BLOW-UP FOR WAVE EQUATION WITH THE SCALE-INVARIANT DAMPING AND COMBINED NONLINEARITIES

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Abstract. In this article, we study the blow-up of the damped wave equation in the *scale-invariant case* and in the presence of two nonlinearities. More precisely, we consider the following equation:

$$u_{tt} - \Delta u + \frac{\mu}{1+t} u_t = |u_t|^p + |u|^q, \quad \text{in } \mathbb{R}^N \times [0,\infty),$$

with small initial data.
For $\mu < \frac{N(q-1)}{2}$ and $\mu \in (0,\mu_*)$, where $\mu_* > 0$ is depending on the nonlinearties' powers and the space dimension (μ_* satisfies $(q-1)\left((N+2\mu_*-1)p-2\right) = 4$), we prove that the wave equation, in this case, behaves like the one without dissipation ($\mu = 0$). Our result completes the previous studies in the case where the dissipation is given by $\frac{\mu}{(1+t)^\beta} u_t$; $\beta > 1$ ([11]), where, contrary to what we obtain in the present work, the effect of the damping is not significant in the dynamics. Interestingly, in our case, the influence of the damping term $\frac{\mu}{1+t} u_t$ is important.

1. Introduction

We consider the following family of semilinear damped wave equations

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{(1+t)^\beta} u_t = a|u_t|^p + b|u|^q, & \text{in } \mathbb{R}^N \times [0,\infty), \\ u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.1}$$

where a and b are nonnegative constants, $\mu \geq 0$ and $\beta > 0$. Moreover, the parameter ε is supposed to be a positive number small enough and f and g are positive functions which are compactly supported on $B_{\mathbb{R}^N}(0,1)$.

Throughout this article, we suppose that $p, q > 1$ and $q \leq \frac{2N}{N-2}$ if $N \geq 3$.

It is worth-mentioning that the presence of two nonlinearities in (1.1) has an interesting effect on the (global) existence or the nonexistence of the solution of (1.1) and its lifespan. Hence, it is natural to study the influence of the nonlinear terms on the behavior of the solution and see whether or not this may produce a kind of competition between these nonlinearities.

2010 *Mathematics Subject Classification.* 35L71, 35B44.
Key words and phrases. scale-invariant damping, nonlinear wave equations, blow-up, lifespan.

It is well-known that in the *scattering* case, $\beta > 1$, the solution, u^L, of the linear equation corresponding to (1.1), namely

$$u^L_{tt} - \Delta u^L + \frac{\mu}{(1+t)^\beta} u^L_t = 0, \tag{1.2}$$

behaves like the one of the wave equation without damping ($\mu = 0$). In particular, this means that the damping term does not play any role. On the other hand, for $\beta < 1$, which corresponds to the *effective* case, the solution of the linear equation (1.2) behaves like the corresponding parabolic equation, namely $\frac{\mu}{(1+t)^\beta} u_t - \Delta u = 0$, see e.g. [20, 21, 22] and the references therein. However, the case $\beta = 1$ corresponds to the *scale-invariant* damping. Indeed, the equation (1.2) is thus invariant under a hyperbolic scaling. The scale-invariant case constitutes, thus, a transition between the parabolic and hyperbolic types. In fact, in this transition, the parameter μ plays a crucial role in determining the behavior of the solution of (1.2), see for example [20].

Coming back to (1.1) and letting $\mu = 0$ and $(a, b) = (0, 1)$, then the equation (1.1) reduces to the classical semilinear wave equation which is somehow related to the Strauss conjecture. This case gives rise to a critical power, denoted by q_S, which is a solution of the following quadratic equation

$$(N-1)q^2 - (N+1)q - 2 = 0, \tag{1.3}$$

and given explicitly by

$$q_S = q_S(N) := \frac{N+1+\sqrt{N^2+10N-7}}{2(N-1)}. \tag{1.4}$$

More precisely, if $q \leq q_S$ then there is no global solution for (1.1) with small initial data, and for $q > q_S$ a global solution exists; see e.g. [8, 16, 23, 24] among many other references.

Now, for the case $\mu = 0$ and $(a, b) = (1, 0)$, the Glassey conjecture states that the critical power p_G is given by

$$p_G = p_G(N) := 1 + \frac{2}{N-1}. \tag{1.5}$$

The above critical value, p_G, gives rise to two regions for p ensuring the existence ($p < p_G$) or the nonexistence ($p \geq p_G$) of a global solution; see e.g. [3, 5, 7, 14, 15, 17, 25].

The case $\mu = 0$ and $a, b \neq 0$ (we can assume without loss of generality that $(a, b) = (1, 1)$) presents a new phenomenon related to the combined nonlinearities. Indeed, in this case, the powers satisfying $p \leq p_G$ or $q \leq q_S$ naturally imply the solution blow-up by a simple adaptation of the proofs in the previous cases $(a, b) = (0, 1)$ or $(1, 0)$. However, the novelty in the present situation consists in the obtaining of an additional region where the solution blows up. This new region is characterized by the following relationship between p and q:

$$\lambda(p,q,N) := (q-1)\left((N-1)p-2\right) < 4. \tag{1.6}$$

We refer the reader to [1, 2, 4, 19] for more details.

Now, we focus on the case $\mu > 0$. First, we recall, as mentioned above, the fact that $\beta > 1$ in (1.1) does not influence the dynamics [9, 10, 18]. However, for the scale-invariant case, $\beta = 1$, we will see that the situation in the present article is totally different. In fact, for $(a,b) = (0,1)$, it is known in the literature that if the weak damping coefficient μ is relatively large, then the equation (1.1) (with $(a,b) = (0,1)$) behaves like the corresponding heat equation. Though, if μ is small, then the behavior of (1.1) is following the one of the corresponding wave equation. More precisely, for μ small, it was proven, in [12] and later on in [6] with a substantial improvement, that the critical power is moving a bit compared to the case without damping, and hence we have for

$$< 0 < \mu < \frac{N^2+N+2}{N+2} \quad \text{and} \quad 1 < q \leq q_S(N+\mu),$$

the blow-up of the solution of (1.1).

On the other hand for $\mu > 0$ and $(a,b) = (1,0)$, the authors prove in [10] a blow-up result for the solution of (1.1) (with $(a,b) = (1,0)$) and they give an upper bound of the lifespan. We stress the fact that in this case there is no restriction for μ in the the blow-up region for p, namely $p \in (1, p_G(N+2\mu))$.

In this work, we consider the following Cauchy problem for the scale-invariant wave equation with combined nonlinearities,

$$\begin{cases} u_{tt} - \Delta u + \dfrac{\mu}{1+t} u_t = |u_t|^p + |u|^q, & \text{in } \mathbb{R}^N \times [0,\infty), \\ u(x,0) = \varepsilon f(x), \ u_t(x,0) = \varepsilon g(x), & x \in \mathbb{R}^N, \end{cases} \tag{1.7}$$

where $\mu > 0$, $N \geq 1$, $\varepsilon > 0$ is a sufficiently small parameter and f, g are chosen in the energy space with compact support.

The emphasis in our work is the study of the Cauchy problem (1.7) for $\mu > 0$ and the influence of this parameter on the blow-up result and the lifespan estimate. For the analogous system of (1.7) with $(\mu/(1+t))u_t$ being replaced by $(\mu/(1+t)^\beta)u_t$ and $\beta > 1$, which corresponds to the scattering case, Lai and Takamura proved in [11] that, comparing to the wave equation without damping, the scattering damping term has no influence in the dynamics. The situation is totally different in the scale-invariant case ($\beta = 1$) where the effect of the weak damping is significant in the study of global existence or blow-up of the solution of (1.7). To overcome the difficulty related to this case, we choose in this work to use the technique of multiplier, and, unlike the scattering case, the multiplier here is not bounded as we can see in (2.2) below. Finally, we stress out that the determination of the threshold for μ and the obtaining of the analogous of the assumption (1.6) constitute the main objectives of this article.

Due to the nature of the problem under consideration here, we notice some interesting challenges. For example, it is natural to look for the critical value of μ where this transition holds. Nevertheless, it is known that, even in the simpler case $(a, b) = (0, 1)$, no critical value is known and it was only conjectured the existence of such critical value; see e.g. [6, 12, 13].

The rest of the article is organized as follows. In Section 2, after giving a sense to the solution of (1.7) in the energy space, we state the main theorem of our work. Then, we state and prove in Section 3 some technical lemmas useful in the proof of the main result which is the subject of Section 4.

2. Main Result

In this section, we will state our main result, but before that, we give the definition of the solution of (1.7) in the corresponding energy space which reads as follows:

Definition 2.1. *We say that u is a weak solution of (1.7) on $[0, T)$ if*

$$u \in \mathcal{C}([0, T), H^1(\mathbb{R}^N)) \cap \mathcal{C}^1([0, T), L^2(\mathbb{R}^N)) \cap \mathcal{C}^1((0, T), L^p(\mathbb{R}^N)),$$

satisfies, for all $\Phi \in \mathcal{C}_0^\infty(\mathbb{R}^N \times [0, T))$ and all $t \in [0, T)$, the following equation:

(2.1)
$$\begin{aligned}
&\int_{\mathbb{R}^N} u_t(x,t)\Phi(x,t)dx - \int_{\mathbb{R}^N} u_t(x,0)\Phi(x,0)dx \\
&- \int_0^t \int_{\mathbb{R}^N} u_t(x,s)\Phi_t(x,s)dx\, ds + \int_0^t \int_{\mathbb{R}^N} \nabla u(x,s) \cdot \nabla \Phi(x,s)dx\, ds \\
&+ \int_0^t \int_{\mathbb{R}^N} \frac{\mu}{1+s} u_t(x,s)\Phi_t(x,s)dx\, ds = \int_0^t \int_{\mathbb{R}^N} \left\{|u_t(x,s)|^p + |u(x,s)|^q\right\} \Phi(x,s)dx\, ds.
\end{aligned}$$

Now, we introduce the following multiplier

$$m(t) := (1+t)^\mu. \tag{2.2}$$

Using the above definition of $m(t)$, we simply observe that

$$\frac{m'(t)}{m(t)} = \frac{\mu}{1+t}.$$

Note that the use of multiplier's technique is useful for the study of the nonlinear damped wave equation in our case.

Hence, with the help of the multiplier $m(t)$, Definition 2.1 can be written in the following equivalent formulation.

Definition 2.2. *We say that u is a weak solution of (1.7) on $[0,T)$ if*

$$u \in \mathcal{C}([0,T), H^1(\mathbb{R}^N)) \cap \mathcal{C}^1([0,T), L^2(\mathbb{R}^N)) \cap \mathcal{C}^1((0,T), L^p(\mathbb{R}^N)),$$

satisfies, for all $\Phi \in \mathcal{C}_0^\infty(\mathbb{R}^N \times [0,T))$ and all $t \in [0,T)$, the following equation:

$$\begin{aligned} & m(t)\int_{\mathbb{R}^N} u_t(x,t)\Phi(x,t)dx - \int_{\mathbb{R}^N} u_t(x,0)\Phi(x,0)dx \\ & - \int_0^t m(s)\int_{\mathbb{R}^N} u_t(x,s)\Phi_t(x,s)dx\,ds + \int_0^t m(s)\int_{\mathbb{R}^N} \nabla u(x,s)\cdot\nabla\Phi(x,s)dx\,ds \\ & = \int_0^t m(s)\int_{\mathbb{R}^N} \{|u_t(x,s)|^p + |u(x,s)|^q\}\,\Phi(x,s)dx\,ds. \end{aligned} \tag{2.3}$$

The main result of this article is then stated in the following theorem.

Theorem 2.3. *Let p, q and $\mu < \frac{N(q-1)}{2}$ be such that*

$$\lambda(p,q,N+2\mu) < 4, \tag{2.4}$$

where the expression of λ is given by (1.6).
Assume that $f \in H^1(\mathbb{R}^N)$ and $g \in L^2(\mathbb{R}^N)$ are non-negative functions which are compactly supported on $B_{\mathbb{R}^N}(0,1)$, and do not vanish everywhere. Let u be an energy solution of (1.7) on $[0,T_\varepsilon)$ such that $\operatorname{supp}(u) \subset \{(x,t) \in \mathbb{R}^N \times [0,\infty) : |x| \le t+1\}$. Then, there exists a constant $\varepsilon_0 = \varepsilon_0(f,g,N,p,q,\mu) > 0$ such that T_ε verifies

$$T_\varepsilon \le C\,\varepsilon^{-\frac{2p(q-1)}{4-\lambda(p,q,N+2\mu)}},$$

where C is a positive constant independent of ε and $0 < \varepsilon \le \varepsilon_0$.

Remark 2.1. Unlike the case with only one nonlinearity ($|u_t(x,s)|^p$ or $|u(x,s)|^q$), one can note, in addition to the two blow-up regions $p \le p_G$ and $q \le q_S$, the obtaining of another blow-up region, characterized by (1.6), which is the result of the interaction of the combined nonlinearities, see [4]. This observation still holds in our case but with (1.6) being replaced by (2.4), otherwise $p_G(N)$ being replaced by $p_G(N+2\mu)$ and $q_S(N)$ by $q_S(N+\mu)$.

Remark 2.2. The assumption (2.4) can be seen as a smallness condition for μ, namely $\mu \in [0,\mu_*)$ where $\mu = \mu_*$ satisfies the equality in (2.4) (otherwise $\mu_* := \frac{q+1}{p(q-1)} - \frac{N-1}{2}$).

Remark 2.3. Note that the results in Theorem 2.3 hold true after replacing the linear damping term in (1.7) $\frac{\mu}{1+t}u_t$ by $\mu b(t)u_t$ with $b(t)$ behaving like $(1+t)^{-1}$ as t goes to ∞. The proof of this generalized damping case can be obtained by following the same steps as in the proof of Theorem 2.3 with the necessary modifications.

3. Some auxiliary results

We introduce the following two positive test functions

$$\psi(x,t) := e^{-t}\phi(x), \quad \phi(x) := \begin{cases} \displaystyle\int_{S^{N-1}} e^{x\cdot\omega} d\omega & \text{for } N \geq 2, \\ e^x + e^{-x} & \text{for } N = 1, \end{cases} \tag{3.1}$$

which was introduced in Yordanov and Zhang [23] and admits the following good properties:

$$\partial_t \psi = -\psi, \quad \partial_{tt}\psi = \Delta\psi = \psi.$$

Moreover, we have the following lemma for the function $\psi(x,t)$.

Lemma 3.1 ([23])**.** *Let $r > 1$. There exists a constant $C = C(N,p,r) > 0$ such that*

$$\int_{|x|\leq t+1} \Big(\psi(x,t)\Big)^r dx \leq C(1+t)^{\frac{(2-r)(N-1)}{2r}}, \quad \forall\ t \geq 0. \tag{3.2}$$

As in the non-perturbed case, we define here the functionals that we will use to prove the blow-up criteria later on:

$$F_1(t) := \int_{\mathbb{R}^N} u(x,t)\psi(x,t)dx, \tag{3.3}$$

and

$$F_2(t) := \int_{\mathbb{R}^N} \partial_t u(x,t)\psi(x,t)dx. \tag{3.4}$$

The next two lemmas give the first lower bounds for $F_1(t)$ and $F_2(t)$, respectively.

Lemma 3.2. *Assume that the assumption in Theorem 2.3 holds. Then, we have*

$$F_1(t) \geq \frac{\varepsilon}{2m(t)} \int_{\mathbb{R}^N} f(x)\phi(x)dx, \quad \textit{for all } t \in [0,T). \tag{3.5}$$

Proof. Using Definition 2.2 and by performing an integration by parts in space in the fourth term in the left-hand side of (2.3), we obtain

$$\begin{aligned} & m(t)\int_{\mathbb{R}^N} u_t(x,t)\Phi(x,t)dx - \varepsilon\int_{\mathbb{R}^N} g(x)\Phi(x,0)dx \\ & -\int_0^t m(s)\int_{\mathbb{R}^N} \left\{u_t(x,s)\Phi_t(x,s) + u(x,s)\Delta\Phi(x,s)\right\} dx\, ds \\ & = \int_0^t m(s)\int_{\mathbb{R}^N} \left\{|u_t(x,s)|^p + |u(x,s)|^q\right\}\Phi(x,s)dx\, ds, \quad \forall\ \Phi \in \mathcal{C}_0^\infty(\mathbb{R}^N\times[0,T)). \end{aligned} \tag{3.6}$$

Now, substituting in (3.6) $\Phi(x,t)$ by $\psi(x,t)$, we infer that

$$
\begin{aligned}
& m(t)\int_{\mathbb{R}^N} u_t(x,t)\psi(x,t)dx - \varepsilon\int_{\mathbb{R}^N} g(x)\psi(x,0)dx \\
& + \int_0^t m(s)\int_{\mathbb{R}^N} \{u_t(x,s)\psi(x,s) - u(x,s)\psi(x,s)\}\, dx\, ds \\
& = \int_0^t m(s)\int_{\mathbb{R}^N} \{|u_t(x,s)|^p + |u(x,s)|^q\}\, \psi(x,s)dx\, ds.
\end{aligned} \tag{3.7}
$$

Using the definition of F_1, as in (3.3), and the fact that

$$\int_0^t m(s)F_1'(s)ds = -\int_0^t m'(s)F_1(s)ds + m(t)F_1(t) - F_1(0),$$

the equation (3.7) yields

$$
\begin{aligned}
& m(t)(F_1'(t) + 2F_1(t)) - \varepsilon C(f,g) \\
& = \int_0^t m'(s)F_1(s)ds + \int_0^t m(s)\int_{\mathbb{R}^N} \{|u_t(x,s)|^p + |u(x,s)|^q\}\, \psi(x,s)dx\, ds,
\end{aligned} \tag{3.8}
$$

where

$$C(f,g) = \int_{\mathbb{R}^N} \{f(x) + g(x)\}\, \phi(x)dx.$$

Dividing (3.8) by $m(t)$ and multiplying the obtained equation by e^{2t}, we deduce after integrating over $[0,t]$ that

$$e^{2t}F_1(t) \geq F_1(0) + \varepsilon C(f,g)\int_0^t \frac{e^{2s}}{m(s)}ds + \int_0^t \frac{\mu e^{2s}}{m(s)}\int_0^s (1+\tau)^{\mu-1}F_1(\tau)d\tau. \tag{3.9}$$

Thanks to (3.9) and the fact that $F_1(0) > 0$, we can easily see that $F_1(t) > 0$. Hence, we have

$$F_1(t) \geq F_1(0)e^{-2t} + \varepsilon C(f,g)\int_0^t \frac{e^{2s-2t}}{m(s)}ds. \tag{3.10}$$

Remember that $m(t)$, given by (2.2), is an increasing function (since here $\mu > 0$), we get

$$F_1(t) \geq F_1(0)e^{-2t} + \frac{\varepsilon C(f,g)}{2m(t)}(1 - e^{-2t}) \geq \varepsilon C(f,0)e^{-2t} + \frac{\varepsilon C(f,0)}{2m(t)}(1 - e^{-2t}). \tag{3.11}$$

Finally, using $m(t) \geq 1$, we obtain (3.5). This ends the proof of Lemma 3.2. $\square$

Now we are in a position to prove the following lemma.

Lemma 3.3. *Under the same assumption of Theorem 2.3, it holds that*

$$F_2(t) \geq \frac{\varepsilon}{2m(t)}\int_{\mathbb{R}^N} g(x)\phi(x)dx, \quad \textit{for all } t \in [0,T). \tag{3.12}$$

Proof. Let $t \in [0,T)$. Hence, using the definition of F_1 and F_2, given respectively by (3.3) and (3.4), and the fact that

$$F_1'(t) + F_1(t) = F_2(t), \tag{3.13}$$

the equation (3.8) yields

$$\begin{aligned} & m(t)(F_2(t)+F_1(t)) - \varepsilon C(f,g) \\ & = \int_0^t m'(s)F_1(s)ds + \int_0^t m(s)\int_{\mathbb{R}^N} \{|u_t(x,s)|^p + |u(x,s)|^q\}\,\psi(x,s)dx\,ds. \end{aligned} \tag{3.14}$$

Differentiating the equation (3.14) in time, we obtain

$$\frac{d}{dt}\{F_2(t)m(t)\} + m(t)\frac{d}{dt}F_1(t) = m(t)\int_{\mathbb{R}^N} \{|u_t(x,t)|^p + |u(x,t)|^q\}\,\psi(x,t)dx. \tag{3.15}$$

Using (3.13), the identity (3.15) becomes

$$\begin{aligned} \frac{d}{dt}\{F_2(t)m(t)\} + 2m(t)F_2(t) = m(t)\{F_1(t)+F_2(t)\} + \\ + m(t)\int_{\mathbb{R}^N} \{|u_t(x,t)|^p + |u(x,t)|^q\}\,\psi(x,t)dx. \end{aligned} \tag{3.16}$$

Thanks to (3.14) and Lemma 3.2, we can easily see that $m(t)(F_2(t)+F_1(t)) \geq \varepsilon C(f,g)$. Then, (3.16) implies that

$$\frac{d}{dt}\left\{F_2(t)m(t)e^{2t}\right\} \geq \varepsilon C(f,g)e^{2t}. \tag{3.17}$$

By integrating in time between 0 and t the inequality (3.17), we obtain

$$F_2(t)m(t)e^{2t} \geq F_2(0) + \varepsilon C(f,g)\int_0^t e^{2s}ds \geq \frac{\varepsilon C(0,g)}{2}e^{2t}. \tag{3.18}$$

So, by (3.18), we have (3.12). This concludes the proof of Lemma 3.3. $\square$

4. Proof of Theorem 2.3

In this section, we will give the proof of the main theorem in this article which states the blow-up result and the lifespan estimate of the solution of (1.7). For that purpose, we will make use of the lemmas proven in Section 3, the multiplier $m(t)$ and a Kato's lemma type.

Throughout this section, we will denote by C a generic positive constant which may depend on the data (p,q,μ,N,f,g) but not on ε and of which the value may change from line to line, but, we keep the same notation to make the presentation simpler.

First, using the hypotheses in Theorem 2.3, we recall that $\mathrm{supp}(u) \subset \{(x,t) \in \mathbb{R}^N \times [0,\infty) : |x| \leq t+1\}$.

Then, we set

$$F(t) := \int_{\mathbb{R}^N} u(x,t)dx. \tag{4.1}$$

Now, by choosing the test function ϕ in (2.3) such that $\phi \equiv 1$ in $\{(x,s) \in \mathbb{R}^N \times [0,t] : |x| \leq s+1\}$[1], we get

$$m(t)\int_{\mathbb{R}^N} u_t(x,t)dx - \int_{\mathbb{R}^N} u_t(x,0)dx = \int_0^t m(s)\int_{\mathbb{R}^N} \{|u_t(x,s)|^p + |u(x,s)|^q\}\, dx\, ds. \tag{4.2}$$

Using the definition of F, (4.2) can be written as

$$m(t)F'(t) = F'(0) + \int_0^t m(s)\int_{\mathbb{R}^N} \{|u_t(x,s)|^p + |u(x,s)|^q\}\, dx\, ds. \tag{4.3}$$

Therefore, by dividing (4.3) by $m(t)$, integrating over $(0,t)$ and using the positivity of $F(0)$ and $F'(0)$, we infer that

$$F(t) \geq \int_0^t \frac{1}{m(s)}\int_0^s m(\tau)\int_{\mathbb{R}^N} \{|u_t(x,\tau)|^p + |u(x,\tau)|^q\}\, dx\, d\tau\, ds. \tag{4.4}$$

By Hölder's inequality and the estimates (3.2) and (3.12), we may bound the nonlinear term as follows:

$$\int_{\mathbb{R}^N} |u_t(x,t)|^p dx \geq F_2^p(t)\left(\int_{|x|\leq t+1}\Big(\psi(x,t)\Big)^{\frac{p}{p-1}}dx\right)^{-(p-1)} \geq C\varepsilon^p(1+t)^{-\mu p - \frac{(N-1)(p-2)}{2}}.$$

Plugging the above inequality into (4.4), we obtain

$$F(t) \geq C\varepsilon^p \int_0^t (1+s)^{-\mu}\int_0^s (1+\tau)^{-\mu(p-1)-\frac{(N-1)(p-2)}{2}} d\tau\, ds. \tag{4.5}$$

A straightforward computation yields

$$F(t) \geq C\varepsilon^p (1+t)^{-\mu p - \frac{p(N-1)}{2}} t^{N+1}. \tag{4.6}$$

On the other hand, we have

$$\Big(\int_{\mathbb{R}^N} u(x,s)dx\Big)^q \leq \int_{|x|\leq t+1} |u(x,s)|^q dx\Big(\int_{|x|\leq t+1} dx\Big)^{q-1}, \tag{4.7}$$

and consequently we deduce that

$$F^q(t) \leq \big(t+1\big)^{N(q-1)}\int_{|x|\leq t+1} |u(x,s)|^q dx. \tag{4.8}$$

Now, by differentiating (4.3) with respect to time, we obtain

$$(m(t)F'(t))' \geq m(t)\int_{\mathbb{R}^N} \{|u_t(x,t)|^p + |u(x,t)|^q\}\, dx \geq m(t)\int_{\mathbb{R}^N} |u(x,t)|^q dx. \tag{4.9}$$

[1]The choice of a test function ϕ which is identically equal to 1 is possible thanks to the fact that the initial data f and g are supported on $B_{\mathbb{R}^N}(0,1)$.

Using (4.8) in (4.9), we infer that

$$(m(t)F'(t))' \geq \frac{F^q(t)}{(1+t)^{N(q-1)-\mu}}. \tag{4.10}$$

Thanks to (4.3) we have $m(t)F_0'(t) > 0$. Then, multiplying (4.10) by $m(t)F_0'(t)$ yields

$$\left\{\Big(m(t)F'(t)\Big)^2\right\}' \geq \frac{2\Big(F^{q+1}(t)\Big)'}{(q+1)(1+t)^{N(q-1)-2\mu}}. \tag{4.11}$$

Integrating the above inequality and using $\mu < \dfrac{N(q-1)}{2}$, we have

$$\Big(m(t)F'(t)\Big)^2 \geq \frac{2F^{q+1}(t)}{(q+1)(1+t)^{N(q-1)-2\mu}} + \left((F'(0))^2 - \frac{2F^{q+1}(0)}{(q+1)}\right). \tag{4.12}$$

Observe that the last term in the right-hand side of (4.12) is positive since we consider here small initial data, and more precisely this holds for ε small enough.
Hence, (4.12) implies that

$$\frac{F'(t)}{F^{1+\delta}(t)} \geq \sqrt{\frac{2}{q+1}}\frac{F^{\frac{q-1}{2}-\delta}(t)}{(1+t)^{\frac{N(q-1)}{2}}}, \tag{4.13}$$

for $\delta > 0$ small enough.

Integrating the inequality (4.13) on $[T_0, t]$, for $T_0 > 1$, and using (4.6), we obtain

$$\frac{1}{\delta}\Big(\frac{1}{F^\delta(T_0)} - \frac{1}{F^\delta(t)}\Big) \geq \sqrt{\frac{2}{q+1}}(C\varepsilon^p)^{\frac{q-1}{2}-\delta}\int_{T_0}^t \frac{(1+s)^{(2-\mu p-\frac{(N-1)(p-2)}{2})(\frac{q-1}{2}-\delta)}}{(1+s)^{\frac{N(q-1)}{2}}}ds. \tag{4.14}$$

Neglecting the second term on the left-hand side in (4.14) which gives

$$\frac{1}{F^\delta(T_0)} \geq \delta\sqrt{\frac{2}{q+1}}(C\varepsilon^p)^{\frac{q-1}{2}-\delta}\int_{T_0}^t (1+s)^{-\frac{\lambda(p,q,N+2\mu)}{4}-\delta\big(2-\mu p-\frac{(N-1)(p-2)}{2}\big)}ds. \tag{4.15}$$

Using the hypothesis (2.4), we have $-\frac{\lambda(p,q,N+2\mu)}{4}+1 > 0$. Hence, we can choose $\delta = \delta_0$ small enough such that $\gamma := -\frac{\lambda(p,q,N+2\mu)}{4} - \delta_0\left(2-\mu p - \frac{(N-1)(p-2)}{2}\right) > -1$. Then, the estimate (4.15) yields

$$\frac{1}{F^{\delta_0}(T_0)} \geq C\varepsilon^{\frac{p(q-1)}{2}-p\delta_0}\left((1+t)^{\gamma+1} - (1+T_0)^{\gamma+1}\right). \tag{4.16}$$

Now, using (4.6) and the fact that $T_0 > 1$, we infer that

$$\varepsilon^{\frac{p(q-1)}{2}}\left((1+t)^{\gamma+1} - (1+T_0)^{\gamma+1}\right) \leq C(1+T_0)^{-2\delta_0+\mu p\delta_0+\frac{(N-1)(p-2)\delta_0}{2}}. \tag{4.17}$$

Consequently, we have

$$\begin{aligned}\varepsilon^{\frac{p(q-1)}{2}}(1+t)^{\gamma+1} \quad &\leq \quad C_0(1+T_0)^{-2\delta_0+\mu p\delta_0+\frac{(N-1)(p-2)\delta_0}{2}} \\ &\quad +\varepsilon^{\frac{p(q-1)}{2}}(1+T_0)^{\gamma+1},\end{aligned} \tag{4.18}$$

where $C_0 = C_0(p, q, \mu, N, f, g)$.
At this level, since $-\frac{\lambda(p,q,N+2\mu)}{4} + 1 > 0$, then for all $\varepsilon > 0$, we choose $T_0 > 1$ such that

$$T_0^{-\frac{\lambda(p,q,N+2\mu)}{4}+1} = C_0 \varepsilon^{-\frac{p(q-1)}{2}}. \tag{4.19}$$

Hence, using (4.19), we deduce from (4.18) that

$$t \leq 2^{\frac{1}{\gamma+1}}(1+T_0) \leq C_1 \varepsilon^{-\frac{2p(q-1)}{4-\lambda(p,q,N+2\mu)}}, \tag{4.20}$$

where $C_1 = C_1(p, q, \mu, N, f, g)$.

This achieves the proof of Theorem 2.3. □

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