also the Green formula for

$$\int_\Omega \varphi^*(\boldsymbol{S})\cdot\mathrm{div}\,\boldsymbol{v} + \mathbb{C}^{-1}\boldsymbol{S}:(\boldsymbol{v}\cdot\nabla)\boldsymbol{S}\,\mathrm{d}x = \int_\Omega \varphi^*(\boldsymbol{S})\cdot\mathrm{div}\,\boldsymbol{v} + \nabla\varphi^*(\boldsymbol{S})\cdot\boldsymbol{v}\,\mathrm{d}x = \int_\Gamma \varphi^*(\boldsymbol{S})(\boldsymbol{v}\cdot\boldsymbol{n})\,\mathrm{d}S = 0 \tag{2.10}$$

and the matrix algebra (2.3) for

$$\begin{aligned}\boldsymbol{E}:\big((\mathrm{skew}\nabla\boldsymbol{v})\boldsymbol{S} + \boldsymbol{S}(\mathrm{skew}\nabla\boldsymbol{v})^\top\big) &= \frac12\boldsymbol{E}:\big((\nabla\boldsymbol{v})\boldsymbol{S} - (\nabla\boldsymbol{v})^\top\boldsymbol{S} - \boldsymbol{S}(\nabla\boldsymbol{v}) + \boldsymbol{S}(\nabla\boldsymbol{v})^\top\big)\\ &= \frac12\big(\boldsymbol{E}\boldsymbol{S}^\top - \boldsymbol{S}^\top\boldsymbol{E}\big):\nabla\boldsymbol{v} - \frac12\boldsymbol{S}:\big((\nabla\boldsymbol{v})\boldsymbol{E} - \boldsymbol{E}(\nabla\boldsymbol{v})^\top\big)\\ &= \frac12\big(\boldsymbol{E}\boldsymbol{S}^\top - \boldsymbol{S}\boldsymbol{E}^\top - \boldsymbol{S}^\top\boldsymbol{E} + \boldsymbol{E}^\top\boldsymbol{S}\big):\nabla\boldsymbol{v} = \mathrm{skew}(\underbrace{\boldsymbol{E}\boldsymbol{S}^\top - \boldsymbol{S}\boldsymbol{E}^\top}_{=\boldsymbol{0}}):\nabla\boldsymbol{v} = \boldsymbol{0} \end{aligned}\tag{2.11}$$

with $\boldsymbol{E} = \mathbb{C}^{-1}\boldsymbol{S}$; here we have to assume that the initial condition for $\boldsymbol{S}$ is valued in $\mathbb{R}^{d\times d}_{\mathrm{sym}}$ and exploit that the Zaremba-Jaumann corotational derivative in (2.4b) keeps symmetry of $\boldsymbol{S}$ during the whole evolution.

Then we should sum up (2.8) and (2.9) and use the boundary conditions (2.5), so that $\int_\Gamma \boldsymbol{T}:(\boldsymbol{v}\otimes\boldsymbol{n})\,\mathrm{d}S = \int_\Gamma \boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}S$ and thus

$$\int_\Omega \mathrm{div}\,\boldsymbol{T}\cdot\boldsymbol{v}\,\mathrm{d}x = \int_\Gamma \boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}S - \int_\Omega \boldsymbol{T}:\nabla\boldsymbol{v}\,\mathrm{d}x = \int_\Gamma \boldsymbol{g}\cdot\boldsymbol{v}\,\mathrm{d}S - \int_\Omega \Big(\varrho\dot{\boldsymbol{v}} + \frac{\varrho}{2}(\mathrm{div}\,\boldsymbol{v})\boldsymbol{v} - \boldsymbol{f}\Big)\cdot\boldsymbol{v}\,\mathrm{d}x\,. \tag{2.12}$$

Thus we eventually obtain the energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \underbrace{\frac{\varrho}{2}|\boldsymbol{v}|^2}_{\substack{\text{kinetic}\\ \text{energy}}} + \underbrace{\frac12\mathbb{C}^{-1}\boldsymbol{S}:\boldsymbol{S}}_{\substack{\text{stored}\\ \text{energy}}}\,\mathrm{d}x + \int_\Omega \underbrace{\mathbb{D}\boldsymbol{e}(\boldsymbol{v}):\boldsymbol{e}(\boldsymbol{v})}_{\text{dissipation rate}}\,\mathrm{d}x = \int_\Omega \underbrace{\boldsymbol{f}\cdot\boldsymbol{v}}_{\substack{\text{power of}\\ \text{external load}}}\,\mathrm{d}x + \int_\Gamma \underbrace{\boldsymbol{g}\cdot\boldsymbol{v}}_{\substack{\text{power of}\\ \text{traction load}}}\,\mathrm{d}S\,. \tag{2.13}$$

Remark 2.1 (*Isotropic materials*)**.** The prominent application of this convective model is for isotropic materials which are invariant for rotations. The elasticity tensor $\mathbb{C}$ is then of the form

$$\mathbb{C}_{ijkl} := K_{\rm E}\delta_{ij}\delta_{kl} + G_{\rm E}\Big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d}\delta_{ij}\delta_{kl}\Big)\,, \tag{2.14}$$

where $K_{\rm E}$ is the bulk elastic modulus and $G_{\rm E}$ is the shear elastic modulus with the physical dimension Pa=J/m^3, while δ is the Kronecker symbol. The viscosity tensor $\mathbb{D}$ for isotropic materials is then in an analogical form

$$\mathbb{D}_{ijkl} := K_{\rm V}\delta_{ij}\delta_{kl} + G_{\rm V}\Big(\delta_{ik}\delta_{jl} + \delta_{il}\delta_{jk} - \frac{2}{d}\delta_{ij}\delta_{kl}\Big) \tag{2.15}$$

with $K_{\rm V}$ and $G_{\rm V}$ the viscosity moduli with the physical dimension Pa s. In this isotropic case, the stored energy in (2.13) reads as

$$\varphi^*(\boldsymbol{S}) = \frac{1}{2}\mathbb{C}^{-1}\boldsymbol{S}{:}\boldsymbol{S} = \frac{1}{2dK_{\rm E}}|{\rm sph}\,\boldsymbol{S}|^2 + \frac{1}{4G_{\rm E}}|{\rm dev}\,\boldsymbol{S}|^2 = \frac{1}{2d^2K_{\rm E}}|{\rm tr}\,\boldsymbol{S}|^2 + \frac{1}{4G_{\rm E}}|{\rm dev}\,\boldsymbol{S}|^2\,, \tag{2.16}$$

cf. [25, Formulas (6.7.9)–(6.7.10)]. In terms of a strain tensor $\boldsymbol{E}$, the stress $\mathbb{C}\boldsymbol{E}$ in this isotropic case looks as $dK_{\rm E}{\rm sph}\,\boldsymbol{E} + 2G_{\rm E}{\rm dev}\,\boldsymbol{E}$. It is important that, taking into account (2.14) and the symmetry of $\boldsymbol{E}$, $\boldsymbol{S} = \mathbb{C}\boldsymbol{E}$ now means $\boldsymbol{S}_{ij} = K_{\rm E}\delta_{ij}\boldsymbol{E}_{kk} + 2G_{\rm E}(\boldsymbol{E}_{ij} - \delta_{ij}\boldsymbol{E}_{kk}/d)$ with the summation convention. Then it is easy to check that $\nabla\boldsymbol{v}\mathbb{C}\boldsymbol{E} = \mathbb{C}(\nabla\boldsymbol{v})\boldsymbol{E}$ and also $(\nabla\boldsymbol{v})^\top\mathbb{C}\boldsymbol{E} = \mathbb{C}(\nabla\boldsymbol{v})^\top\boldsymbol{E}$. Altogether, with some $\boldsymbol{E}$ symmetric tensor and $\mathbb{C}$ homogeneous (i.e. independent of x) isotropic in the sense (2.14), we can see that the strain $\boldsymbol{E}$ is also transported by the Zaremba-Jaumann derivative, i.e.

$$\mathring{\boldsymbol{S}} = \mathbb{C}\mathring{\boldsymbol{E}}\,. \tag{2.17}$$

This will justify the strain/velocity formulation used later in Sections 3–6 where the convex-conjugate energy, i.e. here $\varphi(\boldsymbol{E}) = [\varphi^*]^*(\boldsymbol{E}) = \frac{d}{2}K_{\rm E}|{\rm sph}\,\boldsymbol{E}|^2 + G_{\rm E}|{\rm dev}\,\boldsymbol{E}|^2$, will be used, cf. (4.1) below.

Remark 2.2 (*Semi-compressible fluids*)**.** If the elastic shear modulus $G_{\rm E}$ in (2.14) vanishes (i.e. the elastic response in the solid model on the deviatoric stress vanishes), the medium becomes fluidic. Yet, the elastic response on the spherical stress remains and thus the model describes so-called viscoelastic fluids. Notably, such fluids facilitate propagation of pressure waves, in contrast to the merely viscous fluids. Then $\mathbb{C}\boldsymbol{e}(\boldsymbol{v}) = dK_{\rm E}{\rm sph}\,\boldsymbol{e}(\boldsymbol{v}) = K_{\rm E}({\rm div}\,\boldsymbol{v})\mathbb{I}$. From the last term in (2.16), we can see that ${\rm dev}\,\boldsymbol{S}$ must vanish. Then $\boldsymbol{S}(t,x) = -p(t,x)\mathbb{I}$ for some variable p meaning a pressure (2.4b) turns into

$$\dot{p} + K_{\rm E}{\rm div}\,\boldsymbol{v} = 0\,. \tag{2.18a}$$

Here it is important that, for $\boldsymbol{S} = -p\mathbb{I}$, it holds $\mathring{\boldsymbol{S}} = -\dot{p}\,\mathbb{I}$. The momentum equation (2.4a) degenerates to

$$\varrho\dot{\boldsymbol{v}} = {\rm div}\big(\mathbb{D}\boldsymbol{e}(\boldsymbol{v})\big) - \nabla\Big(p + \frac{p^2}{2K_{\rm E}}\Big) - \frac{\varrho}{2}({\rm div}\,\boldsymbol{v})\,\boldsymbol{v} + \boldsymbol{f}\,. \tag{2.18b}$$

It allows for propagation of *longitudinal* (also called pressure) *waves* (so-called P-waves). In a one-dimensional situation, one can see that longitudinal waves propagate with the velocity $v = v(\lambda) = \sqrt{K_{\rm E}/\varrho - K_{\rm V}^2/(4\varrho^2\lambda^2)}$ with λ denoting the wave length and $K_{\rm V}$ is from (2.15). This shows that, due to the viscosity, this model exhibits a so-called *normal dispersion*, i.e. the speed of waves increases with their wavelength, cf. [37].

Remark 2.3 (*Gradient theories towards analysis*)**.** The existence of weak solutions of the models (2.4) or (2.18) is unfortunately not ensured. To facilitate rigorous analysis, it is seems inevitable to involve some gradient theories into the model. There are several options. The first option it to augment the energy φ^* by $\frac{\varepsilon}{2}\nabla\boldsymbol{S}\,\vdots\,\nabla\mathbb{C}^{-1}\boldsymbol{S}$ and then the Cauchy stress stress in (2.4a) by $-\varepsilon\Delta\boldsymbol{S} - \varepsilon\nabla\boldsymbol{S}\otimes\mathbb{C}^{-1}\nabla\boldsymbol{S}$ with a small $\varepsilon > 0$, which leads to a conservative expansion of the energetics (2.13). The second option is to augment (2.4b) by $-\varepsilon\Delta\boldsymbol{S}$, which leads to a dissipative expansion of the energetics (2.13) by $\varepsilon\nabla\boldsymbol{S}\,\vdots\,\nabla\mathbb{C}^{-1}\boldsymbol{S}$. This is a concept of so-called (small) stress diffusion, which is related with a (large) Péclet (in fluids also called Brenner) number. The third option is to augment the stress in (2.4a) by $-\nu\Delta\boldsymbol{e}(\boldsymbol{v})$, which leads to a dissipative expansion of the energetics (2.13) by $\nu|\nabla\boldsymbol{e}(\boldsymbol{v})|^2$. This is a concept of so-called 2nd-grade nonsimple materials. In what follows, we will employ a nonlinear variant of this third option. These three options can be reflected also in the semi-compressible fluid model (2.18) and have been addressed in [37]. The first and the third options leads to anomalous dispersion while the second one contributes to the normal dispersion of the velocity of the P-waves. For the limit for $K_{\rm E}\to\infty$ (possibly together with $\varepsilon\to 0$ for the second option), one can prove that the semi-compressible fluid turns into the incompressible Navier-Stokes system, which however does not allow for propagation of P-waves.

Remark 2.4 (*Varying mass density*)**.** The above model has used the simplification based on the assumption of a constant mass density ϱ even though the model is not fully incompressible, cf. the calculus (2.7). This is legitimate when, in particular, variations of ϱ during volumetric deformation (which is typically indeed small in liquids and solids, in contrast to gases) can be neglected; e.g. even extremely compressed water in the deepest spot in Earth oceans (i.e. Mariana trench) shrinks only by 5% in volume and rocks varies their volume less than 0.1% by longitudianal waves emitted during even big earthquakes. (Remarkable density variations can occur rather during phase transformations but we neglect it, too.) To keep energy balance while omitting the continuity equation $\dot{\varrho}=-\varrho{\rm div}\,\boldsymbol{v}$ is then compensated by the force $-\varrho({\rm div}\,\boldsymbol{v})\boldsymbol{v}/2$ in (2.4a) and in (3.3a), which is pressumably very small but not Gallilean invariant, as pointed out in [45]. This modeling shortcut simplifies a lot of calculations and allows us to focus on the thermomechanical context of the Stefan problem itself. If the medium would be semi-compressible but inhomogeneous as far as mass density, one should complete the system by a simplified continuity equation $\dot{\varrho}=0$ while still keep the compensating force in the momentum equation. Like for the fully compressible model without the compensating force but with the full continuity equation $\dot{\varrho}=-\varrho{\rm div}\,\boldsymbol{v}$, e.g. (6.12)–(6.14) and (6.39) would become considerably more technical although, because the concept of nonlinear nonsimple material ensures regular velocity fields $\nabla\boldsymbol{v}$ in $L^p_{\rm w*}(I;L^\infty(\varOmega;\mathbb{R}^{d\times d}))$, the analysis would be doable.

3 A thermomechanical model with inelasticity and damage

Now we will enhance the model by some internal variables, namely a scalar-valued *damage* α and an *inelastic* (or plastic) *strain* $\boldsymbol{P}$ which is considered standardly symmetric and isochoric, i.e. ${\rm tr}\,\boldsymbol{P}=0$. Moreover, we enhance it towards full thermodynamics by considering a heat-transfer equation for *temperature* θ coupled with the mechanical part in a thermodynamically consistent way.

Involving damage and creep (or plasticity) into the stress/velocity formulation is quite tricky. Actually, the creep model (arising from a dissipation potential quadratic in terms of the rate $\boldsymbol{\varPi}$ of $\boldsymbol{P}$) can be understood also as the Jeffreys' rheological model.

However, in the convective damage model, this does not seem working properly because the constancy of $\mathbb{C}$ used in (2.10) cannot hold any more because $\mathbb{C}$ must now depend on damage, otherwise there would not be any driving force for damage processes.

In order to eliminate the tensor $\mathbb{C}$ from the formulation of the model, we switch into a *strain/velocity formulation*. We confine ourselves on the *additive*, also called *Green-Naghdi's decomposition* which reads at small strains as $\boldsymbol{e}(\boldsymbol{u})=\boldsymbol{E}+\boldsymbol{P}$, where $\boldsymbol{E}$ denotes the elastic strain and $\boldsymbol{u}$ the displacement. Yet, as we do not want to stay on small displacements, we formulate the additive decomposition rather in terms of rates as

$$\boldsymbol{e}(\boldsymbol{v})=\overset{\circ}{\overline{\boldsymbol{E}+\boldsymbol{P}}}=\overset{\circ}{\boldsymbol{E}}+\boldsymbol{\varPi}\qquad\text{if denoting}\ \ \boldsymbol{\varPi}=\overset{\circ}{\boldsymbol{P}}\,. \tag{3.1}$$

The new variable $\boldsymbol{\varPi}$ is a rate of plastic strain. Thus both $\boldsymbol{u}$ and $\boldsymbol{P}$ will not be eventually used, and the model allows for *large displacements* even though still using linearized elastic strains. For example, after solid has being melted, the resulting liquid may flow on large distances where it can again solidify etc. Usage of the corotational derivative in (3.1) is consistent with Section 2 where, instead of (2.4b), we may use $\overset{\circ}{\boldsymbol{E}} = \boldsymbol{e}(\boldsymbol{v})$ provided the material is isotropic homogeneous, cf. (2.17).

As we want to model also solid-liquid phase transition like the classical Stefan problem in the so-called enthalpy formulation, the deviatoric elastic response will be influenced by temperature and vanishes completely above a melting temperature when the material becomes viscoelastic fluid, cf. Remark 2.2. However, direct involvement of the temperature in the elastic shear modulus would have to be nonlinear and would cause complicated adiabatic effects. We thus devise the model differently by allowing temperature to influence only the dissipative part, cf. Figure 1 below.

The additional variables and data employed in the enhancement and the strain/velocity (instead of stress/velocity) formulation of mechanical merely visco-elastodynamic system (2.4) are summarized in the following table:

$\boldsymbol{E}$ the elastic strain tensor (symmetric),	$\boldsymbol{D}$ dissipative stress,
$\boldsymbol{\Pi}$ the inelastic strain rate tensor ,	$\mathscr{K} = \mathscr{K}(\alpha, w)$ thermal tconductivity,
α damage variable,	$\zeta = \zeta(\alpha, w; \cdot)$ dissipation potential for damage rate,
θ temperature (in K),	$\psi = \psi(\boldsymbol{E}, \alpha, \theta)$ free energy (in Pa=J/m^3),
w enthalpy (internal heat energy in J/m^3),	$\varphi = \varphi(\boldsymbol{E}, \alpha)$ the stored energy (in Pa=J/m^3),
η entropy (in J/m^3K),	$\phi = \phi(\theta)$ thermal part of the free energy (in Pa=J/m^3),
χ water/ice volume fraction,	$\beta = \gamma^{-1}$ with $\gamma(\theta) = \phi(\theta) - \theta\phi'(\theta)$,
$\boldsymbol{T}$ total Cauchy stress (symmetric),	$G_{\rm M} = G_{\rm M}(w)$ Maxwell-viscosity modulus (in Pa s=Nm^{-2}s),
$\boldsymbol{S}$ (first) Piola-Kirchhoff stress,	κ, $\varkappa$ length-scale coefficients,
$\boldsymbol{K}$ Korteweg-like stress,	ν a hyper-viscosity coefficient.

The main ingredient will be the free energy ψ, which expands the stored energy in terms of strains (i.e. the dual to the stored energy φ^* in terms of stresses as used in Section 2). We will consider a gradient theory for damage and, for simplicity, and additive split of mechanical and thermal parts, namely

$$\psi(\boldsymbol{E}, \alpha, \nabla\alpha, \theta) = \varphi(\boldsymbol{E}, \alpha) + \frac{\kappa}{2}|\nabla\alpha|^2 + \phi(\theta)\,. \tag{3.2}$$

We may consider a general stored energy $\varphi = \varphi(\boldsymbol{E}, \alpha)$ possibly non-quadratic in terms of $\boldsymbol{E}$ while, as usual, we are using a linear gradient theory for the damage variable α.

Beside the free energy, another important ingredient is the (pseudo)potential of dissipative forces acting on rates of internal variables, i.e. here of the inelastic strain and the damage. As to the inelastic strain rate $\boldsymbol{\Pi}$, we consider a materially linear Maxwellian creep together with a gradient theory, so that the corresponding potential is quadratic in terms of rate, i.e. $\frac{1}{2}G_{\rm M}|\boldsymbol{\Pi}|^2 + \frac{1}{2}\varkappa|\nabla\boldsymbol{\Pi}|^2$ for some (presumably small) coefficient $\varkappa > 0$; in what follows, $G_{\rm M}$ will depend on temperature (or rather on enthalpy w). A nonlinear generalization will be outlined in Remark 7.2 below. On the other hand, evolution of damage is surely always a very nonlinear phenomenon, in particular that damaging of solid is a fast process while healing is rather slow (except when damaged solid starts melting and healing is to be fast so that α goes fast to 1 in created liquid phase and possible later solidification leads again to an undamaged solid). Thus for the damage rate $\dot{\alpha}$, we will consider a general convex potential depending also on state variables (α, w), let us denote it by $\zeta = \zeta(\alpha, \theta; \cdot)$.

We used the gradient theory for the rate $\boldsymbol{\Pi}$, as suggested in [13], which does not cause any spurious hardening-like effects during large slips particularly along cracks or fluid-solid interface. Simultaneously, as outlined already in Remark 2.3, we will use a higher-order velocity gradient. Here, the goal is to ensure $\nabla\boldsymbol{v} \in L^1_{{\rm w}*}(I; L^\infty(\Omega; \mathbb{R}^{d\times d}))$, which further facilitates regularity of the transport, cf. in particular the estimates (6.30) and (6.33) below. Therefore, another enhancement will be by dissipative gradient terms, exploiting the concept of the so-called *nonsimple fluids*, devised by E. Fried and M. Gurtin [19] and earlier, even more generally and nonlinearly as multipolar fluids, by J. Nečas at al. [3, 29, 30]. More specifically, we use nonlinear 2nd-grade nonsimple fluids, also called *bipolar fluids*, monolithically also in the solid part although making the coefficient ν in (3.3a) dependent on enthalpy could distinguish "intensity" of this concept in the liquid and in the solid areas.

The above introduced model now reads specifically as

$$\varrho\dot{\boldsymbol{v}} = \operatorname{div}\boldsymbol{T} - \frac{\varrho}{2}(\operatorname{div}\boldsymbol{v})\,\boldsymbol{v} + \boldsymbol{f} \quad \text{where} \quad \boldsymbol{T} = \boldsymbol{S} + \psi(\boldsymbol{E},\alpha,\theta)\mathbb{I} + \boldsymbol{K} + \boldsymbol{D}$$
$$\text{with} \quad \boldsymbol{S} = \varphi'_{\boldsymbol{E}}(\boldsymbol{E},\alpha)\,, \quad \boldsymbol{K} = \frac{\kappa}{2}|\nabla\alpha|^2\mathbb{I} - \kappa\nabla\alpha\otimes\nabla\alpha\,,$$
$$\text{and} \quad \boldsymbol{D} = \mathbb{D}\boldsymbol{e}(\boldsymbol{v}) - \operatorname{div}\big(\nu|\nabla\boldsymbol{e}(\boldsymbol{v})|^{p-2}\nabla\boldsymbol{e}(\boldsymbol{v})\big)\,, \tag{3.3a}$$
$$\overset{\circ}{\boldsymbol{E}} = \boldsymbol{e}(\boldsymbol{v}) - \boldsymbol{\Pi}\,, \tag{3.3b}$$

and the equation for creep rate and the flow-rules for damage

$$G_{\rm M}(w)\boldsymbol{\Pi} = \operatorname{dev}\boldsymbol{S} + \varkappa\Delta\boldsymbol{\Pi}\,, \tag{3.3c}$$
$$\partial_{\dot\alpha}\zeta\big(\alpha,w;\dot\alpha\big) + \varphi'_\alpha(\boldsymbol{E},\alpha) \ni \kappa\Delta\alpha\,. \tag{3.3d}$$

Moreover, the system should be thermodynamically closed by the heat-transfer equation, cf. (3.16) below, as

$$\frac{\partial w}{\partial t} + \operatorname{div}\big(\boldsymbol{v}\,w - \mathscr{K}(\alpha,w)\nabla\theta\big) = G_{\rm M}(w)|\boldsymbol{\Pi}|^2 + \xi\big(\alpha,w;\dot\alpha\big) + \mathbb{D}\boldsymbol{e}(\boldsymbol{v}){:}\boldsymbol{e}(\boldsymbol{v}) + \nu|\nabla\boldsymbol{e}(\boldsymbol{v})|^p$$
$$+\,\varkappa|\nabla\boldsymbol{\Pi}|^2 + \phi(\theta)\operatorname{div}\boldsymbol{v} \qquad \text{with} \quad \xi\big(\alpha,w;\dot\alpha\big) = \partial_{\dot\alpha}\zeta\big(\alpha,w;\dot\alpha\big)\dot\alpha \tag{3.3e}$$
$$\text{and with} \quad \theta = \beta(w)\,, \tag{3.3f}$$

where β is related to ϕ by $\beta^{-1}(\theta) = \phi(\theta) - \theta\phi'(\theta)$. Note that (3.3c) is a quasistatic equation from which, when prescribing the boundary conditions (3.4b), $\boldsymbol{\Pi}$ can be explicitly isolated as a function of w and $\boldsymbol{S}$. Moreover, $\boldsymbol{\Pi}$ is always trace-free and, if $\boldsymbol{S}$ is symmetric, $\boldsymbol{\Pi}$ is also symmetric; this last attribute is indeed granted if the initial elastic strain is symmetric, cf. also Remark 3.1.

We complete the system (3.3) by suitable boundary conditions. This is however little delicate due to the 2nd-grade nonsimple fluid concept and naturally needs 2 boundary conditions for the momentum equation (3.3a). One of variationally natural option to to prescribed boundary traction while putting higher derivatives zero, i.e. Neumann-type boundary condition. Here we take it in the tangential direction while combining it with Dirichlet/Neumann condition in the normal direction. As it concerns d-dimensional vector field $\boldsymbol{v}$, we actually prescribe $2d$ conditions on the boundary. More specifically, for the heat flux $h_{\rm ext}$, we consider

$$\boldsymbol{v}{\cdot}\boldsymbol{n} = 0\,, \quad \big[\boldsymbol{T}\boldsymbol{n} + \operatorname{div}_{\rm S}\big(\nu|\nabla\boldsymbol{e}(\boldsymbol{v})|^{p-2}\nabla\boldsymbol{e}(\boldsymbol{v}){\cdot}\boldsymbol{n}\big)\big]_{\rm T} = \boldsymbol{0}\,, \quad \nabla\boldsymbol{e}(\boldsymbol{v}){:}(\boldsymbol{n}\otimes\boldsymbol{n}) = \boldsymbol{0}\,, \tag{3.4a}$$
$$(\boldsymbol{n}{\cdot}\nabla)\boldsymbol{\Pi} = 0, \qquad \nabla\alpha{\cdot}\boldsymbol{n} = 0, \qquad \boldsymbol{n}{\cdot}\mathscr{K}(\alpha,w)\nabla\theta = h_{\rm ext}\,, \tag{3.4b}$$

where $\operatorname{div}_{\rm S} = \operatorname{tr}(\nabla_{\rm S})$ with $\operatorname{tr}(\cdot)$ being the trace of a $(d{-}1){\times}(d{-}1)$-matrix, denotes the $(d{-}1)$-dimensional surface divergence and $\nabla_{\rm S}v = \nabla v - (\nabla v{\cdot}\boldsymbol{n})\boldsymbol{n}$ being the surface gradient of v. In contrast to (2.5), we consider homogeneous mechanical condition rather from the analytical reasons, cf. Remark 7.6.

We will consider an initial-value problem for the evolution boundary-value problem (3.3)–(3.4) and prescribe the corresponding initial conditions enhancing (2.6), namely

$$\boldsymbol{v}|_{t=0} = \boldsymbol{v}_0, \qquad \boldsymbol{S}|_{t=0} = \boldsymbol{S}_0, \qquad \alpha|_{t=0} = \alpha_0, \quad \text{and} \quad w|_{t=0} = w_0\,. \tag{3.5}$$

Let us note that we thus prescribe also the initial condition for $\boldsymbol{E}$, namely

$$\boldsymbol{E}|_{t=0} = \boldsymbol{E}_0 = \big[\varphi'_{\boldsymbol{E}}(\cdot,\alpha_0)\big]^{-1}(\boldsymbol{S}_0)\,. \tag{3.6}$$

The mechanical-energy balance behind the model (3.3a-d) is revealed by adding the momentum equation (3.3a) tested by $\boldsymbol{v}$ as before in Section 2 and now the strain equation (3.3b) tested by the Piola-Kirchhoff stress $\boldsymbol{S}=\varphi'_{\boldsymbol{E}}(\boldsymbol{E},\alpha)$ and further (3.3c) tested by $\boldsymbol{\Pi}$ and (3.3d) tested by $\overset{\bullet}{\alpha}$. At least formally, we use the following calculations:

From (3.3a) tested by $\boldsymbol{v}$ and using again (2.7)–(2.9), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\varrho}{2}|\boldsymbol{v}|^2\,\mathrm{d}x+\int_\Omega\big(\boldsymbol{S}{+}\varphi(\boldsymbol{E},\alpha)\mathbb{I}{+}\boldsymbol{K}{+}\mathbb{D}\boldsymbol{e}(\boldsymbol{v})\big){:}\boldsymbol{e}(\boldsymbol{v})+\nu|\nabla\boldsymbol{e}(\boldsymbol{v})|^p+\phi(\theta)\mathrm{div}\boldsymbol{v}\,\mathrm{d}x=\int_\Omega\boldsymbol{f}{\cdot}\boldsymbol{v}\,\mathrm{d}x\,, \tag{3.7}$$

where we used that $\boldsymbol{S}=\varphi'_{\boldsymbol{E}}(\boldsymbol{E},\alpha)$ is symmetric and also the boundary conditions (3.4) have been used; actually, treatment of the boundary conditions for 2nd-grade nonsimple material is indeed nonsimple and, beside usage of Green formula twice, needs also usage of a surface Green formula, cf. [34, Sect. 2.4.4]. The conservative part of the Cauchy stress is to be treated by using (3.3b) tested by $\boldsymbol{S}$ and (3.3c) tested by $\boldsymbol{\Pi}$. Similarly as in (2.9) we obtain

$$\begin{aligned}\int_\Omega\big(\boldsymbol{S}+\varphi(\boldsymbol{E},\alpha)\mathbb{I}+\boldsymbol{K}\big){:}\boldsymbol{e}(\boldsymbol{v})\,\mathrm{d}x&=\int_\Omega\boldsymbol{S}{:}\overset{\circ}{\boldsymbol{E}}+\mathrm{dev}\boldsymbol{S}{:}\boldsymbol{\Pi}+\boldsymbol{K}{:}\boldsymbol{e}(\boldsymbol{v})+\varphi(\boldsymbol{E},\alpha)\mathrm{div}\boldsymbol{v}\,\mathrm{d}x\\&=\int_\Omega\varphi'_{\boldsymbol{E}}(\boldsymbol{E},\alpha){:}\Big(\frac{\partial\boldsymbol{E}}{\partial t}+(\boldsymbol{v}{\cdot}\nabla)\boldsymbol{E}\Big)+G_{\mathrm{M}}(w)|\boldsymbol{\Pi}|^2+\varkappa|\nabla\boldsymbol{\Pi}|^2+\boldsymbol{K}{:}\boldsymbol{e}(\boldsymbol{v})+\varphi(\boldsymbol{E},\alpha)\mathrm{div}\boldsymbol{v}\,\mathrm{d}x\end{aligned} \tag{3.8}$$

where we used also $\boldsymbol{S}{:}(\mathrm{skew}(\nabla\boldsymbol{v})\boldsymbol{E}-\boldsymbol{E}\mathrm{skew}(\nabla\boldsymbol{v}))=0$ similarly as (2.11), exploiting again the symmetry of $\boldsymbol{E}$ which is now based on the symmetry not only of $\boldsymbol{e}(\boldsymbol{v})$ but also of $\boldsymbol{\Pi}$.

Furthermore, from (3.3d) tested by $\overset{\bullet}{\alpha}$, we obtain

$$\begin{aligned}\int_\Omega\partial_{\overset{\bullet}{\alpha}}\zeta\big(\alpha,w;\overset{\bullet}{\alpha}\big)\overset{\bullet}{\alpha}\,\mathrm{d}x&=\int_\Omega\big(\kappa\Delta\alpha-\varphi'_\alpha(\boldsymbol{E},\alpha)\big)\overset{\bullet}{\alpha}\,\mathrm{d}x\\&=\int_\Gamma\kappa(\boldsymbol{n}{\cdot}\nabla\alpha)\overset{\bullet}{\alpha}\,\mathrm{d}S-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\kappa}{2}|\nabla\alpha|^2\,\mathrm{d}x-\int_\Omega\nabla\alpha{\cdot}\nabla(\boldsymbol{v}{\cdot}\nabla\alpha)+\varphi'_\alpha(\boldsymbol{E},\alpha)\Big(\frac{\partial\alpha}{\partial t}+\boldsymbol{v}{\cdot}\nabla\alpha\Big)\,\mathrm{d}x\\&=\int_\Gamma\kappa(\boldsymbol{n}{\cdot}\nabla\alpha)\overset{\bullet}{\alpha}-\frac{\kappa}{2}|\nabla\alpha|^2(\boldsymbol{v}{\cdot}\boldsymbol{n})\,\mathrm{d}S-\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\frac{\kappa}{2}|\nabla\alpha|^2\,\mathrm{d}x\\&\qquad-\int_\Omega\underbrace{\kappa(\nabla\alpha{\otimes}\nabla\alpha){:}\boldsymbol{e}(\boldsymbol{v})-\frac{\kappa}{2}|\nabla\alpha|^2\mathrm{div}\,\boldsymbol{v}}_{=\boldsymbol{K}:\boldsymbol{e}(\boldsymbol{v})}+\varphi'_\alpha(\boldsymbol{E},\alpha)\Big(\frac{\partial\alpha}{\partial t}+\boldsymbol{v}{\cdot}\nabla\alpha\Big)\,\mathrm{d}x\,,\end{aligned} \tag{3.9}$$

where the boundary integral vanishes due to the boundary conditions $\boldsymbol{n}{\cdot}\nabla\alpha=0$ and $\boldsymbol{v}{\cdot}\boldsymbol{n}=0$. The two convective terms arising in (3.8) and (3.9) are to be handled jointly, which gives

$$\int_\Omega\varphi'_{\boldsymbol{E}}(\boldsymbol{E},\alpha){:}(\boldsymbol{v}{\cdot}\nabla)\boldsymbol{E}+\varphi'_\alpha(\boldsymbol{E},\alpha)(\boldsymbol{v}{\cdot}\nabla\alpha)\,\mathrm{d}x=\int_\Omega\nabla\varphi(\boldsymbol{E},\alpha){\cdot}\boldsymbol{v}\,\mathrm{d}x=\int_\Gamma\varphi(\boldsymbol{E},\alpha)\boldsymbol{v}{\cdot}\boldsymbol{n}\,\mathrm{d}S-\int_\Omega\varphi(\boldsymbol{E},\alpha)\mathrm{div}\,\boldsymbol{v}\,\mathrm{d}x\,. \tag{3.10}$$

This then uses the boundary condition $\boldsymbol{v}{\cdot}\boldsymbol{n}=0$ while the bulk term is balanced with the corresponding pressure-type term in (3.8).

Substituting (3.8) into (3.7) and summing it with (3.9) gives the mechanical energy balance

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega\underbrace{\frac{\varrho}{2}|\boldsymbol{v}|^2}_{\substack{\text{kinetic}\\\text{energy}}}+\underbrace{\varphi(\boldsymbol{E},\alpha)+\frac{\kappa}{2}|\nabla\alpha|^2}_{\text{stored energy}}\,\mathrm{d}x\\&+\int_\Omega\underbrace{G_{\mathrm{M}}(w)|\boldsymbol{\Pi}|^2+\xi\big(\alpha,w;\overset{\bullet}{\alpha}\big)+\mathbb{D}\boldsymbol{e}(\boldsymbol{v}){:}\boldsymbol{e}(\boldsymbol{v})+\nu|\nabla\boldsymbol{e}(\boldsymbol{v})|^p+\varkappa|\nabla\boldsymbol{\Pi}|^2}_{\text{dissipation rate}}\,\mathrm{d}x=\int_\Omega\underbrace{\boldsymbol{f}{\cdot}\boldsymbol{v}-\phi(\theta)\mathrm{div}\,\boldsymbol{v}}_{\substack{\text{power of external load}\\\text{and of adiabatic effects}}}\,\mathrm{d}x\,.\end{aligned} \tag{3.11}$$

Adding (3.3e) tested by 1 and using also the last boundary condition in (3.4b), we obtain the total energy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \underbrace{\frac{\varrho}{2}|\boldsymbol{v}|^2}_{\substack{\text{kinetic}\\ \text{energy}}} + \underbrace{\varphi(\boldsymbol{E},\alpha)+\frac{\kappa}{2}|\nabla\alpha|^2}_{\substack{\text{stored}\\ \text{energy}}} + \underbrace{w}_{\substack{\text{internal heat}\\ \text{energy (enthalpy)}}}\,\mathrm{d}x = \int_\Omega \underbrace{\boldsymbol{f}\cdot\boldsymbol{v}}_{\substack{\text{external}\\ \text{bulk load}}}\,\mathrm{d}x + \int_\Gamma \underbrace{h_{\rm ext}}_{\substack{\text{power of}\\ \text{external heating}}}\,\mathrm{d}S\,. \tag{3.12}$$

The thermodynamical context of this model relies on an additive splitting of the specific free energy ψ into the purely mechanical part and the thermal part ϕ, cf. an example (4.1) below. An important attribute of the model, beside keeping the energetics (3.11)–(3.12), is the entropy balance, and in particular the *Clausius-Duhem inequality* due to non-negativity of the entropy production in (3.14). The specific entropy $\eta = -\psi'_\theta$ is an extensive variable (in $\mathrm{JK}^{-1}\mathrm{m}^{-3}$) and its transport and production is governed by the entropy equation

$$\theta\Big(\frac{\partial\eta}{\partial t}+\mathrm{div}\big(\boldsymbol{v}\,\eta\big)\Big) = \xi - \mathrm{div}\,\boldsymbol{j} \tag{3.13}$$

with ξ denoting the heat production rate (here by mechanical dissipation processes) and $\boldsymbol{j}$ the heat flux (here governed by the Fourier law $\boldsymbol{j} = -\mathscr{K}\nabla\theta$). The ultimate assumptions $\xi \geq 0$ and $\mathscr{K} > 0$ then ensure the entropy balance

$$\frac{\mathrm{d}}{\mathrm{d}t}\int_\Omega \eta\,\mathrm{d}x = \int_\Omega \frac{\xi - \mathrm{div}\,\boldsymbol{j}}{\theta}\,\mathrm{d}x + \int_\Gamma \eta\boldsymbol{v}\cdot\boldsymbol{n}\,\mathrm{d}S = \int_\Omega \underbrace{\frac{\xi}{\theta} + \mathscr{K}\frac{|\nabla\theta|^2}{\theta^2}}_{\text{entropy production}\,\geq\,0}\,\mathrm{d}x + \int_\Gamma \underbrace{(\boldsymbol{v}\eta - \boldsymbol{j}/\theta)}_{\text{entropy flux}}\cdot\boldsymbol{n}\,\mathrm{d}S \tag{3.14}$$

relying on positivity of temperature. Substituting $\eta = -\phi'(\theta)$ into (3.13), we obtain the heat-transfer equation

$$c(\theta)\dot{\theta} = \xi - \mathrm{div}\,\boldsymbol{j} - \theta\eta\,\mathrm{div}\,\boldsymbol{v} \qquad \text{with the heat capacity} \qquad c(\theta) = -\theta\phi''(\theta)\,; \tag{3.15}$$

note that temperature (in K) is an intensive variable and is transported by the material derivative while the adiabatic heat source/sink term $\theta\eta\,\mathrm{div}\,\boldsymbol{v}$ occurs on the right-hand side due to the compressibility of the solid or fluid. Furthermore, the internal energy is given by the Gibbs relation $\psi + \theta\eta$, and splits here into the purely mechanical part and the thermal part $w = \phi(\theta) - \theta\phi'(\theta) =: \gamma(\theta)$. The thermal internal energy w in Jm^{-3} is again an extensive variable and is transported like (3.13), resulting here to the equation

$$\frac{\partial w}{\partial t} + \mathrm{div}\big(\boldsymbol{v}\,w\big) = \xi - \mathrm{div}\,\boldsymbol{j} + \phi(\theta)\mathrm{div}\boldsymbol{v}\,, \tag{3.16}$$

which reveals the structure of (3.3e).

Remark 3.1 (*Inelastic strain $\boldsymbol{P}$).* As already noted, we did not explicitly involve the inelastic strain $\boldsymbol{P}$ in (3.1), neither in the list of initial conditions (3.5). Yet, if an initial condition $\boldsymbol{P}|_{t=0} = \boldsymbol{P}_0$ would be prescribed, we could substitute $\boldsymbol{\Pi} = \mathring{\boldsymbol{P}}$ into (3.3b,c,e). It is important that, formulating the model in terms of $\boldsymbol{P}$ complies with the generally accepted concept of volume preserving (so-called isochoric) inelastic deformations, which is reflected by trace-free inelastic strain $\boldsymbol{P}$. This would be ensured if $\boldsymbol{P}_0$ is valued in $\mathbb{R}^{d\times d}_{\rm dev}$. Then $\boldsymbol{P}(t)$ remains deviatoric (= symmetric trace-free) for all $t > 0$ because

$$\mathrm{tr}\,\mathring{\boldsymbol{P}} = (\mathrm{tr}\,\boldsymbol{P})^{\bullet} \qquad \text{and} \qquad \mathring{\boldsymbol{P}}^\top = (\boldsymbol{P}^\top)^\circ$$

and because $\mathrm{dev}\,\boldsymbol{S}$ which drives the evolution of $\boldsymbol{P}$ in (3.3c) with $\boldsymbol{\Pi} = \mathring{\boldsymbol{P}}$, is symmetric trace-free. This is a general property of the Zaremba-Jaumann derivative, while others, non-corotational (like Oldroyd's or Truesdell's) derivatives would not guarantee this property. Similarly, if $\boldsymbol{E}_0$ is valued in $\mathbb{R}^{d\times d}_{\rm sym}$, then remains symmetric for all $t > 0$ because $\mathring{\boldsymbol{E}}^\top = (\boldsymbol{E}^\top)^\circ$ and because $\boldsymbol{e}(\boldsymbol{v})$ driving the evolution of $\boldsymbol{E}$ in (3.3b) is symmetric.

4 A specialization towards phase transformation and phase-field fracture

For isotropic material as in Remark 2.1 combined with damage, the above model together with the analysis in Section 6 below covers the particular case of the free energy (3.2) as

$$\psi(\boldsymbol{E},\alpha,\nabla\alpha,\theta) = \frac{d}{2}K_{\rm E}|{\rm sph}\,\boldsymbol{E}|^2 + G_{\rm E}(\alpha)|{\rm dev}\,\boldsymbol{E}|^2 + \frac{1}{2\kappa}G_{\rm D}(1{-}\alpha)^2 + \frac{\kappa}{2}|\nabla\alpha|^2 + \phi(\theta)\,. \tag{4.1}$$

The parameters are bulk modulus $K_{\rm E}$ as before, a "damageable" shear modulus $G_{\rm E} = G_{\rm E}(\alpha)$ with $G_{\rm E}(0) = G'_{\rm E}(0) = 0$ and $G_{\rm E}(1) \geq 0$, which is important to keep α valued in $[0,1]$. Further ingredients in (4.1) are a fracture-toughness-like modulus $G_{\rm D} > 0$, and the thermal part ϕ of the free energy. Moreover, the parameter $\kappa > 0$ in (4.1) occurring already in (3.3) is in N=Pa m^2. The $G_{\rm D}$-term gives rise to a driving for healing processes, i.e. $\varphi'_\alpha(0,\alpha) > 0$ if $\alpha < 1$, cf. e.g. [25, Sect. 7.5] or [36]. The most popular choice is $G_{\rm E}(\alpha) = G_{{\rm E},0}(\alpha^2{+}\epsilon^2)$ with some constant $G_{{\rm E},0}$, which is known as an Ambrosio-Tortorelli functional [1]. While $G_{{\rm E},0}$ as well as $K_{\rm E}$ are in Pa=J/m^3, the physical dimension of $G_{\rm D}$ is N/m^2. For κ small, the damage tends to be localized along thin regions, called cracks. Various modifications fitted to both initiation and propagation of cracks are devised in literature and known under the name of *phase-field* fracture, cf. [36, Sect. 3].

Here the bulk modulus $K_{\rm E}$ is considered independent of damage, which can be interpreted as that the material is surely so compressed that no tension (i.e. no opening also called fracture in More I) can occur. This means that the fracture might develop only in Mode II (shearing) or Mode III (tearing). The situations when the material is well compressed are actually what we have implicitly in mind not only for solid part but also for the liquids (modelled here as semicompressible fluids) which can withstand only non-negative (or very small negative) pressures.

The rheology behind the equations (3.3a–c) in this special case can schematically be depicted by Figure 1:

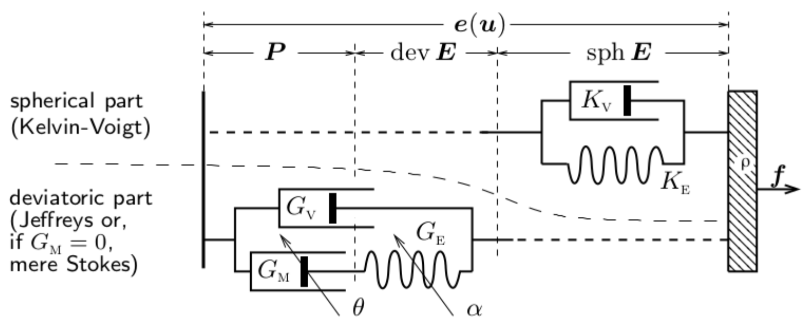

Fig. 1: A schematic 0-dimensional diagramme of the mixed rheology acting differently on spherical (volumetric) and the deviatoric parts. The Jeffreys rheology in the deviatoric part combines Stokes' and Maxwell rheologies in parallel and may degenerate to mere Stokes-type fluid if $G_{\rm M}$ vanishes within melting.

The ansatz (4.1) is thermodynamically simple situation because the mechanical and the thermal parts are coupled additively. This particularly means that the entropy $\eta = -\psi'_\theta = -\phi'(\theta)$ does not depend on mechanical variables and the internal energy $\psi{+}\theta\eta = \varphi(\boldsymbol{E},\alpha){+}\frac{1}{2}\kappa|\nabla\alpha|^2{+}\phi(\theta){-}\theta\phi'(\theta)$ also couples additively the mechanical and the heat parts, cf. (3.12) for $w = \phi(\theta){-}\theta\phi'(\theta)$. This reveals that $\gamma(\theta) = \phi(\theta){-}\theta\phi'(\theta)$ and that the heat capacity $c = c(\theta) = \gamma'(\theta) = -\theta\phi''(\theta)$ in (3.16) also does not depend on mechanical variables. In our Stefan problem, the heat capacity is a distribution containing a Dirac measure at the phase-transition temperature $\theta = \theta_{\rm PT}$, cf. Figure 2:

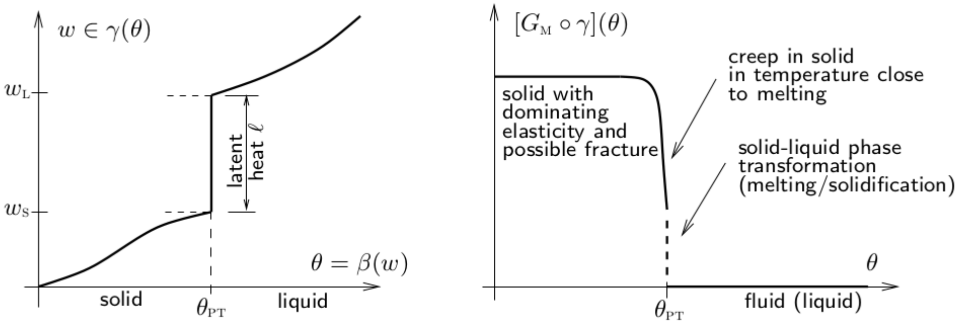

Fig. 2: A basic philosophy of the model: dependence of the enthalpy w on temperature θ in the classical Stefan problem (left) and the dependence of the Maxwellian viscosity shear modulus G_M (right). As γ is multivalued at the solid-liquid phase-transition temperature $\theta = \theta_{PT}$, the heat capacity $\gamma'(\cdot)$ contains a Dirac measure supported at $\theta = \theta_{PT}$.

The dependence on the Maxwellian modulus G_M (controlling creep effects) on temperature in the solid state as schematically depicted on Figure 2-right is in the case of ice is supported by experiments showing that when the ice temperature is close to melting, the creep effects accelerate by several orders, cf. [26, Fig.3]. Across the melting temperature, G_M falls to zero and solid ice becomes fluidic water. This discontinuity in terms of temperature is in fact possible to turn into continuity when expressed in terms of the enthalpy w, cf. Figure 3.

An example for the damage dissipation potential $\zeta(w;\dot\alpha)$ as another important ingredient of the model not explicitly depicted in Figure 1 can be

$$\zeta\big(w;\dot\alpha\big) = -\sigma_F \min(0,\dot\alpha) + A(w)\max(0,\dot\alpha)^2 + \epsilon\dot\alpha^2 \tag{4.2}$$

with $A:\mathbb{R}\to\mathbb{R}$ continuous non-increasing very large for solid regions (i.e. for $w \le w_S$) while zero for liquid regions (i.e. for $w \ge w_L$). The meaning of $\sigma_F > 0$ is a so-called fracture toughness, i.e. the energy per volume needed to damage the solid material completely, while $A = A(w) \ge 0$ influences the healing rate and, if $A \to 0$, then α tends to come fast to 1 (i.e. to undamaged material) due to the term $G_D(1-\alpha)^2/(2\kappa)$ in the stored energy in (4.1). This holds if $\epsilon = 0$, while later we will assume $\epsilon > 0$ small for the analytical reasons, cf. (6.1e) below. Thus, in liquid phase, $\alpha \sim 1$ and therefore the role of damage is naturally eliminated and solidification leads to the originally undamaged solid. In general, ζ may depend also on α itself through an α-dependence of both σ_F and A.

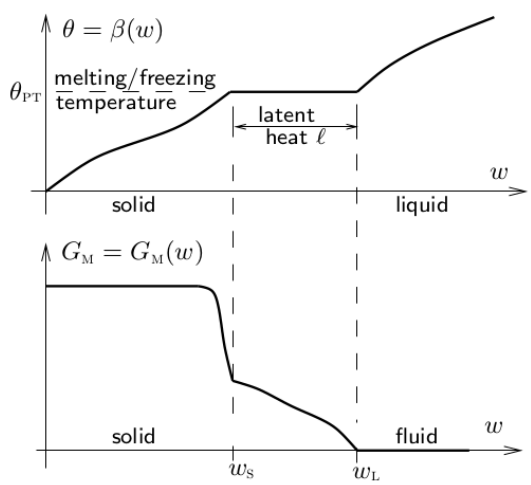

Fig. 3: Temperature θ and the Maxwellian viscous creep modulus G_M depending continuously on the enthalpy w.

Remark 4.1 (*Semi-compressible fluids II*)**.** When the deviatoric elastic response vanishes so that $\varphi(\boldsymbol{E}) = dK_{\rm E}|{\rm sph}\,\boldsymbol{E}|^2/2$, we again obtain the model for semi-compressible fluids as in Remark 2.2 except that the momentum balance (2.18b) is augmented by the higher-gradient term ${\rm div}^2(\nu|\nabla\boldsymbol{e}(\boldsymbol{v})|^{p-2}\nabla\boldsymbol{e}(\boldsymbol{v}))$, related with the mentioned 2nd-grade nonsimple fluid concept. Indeed, if $G_{\rm M}(0) = 0$, then ${\rm dev}\,\boldsymbol{E} = 0$ and the elastic stress simplifies to $\boldsymbol{S} = \varphi'_{\boldsymbol{E}}(\boldsymbol{E},\alpha) = dK_{\rm E}{\rm sph}\,\boldsymbol{E} = K_{\rm E}({\rm tr}\,\boldsymbol{E})\mathbb{I} = p\mathbb{I}$ when denoting $p = K_{\rm E}{\rm tr}\,\boldsymbol{E}$, and then (3.3b) yields $\dot{p}/K_{\rm E} = {\rm div}\,\boldsymbol{v}$, which is just (2.18a). Also, (3.3a) with omitted the internal variable α (as indeed natural in fluids) gives (2.18b). Here, the fluidic region can be, at least formally, combined with the solid regions in one "monolithic" model. Pressure (longitudial) waves can propagate thorough solid and liquid regions, being possibly refracted and partly reflected on the solid-liquid interfaces, while shear waves can propagate only in solid regions, being fully reflected on the solid-liquid interfaces or possibly partly transformed to longitudial waves in fluidic regions. Of course, these solid-liquid interfaces may evolve during solidification or melting. Moreover, the mentioned higher-gradient term likely causes an *anomalous dispersion* of these waves (like if the 2nd-grade nonsimple term would act in the stored energy, cf. [25, Remark 6.3.6]).

5 Two-phase Stefan problem involving supercooling/superheating by phase relaxation

The basic Stefan problem was enriched by various fine effects, in particular towards a kinetic supercooling and superheating (sometimes referred as undercooling and overheating) situations [10, 11, 12, 41, 46, 47], relying on that melting or solidification proceed at a rate that increases with the difference between the actual temperature and the equilibrium (= phase transformation) temperature [9]. This enrichment (also called a relaxed Stefan problem) will also bring analytical benefits especially in the context of mechanical coupling with L^1-heat sources. In particular, it allows for strong convergence of enthalpy w so that the mechanical properties may depend directly on w and may thus exhibit jumps when temperature crosses the phase-transformation threshold $\theta_{\rm PT}$. Even more important, the usual limit passage in the relation $\theta = \beta(w)$ relying on estimation of $\nabla w = \gamma'(\theta)\nabla\theta$ from an estimate of $\nabla\theta$ does not work for Stefan problem where $\gamma'(\theta)$ is unbounded while the estimate of $\frac{\partial}{\partial t}\theta$ as in [25, Remark 8.2.4] does not work.

Here, of course, we present the supercooling/superheating in the convective variant and in the context of the L^1-theory for heat-transfer problem.

Having in mind the situation from Fig. 2, i.e. the so-called two-phase Stefan problem with only one phase-transformation temperature $\theta_{\rm PT}$, we split the set-valued function γ into a continuous part and the multi-valued Heaviside function H, i.e.

$$\gamma(\theta) = \widetilde{\gamma}(\theta) + \ell H\Big(\frac{\theta}{\theta_{\rm PT}} - 1\Big) \quad \text{with} \quad H(\theta) = \begin{cases} 0 & \text{if } \ \theta < 0\,, \quad \textsf{(complete solid)} \\ [0,1] & \text{if } \ \theta = 0\,, \quad \textsf{(in between solid and fluid)} \\ 1 & \text{if } \ \theta > 0\,, \quad \textsf{(complete fluid)} \end{cases} \tag{5.1a}$$

and correspondingly we split also the heat part of the free energy and the internal energy (enthalpy) as

$$\phi(\theta) = \widetilde{\phi}(\theta) - \ell\Big(\frac{\theta}{\theta_{\rm PT}} - 1\Big)^+ \quad \text{and} \tag{5.1b}$$

$$w = \vartheta + \ell\chi \quad \text{with} \quad \vartheta = \widetilde{\gamma}(\theta) = \widetilde{\phi}(\theta) - \theta\widetilde{\phi}'(\theta) \tag{5.1c}$$

$$\text{and with } \ \chi \in H\Big(\frac{\theta}{\theta_{\rm PT}} - 1\Big), \quad \text{or equivalently } \ H^{-1}(\chi) \ni \frac{\theta}{\theta_{\rm PT}} - 1\,, \tag{5.1d}$$

where $(\cdot)^+$ denotes the positive part of its argument. Now $\widetilde{\gamma}(\cdot)$ is a continuous single-value function, in contrast to γ. The new variable χ has a meaning of a volume fraction.

The model from Sections 3–4 will now be slightly modified by "relaxing" the last inclusion in (5.1d) as

$$\omega\overset{\bullet}{\chi} + H^{-1}(\chi) \ni \Upsilon\Big(\frac{\theta}{\theta_{\rm PT}} - 1\Big) \tag{5.2}$$

with $\omega > 0$ a (typically small) relaxation time and with Υ some continuous increasing function with $\Upsilon(0) = 0$. For $\omega = 0$, we obtain exactly the Stefan model (5.1). In literature, usually $\Upsilon(\theta) = \theta$ but here we consider a general Υ with some sublinear growth, cf. (6.1d) below, which will be needed for the test (6.23) which is then needed for the strong convergence of χ's.

Let us note that we used the convective time derivative in (5.2), which is related with the attribute of the "volume-fraction" variable χ as an intensive variable taking values in $[0, 1]$. In contrast to it, $\ell\chi$ in (5.1d) in the dimension $\mathrm{J/m^3}$ is an extensive variable and can be summed up with ϑ, giving w which should be transported as w in (3.16).

Altogether, we now consider the system (3.3a-e) with ϕ from (5.1b) and, instead of (3.3f), now with w from (5.1c) completed with the volume-fraction equation (5.2).

The mechanical-energy balance (3.11) is to be obtained as before by testing the four equations/inclusion (3.3a–d) respectively by $\boldsymbol{v}$, $\boldsymbol{S}$, $\boldsymbol{\Pi}$, and $\overset{\bullet}{\alpha}$. The total-energy conservation (3.12) is again by summing it with (3.3e) tested by 1. The volume-fraction equation (5.2) is not directly included in the energetics. It also needs its own initial condition $\chi|_{t=0} = \chi_0$. Then we can prescribe the initial temperature $\theta|_{t=0} = \theta_0$ because the initial enthalpy is the uniquely determined as $w|_{t=0} = \widetilde{\gamma}(\theta_0) + \ell\chi_0$.

Remark 5.1 (*Multi-phase Stefan problem*)**.** *Actually, a generalization to more than two phases in materials undergoing several phase transformations at several phase-transformation temperatures might be routinely possible by using more than one volume fraction χ'. This occurs in metals when a recrystallization with a certain latent heat (and with changing mechanical properties) occurs still in the solid phase at temperature below an actual solid-fluid transformation (i.e. melting/solidification), e.g. iron recrystalizes at $\sim 450^\circ C$ (i.e. a Stefan problem in solid phase) while melts at $1538^\circ C$ (i.e. a Stefan problem accompained by solid-fluid transition). Rather for notational simplicity, we are considering only the most interesting scenario, i.e. an only two-phase problem with a solid-fluid transition.*

6 Analysis for incomplete melting by time discretisation

Although the semi-compressible fluids resulting after the complete melting are well amenable to analysis, the transition from solid to completely melted fluid (or reversely from the semi-compressible fluid through freezing to solid) is analytically difficult, cf. Remark 4.1. These troubles are not surprising and occur in other similar situations where some coercivity continuously degenerates like in contact mechanics (Coulomb friction). We thus perform the analysis only for incomplete melting, cf. the assumption (6.1c) below, which can be understood as a certain regularization of the desired model. It may be particularly relevant in situations when $G_{\rm M}$ drops by many orders within solid-fluid transition but anyhow stays positive; e.g. in rocks $G_{\rm M} \sim 10^{18-25}\mathrm{Pa\,s}$ while in magma $G_{\rm M} \sim 10^{1-7}\mathrm{Pa\,s}$. Note that, if $G_{\rm M} = 0$, $\boldsymbol{\Pi}$ is not controlled (except its gradient but the coefficient $\varkappa > 0$ is expected small) and $\boldsymbol{E}$ is not linked with $\boldsymbol{e}(\boldsymbol{v})$ through (3.3b) so that, naturally, $\boldsymbol{E} \sim \boldsymbol{0}$ as expected in fluids and $\boldsymbol{\Pi} \sim \boldsymbol{e}(\boldsymbol{v})$. In other words, the incomplete melting is expectedly an acceptable modelling approximation if $G_{\rm M}$ drops much below $G_{\rm V}$, cf. Figure 1, or if the flow is not too fast (i.e. not exhibiting too big shear rates). On the other hand, the nonsimple-material concept allows us to analyse even a complete damage as far as the elastic response, which otherwise is possible only in rather special rate-independent situations, cf. [6] or [27, Proposition 4.3.22]. In fact, this is possible due to the mentioned viscous response which (even if generalized for being damage dependent) is to stay always incomplete.

We will use the standard notation concerning the Lebesgue and the Sobolev spaces, namely $L^p(\Omega;\mathbb{R}^n)$ for Lebesgue measurable functions $\Omega \to \mathbb{R}^n$ whose Euclidean norm is integrable with p-power, and $W^{k,p}(\Omega;\mathbb{R}^n)$

for functions from $L^p(\Omega;\mathbb{R}^n)$ whose all derivative up to the order k have their Euclidean norm integrable with p-power. We also write briefly $H^k = W^{k,2}$. The notation 2^* will denote the exponent from the embedding $H^1(\Omega)\subset L^{2^*}(\Omega)$, i.e. $2^* = 2d/(d{-}2)$. Moreover, for a Banach space X and for $I=[0,T]$, we will use the notation $L^p(I;X)$ for the Bochner space of Bochner measurable functions $I\to X$ whose norm is in $L^p(I)$, and $H^1(I;X)$ for functions $I\to X$ whose distributional derivative is in $L^2(I;X)$. Furthermore, $C_{\rm w}(I;X)$ will denote the Banach space of weakly continuous functions $I\to X$, and $C_{{\rm w}*}(I;X)$ of weakly* continuous if X has a predual, i.e. there is X' such that $X=(X')^*$ where $(\cdot)^*$ denotes the dual space. Occasionally, we will use $L^p_{{\rm w}*}(I;X)$ the space of weakly* measurable mappings $I\to X$; recall that $L^p_{{\rm w}*}(I;X)=L^p(I;X)$ if X is separable reflexive. Also, ${\rm Meas}(\overline\Omega)=C(\overline\Omega)^*$ is a space of Borel measures on the closure $\overline\Omega$ of Ω.

We will assume, with some $\epsilon>0$ arbitrarily small, that

$$\begin{aligned}
&\varphi:\mathbb{R}^{d\times d}\times[0,1]\to\mathbb{R}\ \text{ twice continuously differentiable, bounded from below with}\\
&\qquad\forall(\boldsymbol{E},\alpha)\in\mathbb{R}^{d\times d}_{\rm sym}\times[0,1]:\quad |\varphi'_{\boldsymbol{E}}(\boldsymbol{E},\alpha)|^2\le\varphi(\boldsymbol{E},\alpha)/\epsilon,\\
&\qquad\qquad|\varphi'_\alpha(\boldsymbol{E},\alpha)|\le(1{+}|\boldsymbol{E}|^2)/\epsilon,\quad \varphi'_\alpha(\boldsymbol{E},0)\le0,\quad \varphi'_\alpha(\boldsymbol{E},1)\ge0,
\end{aligned}\tag{6.1a}$$

$$\begin{aligned}
&\widetilde\phi:\mathbb{R}\to\mathbb{R}\ \text{ continuously differentiable, }\ \widetilde\phi(0)=0\,,\\
&\qquad\widetilde\gamma:\theta\mapsto\widetilde\phi(\theta){-}\theta\widetilde\phi'(\theta)\ \text{and}\ \widetilde\gamma^{-1}(\cdot)\ \text{have at most linear growth,}
\end{aligned}\tag{6.1b}$$

$$G_{\rm M}:\mathbb{R}\to\mathbb{R}\ \text{ continuous and bounded with }\ \inf G_{\rm M}(\cdot)>0\,,\tag{6.1c}$$

$$\Upsilon:\mathbb{R}\to\mathbb{R}\ \text{ continuous, increasing, }\ \Upsilon(0)=0,\ \sup_{\mathbb{R}}\frac{|\Upsilon(\cdot)|}{1{+}|\cdot|^{1/2}}<\infty\,,\tag{6.1d}$$

$$\begin{aligned}
&\zeta:[0,1]\times\mathbb{R}\times\mathbb{R}\to\mathbb{R}\ \text{ continuous with }\ \zeta(\alpha,w;\cdot):\mathbb{R}\to\mathbb{R}\ \text{ convex and smooth on }\mathbb{R}\setminus\{0\},\\
&\qquad\epsilon\big|\dot\alpha\big|^2\le\zeta\big(\alpha,w;\dot\alpha\big)\le\big(1+\big|\dot\alpha\big|^2\big)/\epsilon\,,
\end{aligned}\tag{6.1e}$$

$$\beta:\mathbb{R}\to\mathbb{R}\ \text{ continuous, nondecreasing, with at most linear growth,}\tag{6.1f}$$

$$\mathscr{K}:\mathbb{R}^2\to\mathbb{R}\ \text{ continuous, bounded,}\quad \inf\mathscr{K}>0\,,\tag{6.1g}$$

$$p>d,\ \ \varrho,\ \kappa,\ \varkappa,\ \nu,\ \theta_{\rm PT}>0,\ \mathbb{D}\in\mathbb{R}^{d^4}\ \text{ symmetric positive definite,}\tag{6.1h}$$

$$\begin{aligned}
&\boldsymbol{v}_0\in L^2(\Omega;\mathbb{R}^d),\quad \boldsymbol{E}_0\in H^1(\Omega;\mathbb{R}^{d\times d}_{\rm sym}),\quad \alpha_0\in H^1(\Omega),\quad 0\le\alpha_0\le1,\\
&\qquad\theta_0\in L^1(\Omega),\ \ \theta_0\ge0,\ \ \chi_0\in W^{1,q}(\Omega),\ \ 0\le\chi_0\le1\ \text{ with }1\le q<(d{+}2)/(d{+}1),
\end{aligned}\tag{6.1i}$$

$$\boldsymbol{f}\in L^1(I;L^2(\Omega;\mathbb{R}^d))\,,\qquad h_{\rm ext}\in L^1(I{\times}\Gamma)\,,\qquad h_{\rm ext}\ge0\,.\tag{6.1j}$$

We have formulated our growth assumption (6.1a) to be valid for $d=3$ and $d=2$ too, but especially for the latter case they can be weakened. Let us emphasize that we do not assume φ convex, which allows for realistic damage models where φ is intentionally always nonconvex.

For our time-discretisation method used in below, we will still add a certain mild qualification of φ, cf. the Remark 6.5 below, namely that

$$\forall\varepsilon>0\ \exists\varphi_\varepsilon:\mathbb{R}^{d\times d}_{\rm sym}\times[0,1]\to\mathbb{R}\ \text{ continuously differentiable, bounded from below with}$$

$$\forall(\boldsymbol{E},\alpha)\in\mathbb{R}^{d\times d}_{\rm sym}\times[0,1]:\quad |[\varphi_\varepsilon]'_{\boldsymbol{E}}(\boldsymbol{E},\alpha)|^2\le\varphi_\varepsilon(\boldsymbol{E},\alpha)/\epsilon,\tag{6.2a}$$

$$|[\varphi_\varepsilon]'_\alpha(\boldsymbol{E},\alpha)|^2\le(1{+}|\boldsymbol{E}|^2)/\epsilon,\quad [\varphi_\varepsilon]'_\alpha(\boldsymbol{E},0)\le0,\quad [\varphi_\varepsilon]'_\alpha(\boldsymbol{E},1)\ge0,\tag{6.2b}$$

$$\exists K=K(\varepsilon):\qquad (\boldsymbol{E},\alpha)\mapsto\varphi_\varepsilon(\boldsymbol{E},\alpha)+\frac12K|\alpha|^2\ \text{ is convex,}\tag{6.2c}$$

$$\forall\,\boldsymbol{E}\in\mathbb{R}^{d\times d}_{\rm sym},\ \alpha\in[0,1]:\quad \big|\varphi_\varepsilon(\boldsymbol{E},\alpha)-\varphi(\boldsymbol{E},\alpha)\big|\le o(\varepsilon)(1+|\boldsymbol{E}|^2)\,,\tag{6.2d}$$

$$\big|[\varphi_\varepsilon]'_{\boldsymbol{E}}(\boldsymbol{E},\alpha)-\varphi'_{\boldsymbol{E}}(\boldsymbol{E},\alpha)\big|\le o(\varepsilon)(1+|\boldsymbol{E}|)\,,\quad\text{and}\tag{6.2e}$$

$$\big|[\varphi_\varepsilon]'_\alpha(\boldsymbol{E},\alpha)-\varphi'_\alpha(\boldsymbol{E},\alpha)\big|\le o(\varepsilon)(1+|\boldsymbol{E}|^2)\,.\tag{6.2f}$$

We should realize that H^{-1} in (5.2) is the normal cone to $[0,1]$, i.e. the subdifferential to the indicator function to the interval $[0,1]$, and thus (5.2) means χ is valued in $[0,1]$ and satisfies the variational inequality

$$\Big(\Upsilon\big(\frac{\theta}{\theta_{\rm PT}}-1\big)-\omega\frac{\partial\chi}{\partial t}-\omega\boldsymbol{v}{\cdot}\nabla\chi\Big)(\widetilde\chi-\chi)\ge0\tag{6.3}$$

to be valid a.e. on $I\times\Omega$ for all $\widetilde\chi$ valued in $[0,1]$. As χ is however not smooth in space/time, we have to understand (6.3) only weakly by using Green's formula $\int_\Omega \boldsymbol{v}\cdot\nabla\chi(\widetilde\chi-\chi)\,\mathrm{d}x = \int_\Omega \frac12(\mathrm{div}\,\boldsymbol{v})\chi^2-\chi\mathrm{div}(\boldsymbol{v}\widetilde\chi)\,\mathrm{d}x$ and by-part integration in time for the term $\frac{\partial}{\partial t}\chi(\widetilde\chi-\chi)$, resulting thus to (6.5c) in the following:

Definition 6.1 (Weak solutions to the system (3.3a-e)–(3.4)–(3.5) with (5.1b,c)–(5.2).)**.** *A six-tuple* $(\boldsymbol{v},\boldsymbol{E},\boldsymbol{\Pi},\alpha,\theta,\chi)$ *with*

$$\boldsymbol{v}\in C_{\rm w}(I;L^2(\Omega;\mathbb{R}^d))\cap L^p(I;W^{2,p}(\Omega;\mathbb{R}^d))\quad\text{with}\quad \tfrac{\partial}{\partial t}\boldsymbol{v}\in L^{p'}(I;W^{2,p}(\Omega;\mathbb{R}^d)^*)+L^1(I;L^2(\Omega;\mathbb{R}^d)), \tag{6.4a}$$

$$\boldsymbol{E}\in C_{\rm w}(I;H^1(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))\cap H^1(I,L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))\,, \tag{6.4b}$$

$$\boldsymbol{\Pi}\in L^2(I;H^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))\,, \tag{6.4c}$$

$$\alpha\in C_{\rm w}(I;H^1(\Omega))\cap H^1(I;L^2(\Omega))\quad\text{and}\quad 0\le\alpha\le1\ \text{ a.e. on } I\times\Omega\,, \tag{6.4d}$$

$$\theta\in C_{\rm w}(I;L^1(\Omega))\cap L^r(I;W^{1,r}(\Omega))\quad\text{with } 1\le r<\tfrac{d+2}{d+1},\ \text{ and } \theta\ge0\ \text{ a.e. on } I\times\Omega\,,\quad\text{and} \tag{6.4e}$$

$$\chi\in L^\infty(I\times\Omega)\cap H^1(I;H^1(\Omega)^*)\quad\text{with}\quad 0\le\chi\le1 \text{ a.e. on } I\times\Omega \tag{6.4f}$$

will be called a weak solution to the boundary-value problem (3.3a-e)–(3.4) *with (5.1b,c)–*(5.2) *and with the initial conditions* (3.5) *and* $\chi|_{t=0}=\chi_0$ *if* $\boldsymbol{S}=\varphi'_{\boldsymbol{E}}(\boldsymbol{E},\alpha)\in L^2(I;H^1(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))$*,* $\varphi'_\alpha(\boldsymbol{E},\alpha)\in L^2(I\times\Omega)$*,* $\Delta\alpha\in L^2(I\times\Omega)$*,* $\boldsymbol{n}\cdot\boldsymbol{v}=0$ *on* $I\times\Gamma$*,* $w=\widetilde\gamma(\theta)+\ell\chi\in C_{\rm w}(I;L^1(\Omega))$ *with* $\frac{\partial}{\partial t}w\in L^1(I;H^{d+1}(\Omega)^*)$*, and if*

$$\int_0^T\!\!\int_\Omega\Big(\varrho\Big((\boldsymbol{v}\cdot\nabla)\boldsymbol{v}+\frac12(\mathrm{div}\,\boldsymbol{v})\boldsymbol{v}\Big)\cdot\widetilde{\boldsymbol{v}}+\big(\boldsymbol{S}+\mathbb{D}\boldsymbol{e}(\boldsymbol{v})+\boldsymbol{K}\big):\boldsymbol{e}(\widetilde{\boldsymbol{v}})+\psi(\boldsymbol{E},\alpha,\theta)\mathrm{div}\widetilde{\boldsymbol{v}}$$
$$+\nu|\nabla\boldsymbol{e}(\boldsymbol{v})|^{p-2}\nabla\boldsymbol{e}(\boldsymbol{v})\,\vdots\,\nabla\boldsymbol{e}(\widetilde{\boldsymbol{v}})-\varrho\boldsymbol{v}\cdot\frac{\partial\widetilde{\boldsymbol{v}}}{\partial t}\Big)\,\mathrm{d}x\mathrm{d}t=\int_\Omega\varrho\boldsymbol{v}_0\cdot\widetilde{\boldsymbol{v}}(0)\,\mathrm{d}x+\int_0^T\!\!\int_\Omega\boldsymbol{f}\cdot\widetilde{\boldsymbol{v}}\,\mathrm{d}x\mathrm{d}t \tag{6.5a}$$

with $\boldsymbol{K}$ *from* (3.3a) *for all* $\widetilde{\boldsymbol{v}}\in H^1(I;L^2(\Omega;\mathbb{R}^d))\cap L^p(I;W^{2,p}(\Omega;\mathbb{R}^d))$ *with* $\boldsymbol{n}\cdot\widetilde{\boldsymbol{v}}=0$ *on* $I\times\Gamma$ *and* $\widetilde{\boldsymbol{v}}(T)=0$*, and if*

$$\int_0^T\!\!\int_\Omega\big(G_{\rm M}(w)\boldsymbol{\Pi}+\mathrm{dev}\,\boldsymbol{S}\big):\widetilde{\boldsymbol{\Pi}}+\varkappa\nabla\boldsymbol{\Pi}\,\vdots\,\nabla\widetilde{\boldsymbol{\Pi}}\,\mathrm{d}x\mathrm{d}t=0 \tag{6.5b}$$

holds for all $\widetilde{\boldsymbol{\Pi}}\in H^1(I\times\Omega;\mathbb{R}^{d\times d}_{\rm dev})$*, and if*

$$\int_0^T\!\!\int_\Omega\zeta(\alpha,w;\widetilde\alpha)+\varphi'_\alpha(\boldsymbol{E},\alpha)\big(\widetilde\alpha-\overset{\bullet}{\alpha}\big)+\kappa\nabla\alpha\cdot\nabla\widetilde\alpha+\kappa\Delta\alpha(\boldsymbol{v}\cdot\nabla\alpha)\,\mathrm{d}x\mathrm{d}t$$
$$+\int_\Omega\frac\kappa2|\nabla\alpha_0|^2\,\mathrm{d}x\ge\int_\Omega\frac\kappa2|\nabla\alpha(T)|^2\,\mathrm{d}x+\int_0^T\!\!\int_\Omega\zeta\big(\alpha,w;\overset{\bullet}{\alpha}\big)\,\mathrm{d}x\mathrm{d}t \tag{6.5c}$$

holds for all $\widetilde\alpha\in L^2(I;H^1(\Omega))$*, and*

$$\int_0^T\!\!\int_\Omega\big(\mathscr{K}(\alpha,w)\nabla\theta-\boldsymbol{v}\,w\big)\cdot\nabla\widetilde w-\Big(G_{\rm M}(w)|\boldsymbol{\Pi}|^2+\mathbb{D}\boldsymbol{e}(\boldsymbol{v}):\boldsymbol{e}(\boldsymbol{v})+\nu|\nabla\boldsymbol{e}(\boldsymbol{v})|^p+\phi(\theta)\mathrm{div}\,\boldsymbol{v}\Big)\widetilde w-w\frac{\partial\widetilde w}{\partial t}\,\mathrm{d}x\mathrm{d}t$$
$$=\int_\Omega\big(\widetilde\gamma(\theta_0)+\ell\chi_0\big)\widetilde w(0)\,\mathrm{d}x+\int_0^T\!\!\int_\Gamma h_{\rm ext}\widetilde w\,\mathrm{d}S\mathrm{d}t \tag{6.5d}$$

for any $\widetilde w\in W^{1,\infty}(I\times\Omega)$ *with* $\widetilde w(T)=0$*, and furthermore*

$$\int_0^T\!\!\int_\Omega\Upsilon\Big(\frac{\theta}{\theta_{\rm PT}}-1\Big)(\widetilde\chi-\chi)+\omega\chi\frac{\partial\widetilde\chi}{\partial t}-\frac\omega2(\mathrm{div}\,\boldsymbol{v})\chi^2+\omega\chi\mathrm{div}(\boldsymbol{v}\widetilde\chi)\,\mathrm{d}x\mathrm{d}t$$
$$+\int_\Omega\frac\omega2\chi_0^2-\omega\chi_0\chi(0)\,\mathrm{d}x\ge\int_\Omega\frac\omega2\chi(T)^2-\omega\chi(T)\widetilde\chi(T)\,\mathrm{d}x \tag{6.5e}$$

for any $\widetilde\chi\in W^{1,\infty}(I\times\Omega)$ *with* $0\le\widetilde\chi\le1$ *a.e. on* $I\times\Omega$*, and eventually also* $\overset{\circ}{\boldsymbol{E}}+\boldsymbol{\Pi}=\boldsymbol{e}(\boldsymbol{v})$ *holds a.e. on* $I\times\Omega$ *and* $\boldsymbol{E}(0)=\boldsymbol{E}_0$ *a.e. on* Ω*.*

Proposition 6.2. *Let* (6.1) *and* (6.2) *be valid. Then the initial-boundary-value problem* (3.3a-e)–(3.4)–(3.5) *with* (5.1b,c)–(5.2) *has a weak solution* $(\boldsymbol{v},\boldsymbol{E},\boldsymbol{\Pi},\alpha,\theta,\chi)$ *according Definition 6.1 and every such solution also satisfies the mechanical-energy balance* (3.11) *and conserves the total energy in the sense* (3.12).

Proof. For lucidity, we divide the proof into five steps.

Step 1: Approximation by time discretisation. As we need testing by convective (but not mere partial) time derivatives, using of Galerkin method would be very technical (if not just impossible). Therefore, we use the Rothe method, i.e. the fully implicit time discretisation with an equidistant partition of the time interval I with the time step $\tau>0$. This approximation is also rather technical because the damage-dependent stored energy φ is necessarily nonconvex. In addition to time discretisation, we thus use also an approximation of the stored energy φ by a semi-convex φ_ε satisfying (6.2).

We denote by $\boldsymbol{v}_{\varepsilon\tau}^k$, $\boldsymbol{E}_{\varepsilon\tau}^k$, $\boldsymbol{S}_{\varepsilon\tau}^k$ ... the approximate values of $\boldsymbol{v}$, $\boldsymbol{E}$, $\boldsymbol{S}$ ... at time $k\tau$ with $k=1,2,...,T/\tau$. We introduce a shorthand notation for the bi-linear operator

$$\boldsymbol{B}_{\rm ZJ}(\boldsymbol{v},\boldsymbol{E})=(\boldsymbol{v}\cdot\nabla)\boldsymbol{E}-\mathrm{skew}(\nabla\boldsymbol{v})\boldsymbol{E}+\boldsymbol{E}\,\mathrm{skew}(\nabla\boldsymbol{v})\,. \tag{6.6}$$

We will then use the following recursive regularized time-discrete scheme

$$\begin{aligned}
&\varrho\Big(\frac{\boldsymbol{v}_{\varepsilon\tau}^k-\boldsymbol{v}_{\varepsilon\tau}^{k-1}}{\tau}+(\boldsymbol{v}_{\varepsilon\tau}^k\cdot\nabla)\boldsymbol{v}_{\varepsilon\tau}^k\Big)=\mathrm{div}\boldsymbol{T}_{\varepsilon\tau}^k-\frac{\varrho}{2}(\mathrm{div}\,\boldsymbol{v}_{\varepsilon\tau}^k)\,\boldsymbol{v}_{\varepsilon\tau}^k+\boldsymbol{f}_\tau^k\\
&\qquad\text{with}\quad \boldsymbol{T}_{\varepsilon\tau}^k=\boldsymbol{S}_{\varepsilon\tau}^k+\big(\varphi_\varepsilon(\boldsymbol{E}_{\varepsilon\tau}^k,\alpha_{\varepsilon\tau}^k)+\phi(\theta_{\varepsilon\tau}^k)\big)\mathbb{I}+\boldsymbol{K}_{\varepsilon\tau}^k+\mathbb{D}\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k)-\mathrm{div}\big(\nu|\nabla\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k)|^{p-2}\nabla\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k)\big)\,,\\
&\qquad\text{and}\quad \boldsymbol{S}_{\varepsilon\tau}^k=[\varphi_\varepsilon]_{\boldsymbol{E}}'(\boldsymbol{E}_{\varepsilon\tau}^k,\alpha_{\varepsilon\tau}^k)\ \ \text{and}\ \ \boldsymbol{K}_{\varepsilon\tau}^k=\frac{\kappa}{2}|\nabla\alpha_{\varepsilon\tau}^k|^2\mathbb{I}-\kappa\nabla\alpha_{\varepsilon\tau}^k\otimes\nabla\alpha_{\varepsilon\tau}^k\,, && (6.7a)\\
&\frac{\boldsymbol{E}_{\varepsilon\tau}^k-\boldsymbol{E}_{\varepsilon\tau}^{k-1}}{\tau}+\boldsymbol{B}_{\rm ZJ}(\boldsymbol{v}_{\varepsilon\tau}^k,\boldsymbol{E}_{\varepsilon\tau}^k)=\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k)-\boldsymbol{\Pi}_{\varepsilon\tau}^k\,, && (6.7b)\\
&G_{\rm M}(w_{\varepsilon\tau}^{k-1})\boldsymbol{\Pi}_{\varepsilon\tau}^k=\mathrm{dev}\,\boldsymbol{S}_{\varepsilon\tau}^k+\varkappa\Delta\boldsymbol{\Pi}_{\varepsilon\tau}^k\,, && (6.7c)\\
&\partial_{\dot\alpha}\zeta\Big(\alpha_{\varepsilon\tau}^{k-1},w_{\varepsilon\tau}^{k-1};\frac{\alpha_{\varepsilon\tau}^k-\alpha_{\varepsilon\tau}^{k-1}}{\tau}+\boldsymbol{v}_{\varepsilon\tau}^k\cdot\nabla\alpha_{\varepsilon\tau}^k\Big)+[\varphi_\varepsilon]_\alpha'(\boldsymbol{E}_{\varepsilon\tau}^k,\alpha_{\varepsilon\tau}^k)+\frac{\alpha_{\varepsilon\tau}^k-\alpha_{\varepsilon\tau}^{k-1}}{\sqrt{\tau}}\ni\kappa\Delta\alpha_{\varepsilon\tau}^k\,, && (6.7d)\\
&\frac{w_{\varepsilon\tau}^k-w_{\varepsilon\tau}^{k-1}}{\tau}+\mathrm{div}\big(\boldsymbol{v}_{\varepsilon\tau}^k\,w_{\varepsilon\tau}^k-\mathscr{K}(\alpha_{\varepsilon\tau}^{k-1},w_{\varepsilon\tau}^{k-1})\nabla\theta_{\varepsilon\tau}^k\big)=G_{\rm M}(w_{\varepsilon\tau}^{k-1})|\boldsymbol{\Pi}_{\varepsilon\tau}^k|^2+\mathbb{D}\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k){:}\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k)\\
&\qquad+(1-\sqrt[4]{\tau})\xi\Big(\alpha_{\varepsilon\tau}^{k-1},w_{\varepsilon\tau}^{k-1};\frac{\alpha_{\varepsilon\tau}^k-\alpha_{\varepsilon\tau}^{k-1}}{\tau}+\boldsymbol{v}_{\varepsilon\tau}^k\cdot\nabla\alpha_{\varepsilon\tau}^k\Big)+\nu|\nabla\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k)|^p\\
&\qquad+\varkappa|\nabla\boldsymbol{\Pi}_{\varepsilon\tau}^k|^2+\phi(\theta_{\varepsilon\tau}^k)\mathrm{div}\,\boldsymbol{v}_{\varepsilon\tau}^k\qquad\text{with}\qquad w_{\varepsilon\tau}^k=\widetilde\gamma(\theta_{\varepsilon\tau}^k)+\ell\chi_{\varepsilon\tau}^k\,,\quad\text{and} && (6.7e)\\
&\omega\frac{\chi_{\varepsilon\tau}^k-\chi_{\varepsilon\tau}^{k-1}}{\tau}+\omega\boldsymbol{v}_{\varepsilon\tau}^k\cdot\nabla\chi_{\varepsilon\tau}^k+H^{-1}(\chi_{\varepsilon\tau}^k)\ni\Upsilon\Big(\frac{\theta_{\varepsilon\tau}^k}{\theta_{\rm PT}}-1\Big) && (6.7f)
\end{aligned}$$

with $\xi=\xi(\alpha,w;\dot\alpha)$ from (3.3e) and with $\widetilde\gamma$ from (5.1c). We complete the system (6.7) by the corresponding boundary conditions, i.e.

$$\begin{aligned}
&\boldsymbol{v}_{\varepsilon\tau}^k\cdot\boldsymbol{n}=0\,,\qquad\big[\boldsymbol{T}_{\varepsilon\tau}^k\boldsymbol{n}+\mathrm{div}_{\rm S}(\nu|\nabla\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k)|^{p-2}\nabla\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k)\boldsymbol{n})\big]_{\rm T}=0\,, && (6.8a)\\
&\nabla\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k){:}(\boldsymbol{n}\otimes\boldsymbol{n})=0\,,\quad(\boldsymbol{n}\cdot\nabla)\boldsymbol{\Pi}_{\varepsilon\tau}^k=0\,,\quad\nabla\alpha_{\varepsilon\tau}^k\cdot\boldsymbol{n}=0,\quad\text{and} && (6.8b)\\
&\mathscr{K}(\alpha_{\varepsilon\tau}^{k-1},w_{\varepsilon\tau}^{k-1}){:}(\nabla\theta_{\varepsilon\tau}^k\otimes\boldsymbol{n})=h_{{\rm ext},\tau}^k\,. && (6.8c)
\end{aligned}$$

Here we used $\boldsymbol{f}_\tau^k:=\int_{(k-1)\tau}^{k\tau}\boldsymbol{f}(t)\,\mathrm{d}t$ and similarly also for $\boldsymbol{g}_\tau^k$ and $h_{{\rm ext},\tau}^k$. Let us point out that the term $(\alpha_{\varepsilon\tau}^k-\alpha_{\varepsilon\tau}^{k-1})/\sqrt{\tau}$ in (6.7d) is devised to convexify φ_ε due to (6.2c) for small $\tau>0$ and vanishes in the limit. This system of boundary-value problems is to be solved recursively for $k=1,2,...,T/\tau$, starting with the initial conditions for $k=1$:

$$\boldsymbol{v}_{\varepsilon\tau}^0=\boldsymbol{v}_0,\qquad\boldsymbol{E}_{\varepsilon\tau}^0=\boldsymbol{E}_0,\qquad\alpha_{\varepsilon\tau}^0=\alpha_0,\qquad w_{\varepsilon\tau}^0=w_0\,,\quad\text{and}\quad\chi_{\varepsilon\tau}^0=\chi_0\,. \tag{6.9}$$

The existence of a weak solution $(\boldsymbol{v}_{\varepsilon\tau}^k, \boldsymbol{E}_{\varepsilon\tau}^k, \boldsymbol{\Pi}_{\varepsilon\tau}^k, \alpha_{\varepsilon\tau}^k, \chi_{\varepsilon\tau}^k, w_{\varepsilon\tau}^k) \in W^{2,p}(\Omega;\mathbb{R}^d)\times L^2(\Omega;\mathbb{R}_{\rm sym}^{d\times d})\times H^1(\Omega;\mathbb{R}_{\rm sym}^{d\times d})\times H^1(\Omega)\times L^\infty(\Omega)\times W^{1,1}(\Omega)$ of the coupled quasi-linear boundary-value problem (6.7)–(6.8) can thus be seen by a combination of the quasilinear technique for (6.7a) involving the quasilinear term ${\rm div}^2(\nu|\nabla e(\boldsymbol{v}_{\varepsilon\tau}^k)|^{p-2}\nabla e(\boldsymbol{v}_{\varepsilon\tau}^k))$, with the usual semi-linear technique for (6.7c–d), with the L^1-technique for the heat equation (6.7e), and with the set-valued inclusion (in fact a variational inequality) (6.7f) provided $\boldsymbol{v}_{\varepsilon\tau}^{k-1}$, $\boldsymbol{E}_{\varepsilon\tau}^{k-1}$, $\alpha_{\varepsilon\tau}^{k-1}$, $w_{\varepsilon\tau}^{k-1}$, and $\chi_{\varepsilon\tau}^{k-1}$ are known from the previous time step. Actually, $w_{\varepsilon\tau}^k \in W^{1,q}(\Omega)$ for any $q<d'$. Let us note that the system (6.7) is indeed fully coupled due to the convective derivatives and due to the adiabatic effects, cf. the term $\phi(\theta_{\varepsilon\tau}^k)$ in (6.7a), and it seems not possible to devise some decoupled (staggered) discrete scheme which would allow for some reasonable estimation strategy of the recursive scheme. The mentioned coercivity is a particular consequence of the a-priori estimates derived below. Thus also $\boldsymbol{\Pi}_{\varepsilon\tau}^k \in H^1(\Omega;\mathbb{R}^{d\times d})$ is obtained. Let us note that, due to the convective terms, this system does not have any potential so the rather nonconstructive Brouwer fixed-point arguments combined with the Galerkin approximation are to be used. Here we rely on strict monotonicity of the main parts of (6.7a–d) so that the approximated right-hand side of the semilinear equation (6.7e) can be shown to converge strongly in $L^1(\Omega)$. In general, one cannot expect any uniqueness of this solution.

Since we assume $[\varphi_\varepsilon]_\alpha'(\boldsymbol{E},0)\le 0$ and $[\varphi_\varepsilon]_\alpha'(\boldsymbol{E},1)\ge 0$, the damage $\alpha_{\varepsilon\tau}^k$ remains valued in $[0,1]$ on a.e. Ω provided α_0 is so. Similarly, also $\chi_{\varepsilon\tau}^k$ remains valued in $[0,1]$ on a.e. Ω due to the constraints involved in $H^{-1}(\cdot)$ in (6.7f). Moreover, $w_{\varepsilon\tau}^k\ge 0$ a.e. on Ω for at least one weak solution. To this goal, we test (6.7e) by the negative part of $\theta_{\varepsilon\tau}^k$. Here we exploit also $\overset{\bullet}{\chi}\min(0,\theta)=0$ on the discrete level (6.7f) when $\theta_{\rm PT}>0$ is taken into account, we prove that $\theta_{\varepsilon\tau}^k\ge 0$. Using also $\chi_{\varepsilon\tau}^k\ge 0$ and $\widetilde\gamma([0,+\infty))\ge 0$, then also $w_{\varepsilon\tau}^k=\widetilde\gamma(\theta_{\varepsilon\tau}^k)+\ell\chi_{\varepsilon\tau}^k\ge 0$.

Using the values $(\boldsymbol{v}_{\varepsilon\tau}^k)_{k=0}^{T/\tau}$, we define the piecewise constant and the piecewise affine interpolants respectively as

$$\bar{\boldsymbol{v}}_{\varepsilon\tau}(t):=\boldsymbol{v}_{\varepsilon\tau}^k,\quad \underline{\boldsymbol{v}}_{\varepsilon\tau}(t):=\boldsymbol{v}_{\varepsilon\tau}^{k-1},\ \text{ and }\ \boldsymbol{v}_{\varepsilon\tau}(t):=\Big(\frac{t}{\tau}-k+1\Big)\boldsymbol{v}_{\varepsilon\tau}^k+\Big(k-\frac{t}{\tau}\Big)\boldsymbol{v}_{\varepsilon\tau}^{k-1}\quad\text{for}\quad (k-1)\tau<t\le k\tau \tag{6.10}$$

for $k=0,1,...,T/\tau$. Analogously, we define also $\boldsymbol{E}_{\varepsilon\tau}$, $\bar{\boldsymbol{E}}_{\varepsilon\tau}$, $\bar{\boldsymbol{\Pi}}_{\varepsilon\tau}$, $\underline{w}_{\varepsilon\tau}$, etc. Thus, (6.7) holding a.e. on Ω for $k=1,...,T/\tau$ can be written "compactly" as

$$\varrho\Big(\frac{\partial\boldsymbol{v}_{\varepsilon\tau}}{\partial t}+(\bar{\boldsymbol{v}}_{\varepsilon\tau}\cdot\nabla)\bar{\boldsymbol{v}}_{\varepsilon\tau}\Big)={\rm div}\bar{\boldsymbol{T}}_{\varepsilon\tau}-\frac{\varrho}{2}({\rm div}\,\bar{\boldsymbol{v}}_{\varepsilon\tau})\,\bar{\boldsymbol{v}}_{\varepsilon\tau}+\boldsymbol{f}_\tau^k$$
$$\text{with}\quad \bar{\boldsymbol{T}}_{\varepsilon\tau}=\bar{\boldsymbol{S}}_{\varepsilon\tau}+\big(\varphi_\varepsilon(\bar{\boldsymbol{E}}_{\varepsilon\tau},\bar\alpha_{\varepsilon\tau})+\phi(\bar\theta_{\varepsilon\tau})\big)\mathbb{I}+\mathbb{D}\boldsymbol{e}(\bar{\boldsymbol{v}}_{\varepsilon\tau})+\bar{\boldsymbol{K}}_{\varepsilon\tau}-{\rm div}\big(\nu|\nabla e(\bar{\boldsymbol{v}}_{\varepsilon\tau})|^{p-2}\nabla e(\bar{\boldsymbol{v}}_{\varepsilon\tau})\big)\,,$$
$$\text{and}\quad \bar{\boldsymbol{S}}_{\varepsilon\tau}=[\varphi_\varepsilon]_{\boldsymbol{E}}'(\bar{\boldsymbol{E}}_{\varepsilon\tau},\bar\alpha_{\varepsilon\tau})\quad\text{and}\quad \bar{\boldsymbol{K}}_{\varepsilon\tau}=\frac{\kappa}{2}|\nabla\bar\alpha_{\varepsilon\tau}|^2\mathbb{I}-\kappa\nabla\bar\alpha_{\varepsilon\tau}\otimes\nabla\bar\alpha_{\varepsilon\tau}\,, \tag{6.11a}$$

$$\frac{\partial\boldsymbol{E}_{\varepsilon\tau}}{\partial t}+\boldsymbol{B}_{\rm ZJ}(\bar{\boldsymbol{v}}_{\varepsilon\tau},\bar{\boldsymbol{E}}_{\varepsilon\tau})=\boldsymbol{e}(\bar{\boldsymbol{v}}_{\varepsilon\tau})-\bar{\boldsymbol{\Pi}}_{\varepsilon\tau}\,, \tag{6.11b}$$

$$G_{\rm M}(\underline{w}_{\varepsilon\tau})\bar{\boldsymbol{\Pi}}_{\varepsilon\tau}={\rm dev}\,\bar{\boldsymbol{S}}_{\varepsilon\tau}+\varkappa\Delta\bar{\boldsymbol{\Pi}}_{\varepsilon\tau}\,, \tag{6.11c}$$

$$\partial_{\overset{\bullet}{\alpha}}\zeta\Big(\underline{\alpha}_{\varepsilon\tau},\underline{w}_{\varepsilon\tau};\frac{\partial\alpha_{\varepsilon\tau}}{\partial t}+\bar{\boldsymbol{v}}_{\varepsilon\tau}\cdot\nabla\bar\alpha_{\varepsilon\tau}\Big)+[\varphi_\varepsilon]_\alpha'(\bar{\boldsymbol{E}}_{\varepsilon\tau},\bar\alpha_{\varepsilon\tau})+\sqrt{\tau}\frac{\partial\alpha_{\varepsilon\tau}}{\partial t}\ni\kappa\Delta\bar\alpha_{\varepsilon\tau}\,, \tag{6.11d}$$

$$\frac{\partial w_{\varepsilon\tau}}{\partial t}+{\rm div}\big(\bar{\boldsymbol{v}}_{\varepsilon\tau}\,\bar w_{\varepsilon\tau}-\mathscr{K}(\underline{\alpha}_{\varepsilon\tau},\underline{w}_{\varepsilon\tau})\nabla\bar\theta_{\varepsilon\tau}\big)=G_{\rm M}(\underline{w}_{\varepsilon\tau})|\bar{\boldsymbol{\Pi}}_{\varepsilon\tau}|^2+\mathbb{D}\boldsymbol{e}(\bar{\boldsymbol{v}}_{\varepsilon\tau}):\boldsymbol{e}(\bar{\boldsymbol{v}}_{\varepsilon\tau})$$
$$+(1-\sqrt[4]{\tau})\xi\Big(\underline{\alpha}_{\varepsilon\tau},\underline{w}_{\varepsilon\tau};\frac{\partial\alpha_{\varepsilon\tau}}{\partial t}+\bar{\boldsymbol{v}}_{\varepsilon\tau}\cdot\nabla\bar\alpha_{\varepsilon\tau}\Big)$$
$$+\varkappa|\nabla\bar{\boldsymbol{\Pi}}_{\varepsilon\tau}|^2+\nu|\nabla\boldsymbol{e}(\bar{\boldsymbol{v}}_{\varepsilon\tau})|^p+\phi(\bar\theta_{\varepsilon\tau}){\rm div}\,\bar{\boldsymbol{v}}_{\varepsilon\tau}\quad\text{with}\quad \bar w_{\varepsilon\tau}=\widetilde\gamma(\bar\theta_{\varepsilon\tau})+\ell\bar\chi_{\varepsilon\tau}\,,\quad\text{and} \tag{6.11e}$$

$$\omega\frac{\partial\chi_{\varepsilon\tau}}{\partial t}+\omega\bar{\boldsymbol{v}}_{\varepsilon\tau}\cdot\nabla\bar\chi_{\varepsilon\tau}+H^{-1}(\bar\chi_{\varepsilon\tau})\ni\Upsilon\Big(\frac{\bar\theta_{\varepsilon\tau}}{\theta_{\rm PT}}-1\Big) \tag{6.11f}$$

holding on $I\times\Omega$ either a.e. or in a weak sense involving also the boundary conditions (6.8) which are to be written analogously in terms of the above introduced interpolants.

Step 2: A-priori estimates. The a-priori estimation is based on the energy test for the mechanical part combined with the heat problem. This means here the test of (6.7a) by $\boldsymbol{v}_{\varepsilon\tau}^k$ while using also (6.7b) tested by $\boldsymbol{S}_{\varepsilon\tau}^k = [\varphi_\varepsilon]'_{\boldsymbol{E}}(\boldsymbol{E}_{\varepsilon\tau}^k, \alpha_{\varepsilon\tau}^k)$, then the test (6.7c) by $\boldsymbol{\Pi}_{\varepsilon\tau}^k$ and the inclusion (6.7d) by $(\alpha_{\varepsilon\tau}^k - \alpha_{\varepsilon\tau}^{k-1})/\tau + \boldsymbol{v}_{\varepsilon\tau}^k \cdot \nabla \alpha_{\varepsilon\tau}^k$, and (6.7e) by 1. We thus obtain an energy-like inequality for the time-discrete approximation, but in contrast to (3.12), we still keep part of the dissipation rate like in (3.11).

More specifically, the terms related to inertia in (6.7a) uses the calculus

$$\Big(\varrho\frac{\boldsymbol{v}_{\varepsilon\tau}^k - \boldsymbol{v}_{\varepsilon\tau}^{k-1}}{\tau} + \varrho(\boldsymbol{v}_{\varepsilon\tau}^k\cdot\nabla)\boldsymbol{v}_{\varepsilon\tau}^k + \frac{\varrho}{2}(\operatorname{div}\boldsymbol{v}_{\varepsilon\tau}^k)\boldsymbol{v}_{\varepsilon\tau}^k\Big)\cdot\boldsymbol{v}_{\varepsilon\tau}^k = \frac{\varrho}{2}\frac{|\boldsymbol{v}_{\varepsilon\tau}^k|^2 - |\boldsymbol{v}_{\varepsilon\tau}^{k-1}|^2}{\tau} + \varrho(\boldsymbol{v}_{\varepsilon\tau}^k\cdot\nabla)\boldsymbol{v}_{\varepsilon\tau}^k\cdot\boldsymbol{v}_{\varepsilon\tau}^k + \frac{\varrho}{2}(\operatorname{div}\boldsymbol{v}_{\varepsilon\tau}^k)|\boldsymbol{v}_{\varepsilon\tau}^k|^2 + \tau\frac{\varrho}{2}\Big|\frac{\boldsymbol{v}_{\varepsilon\tau}^k - \boldsymbol{v}_{\varepsilon\tau}^{k-1}}{\tau}\Big|^2. \tag{6.12}$$

This holds pointwise and, when integrated over Ω, we further use also

$$\begin{aligned}\int_\Omega \varrho(\boldsymbol{v}_{\varepsilon\tau}^k\cdot\nabla)\boldsymbol{v}_{\varepsilon\tau}^k\cdot\boldsymbol{v}_{\varepsilon\tau}^k\,\mathrm{d}x &= -\int_\Omega \frac{\varrho}{2}|\boldsymbol{v}_{\varepsilon\tau}^k|^2(\operatorname{div}\boldsymbol{v}_{\varepsilon\tau}^k)\,\mathrm{d}x + \int_\Gamma \frac{\varrho}{2}|\boldsymbol{v}_{\varepsilon\tau}^k|^2(\boldsymbol{v}_{\varepsilon\tau}^k\cdot\boldsymbol{n})\,\mathrm{d}S\\ &= -\int_\Omega\Big(\frac{\varrho}{2}|\boldsymbol{v}_{\varepsilon\tau}^k|^2\mathbb{I}\Big):\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k)\,\mathrm{d}x + \int_\Gamma \frac{\varrho}{2}|\boldsymbol{v}_{\varepsilon\tau}^k|^2(\boldsymbol{v}_{\varepsilon\tau}^k\cdot\boldsymbol{n})\,\mathrm{d}S\,.\end{aligned} \tag{6.13}$$

The last term in (6.12) is non-negative and will simply be forgotten, which will give the inequality

$$\begin{aligned}&\int_\Omega\Big(\varrho\frac{\boldsymbol{v}_{\varepsilon\tau}^k - \boldsymbol{v}_{\varepsilon\tau}^{k-1}}{\tau} + \varrho(\boldsymbol{v}_{\varepsilon\tau}^k\cdot\nabla)\boldsymbol{v}_{\varepsilon\tau}^k + \frac{\varrho}{2}(\operatorname{div}\boldsymbol{v}_{\varepsilon\tau}^k)\boldsymbol{v}_{\varepsilon\tau}^k\Big)\cdot\boldsymbol{v}_{\varepsilon\tau}^k\,\mathrm{d}x \overset{(6.12)}{\geq} \int_\Omega\Big(\frac{\varrho}{2}\frac{|\boldsymbol{v}_{\varepsilon\tau}^k|^2 - |\boldsymbol{v}_{\varepsilon\tau}^{k-1}|^2}{\tau}\\ &+ \varrho(\boldsymbol{v}_{\varepsilon\tau}^k\cdot\nabla)\boldsymbol{v}_{\varepsilon\tau}^k\cdot\boldsymbol{v}_{\varepsilon\tau}^k + \frac{\varrho}{2}(\operatorname{div}\boldsymbol{v}_{\varepsilon\tau}^k)|\boldsymbol{v}_{\varepsilon\tau}^k|^2\Big)\,\mathrm{d}x \overset{(6.13)}{=} \int_\Omega\frac{\varrho}{2}\frac{|\boldsymbol{v}_{\varepsilon\tau}^k|^2 - |\boldsymbol{v}_{\varepsilon\tau}^{k-1}|^2}{\tau}\,\mathrm{d}x + \int_\Gamma\frac{\varrho}{2}|\boldsymbol{v}_{\varepsilon\tau}^k|^2(\boldsymbol{v}_{\varepsilon\tau}^k\cdot\boldsymbol{n})\,\mathrm{d}S\,.\end{aligned} \tag{6.14}$$

The last term vanishes due to the boundary condition (6.8a). The further term in (6.7a) uses the calculus

$$\begin{aligned}&\int_\Omega \operatorname{div}\boldsymbol{S}_{\varepsilon\tau}^k\cdot\boldsymbol{v}_{\varepsilon\tau}^k\,\mathrm{d}x = \int_\Gamma \boldsymbol{S}_{\varepsilon\tau}^k{:}(\boldsymbol{v}_{\varepsilon\tau}^k\otimes\boldsymbol{n})\,\mathrm{d}S - \int_\Omega \boldsymbol{S}_{\varepsilon\tau}^k{:}\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k)\,\mathrm{d}x\\ &\overset{(6.7b)}{=} \int_\Gamma \boldsymbol{S}_{\varepsilon\tau}^k{:}(\boldsymbol{v}_{\varepsilon\tau}^k\otimes\boldsymbol{n})\,\mathrm{d}S - \int_\Omega \boldsymbol{S}_{\varepsilon\tau}^k{:}\Big(\frac{\boldsymbol{E}_{\varepsilon\tau}^k - \boldsymbol{E}_{\varepsilon\tau}^{k-1}}{\tau} + \boldsymbol{B}_{\mathrm{ZJ}}(\boldsymbol{v}_{\varepsilon\tau}^k, \boldsymbol{E}_{\varepsilon\tau}^k) - \boldsymbol{\Pi}_{\varepsilon\tau}^k\Big)\,\mathrm{d}x\\ &\overset{(6.7a,c)}{=} \int_\Gamma \boldsymbol{S}_{\varepsilon\tau}^k{:}(\boldsymbol{v}_{\varepsilon\tau}^k\otimes\boldsymbol{n})\,\mathrm{d}S + \int_\Omega\bigg(G_{\mathrm{M}}(w_{\varepsilon\tau}^{k-1})|\boldsymbol{\Pi}_{\varepsilon\tau}^k|^2 + \varkappa|\nabla\boldsymbol{\Pi}_{\varepsilon\tau}^k|^2\\ &\qquad - [\varphi_\varepsilon]'_{\boldsymbol{E}}(\boldsymbol{E}_{\varepsilon\tau}^k, \alpha_{\varepsilon\tau}^k){:}\Big(\frac{\boldsymbol{E}_{\varepsilon\tau}^k - \boldsymbol{E}_{\varepsilon\tau}^{k-1}}{\tau} + \boldsymbol{B}_{\mathrm{ZJ}}(\boldsymbol{v}_{\varepsilon\tau}^k, \boldsymbol{E}_{\varepsilon\tau}^k)\Big)\bigg)\,\mathrm{d}x\,,\end{aligned} \tag{6.15}$$

where we used also (6.7c) tested by $\boldsymbol{\Pi}_{\varepsilon\tau}^k$.

Testing (6.7d) by the discrete convective derivative of α, we obtain

$$\begin{aligned}\int_\Omega\bigg(\xi\Big(\alpha_{\varepsilon\tau}^{k-1}, w_{\varepsilon\tau}^{k-1}; \frac{\alpha_{\varepsilon\tau}^k - \alpha_{\varepsilon\tau}^{k-1}}{\tau} + \boldsymbol{v}_{\varepsilon\tau}^k\cdot\nabla\alpha_{\varepsilon\tau}^k\Big) + \Big([\varphi_\varepsilon]'_\alpha(\boldsymbol{E}_{\varepsilon\tau}^k, \alpha_{\varepsilon\tau}^k) + \frac{\alpha_{\varepsilon\tau}^k - \alpha_{\varepsilon\tau}^{k-1}}{\sqrt{\tau}}\Big)\frac{\alpha_{\varepsilon\tau}^k - \alpha_{\varepsilon\tau}^{k-1}}{\tau}&\\ + \frac{\kappa}{2\tau}|\nabla\alpha_{\varepsilon\tau}^k|^2\bigg)\,\mathrm{d}x \leq \int_\Omega \frac{\kappa}{2\tau}|\nabla\alpha_{\varepsilon\tau}^{k-1}|^2 - \Big(\Delta\alpha_{\varepsilon\tau}^k + \frac{\alpha_{\varepsilon\tau}^k - \alpha_{\varepsilon\tau}^{k-1}}{\sqrt{\tau}}\Big)\boldsymbol{v}_{\varepsilon\tau}^k\cdot\nabla\alpha_{\varepsilon\tau}^k\,.&\end{aligned} \tag{6.16}$$

For summing the φ-terms in (6.15) and (6.16), we use the semi-convexity (6.2c) of φ_ε to estimate

$$\begin{aligned}&\boldsymbol{S}_\tau^k{:}\frac{\boldsymbol{E}_{\varepsilon\tau}^k - \boldsymbol{E}_{\varepsilon\tau}^{k-1}}{\tau} + \Big([\varphi_\varepsilon]'_\alpha(\boldsymbol{E}_{\varepsilon\tau}^k, \alpha_{\varepsilon\tau}^k) + \frac{\alpha_{\varepsilon\tau}^k - \alpha_{\varepsilon\tau}^{k-1}}{\sqrt{\tau}}\Big)\frac{\alpha_{\varepsilon\tau}^k - \alpha_{\varepsilon\tau}^{k-1}}{\tau}\\ &= [\varphi_\varepsilon]'_{\boldsymbol{E}}(\boldsymbol{E}_{\varepsilon\tau}^k, \alpha_{\varepsilon\tau}^k){:}\frac{\boldsymbol{E}_{\varepsilon\tau}^k - \boldsymbol{E}_{\varepsilon\tau}^{k-1}}{\tau} + \Big([\varphi_\varepsilon]'_\alpha(\boldsymbol{E}_{\varepsilon\tau}^k, \alpha_{\varepsilon\tau}^k) + \frac{\alpha_{\varepsilon\tau}^k}{\sqrt{\tau}}\Big)\frac{\alpha_{\varepsilon\tau}^k - \alpha_{\varepsilon\tau}^{k-1}}{\tau} - \frac{\alpha_{\varepsilon\tau}^{k-1}}{\sqrt{\tau}}\,\frac{\alpha_{\varepsilon\tau}^k - \alpha_{\varepsilon\tau}^{k-1}}{\tau}\\ &\geq \frac{\varphi_\varepsilon(\boldsymbol{E}_{\varepsilon\tau}^k, \alpha_{\varepsilon\tau}^k) - \varphi_\varepsilon(\boldsymbol{E}_{\varepsilon\tau}^{k-1}, \alpha_{\varepsilon\tau}^{k-1})}{\tau} + \frac{1}{2\sqrt{\tau}}\,\frac{|\alpha_{\varepsilon\tau}^k|^2 - |\alpha_{\varepsilon\tau}^{k-1}|^2}{\tau} - \frac{\alpha_{\varepsilon\tau}^{k-1}}{\sqrt{\tau}}\,\frac{\alpha_{\varepsilon\tau}^k - \alpha_{\varepsilon\tau}^{k-1}}{\tau}\end{aligned}$$

$$= \frac{\varphi_\varepsilon(\boldsymbol{E}_{\varepsilon\tau}^k,\alpha_{\varepsilon\tau}^k)-\varphi_\varepsilon(\boldsymbol{E}_{\varepsilon\tau}^{k-1},\alpha_{\varepsilon\tau}^{k-1})}{\tau} - \frac{\sqrt{\tau}}{2}\Big|\frac{\alpha_{\varepsilon\tau}^k-\alpha_{\varepsilon\tau}^{k-1}}{\tau}\Big|^2, \tag{6.17}$$

cf. also the calculation in [34, Remark 8.24]. This holds a.e. on Ω and is to be integrated over Ω. For the remaining two convective terms arising from these tests, we use the calculus

$$\int_\Omega\Big([\varphi_\varepsilon]'_{\boldsymbol{E}}(\boldsymbol{E}_{\varepsilon\tau}^k,\alpha_{\varepsilon\tau}^k){:}(\boldsymbol{v}_{\varepsilon\tau}^k{\cdot}\nabla)\boldsymbol{E}_{\varepsilon\tau}^k+[\varphi_\varepsilon]'_\alpha(\boldsymbol{E}_{\varepsilon\tau}^k,\alpha_{\varepsilon\tau}^k){\cdot}(\boldsymbol{v}_{\varepsilon\tau}^k{\cdot}\nabla\alpha_{\varepsilon\tau}^k)\Big)\,\mathrm{d}x=\int_\Omega\nabla\varphi_\varepsilon(\boldsymbol{E}_{\varepsilon\tau}^k,\alpha_{\varepsilon\tau}^k){\cdot}\boldsymbol{v}_{\varepsilon\tau}^k\,\mathrm{d}x$$
$$=\int_\Gamma\varphi_\varepsilon(\boldsymbol{E}_{\varepsilon\tau}^k,\alpha_{\varepsilon\tau}^k)\boldsymbol{v}_{\varepsilon\tau}^k\cdot\boldsymbol{n}\,\mathrm{d}S-\int_\Omega\varphi_\varepsilon(\boldsymbol{E}_{\varepsilon\tau}^k,\alpha_{\varepsilon\tau}^k)\mathrm{div}\,\boldsymbol{v}_{\varepsilon\tau}^k\,\mathrm{d}x=-\int_\Gamma\varphi_\varepsilon(\boldsymbol{E}_{\varepsilon\tau}^k,\alpha_{\varepsilon\tau}^k)\mathbb{I}{:}\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k)\,\mathrm{d}x\,, \tag{6.18}$$

which cancels with the pressure-type stress contribution $\varphi_\varepsilon(\boldsymbol{E}_{\varepsilon\tau}^k,\alpha_{\varepsilon\tau}^k)\mathbb{I}$, cf. (3.10). Noteworthy, by the test of the regularizing term $(\alpha_{\varepsilon\tau}^k{-}\alpha_{\varepsilon\tau}^{k-1})/\sqrt{\tau}$ by the convective time difference $(\alpha_{\varepsilon\tau}^k{-}\alpha_{\varepsilon\tau}^{k-1})/\tau+\boldsymbol{v}_{\varepsilon\tau}^k{\cdot}\nabla\alpha_{\varepsilon\tau}^k$, as in [39], we obtain still the term

$$\int_\Omega\frac{\alpha_{\varepsilon\tau}^k-\alpha_{\varepsilon\tau}^{k-1}}{\sqrt{\tau}}\,\boldsymbol{v}_{\varepsilon\tau}^k{\cdot}\nabla\alpha_{\varepsilon\tau}^k\,\mathrm{d}x=\sqrt{\tau}\int_\Omega\frac{\alpha_{\varepsilon\tau}^k-\alpha_{\varepsilon\tau}^{k-1}}{\tau}\,\boldsymbol{v}_{\varepsilon\tau}^k{\cdot}\nabla\alpha_{\varepsilon\tau}^k\,\mathrm{d}x$$
$$=\sqrt{\tau}\int_\Omega\frac{\alpha_{\varepsilon\tau}^k-\alpha_{\varepsilon\tau}^{k-1}}{\tau}\Big(\frac{\alpha_{\varepsilon\tau}^k-\alpha_{\varepsilon\tau}^{k-1}}{\tau}+\boldsymbol{v}_{\varepsilon\tau}^k{\cdot}\nabla\alpha_{\varepsilon\tau}^k\Big)-\Big|\frac{\alpha_{\varepsilon\tau}^k-\alpha_{\varepsilon\tau}^{k-1}}{\tau}\Big|^2\,\mathrm{d}x$$
$$\le\frac{\sqrt{\tau}}{2}\Big\|\frac{\alpha_{\varepsilon\tau}^k-\alpha_{\varepsilon\tau}^{k-1}}{\tau}+\boldsymbol{v}_{\varepsilon\tau}^k{\cdot}\nabla\alpha_{\varepsilon\tau}^k\Big\|_{L^2(\Omega)}^2-\frac{\sqrt{\tau}}{2}\Big\|\frac{\alpha_{\varepsilon\tau}^k-\alpha_{\varepsilon\tau}^{k-1}}{\tau}\Big\|_{L^2(\Omega)}^2. \tag{6.19}$$

The last term advantageously absorbs the last term in (6.17) while the penultimate term in (6.19) can be absorbed for sufficiently small τ's in the dissipation term $\xi(\alpha_{\varepsilon\tau}^k,w_{\varepsilon\tau}^k;\cdot)$ later in (6.20).

The other stress contributions, i.e. $\boldsymbol{K}_{\varepsilon\tau}^k{+}\mathbb{D}\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k)-\mathrm{div}(\nu|\nabla\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k)|^{p-2}\nabla\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k))$, can be treated by adapting the calculus (3.8) and (3.9).

Altogether, after summation over $k=1,2,....$, we obtain (2.13) as an upper estimate up to an error term which is small for $\tau>0$ small, so that it can be used for a-priori estimates. More specifically, we thus obtain the discrete mechanical-energy balance as an inequality

$$\int_\Omega\frac{\varrho}{2}|\boldsymbol{v}_{\varepsilon\tau}^k|^2+\varphi_\varepsilon(\boldsymbol{E}_{\varepsilon\tau}^k,\alpha_{\varepsilon\tau}^k)+\frac{\kappa}{2}|\nabla\alpha_{\varepsilon\tau}^k|^2\,\mathrm{d}x+\tau\int_\Omega\bigg(G_{\rm M}(w_{\varepsilon\tau}^{k-1})|\boldsymbol{\Pi}_{\varepsilon\tau}^k|^2+\mathbb{D}\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k){:}\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k)$$
$$+\nu|\nabla\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k)|^p+\xi\Big(\alpha_{\varepsilon\tau}^{k-1},w_{\varepsilon\tau}^{k-1};\frac{\alpha_{\varepsilon\tau}^k-\alpha_{\varepsilon\tau}^{k-1}}{\tau}+\boldsymbol{v}_{\varepsilon\tau}^k{\cdot}\nabla\alpha_{\varepsilon\tau}^k\Big)+\varkappa|\nabla\boldsymbol{\Pi}_{\varepsilon\tau}^k|^2\bigg)\,\mathrm{d}x$$
$$\le\int_\Omega\frac{\varrho}{2}|\boldsymbol{v}_{\varepsilon\tau}^{k-1}|^2+\varphi_\varepsilon(\boldsymbol{E}_{\varepsilon\tau}^{k-1},\alpha_{\varepsilon\tau}^{k-1})+\frac{\kappa}{2}|\nabla\alpha_{\varepsilon\tau}^{k-1}|^2+\tau\boldsymbol{f}_\tau^k{\cdot}\boldsymbol{v}_{\varepsilon\tau}^k-\tau\phi(\theta_{\varepsilon\tau}^k)\mathrm{div}\boldsymbol{v}_{\varepsilon\tau}^k\,\mathrm{d}x$$
$$+\frac{\sqrt{\tau}}{2}\Big\|\frac{\alpha_{\varepsilon\tau}^k-\alpha_{\varepsilon\tau}^{k-1}}{\tau}+\boldsymbol{v}_{\varepsilon\tau}^k{\cdot}\nabla\alpha_{\varepsilon\tau}^k\Big\|_{L^2(\Omega)}^2. \tag{6.20}$$

Then we test the discrete heat-transfer equation (6.7e) by 1 and add (6.20), we can cancel the adiabatic terms $\pm\tau\phi(\theta_{\varepsilon\tau}^k)\mathrm{div}\boldsymbol{v}_{\varepsilon\tau}^k$. Summing it for $k=1,...,l$ with $l\le T/\tau$, we obtain a discrete energy-like inequality:

$$\int_\Omega\frac{\varrho}{2}|\boldsymbol{v}_{\varepsilon\tau}^l|^2+\varphi_\varepsilon(\boldsymbol{E}_{\varepsilon\tau}^l,\alpha_{\varepsilon\tau}^l)+\frac{\kappa}{2}|\nabla\alpha_{\varepsilon\tau}^l|^2+w_{\varepsilon\tau}^l\,\mathrm{d}x\le\int_\Omega\frac{\varrho}{2}|\boldsymbol{v}_0|^2+\varphi_\varepsilon(\boldsymbol{E}_0,\alpha_0)+\frac{\kappa}{2}|\nabla\alpha_0|^2+w_0\,\mathrm{d}x$$
$$+\tau\sum_{k=1}^l\Big(\int_\Omega\boldsymbol{f}_\tau^k{\cdot}\boldsymbol{v}_{\varepsilon\tau}^k\,\mathrm{d}x+\int_\Gamma h_{\mathrm{ext},\tau}^k\mathrm{d}S\Big) \tag{6.21}$$

provided $\tau\le 2^8\epsilon^4$ with ϵ from (6.1e); here the smallness of τ allows for absorption of the last term in (6.20) when using the uniform convexity of the rest of the dissipation potential $\sqrt[4]{\tau}\zeta(\alpha,w;\cdot)$ which also implies the uniform convexity of the corresponding dissipation rate $\sqrt[4]{\tau}\xi(\alpha,w;\cdot)$. This reveals why we used the factor $1{-}\sqrt[4]{\tau}$ in (6.7e).

Then, by the discrete Gronwall inequality, we obtain the a-priori estimates

$$\|\boldsymbol{v}_{\varepsilon\tau}\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^d))} \le C, \tag{6.22a}$$

$$\|\boldsymbol{S}_{\varepsilon\tau}\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{d\times d}_{\rm sym}))} \le C\,, \tag{6.22b}$$

$$\|\boldsymbol{\Pi}_{\varepsilon\tau}\|_{L^2(I;H^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))} \le C\,. \tag{6.22c}$$

$$\|\alpha_{\varepsilon\tau}\|_{L^\infty(I;H^1(\Omega))\,\cap\, L^\infty(I\times\Omega)} \le C, \tag{6.22d}$$

$$\|w_{\varepsilon\tau}\|_{L^\infty(I;L^1(\Omega))} \le C \quad\text{and}\quad \|\theta_{\varepsilon\tau}\|_{L^\infty(I;L^1(\Omega))} \le C\,. \tag{6.22e}$$

Actually, from (6.21) we can see $\|\varphi_\varepsilon(\overline{\boldsymbol{E}}_{\varepsilon\tau},\overline{\alpha}_{\varepsilon\tau})\|_{L^\infty(I;L^1(\Omega))}\le C$ and then from (6.2a) we obtain the estimate $\|[\varphi_\varepsilon]'_{\boldsymbol{E}}(\overline{\boldsymbol{E}}_{\varepsilon\tau},\overline{\alpha}_{\varepsilon\tau})\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{d\times d}))}\le C$, i.e. (6.22b).

Furthermore, we test the volume-fraction equation (6.11f) by the convective derivative of χ, i.e. in the discrete form by $\frac{\partial}{\partial t}\chi_{\varepsilon\tau}+\overline{\boldsymbol{v}}_{\varepsilon\tau}{\cdot}\nabla\overline{\chi}_{\varepsilon\tau}$. We obtain

$$\int_\Omega \omega\Big|\frac{\partial\chi_{\varepsilon\tau}}{\partial t}+\overline{\boldsymbol{v}}_{\varepsilon\tau}{\cdot}\nabla\overline{\chi}_{\varepsilon\tau}\Big|^2 + H^{-1}(\overline{\chi}_{\varepsilon\tau})\frac{\partial\chi_{\varepsilon\tau}}{\partial t}+H^{-1}(\overline{\chi}_{\varepsilon\tau})\overline{\boldsymbol{v}}_{\varepsilon\tau}{\cdot}\nabla\overline{\chi}_{\varepsilon\tau}\,\mathrm{d}x$$
$$=\int_\Omega \Upsilon\Big(\frac{\overline{\theta}_{\varepsilon\tau}}{\theta_{\rm PT}}-1\Big)\Big(\frac{\partial\chi_{\varepsilon\tau}}{\partial t}+\overline{\boldsymbol{v}}_{\varepsilon\tau}{\cdot}\nabla\overline{\chi}_{\varepsilon\tau}\Big)\,\mathrm{d}x\,. \tag{6.23}$$

Realizing that $H^{-1}(\overline{\chi}_{\varepsilon\tau})\frac{\partial}{\partial t}\chi_{\varepsilon\tau}=\frac{\partial}{\partial t}\delta_{[0,1]}(\chi_{\varepsilon\tau})=0$ and $H^{-1}(\overline{\chi}_{\varepsilon\tau})\overline{\boldsymbol{v}}_{\varepsilon\tau}{\cdot}\nabla\overline{\chi}_{\varepsilon\tau}=\overline{\boldsymbol{v}}_{\varepsilon\tau}{\cdot}\nabla\delta_{[0,1]}(\overline{\chi}_{\varepsilon\tau})=0$ because $\overline{\chi}_{\varepsilon\tau}$ is valued in $[0,1]$ so that $\delta_{[0,1]}(\overline{\chi}_{\varepsilon\tau})=0$, we obtain that $\frac{\partial}{\partial t}\chi_{\varepsilon\tau}{+}\overline{\boldsymbol{v}}_{\varepsilon\tau}{\cdot}\nabla\overline{\chi}_{\varepsilon\tau}$ bounded in $L^2(I{\times}\Omega)$. Here we also used that $\Upsilon(\overline{\theta}_{\varepsilon\tau}/\theta_{\rm PT}{-}1)$ is surely bounded in $L^2(I{\times}\Omega)$ due to the already proved bound for $\overline{\theta}_{\varepsilon\tau}\in L^\infty(I;L^1(\Omega))$ and due to the growth assumption (6.1d).

As the next step, we use the L^1-technique to estimate of temperature gradient developed by Boccardo and Gallouët [5] exploiting sophistically Gagliardo-Nirenberg inequality, cf. [25, Prop. 8.2.1]. The essence is to test the heat-transfer equation (6.11e) by a smoothened the Heaviside function H from (5.1a), say $\omega(\theta):=1-(1{+}\theta)^{-\epsilon}$ for $\epsilon>0$, as suggested in [16]. The modification in comparison with the usual "heat operator" in the form $\frac{\partial}{\partial t}\vartheta-\mathrm{div}(\mathscr{K}\nabla\theta)$ with $\vartheta=\widetilde{\gamma}(\theta)$ and with an L^1-right-hand side consists in that $\dot{w}$ contains two other terms more, namely $\dot{w}=\frac{\partial}{\partial t}\widetilde{\gamma}(\theta)+\ell\dot{\chi}+\mathrm{div}(\widetilde{\gamma}(\theta)\boldsymbol{v})$. In the discrete form, these additional terms are $\ell(\frac{\partial}{\partial t}\chi_{\varepsilon\tau}{+}\overline{\boldsymbol{v}}_{\varepsilon\tau}{\cdot}\nabla\overline{\chi}_{\varepsilon\tau})+\mathrm{div}(\widetilde{\gamma}(\overline{\theta}_{\varepsilon\tau})\overline{\boldsymbol{v}}_{\varepsilon\tau})$. As for $\ell(\frac{\partial}{\partial t}\chi_{\varepsilon\tau}{+}\overline{\boldsymbol{v}}_{\varepsilon\tau}{\cdot}\nabla\overline{\chi}_{\varepsilon\tau})$, we have it already estimated in $L^2(I{\times}\Omega)$ and it remains so after being tested by $\omega(\overline{\theta}_{\varepsilon\tau})$. As for the convective term $\mathrm{div}(\widetilde{\gamma}(\overline{\theta}_{\varepsilon\tau})\overline{\boldsymbol{v}}_{\varepsilon\tau})$ tested by $\omega(\overline{\theta}_{\varepsilon\tau})$, we can estimate it "on the right-hand side" as

$$-\int_\Omega \mathrm{div}(\widetilde{\gamma}(\overline{\theta}_{\varepsilon\tau})\overline{\boldsymbol{v}}_{\varepsilon\tau})\omega(\overline{\theta}_{\varepsilon\tau})\,\mathrm{d}x=\int_\Omega \overline{\boldsymbol{v}}_{\varepsilon\tau}{\cdot}\widetilde{\gamma}(\overline{\theta}_{\varepsilon\tau})\omega'(\overline{\theta}_{\varepsilon\tau})\nabla\overline{\theta}_{\varepsilon\tau}\,\mathrm{d}x$$
$$\le\int_\Omega \frac{1}{\epsilon}|\overline{\boldsymbol{v}}_{\varepsilon\tau}|^2\widetilde{\gamma}(\overline{\theta}_{\varepsilon\tau})^2\omega'(\overline{\theta}_{\varepsilon\tau})\,\mathrm{d}x+\epsilon\int_\Omega \omega'(\overline{\theta}_{\varepsilon\tau})|\nabla\overline{\theta}_{\varepsilon\tau}|^2\,\mathrm{d}x\,. \tag{6.24}$$

As $\widetilde{\gamma}(\theta)=\mathscr{O}(\theta)$ while $\omega'(\theta)=\mathscr{O}(1/\theta)$, we have $[\widetilde{\gamma}^2\omega'](\theta)=\mathscr{O}(\theta)$ so that $\widetilde{\gamma}(\overline{\theta}_{\varepsilon\tau})^2\omega'(\overline{\theta}_{\varepsilon\tau})$ is bounded in $L^\infty(I;L^1(\Omega))$ while $|\overline{\boldsymbol{v}}_{\varepsilon\tau}|^2$ is surely bounded in $L^{p/2}_{\rm w*}(I;L^\infty(\Omega))$, so that the integrand in the penultimate integral is bounded in $L^{p/2}(I;L^1(\Omega))$. For the last integral, this is exactly fitted with the estimation in the L^1-theory, and for $\epsilon>0$ sufficiently small can be absorbed in the respective estimation, cf. [25, Formula (8.2.17)]. Using also the already obtained estimate (6.22e), the resulted "prefabricated" estimate is that $\|\theta_{\varepsilon\tau}\|^s_{L^s(I\times\Omega)}$ and $\|\nabla\theta_{\varepsilon\tau}\|^r_{L^r(I\times\Omega;\mathbb{R}^d)}$ is estimated by the $L^1(I{\times}\Omega)$-norm of the right-hand side of (6.11e) provided $s<1+2/d$ and $r<(d{+}2)/(d{+}1)$, cf. [25, Formula (8.2.6b)].

Then, to estimate the dissipation rate, we come back to (6.20) and estimate $|\int_\Omega\phi(\theta^k_{\varepsilon\tau})\mathrm{div}\boldsymbol{v}^k_{\varepsilon\tau}\,\mathrm{d}x|\le C\|\theta^k_{\varepsilon\tau}\|_{L^1(\Omega)}\|\boldsymbol{v}^k_{\varepsilon\tau}\|_{W^{2,p}(\Omega;\mathbb{R}^d)}$, exploiting $p>d$ and (6.22e). Actually, we have now even a better estimate of $\theta^k_{\varepsilon\tau}$ than $L^1(\Omega)$ but we do not need it.

To summarize, in addition to (6.22), we have proved

$$\|\boldsymbol{v}_{\varepsilon\tau}\|_{L^p(I;W^{2,p}(\Omega;\mathbb{R}^d))} \le C, \tag{6.25a}$$

$$\|\alpha_{\varepsilon\tau}\|_{H^1(I;L^2(\Omega))} \le C, \tag{6.25b}$$

$$\|\overline{\theta}_{\varepsilon\tau}\|_{L^r(I;W^{1,r}(\Omega))} \le C_r \qquad \text{with } \ 1\le r<\frac{d+2}{d+1}, \ \text{ and} \tag{6.25c}$$

$$\|\overline{\vartheta}_{\varepsilon\tau}\|_{L^r(I;W^{1,r}(\Omega))\,\cap\, L^s(I\times\Omega)} \le C_{r,s} \qquad \text{with } \ 1\le s<1+\frac{2}{d}. \tag{6.25d}$$

The L^r-estimate (6.25d) of $\nabla\overline{\vartheta}_{\varepsilon\tau}$ can be read from (6.25c) due to $\nabla\overline{\vartheta}_{\varepsilon\tau} = \nabla\widetilde{\gamma}(\overline{\theta}_{\varepsilon\tau}) = \widetilde{\gamma}'(\overline{\theta}_{\varepsilon\tau})\nabla\overline{\theta}_{\varepsilon\tau}$. The L^s-estimate of $\overline{\vartheta}_{\varepsilon\tau}$ is to be read by the Gagliardo-Nirenberg interpolation of the first estimate in (6.25d) with (6.22e). For (6.25b), we use the bound for $\frac{\partial}{\partial t}\alpha_{\varepsilon\tau} + \bar{\boldsymbol{v}}_{\varepsilon\tau}\cdot\nabla\bar{\alpha}_{\varepsilon\tau}$ in $L^2(I\times\Omega)$ obtained from (6.21) and that $\bar{\boldsymbol{v}}_{\varepsilon\tau}\cdot\nabla\bar{\alpha}_{\varepsilon\tau}$ is bounded in $L^2(I\times\Omega)$ due to (6.22a) with the former part of (6.25b). By comparison from $\Delta\bar{\alpha}_{\varepsilon\tau} \in (\partial_{\dot\alpha}\zeta(\underline{\alpha}_{\varepsilon\tau},\underline{w}_{\varepsilon\tau};\frac{\partial}{\partial t}\alpha_{\varepsilon\tau}+\bar{\boldsymbol{v}}_{\varepsilon\tau}\cdot\nabla\bar{\alpha}_{\varepsilon\tau}) + [\varphi_\varepsilon]'_\alpha(\overline{\boldsymbol{E}}_{\varepsilon\tau},\bar{\alpha}_{\varepsilon\tau}) + \tau\frac{\partial}{\partial t}\alpha_{\varepsilon\tau})/\kappa$, we also obtain

$$\|\Delta\bar{\alpha}_{\varepsilon\tau}\|_{L^2(I\times\Omega)} \le C\,. \tag{6.26}$$

An important attribute of the model is that the convective transport of variables via the velocity field $\boldsymbol{v}\in L^1_{\mathrm{w}*}(I;W^{1,\infty}(\Omega;\mathbb{R}^d))$ or, in the discrete variant by $\overline{\boldsymbol{v}}_\tau$ bounded in $L^1(I;W^{1,\infty}(\Omega;\mathbb{R}^d))$, qualitatively well copies regularity properties of the initial conditions. We use this phenomenon particularly for the Zaremba-Jaumann time difference of the elastic strain $\boldsymbol{E}=[E_{ij}]$ and also for the convective time-difference of the volume fraction χ. Let us consider a general tensor-valued source $\boldsymbol{F}\in L^2(I\times\Omega;\mathbb{R}^{d\times d})$ for $\overset{\circ}{\boldsymbol{E}}=\boldsymbol{F}$, i.e. in the difference variant

$$\frac{\boldsymbol{E}^k_\tau-\boldsymbol{E}^{k-1}_\tau}{\tau} + B_{\rm ZJ}(\boldsymbol{v}^k_\tau,\boldsymbol{E}^k_\tau) = \boldsymbol{F}^k_\tau\,. \tag{6.27}$$

For $\sigma>1$, we use the following calculus exploiting the Green formula with the boundary condition $\boldsymbol{v}\cdot\boldsymbol{n}=0$:

$$\int_\Omega(\boldsymbol{v}\cdot\nabla z)|z|^{\sigma-2}z\,\mathrm{d}x = \int_\Gamma|z|^\sigma(\boldsymbol{v}\cdot\boldsymbol{n})\,\mathrm{d}S - \int_\Omega(\sigma{-}1)|z|^{\sigma-2}z(\boldsymbol{v}\cdot\nabla z) + (\mathrm{div}\,\boldsymbol{v})|z|^\sigma\mathrm{d}x = -\frac{1}{\sigma}\int_\Omega(\mathrm{div}\,\boldsymbol{v})|z|^\sigma\mathrm{d}x\,. \tag{6.28}$$

We test (6.27) by $|\boldsymbol{E}^k_\tau|^{\sigma-2}\boldsymbol{E}^k_\tau$, which gives

$$\frac{1}{\sigma}\int_\Omega\frac{|\boldsymbol{E}^k_\tau|^\sigma-|\boldsymbol{E}^{k-1}_\tau|^\sigma}{\tau}\,\mathrm{d}x \le \int_\Omega\frac{\mathrm{div}\,\boldsymbol{v}^k_\tau}{\sigma}|\boldsymbol{E}^k_\tau|^\sigma + 2|\mathrm{skew}(\nabla\boldsymbol{v}^k_\tau)|\,|\boldsymbol{E}^k_\tau|^\sigma + |\boldsymbol{E}^k_\tau|^{\sigma-2}\boldsymbol{E}^k_\tau{:}\boldsymbol{F}^k_\tau\,\mathrm{d}x\,, \tag{6.29}$$

where we used (6.28) for each component $z=E_{ij}$. From (6.29), by the Young and the discrete Gronwall inequalities, we obtain the estimate

$$\|\boldsymbol{E}^k_\tau\|^\sigma_{L^\sigma(\Omega;\mathbb{R}^{d\times d})} \le C\mathrm{e}^{1+2k\tau\max_{l=1,\dots,k}\|\nabla\boldsymbol{v}^k_\tau\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}}\bigg(\|\boldsymbol{E}^0_\tau\|^\sigma_{L^\sigma(\Omega;\mathbb{R}^{d\times d})} + \tau\sum_{l=1}^k\|\boldsymbol{F}^l_\tau\|^{\sigma'}_{L^{\sigma'}(\Omega;\mathbb{R}^{d\times d})}\bigg) \tag{6.30}$$

for some C and for $\tau>0$ sufficiently small, namely for all $\tau\le 1/((2{+}4\sigma)\max_{l=1,\dots,k}\|\nabla\boldsymbol{v}^k_\tau\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}+2\sigma)$.

Moreover, we can also test (6.27) by the q-Laplacian $-\mathrm{div}(|\nabla\boldsymbol{E}^k_\tau|^{q-2}\nabla\boldsymbol{E}^k_\tau)$. More specifically, we can apply the ∇-operator to (6.27) and test it by $|\nabla\boldsymbol{E}^k_\tau|^{q-2}\nabla\boldsymbol{E}^k_\tau$. Instead of (6.28), we use the calculus

$$\begin{aligned}\int_\Omega\nabla(\boldsymbol{v}\cdot\nabla z)\cdot|\nabla z|^{q-2}\nabla z\,\mathrm{d}x &= \int_\Omega|\nabla z|^{q-2}\nabla\boldsymbol{v}{:}(\nabla z\otimes\nabla z) + (\boldsymbol{v}\cdot\nabla^2 z)\cdot|\nabla z|^{q-2}\nabla z\,\mathrm{d}x\\ &= \int_\Gamma|\nabla z|^q(\boldsymbol{v}\cdot\boldsymbol{n})\,\mathrm{d}S - \int_\Omega|\nabla z|^{q-2}\nabla\boldsymbol{v}{:}(\nabla z\otimes\nabla z) + (q{-}1)|\nabla z|^{q-2}\nabla z\cdot(\boldsymbol{v}\cdot\nabla^2 z) + (\mathrm{div}\,\boldsymbol{v})|\nabla z|^q\,\mathrm{d}x\\ &= \int_\Omega|\nabla z|^{q-2}\nabla\boldsymbol{v}{:}(\nabla z\otimes\nabla z) - \frac{1}{q}(\mathrm{div}\,\boldsymbol{v})|\nabla z|^q\,\mathrm{d}x\,.\end{aligned} \tag{6.31}$$

In the tensorial situation (6.27), we use it again for $z=E_{ij}$ and then we use also

$$\nabla\Big(\mathrm{skew}(\nabla\boldsymbol{v}^k_\tau)\boldsymbol{E}^k_\tau - \boldsymbol{E}^k_\tau\mathrm{skew}(\nabla\boldsymbol{v}^k_\tau)\Big)\,\vdots\,|\nabla\boldsymbol{E}^k_\tau|^{q-2}\nabla\boldsymbol{E}^k_\tau \le 2|\nabla\boldsymbol{v}^k_\tau|\,|\nabla\boldsymbol{E}^k_\tau|^q + 2|\nabla^2\boldsymbol{v}^k_\tau|\,|\boldsymbol{E}^k_\tau|\,|\nabla\boldsymbol{E}^k_\tau|^{q-1}. \tag{6.32}$$

Instead of (6.29), this gives

$$
\begin{aligned}
\frac{1}{q}\int_\Omega \frac{|\nabla \boldsymbol{E}_\tau^k|^q - |\nabla \boldsymbol{E}_\tau^{k-1}|^q}{\tau}\,\mathrm{d}x \le{}& \int_\Omega\bigg(\frac{\operatorname{div}\boldsymbol{v}_\tau^k}{q}|\nabla \boldsymbol{E}_\tau^k|^q + \Big(2+\frac{1}{q}\Big)|\nabla \boldsymbol{v}_\tau^k|\,|\nabla \boldsymbol{E}_\tau^k|^q \\
&\qquad + |\nabla \boldsymbol{E}_\tau^k|^{q-2}\nabla \boldsymbol{E}_\tau^k \vdots \nabla \boldsymbol{F}_\tau^k + 2|\nabla^2 \boldsymbol{v}_\tau^k|\,|\boldsymbol{E}_\tau^k|\,|\nabla \boldsymbol{E}_\tau^k|^{q-1}\bigg)\,\mathrm{d}x \\
\le{}& C + C\big(1+\|\nabla \boldsymbol{v}_\tau^k\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}\big)\|\nabla \boldsymbol{E}_\tau^k\|_{L^q(\Omega;\mathbb{R}^{d\times d\times d})}^q \\
&\qquad + \|\nabla \boldsymbol{F}_\tau^k\|_{L^q(\Omega;\mathbb{R}^{d\times d\times d})}^q + \|\nabla^2 \boldsymbol{v}_\tau^k\|_{L^p(\Omega;\mathbb{R}^{d\times d\times d})}^p + \|\boldsymbol{E}_\tau^k\|_{L^\sigma(\Omega;\mathbb{R}^{d\times d\times d})}^\sigma\,.
\end{aligned}
\tag{6.33}
$$

for some C sufficiently large, provided $1/p + 1/\sigma + 1/q' \le 1$ with p from (6.25a). By a discrete Gronwall inequality like (6.30), provided τ is sufficiently small, we obtain

$$
\|\nabla \boldsymbol{E}_\tau^k\|_{L^q(\Omega;\mathbb{R}^{d\times d\times d})}^q \le C\bigg(\|\nabla \boldsymbol{E}_\tau^0\|_{L^q(\Omega;\mathbb{R}^{d\times d\times d})}^q + \tau\sum_{l=1}^k \|\nabla \boldsymbol{F}_\tau^l\|_{L^q(\Omega;\mathbb{R}^{d\times d\times d})}^q\bigg)
\tag{6.34}
$$

with some C depending on $\max_{l=1,\dots,k}\|\nabla \boldsymbol{v}_\tau^k\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}$ and on $\tau\sum_{l=1}^k \|\nabla^2 \boldsymbol{v}_\tau^k\|_{L^p(\Omega)}^p + \|\boldsymbol{E}_\tau^k\|_{L^\sigma(\Omega)}^\sigma$.

Considering $\boldsymbol{v}_\tau^k = \boldsymbol{v}_{\varepsilon\tau}^k$ and $\boldsymbol{F}_\tau^k = \boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k) - \boldsymbol{\Pi}_{\varepsilon\tau}^k$ and using the already obtained estimate (6.22c) and the initial condition $\boldsymbol{E}_0 \in H^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev})$, we use the calculus (6.29)–(6.30) with $\sigma = 2^*$ and $q = 2$ for (6.7b). Thus we obtain

$$
\|\nabla \overline{\boldsymbol{E}}_{\varepsilon\tau}\|_{L^\infty(I;L^2(\Omega;\mathbb{R}^{d\times d\times d}))} \le C\,.
\tag{6.35}
$$

The arguments (6.29)–(6.30) and (6.33)–(6.34) hold even simplified for the convective transport of scalar variables; in particular (6.32) and last two terms in (6.33) would be omitted. Here, having now $\nabla\overline{\theta}_{\varepsilon\tau}$ estimated, by using the calculus (6.31) with $z = \chi$, from the discrete volume-fraction evolution (6.7f) we thus obtain

$$
\begin{aligned}
\frac{1}{q}\int_\Omega \frac{|\nabla \chi_{\varepsilon\tau}^k|^q - |\nabla \chi_{\varepsilon\tau}^{k-1}|^q}{\tau}\,\mathrm{d}x &\le \int_\Omega\Big(\frac{\nabla\theta_{\varepsilon\tau}^k}{\theta_{\rm PT}} - \boldsymbol{v}_{\varepsilon\tau}^k\cdot\nabla\chi_{\varepsilon\tau}^k\Big)\cdot|\nabla\chi_{\varepsilon\tau}^k|^{q-2}\nabla\chi_{\varepsilon\tau}^k\,\mathrm{d}x \\
&= \int_\Omega \frac{\nabla\theta_{\varepsilon\tau}^k}{\theta_{\rm PT}}\cdot|\nabla\chi_{\varepsilon\tau}^k|^{q-2}\nabla\chi_{\varepsilon\tau}^k + |\nabla\chi_{\varepsilon\tau}^k|^{q-2}\big(\nabla\chi_{\varepsilon\tau}^k\otimes\nabla\chi_{\varepsilon\tau}^k\big):\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k) - \frac{1}{q}|\nabla\chi_{\varepsilon\tau}^k|^q\operatorname{div}\boldsymbol{v}_{\varepsilon\tau}^k\,\mathrm{d}x \\
&\le \theta_{\rm PT}^{-q}\|\nabla\theta_{\varepsilon\tau}^k\|_{L^q(\Omega;\mathbb{R}^d)}^q + \big(1+\|\boldsymbol{e}(\boldsymbol{v}_{\varepsilon\tau}^k)\|_{L^\infty(\Omega;\mathbb{R}^{d\times d})}\big)\|\nabla\chi_{\varepsilon\tau}^k\|_{L^q(\Omega;\mathbb{R}^d)}^q\,,
\end{aligned}
\tag{6.36}
$$

where we again used $\boldsymbol{v}_{\varepsilon\tau}^k\cdot\boldsymbol{n} = 0$. The first inequality in (6.36) follows from the convexity of $|\cdot|^q$ and from that, written formally, $\nabla H^{-1}(\chi_{\varepsilon\tau}^k)\cdot|\nabla\chi_{\varepsilon\tau}^k|^{q-2}\nabla\chi_{\varepsilon\tau}^k = \partial^2\delta_{[0,1]}(\chi_{\varepsilon\tau}^k)|\nabla\chi_{\varepsilon\tau}^k|^q \ge 0$, where $\partial^2\delta_{[0,1]}$ denotes the (generalized) Hessian of the convex indicator function of the interval $[0,1]$. When $q < (d{+}2)/(d{+}1)$, we can use (6.25a) and the assumption (6.1i) on χ_0 and, by the discrete Gronwall inequality, we obtain a bound for $\nabla\overline{\chi}_{\varepsilon\tau}$ in $L^\infty(I;L^q(\Omega;\mathbb{R}^d))$.

Since now we have $\nabla\overline{\chi}_{\varepsilon\tau}$ estimated, we can see that $\overline{\boldsymbol{v}}_{\varepsilon\tau}\cdot\nabla\overline{\chi}_{\varepsilon\tau}$ is surely bounded in $L^1(I{\times}\Omega)$. Since $\frac{\partial}{\partial t}\chi_{\varepsilon\tau}{+}\overline{\boldsymbol{v}}_{\varepsilon\tau}\cdot\nabla\overline{\chi}_{\varepsilon\tau}$ has already been proved bounded in $L^2(I{\times}\Omega)$, we thus obtain a bound for $\frac{\partial}{\partial t}\chi_{\varepsilon\tau} = (\frac{\partial}{\partial t}\chi_{\varepsilon\tau}{+}\overline{\boldsymbol{v}}_{\varepsilon\tau}\cdot\nabla\overline{\chi}_{\varepsilon\tau}) - \overline{\boldsymbol{v}}_{\varepsilon\tau}\cdot\nabla\overline{\chi}_{\varepsilon\tau}$ in $L^1(I{\times}\Omega)$, and $\frac{\partial}{\partial t}\overline{\chi}_{\varepsilon\tau}$ in $\mathrm{BV}(I;L^1(\Omega))$. To summarize, we proved

$$
\|\overline{\chi}_{\varepsilon\tau}\|_{L^\infty(I\times\Omega)} \le C \quad\text{ and }\quad \|\overline{\chi}_{\varepsilon\tau}\|_{L^\infty(I;W^{1,q}(\Omega))\cap\mathrm{BV}(I;L^1(\Omega))} \le C_q \quad\text{ with }\ q < \frac{d{+}2}{d{+}1}\,.
\tag{6.37}
$$

Step 3: convergence in the mechanical part for $\tau\to 0$. We now consider $\varepsilon > 0$ fixed. By the Banach selection principle, we obtain a subsequence converging weakly* with respect to the topologies indicated in (6.22) and (6.25) to some limit $(\boldsymbol{v}_\varepsilon, \boldsymbol{E}_\varepsilon, \boldsymbol{\Pi}_\varepsilon, \alpha_\varepsilon, \theta_\varepsilon, \chi_\varepsilon)$.

By the uniform monotonicity of the operators $\frac{\partial}{\partial t}$ and $\mathrm{div}(\mathrm{div}(\nu|\nabla \boldsymbol{e}(\cdot)|^{p-2}\nabla \boldsymbol{e}(\cdot)) - \mathbb{D}\boldsymbol{e}(\cdot))$, we obtain the strong convergence

$$\bar{\boldsymbol{v}}_{\varepsilon\tau} \to \boldsymbol{v}_\varepsilon \qquad \text{in } \ L^p(I;W^{2,p}(\Omega;\mathbb{R}^d)) \quad \text{and} \tag{6.38a}$$

$$\boldsymbol{v}_{\varepsilon\tau}(T) \to \boldsymbol{v}_\varepsilon(T) \qquad \text{in } \ L^2(\Omega;\mathbb{R}^d)\,. \tag{6.38b}$$

More in detail, we use the discrete momentum equation tested by $\bar{\boldsymbol{v}}_{\varepsilon\tau}-\widehat{\boldsymbol{v}}_\tau$ with some $\widehat{\boldsymbol{v}}_\tau$ piecewise constant on the time-partition of the times step τ and converging strongly to $\boldsymbol{v}_\varepsilon$ in $L^p(I;W^{2,p}(\Omega;\mathbb{R}^d))\cap L^\infty(I;L^2(\Omega;\mathbb{R}^d))$. After integration over time, we obtain

$$\begin{aligned}
&c\|\boldsymbol{v}_{\varepsilon\tau}(T)-\widehat{\boldsymbol{v}}_\tau(T)\|^2_{L^2(\Omega;\mathbb{R}^d)} + c\|\nabla\boldsymbol{e}(\bar{\boldsymbol{v}}_{\varepsilon\tau}-\widehat{\boldsymbol{v}}_\tau)\|^p_{L^p(I\times\Omega;\mathbb{R}^{d\times d\times d})} \le \int_\Omega \frac{\varrho}{2}|\boldsymbol{v}_{\varepsilon\tau}(T)-\widehat{\boldsymbol{v}}_\tau(T)|^2\mathrm{d}x\\
&\ + \int_0^T\!\!\int_\Omega \mathbb{D}\boldsymbol{e}(\bar{\boldsymbol{v}}_{\varepsilon\tau}-\widehat{\boldsymbol{v}}_\tau){:}\boldsymbol{e}(\bar{\boldsymbol{v}}_{\varepsilon\tau}-\widehat{\boldsymbol{v}}_\tau) + \nu\big(|\nabla\boldsymbol{e}(\bar{\boldsymbol{v}}_{\varepsilon\tau})|^{p-2}\nabla\boldsymbol{e}(\bar{\boldsymbol{v}}_{\varepsilon\tau}) - |\nabla\boldsymbol{e}(\widehat{\boldsymbol{v}}_\tau)|^{p-2}\nabla\boldsymbol{e}(\widehat{\boldsymbol{v}}_\tau)\big)\,\vdots\,\nabla\boldsymbol{e}(\bar{\boldsymbol{v}}_{\varepsilon\tau}-\widehat{\boldsymbol{v}}_\tau)\,\mathrm{d}x\mathrm{d}t\\
&\ \le \int_0^T\!\!\int_\Omega \overline{\boldsymbol{f}}_\tau\cdot(\bar{\boldsymbol{v}}_{\varepsilon\tau}-\widehat{\boldsymbol{v}}_\tau) - \big(\overline{\boldsymbol{S}}_{\varepsilon\tau}+\overline{\boldsymbol{K}}_{\varepsilon\tau}\big) : \boldsymbol{e}(\bar{\boldsymbol{v}}_{\varepsilon\tau}-\widehat{\boldsymbol{v}}_\tau) - \nu|\nabla\boldsymbol{e}(\widehat{\boldsymbol{v}}_\tau)|^{p-2}\nabla\boldsymbol{e}(\widehat{\boldsymbol{v}}_\tau)\,\vdots\,\nabla\boldsymbol{e}(\bar{\boldsymbol{v}}_{\varepsilon\tau}-\widehat{\boldsymbol{v}}_\tau)\\
&\qquad\qquad - \mathbb{D}\boldsymbol{e}(\widehat{\boldsymbol{v}}_\tau){:}\boldsymbol{e}(\bar{\boldsymbol{v}}_{\varepsilon\tau}-\widehat{\boldsymbol{v}}_\tau)\,\mathrm{d}x\mathrm{d}t - \int_\Omega \varrho\widehat{\boldsymbol{v}}_\tau(T)\cdot(\boldsymbol{v}_{\varepsilon\tau}(T)-\widehat{\boldsymbol{v}}_\tau(T))\,\mathrm{d}x
\end{aligned}\tag{6.39}$$

with some small $c>0$. As the right-hand side of (6.39) converges to 0, we obtain (6.38).

Moreover, we use the Aubin-Lions compact-embedding theorem to show the strong convergence

$$\overline{\boldsymbol{E}}_{\varepsilon\tau} \to \boldsymbol{E}_\varepsilon \quad \text{and} \quad \overline{\boldsymbol{S}}_{\varepsilon\tau} \to \boldsymbol{S}_\varepsilon = [\varphi_\varepsilon]'_{\boldsymbol{E}}(\boldsymbol{E}_\varepsilon,\alpha_\varepsilon) \qquad \text{in } \ L^2(I{\times}\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})\,, \tag{6.40a}$$

$$\bar{\alpha}_{\varepsilon\tau} \to \alpha_\varepsilon \qquad \text{in } \ L^q(I{\times}\Omega), \quad q<\infty\,; \tag{6.40b}$$

in fact, (6.40a) holds even in better spaces (i.e. in stronger modes). For this, we need some information about time derivatives. It is directly at disposal for (6.40b) from the a-priori estimate (6.37). For the former convergence in (6.40a), we can see some information by comparison from (6.11b), from which we can see that the sequence $\{\frac{\partial}{\partial t}\boldsymbol{E}_{\varepsilon\tau}\}_{\tau>0}$ is bounded in $L^2(I;H^1(\Omega;\mathbb{R}^{d\times d}_{\mathrm{sym}})^*)$. Actually, usage of the mentioned Aubin-Lions theorem for piece-wise constant functions in time needs its generalized version for functions whose time-derivatives are measures, cf. [34, Cor.7.9]. The later strong convergence in (6.40a) is then simply by continuity of the Nemytskiĭ operator induced by $[\varphi_\varepsilon]'_{\boldsymbol{E}}$, even without having any information about the time derivative.

As $\nabla\alpha$ occurs nonlinearly in the stress $\boldsymbol{K}$ and also in the weak formulation (6.5c) multiplied by $\Delta\alpha$, we need to prove also the strong convergence

$$\nabla\bar{\alpha}_{\varepsilon\tau} \to \nabla\alpha_\varepsilon \qquad \text{strongly in } \ L^2(I{\times}\Omega;\mathbb{R}^d)\,. \tag{6.41}$$

To prove it, we take a sequence $\{\widetilde{\alpha}_\tau\}_{\tau>0}$ piecewise constant in time with respect to the partition with the time step τ and converging strongly towards α_ε for $\tau\to0$. Then, using the variational inequality arising from the inclusion (6.11d) tested by $\nabla(\bar{\alpha}_{\varepsilon\tau}-\widetilde{\alpha}_\tau)$ we can see that, written in terms of the interpolants,

$$\begin{aligned}
\int_0^T\!\!\int_\Omega \kappa|\nabla(\bar{\alpha}_{\varepsilon\tau}-\widetilde{\alpha}_\tau)|^2\,\mathrm{d}x\mathrm{d}t \le -\int_0^T\!\!\int_\Omega\bigg(\partial_{\dot\alpha}\zeta\Big(\underline{\alpha}_{\varepsilon\tau},\underline{w}_{\varepsilon\tau};\frac{\partial\alpha_{\varepsilon\tau}}{\partial t}+\bar{\boldsymbol{v}}_{\varepsilon\tau}\cdot\nabla\bar{\alpha}_{\varepsilon\tau}\Big) + \sqrt{\tau}\frac{\partial\alpha_{\varepsilon\tau}}{\partial t}\\
+ [\varphi_\varepsilon]'_\alpha(\overline{\boldsymbol{E}}_{\varepsilon\tau},\bar{\alpha}_{\varepsilon\tau})\bigg)(\bar{\alpha}_{\varepsilon\tau}-\widetilde{\alpha}_\tau) + \kappa\nabla\widetilde{\alpha}_{\varepsilon\tau}\cdot\nabla(\bar{\alpha}_{\varepsilon\tau}-\widetilde{\alpha}_\tau)\,\mathrm{d}x\mathrm{d}t \to 0\,.
\end{aligned}\tag{6.42}$$

Here we used (6.1b) with (6.40a) so that $[\varphi_\varepsilon]'_\alpha(\overline{\boldsymbol{E}}_{\varepsilon\tau},\bar{\alpha}_{\varepsilon\tau})$ is bounded in $L^2(I{\times}\Omega)$ while $\bar{\alpha}_{\varepsilon\tau}-\widetilde{\alpha}_\tau\to0$ strongly in $L^q(I{\times}\Omega)$ with any $q<+\infty$; this is due to an interpolated Aubin-Lions theorem, relying on (6.22d) with (6.25b). In (6.42), we used that $\partial_{\dot\alpha}\zeta(\underline{\alpha}_{\varepsilon\tau},\underline{w}_{\varepsilon\tau};\frac{\partial}{\partial t}\alpha_{\varepsilon\tau}+\bar{\boldsymbol{v}}_{\varepsilon\tau}\cdot\nabla\bar{\alpha}_{\varepsilon\tau})$ is bounded in $L^2(I{\times}\Omega)$ and, moreover, that $\|\sqrt{\tau}\frac{\partial}{\partial t}\alpha_{\varepsilon\tau}\|_{L^2(I\times\Omega)} = \mathscr{O}(\sqrt{\tau})\to0$ due to (6.25b).

As $G_{\mathrm{M}}(\cdot)$ and $\zeta(\alpha,\cdot;\dot{\alpha})$ depend nonlinearly on the enthalpy $w = \vartheta + \ell\chi$, we need a strong convergence of both $\underline{\vartheta}_{\varepsilon\tau}$ and $\underline{\chi}_{\varepsilon\tau}$. The former one follows from the estimates on the gradient (6.25d) and from some information about time derivative of $\underline{\vartheta}_{\varepsilon\tau}$ from the discrete equations (6.11e) by the (generalized) Aubin-Lions compact embedding theorem. More in detail, from (6.11e) we can see that $\frac{\partial}{\partial t}\underline{w}_{\varepsilon\tau} \in \mathrm{BV}(I;H^2(\Omega)^*)$ so that, taking $\frac{\partial}{\partial t}\underline{\chi}_{\varepsilon\tau} \in \mathrm{BV}(I;L^1(\Omega))$ proved already before in (6.37), we obtain $\frac{\partial}{\partial t}\underline{\vartheta}_{\varepsilon\tau} = \frac{\partial}{\partial t}\underline{w}_{\varepsilon\tau} - \ell\frac{\partial}{\partial t}\underline{\chi}_{\varepsilon\tau} \in \mathrm{BV}(I;H^2(\Omega)^*)$. Thus we proved the strong convergence

$$\underline{w}_{\varepsilon\tau} = \widetilde{\gamma}(\underline{\theta}_{\varepsilon\tau}) + \ell\underline{\chi}_{\varepsilon\tau} \to \widetilde{\gamma}(\theta_\varepsilon) + \ell\chi_\varepsilon = w_\varepsilon \ \text{ strongly in } \ L^s(I\times\Omega) \tag{6.43}$$

with s from (6.25d). For the latter one, i.e. for $\underline{\chi}_{\varepsilon\tau} \to \chi_\varepsilon$ strongly in $L^q(I\times\Omega)$ for any $q < +\infty$, we use an interpolated (and generalized) Aubin-Lions compact embedding theorem exploiting the a-priori estimates (6.37).

The limit passage in the variational inequality for $\overline{\alpha}_{\varepsilon\tau}$ which is behind the inclusion (6.11d), cf. (6.5c), exploits $[\varphi_\varepsilon]'_\alpha(\overline{\boldsymbol{E}}_{\varepsilon\tau},\overline{\alpha}_{\varepsilon\tau}) \to [\varphi_\varepsilon]'_\alpha(\boldsymbol{E}_\varepsilon,\alpha_\varepsilon)$ strongly in $L^2(I\times\Omega)$.

The limit passage the variational inequality, which is behind the volume-fraction equation (6.11f), towards the variational inequality (6.5e) written for the ε-solution is based on the a-priori estimates (6.37). From them, as we already used for $\underline{\chi}_{\varepsilon\tau}$, using the Aubin-Lions theorem (again generalized for functions with measure time derivatives as in [34, Cor.7.9]) we obtain that $\overline{\chi}_{\varepsilon\tau} \to \chi_\varepsilon$ in $L^q(I\times\Omega)$ to be used for the term $\frac{1}{2}(\mathrm{div}\,\overline{\boldsymbol{v}}_{\varepsilon\tau})\overline{\chi}^2_{\varepsilon\tau}$ with any $q < +\infty$. Also we use the weak convergence of $\overline{\chi}_{\varepsilon\tau}(T) \to \chi_\varepsilon(T)$ in $L^q(\Omega)$ with any $q < +\infty$.

Using also (6.40a,b), we can see that the Korteweg-like stress converges even strongly, namely

$$\overline{\boldsymbol{K}}_{\varepsilon\tau} \to \boldsymbol{K}\varepsilon \qquad \text{in} \ \ L^s(I;L^1(\Omega;\mathbb{R}^{d\times d})), \ \ 1 < s < \infty\,. \tag{6.44}$$

The latter relation in (6.11b) easily converges to $\boldsymbol{\Pi}_\varepsilon$ exploiting that $\nabla\overline{\boldsymbol{v}}_{\varepsilon\tau} \to \nabla\boldsymbol{v}_\varepsilon$ strongly in $L^p_{\mathrm{w}*}(I;L^\infty(\Omega;\mathbb{R}^{d\times d}))$.

Step 4: Convergence of the dissipation rate and of the heat-transfer equation for $\tau \to 0$. Here we use the already proved convergence in the mechanical part together with the mechanical-energy conservation. We have the chain of estimates:

$$\begin{aligned}
&\int_0^T\!\!\int_\Omega G_{\mathrm{M}}(w_\varepsilon)|\boldsymbol{\Pi}_\varepsilon|^2 + \mathbb{D}\boldsymbol{e}(\boldsymbol{v}_\varepsilon){:}\boldsymbol{e}(\boldsymbol{v}_\varepsilon) + \nu|\nabla\boldsymbol{e}(\boldsymbol{v}_\varepsilon)|^p + \xi\big(\alpha_\varepsilon, w_\varepsilon;\dot{\alpha}_\varepsilon\big) + \varkappa|\nabla\boldsymbol{\Pi}_\varepsilon|^2\,\mathrm{d}x\mathrm{d}t\\
&\quad\le \liminf_{\tau\to0}\int_0^T\!\!\int_\Omega G_{\mathrm{M}}(\underline{w}_{\varepsilon\tau})|\overline{\boldsymbol{\Pi}}_{\varepsilon\tau}|^2 + \mathbb{D}\boldsymbol{e}(\overline{\boldsymbol{v}}_{\varepsilon\tau}){:}\boldsymbol{e}(\overline{\boldsymbol{v}}_{\varepsilon\tau}) + \nu|\nabla\boldsymbol{e}(\overline{\boldsymbol{v}}_{\varepsilon\tau})|^p\\
&\qquad\qquad\qquad + \xi\Big(\underline{\alpha}_{\varepsilon\tau}, \underline{w}_{\varepsilon\tau}; \frac{\partial\alpha_{\varepsilon\tau}}{\partial t} + \overline{\boldsymbol{v}}_{\varepsilon\tau}{\cdot}\nabla\overline{\alpha}_{\varepsilon\tau}\Big) + \varkappa|\nabla\overline{\boldsymbol{\Pi}}_{\varepsilon\tau}|^2\,\mathrm{d}x\mathrm{d}t\\
&\quad\overset{(6.20)}{\le} \int_\Omega \frac{\varrho}{2}|\boldsymbol{v}_0|^2 + \varphi_\varepsilon(\boldsymbol{E}_0,\alpha_0) + \frac{\kappa}{2}|\nabla\alpha_0|^2\,\mathrm{d}x - \liminf_{\tau\to0}\int_\Omega\frac{\varrho}{2}|\boldsymbol{v}_{\varepsilon\tau}(T)|^2 + \varphi_\varepsilon(\boldsymbol{E}_{\varepsilon\tau}(T),\alpha_{\varepsilon\tau}(T)) + \frac{\kappa}{2}|\nabla\alpha_{\varepsilon\tau}(T)|^2\,\mathrm{d}x\\
&\qquad\qquad + \lim_{\tau\to0}\int_0^T\!\!\int_\Omega \overline{\boldsymbol{f}}_\tau\cdot\overline{\boldsymbol{v}}_{\varepsilon\tau} - \phi(\overline{\theta}_{\varepsilon\tau})\mathrm{div}\,\overline{\boldsymbol{v}}_{\varepsilon\tau} + \frac{\sqrt{\tau}}{2}\Big|\frac{\partial\alpha_{\varepsilon\tau}}{\partial t} + \overline{\boldsymbol{v}}_{\varepsilon\tau}{\cdot}\nabla\overline{\alpha}_{\varepsilon\tau}\Big|\,\mathrm{d}x\mathrm{d}t\\
&\quad\le \int_\Omega\frac{\varrho}{2}|\boldsymbol{v}_0|^2 + \varphi_\varepsilon(\boldsymbol{E}_0,\alpha_0) + \frac{\kappa}{2}|\nabla\alpha_0|^2 - \frac{\varrho}{2}|\boldsymbol{v}_\varepsilon(T)|^2 - \varphi_\varepsilon(\boldsymbol{E}_\varepsilon(T),\alpha_\varepsilon(T)) - \frac{\kappa}{2}|\nabla\alpha_\varepsilon(T)|^2\,\mathrm{d}x\\
&\qquad\qquad + \int_0^T\!\!\int_\Omega \boldsymbol{f}{\cdot}\boldsymbol{v}_\varepsilon - \phi(\theta_\varepsilon)\mathrm{div}\boldsymbol{v}_\varepsilon\,\mathrm{d}x\mathrm{d}t\\
&\quad= \int_0^T\!\!\int_\Omega G_{\mathrm{M}}(w_\varepsilon)|\boldsymbol{\Pi}_\varepsilon|^2 + \mathbb{D}\boldsymbol{e}(\boldsymbol{v}_\varepsilon){:}\boldsymbol{e}(\boldsymbol{v}_\varepsilon) + \nu|\nabla\boldsymbol{e}(\boldsymbol{v}_\varepsilon)|^p + \xi\big(\alpha_\varepsilon, w_\varepsilon;\dot{\alpha}_\varepsilon\big) + \varkappa|\nabla\boldsymbol{\Pi}_\varepsilon|^2\,\mathrm{d}x\mathrm{d}t\,.
\end{aligned}\tag{6.45}$$

The first inequality is due to the weak lower semicontinuity. The last equality in (6.45) is just the mechanical-energy balance (3.11) written for the ε-solution.

This mechanical-energy balance follows from the tests as in the (formal) calculations (3.7)–(3.10) written for the ε-solution. The validity of this balance is not automatic and the rigorous prove needs to have granted that testing the particular equations by $\boldsymbol{v}_\varepsilon$, $\boldsymbol{S}_\varepsilon$, and $\boldsymbol{\Pi}_\varepsilon$ is indeed legal. Here, in particular it is important that $\boldsymbol{K}_\varepsilon{:}\nabla\boldsymbol{v}_\varepsilon \in L^1(I{\times}\Omega)$ because $\boldsymbol{K}_\varepsilon \in L^\infty(I;L^1(\Omega;\mathbb{R}^{d\times d}))$ and $\nabla\boldsymbol{v}_\varepsilon \in L^p_{\mathrm{w}*}(I;L^\infty(\Omega;\mathbb{R}^{d\times d}))$. Also, $\kappa\Delta\alpha_\varepsilon \in \partial_{\dot\alpha}\zeta(\alpha_\varepsilon,w_\varepsilon;\dot\alpha_\varepsilon)+[\varphi_\varepsilon]'_\alpha(\boldsymbol{E}_\varepsilon,\alpha_\varepsilon)$ holds pointwise a.e. in the sense of $L^2(I{\times}\Omega)$, we can legitimately test it by $\dot\alpha_\varepsilon \in L^2(I{\times}\Omega)$. Since $\partial_{\dot\alpha}\zeta(\alpha_\varepsilon,w_\varepsilon;\cdot)$ is single-valued except 0, cf. (6.1e), $\partial_{\dot\alpha}\zeta(\alpha_\varepsilon,w_\varepsilon;\dot\alpha_\varepsilon)\dot\alpha_\varepsilon$ is single-valued, being equal to $\xi(\alpha_\varepsilon,w_\varepsilon;\dot\alpha_\varepsilon) \in L^1(I{\times}\Omega)$.

Altogether, this reveals that there are actually equalities in (6.45). Since the dissipation rate is uniformly convex in terms of rates on the uniformly convex L^2-spaces, these rates converge not only weakly but even strongly in these L^2-spaces. Thus the dissipation rate itself converges strongly in $L^1(I{\times}\Omega)$.

The limit passage in the resting semilinear terms in (6.11e) is then easy.

Step 5: convergence for $\varepsilon \to 0$. This final convergence towards the weak solution due to Definition 6.1 copies the arguments in the Steps 2-4 above. Actually, derivation of the estimates (6.22) and (6.25a,b) in Step 2 is even simplified because the time-continuous problem can rely directly on the calculus from Section 3. The manipulation exploiting the semi-convexity (6.2c) of ϕ_ε is no longer needed.

By (6.2d), we have $\varphi_\varepsilon(\boldsymbol{E}_\varepsilon,\alpha_\varepsilon) = \varphi(\boldsymbol{E}_\varepsilon,\alpha_\varepsilon) + o(\varepsilon)(1{+}\|\boldsymbol{E}_\varepsilon\|_{L^2(I\times\Omega;\mathbb{R}^{d\times d})}) \to \varphi(\boldsymbol{E},\alpha)$ in $L^1(I{\times}\Omega)$ to be used for the convergence in the Korteweg-like stress, cf. (6.18) written for ε-solution. Also we have $|\varphi_\varepsilon(\boldsymbol{E}_\varepsilon(T),\alpha_\varepsilon(T)) - \varphi(\boldsymbol{E}(T),\alpha(T))| \le o(\varepsilon)(1{+}|\boldsymbol{E}(T)|^2)$ so that

$$\liminf_{\varepsilon\to0}\int_\Omega \varphi_\varepsilon(\boldsymbol{E}_\varepsilon(T),\alpha_\varepsilon(T))\,\mathrm{d}x \ge \liminf_{\varepsilon\to0}\int_\Omega \varphi(\boldsymbol{E}_\varepsilon(T),\alpha_\varepsilon(T))\,\mathrm{d}x \\ - o(\varepsilon)\big(1{+}\|\boldsymbol{E}(T)\|^2_{L^2(\Omega;\mathbb{R}^{d\times d})}\big) \ge \int_\Omega \varphi(\boldsymbol{E}(T),\alpha(T))\,\mathrm{d}x \tag{6.46}$$

to be used for (6.45) written between the ε-solutions and their limit.

Furthermore, from (6.2e) we have $[\varphi_\varepsilon]'_{\boldsymbol{E}}(\boldsymbol{E}_\varepsilon,\alpha_\varepsilon) \to \varphi'_{\boldsymbol{E}}(\boldsymbol{E},\alpha)$ in $L^{1/\delta}(I;L^{2^*-\delta}(\Omega;\mathbb{R}^{d\times d}))$ for any $\delta > 0$ to be used e.g. in (6.40a) written for ε-solutions. Eventually, from (6.2f) we have $[\varphi_\varepsilon]'_\alpha(\boldsymbol{E}_\varepsilon,\alpha_\varepsilon) \to \varphi'_\alpha(\boldsymbol{E},\alpha)$ in $L^{1/\delta}(I;L^{2^*/2-\delta}(\Omega))$, which is to be used in (6.42) written between ε-solution and the limit.

The weak-continuity (6.4e) follows from the estimate (6.22e) inherited for θ first as the estimate in $L^\infty_{\mathrm{w}*}(I;\mathrm{Meas}(\overline\Omega))$. Further, due to the estimate for $\frac{\partial}{\partial t}\theta$, we have also $C_{\mathrm{w}*}(I;\mathrm{Meas}(\overline\Omega))$. Yet, since $\theta(t) \in W^{1,r}(\Omega) \subset L^1(\Omega)$ for a.a. $t \in I$, the weak* continuity of $\theta : I \to \mathrm{Meas}(\overline\Omega)$ turns into the weak continuity of $\theta : I \to L^1(\Omega)$. $\square$

Remark 6.3 (*Inelastic strain*)**.** *Given an additional initial condition $\boldsymbol{P}_0$, the inelastic strain $\boldsymbol{P}$ can be reconstructed from $\mathring{\boldsymbol{P}} = \boldsymbol{\Pi}$, cf. Remark 3.1. Since $\boldsymbol{\Pi} \in L^2(I;H^1(\Omega;\mathbb{R}^{d\times d}))$, assuming $\boldsymbol{P}_0 \in L^q(\Omega;\mathbb{R}^{d\times d})$, using the estimation as in* (6.29)–(6.30)*, one can see that $\boldsymbol{P} \in L^\infty(I;L^q(\Omega;\mathbb{R}^{d\times d}))$. If even $\boldsymbol{P}_0 \in H^1(\Omega;\mathbb{R}^{d\times d})$, the estimation as in* (6.33)–(6.34) *gives $\boldsymbol{P} \in L^\infty(I;H^1(\Omega;\mathbb{R}^{d\times d}))$.*

Remark 6.4 (*Regularity*)**.** On smooth or convex domains Ω, one can use the H^2-regularity for damage to be seen by comparison from $\Delta\alpha \in (\partial_{\dot\alpha}\zeta(\alpha,w;\dot\alpha) + [\varphi_\varepsilon]'_\alpha(\boldsymbol{E},\alpha))/\kappa \subset L^2(I{\times}\Omega)$ bounded. Knowing $\alpha \in L^2(I;H^2(\Omega))$, we have also a better integrability of the Korteweg-like stress, namely $\boldsymbol{K} \in L^\infty(I;L^1(\Omega;\mathbb{R}^{d\times d})) \cap L^{1+2/d}(I{\times}\Omega;\mathbb{R}^{d\times d})$. To show (6.41), one can then use the Aubin-Lions theorem instead of (6.42).

Remark 6.5 (Regularization of the stored energy)**.** In the case of φ from (4.1), the regularization satisfying the condition (6.2) can be done as

$$\varphi_\varepsilon(\boldsymbol{E},\alpha) = \frac{d}{2}K_{\mathrm{E}}|\mathrm{sph}\,\boldsymbol{E}|^2 + G_{\mathrm{E}}(\alpha)\,\frac{|\mathrm{dev}\,\boldsymbol{E}|^2}{\sqrt{1{+}\varepsilon|\mathrm{dev}\,\boldsymbol{E}|^2}} + \frac{1}{2\kappa}G_{\mathrm{D}}(1{-}\alpha)^2\,. \tag{6.47}$$

This ensures semiconvexity (6.2c) of φ_ε. Simultaneously, the conditions (6.2a,c-e) hold.

7 Concluding remarks

We close this article by several remarks to the possible modifications, generalization, and expansion of the model and its applications. Anyhow, several important aspects still remain open - in particular a pressure-dependent phase-transformation temperature.

Remark 7.1 (*Omitting damage in solid part — stress formulation*)**.** When damage (or the phase-field fracture) would not be considered so that the elastic moduli are fixed, one can come back to the stress/velocity formulation like in Sect. 2. Confining ourselves to an isotropic material like in (2.14), the "undamageable" stored energy in terms of stresses can be considered again as (2.16) but now together with the additive decomposition of the total stress $\boldsymbol{\Sigma} = \boldsymbol{S} + \boldsymbol{B}$ to the actual elastic stress $\boldsymbol{S}$ and the deviatoric "back stress" $\boldsymbol{B}$ and the stored energy φ^* from (2.16). In terms of a back stress rate $\boldsymbol{R} = \overset{\circ}{\boldsymbol{B}}$, the resulting system then reads as

$$
\begin{aligned}
&\varrho\overset{\bullet}{\boldsymbol{v}} = \mathrm{div}\big(\boldsymbol{S} + \varphi^*(\boldsymbol{S})\mathbb{I} + \boldsymbol{D}\big) - \frac{\varrho}{2}(\mathrm{div}\,\boldsymbol{v})\,\boldsymbol{v} + \boldsymbol{f} \quad\text{with}\quad \boldsymbol{D} = \mathbb{D}\boldsymbol{e}(\boldsymbol{v}) - \mathrm{div}\big(\nu|\nabla\boldsymbol{e}(\boldsymbol{v})|^{p-2}\nabla\boldsymbol{e}(\boldsymbol{v})\big)\,, && (7.1a)\\
&\overset{\circ}{\boldsymbol{S}} = \big[[\varphi^*]'\big]^{-1}\boldsymbol{e}(\boldsymbol{v}) - \boldsymbol{R}\,, && (7.1b)\\
&G_{\rm M}(w)\boldsymbol{R} = 2G_{\rm E}\mathrm{dev}\boldsymbol{S} + \varkappa\Delta\boldsymbol{R}\,, && (7.1c)\\
&\frac{\partial w}{\partial t} + \mathrm{div}\big(\boldsymbol{v}\,w - \mathscr{K}(\alpha,w)\nabla\theta\big) = \frac{G_{\rm M}(w)}{4G_{\rm E}^2}|\boldsymbol{R}|^2 + \frac{\varkappa}{4G_{\rm E}^2}|\nabla\boldsymbol{R}|^2 + \mathbb{D}\boldsymbol{e}(\boldsymbol{v}){:}\boldsymbol{e}(\boldsymbol{v})\\
&\qquad\qquad + \nu|\nabla\boldsymbol{e}(\boldsymbol{v})|^p + \phi(\theta)\mathrm{div}\,\boldsymbol{v} \qquad\text{with}\quad w = \widetilde{\gamma}(\theta) + \ell\chi\,, && (7.1d)\\
&\omega\overset{\bullet}{\chi} + H^{-1}(\chi) \ni \frac{\theta_{\varepsilon\tau}^k}{\theta_{\rm PT}} - 1\,; && (7.1e)
\end{aligned}
$$

in (7.1b), $[[\varphi^*]']^{-1} = \varphi'$ with the convex conjugate φ to φ^*. The energetics in the mechanical part is now revealed by testing (7.1a) by $\boldsymbol{v}$, (7.1b) by the strain $[\varphi^*]'(\boldsymbol{S})$, and (7.1c) by $\boldsymbol{R}$. When added still (7.1d) tested by 1, one obtain the total energy conservation. Altogether, we obtain (3.11)–(3.12) but with $\varphi^*(\boldsymbol{S})$ instead of $\varphi(\boldsymbol{E},\alpha)$, with ζ in (3.11) omitted, and with $G_{\rm M}(w)|\boldsymbol{\Pi}|^2 + \varkappa|\nabla\boldsymbol{\Pi}|^2$ replaced by $G_{\rm M}(w)|\boldsymbol{R}|^2/(2G_{\rm E}) + \varkappa|\nabla\boldsymbol{R}|^2/(2G_{\rm E})$. For an isothermal incompressible variant with nonlinear creep (involving possibly also plasticity) and with $\boldsymbol{B}$ eliminated see [15].

Remark 7.2 (*Nonlinear or even activated creep*)**.** The (materially) linear creep law (3.3c) is an idealization and in some materials such linear Maxwellian rheology may not be realistic. Then a non-quadratic (or even nonsmooth at 0) dissipation potential $\zeta_{\rm crp} : \mathbb{R}^{d\times d}_{\rm dev} \to R$ should be considered and (3.3c) is to be generalized as

$$G_{\rm M}(w)\partial\zeta_{\rm crp}\big(\boldsymbol{\Pi}\big) \ni \mathrm{dev}\,\boldsymbol{S} + \varkappa\Delta\boldsymbol{\Pi}\,, \tag{7.2}$$

where $\partial\zeta_{\rm crp}$ denotes the convex subdifferential of $\zeta_{\rm crp}$. By the definition of the convex subdifferential, (7.2) integrated over $I\times\Omega$ means the variational inequality

$$\int_0^T\!\!\int_\Omega \zeta_{\rm crp}\big(\widetilde{\boldsymbol{\Pi}}\big) - \boldsymbol{S}{:}\big(\widetilde{\boldsymbol{\Pi}}{-}\boldsymbol{\Pi}\big) + \varkappa\nabla\boldsymbol{\Pi}\,\vdots\,\nabla\widetilde{\boldsymbol{\Pi}}\,\mathrm{d}x\mathrm{d}t \geq \int_0^T\!\!\int_\Omega \zeta_{\rm crp}\big(\boldsymbol{\Pi}\big) + \varkappa|\nabla\boldsymbol{\Pi}|^2\,\mathrm{d}x\mathrm{d}t \tag{7.3}$$

to hold for any $\widetilde{\boldsymbol{\Pi}} \in L^q(I{\times}\Omega;\mathbb{R}^{d\times d}_{\rm dev}) \cap L^2(I;H^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))$ if we assume coercivity $\zeta_{\rm crp}(\cdot) \geq \epsilon|\cdot|^q$ with some $q > 1$. Then $\boldsymbol{\Pi} \in L^q(I{\times}\Omega;\mathbb{R}^{d\times d}_{\rm dev}) \cap L^2(I;H^1(\Omega;\mathbb{R}^{d\times d}_{\rm dev}))$. To converge with the approximate solution in (7.3), we use the strong convergence (6.40a) of $\boldsymbol{S} = [\varphi_\varepsilon]'_{\boldsymbol{E}}(\boldsymbol{E},\alpha)$ and weak convergence of $\boldsymbol{\Pi}$ together with convexity of $\zeta_{\rm crp} + \varkappa|\cdot|^2$.

Remark 7.3 (*A phenomenological model of ice creep*)**.** An example of a nonlinear creep from Remark 7.2 is a phenomenological Glen's law [22] for polycrystalline ice: strain rate is proportional to the n-power of stress (with n around 3). This fits with the potential $\zeta_{\rm crp}(\cdot) = G_0|\cdot|^q$ with $q = 1 + 1/n$. When $\zeta_{\rm crp}$ is nonsmooth at 0, we obtain an activated creep (plasticity), which is also a phenomenon sometimes considered for ice.

Remark 7.4 (*Rate-and-state friction model*)**.** Letting the dissipation potential $\zeta_{\rm crp}$ state dependent, i.e. dependent on α and w, allows for mimicking the popular Dieterich-Ruina's [14, 40] rate-and-state friction model for sliding of rocks along tectonic faults, cf. [35, Sect. 6].

Remark 7.5 (*Adiabatic effects: buoyancy*)**.** Another example of the adiabatic effects is the Oberbeck-Boussinesq' simplified buoyancy model used for incompressible media in an external (typically gravitational) field. Although these media are modelled incompressible, they anyhow exhibit a slight thermal expansibility, which gives rise to the extra force by replacing $\boldsymbol{f}$ with $\boldsymbol{f}(1-b(\theta))$ in (3.3a) with $b(\cdot)$ continuous.Then (3.3e) expands by the adiabatic heat source/sink $b(\theta)\boldsymbol{v}\cdot\boldsymbol{f}$. Like for the term $\phi(\theta)\mathrm{div}\,\boldsymbol{v}$, the a-priori estimation strategy uses several-times the Gagliardo-Nirenberg interpolation, and essentially relies on that, if $\boldsymbol{f}$ assumed bounded and $b(\cdot)$ has a certain at most polynomial (even superlinear) growth, $\theta \in L^{1+2/d-\epsilon}(I\times\Omega;\mathbb{R}^d)$ is well in duality with $\boldsymbol{v} \in L^\infty(I;L^2(\Omega;\mathbb{R}^d)) \cap L^p(I;W^{2,p}(\Omega;\mathbb{R}^d))$, cf. the estimates (6.22a) and (6.25a), which actually can be proved for this adiabatic coupling, too. Here, during solid-fluid phase transformation, the density (approximated as constant) may in fact jump up, which leads to discontinuity of b at $\theta_{\rm PT}$. E.g. water is about 8% heavier than ice at $\theta_{\rm PT} = 273°$C. Such a discontinuity may easily be treated by making the buoyancy b dependent continuously on w rather than on θ, similarly as we did for $G_{\rm M}$, cf. Figure 3. Such jump is e.g. during ice/water transition and makes ice floats (icebergs) floating rather on the sea surface because ice is lighter than water even if both are on the same melting/freezing temperature, which comes back to the original motivation of Josef Stefan in melting of arctic floats (that time without any mechanical context, of course).

Remark 7.6 (*Traction load*)**.** Considering nonhomogeneous mechanical boundary condition with the load $\boldsymbol{g}$ as (2.5) also in (3.4) would bring technical difficulties in estimation strategy. Note that the term $\int_\Gamma \boldsymbol{g}_\tau^k\cdot\boldsymbol{v}_{\varepsilon\tau}^k\,\mathrm{d}S$ arising in (6.21) would not be well controlled. One would need to add heat equation tested by 1/2 (instead of 1) to (6.20) to see some dissipation term to be able to control the traces of $\boldsymbol{v}_{\varepsilon\tau}^k$ on Γ but, on the other hand, an estimation of the resting adiabatic term $\phi(\theta_{\varepsilon\tau}^k)\mathrm{div}\boldsymbol{v}_{\varepsilon\tau}^k/2$ is to be performed, for which we would need to impose some polynomial growth of heat capacity for $\theta \to \infty$.

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