

AN ECO-EPIDEMIOLOGICAL MODEL WITH GENERAL FUNCTIONAL RESPONSE OF PREDATOR TO PREY

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ABSTRACT. We consider a nonautonomous eco-epidemiological model with general functions for predation on infected and uninfected preys as well as general functions associated to the vital dynamics of the susceptible prey and predator populations. We obtain persistence and extinction results for the infected prey based on assumptions on auxiliary systems constructed from the disease-free system. We moreover consider an iterative process that can improve the extinction results. We apply our results to general eco-epidemiological models that include several examples existent in the literature.

1. INTRODUCTION

The description of the dynamics of eco-epidemiological systems is a subject that have been receiving increasing attention by the researchers interested in mathematical biology. The inclusion of infected classes in predator-prey models has shown that eco-epidemiological dynamics can show several differences to the original models. In particular, the inclusion of a disease in the preys or in the predators have impact on the population size of the predator-prey community [5, 7].

Additionally, to make models more realistic, it is important, in many situations, to consider time varying parameters. For instance, it is well known in epidemiology that incidence rates are seldom subject to periodic seasonal fluctuations. In the context of eco-epidemiological models, several periodic systems have been studied in the literature [1, 2, 7, 8, 10]. In [10] a class of general non-autonomous eco-epidemiological models with disease in the prey, containing the periodic case as a very particular situation, is considered and threshold conditions for the extinction and persistence of the infected preys are obtained. Related to the periodic version of this model, in [12], it is proved the existence of an endemic periodic orbit. In [10, 12], it is assumed that only infected preys are predated. More recently, based on this model, [9] proposed a family of models that include predation on uninfected preys described by a bilinear functional response and obtained threshold conditions for the extinction and persistence of the disease.

In the previous papers, the functional response of the predator to prey is given by some particular function. Also the vital dynamics of predator and prey is assumed to follow some particular law. In this paper we generalize the models in [9, 10] by considering general functions corresponding to the predation on infected and

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uninfected prey and also to the vital dynamics of uninfected prey and predator populations. Namely, we consider the following eco-epidemiological model:

$$\begin{cases} S' = G(t, S) - a(t)f(S, I, P)P - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)g(S, I, P)I - c(t)I \\ P' = h(t, P)P + \gamma(t)a(t)f(S, I, P)P + \theta(t)\eta(t)g(S, I, P)I \end{cases}, \quad (1)$$

where S , I and P correspond, respectively, to the susceptible prey, infected prey and predator, $\beta(t)$ is the incidence rate of the disease, $\eta(t)$ is the predation rate of infected prey, $c(t)$ is the death rate in the infective class, $\gamma(t)$ is the rate converting susceptible prey into predator (biomass transfer), $\theta(t)$ is the rate of converting infected prey into predator, $G(t, S)$ and $h(t, P)P$ represent the vital dynamics of the susceptible prey and predator populations, respectively, $a(t)f(S, I, P)$ is the predation of susceptible prey and $\eta(t)g(S, I, P)$ represent the predation of infected prey. It is assumed that only susceptible preys S are capable of reproducing, i.e., the infected prey is removed by death (including natural and disease-related death) or by predation before having the possibility of reproducing.

The objective of this work is to discuss the uniform strong persistence and extinction of the infectives I in system (1). Recall that the infectives are *uniformly strong persistent* in system (1) if there exist $0 < m_1 < m_2$ such that for every solution $(S(t), I(t), P(t))$ of (1) with positive initial conditions $S(t_0), I(t_0), P(t_0) > 0$, we have

$$m_1 < \liminf_{t \rightarrow \infty} I(t) \leq \limsup_{t \rightarrow \infty} I(t) < m_2,$$

and we say that the infectives I go to *extinction* in system (1) if

$$\lim_{t \rightarrow +\infty} I(t) = 0,$$

for all solutions of (1) with positive initial conditions. For biological reasons we will only consider for system (1) solutions with initial conditions in the set $(\mathbb{R}_0^+)^3$.

Our approach is very different to the one in [10] and [9]. In fact, we want to discuss the extinction and strong persistence of the infectives in system (1), having as departure point some prescribed behaviour of subsystems related to the dynamics of preys and predators in the absence of disease. We will assume that we have global asymptotic stability of solutions of these special bi-dimensional subsystems (see condition S6) in Section 2). Thus, to apply our results to specific situations in the literature, one must first verify that the underlying referred subsystems satisfy our assumptions or, looking at our results differently, we can construct an eco-epidemiological model from a previously studied predator-prey model (the uninfected subsystem) that satisfies our assumptions. We believe that this approach is interesting since it highlights the relation of the dynamics of the eco-epidemiological model with the behaviour of the predator-prey model used in its construction.

We note that, similarly to the thresholds obtained in [9], our thresholds for extinction and uniform strong persistence are not sharp. In spite of this, unlike the conditions for extinction and strong persistence in [9], that rely on parameters that can not, in principle, be computed explicitly (note that conditions (22) and (43) in [9] depend on q_1), our thresholds in Theorem 1 and Theorem 2 can be directly obtained from the parameters and the limit behavior of the predator-(uninfected) prey subsystem.

To illustrate this results, in Section 3 some models available in the literature, satisfying our assumptions, are considered and thresholds conditions for the corresponding eco-epidemiological model automatically obtained from our results: in Section 3.1 we consider the situation where $f \equiv 0$ in system (1), that is with no predation on uninfected preys, corresponding to a generalized version of the situation studied in [10]; in Section 3.2 we obtain a particular form for the threshold conditions in the context of periodic models and particularize our result for a model constructed from the predator-prey model in [4]; in Section 3.3 we consider a model with Michaelis-Menten (or Holling-type I) functional response of predator to infected prey and a Holling-type II functional response of predator to susceptible prey; finally, in Section 3.4 we consider the eco-epidemiological model obtained from an uninfected subsystem with ratio-dependent functional response of predator to prey, a type interaction considered as an attempt to overcome some know biological paradoxes observed in models with Gause-type interaction and again obtain the corresponding results for the eco-epidemiological model, based on the discussion of ratio-dependent predator-prey systems in [6]. For all these examples we present some simulation that illustrate our conclusions.

In Theorem 3 we provide an iterative process that can be used to improve the extinction estimates in Theorem 1. In Section 3.5 we provide an example in order to illustrate this iterative scheme in a simple situation.

2. ECO-EPIDEMIOLOGICAL MODELS WITH ASYMPTOTICALLY STABLE BEHAVIOR IN THE PREDATOR-UNINFECTED PREY SUBSPACE

We will assume the following hypothesis concerning the parameter functions and the functions f , g , G and h appearing in our model (1):

- S1) The real valued functions a , β , η , c , γ and θ are bounded, nonnegative and continuous;
- S2) The real valued functions f , g , G and $H(t, x) = h(t, x)x$ are locally Lipschitz and functions f and g are nonnegative and $f(0, 0, z) = 0$, for every $z \geq 0$. For fixed $x, z \geq 0$, functions $y \mapsto f(x, y, z)$ and $y \mapsto g(x, y, z)$ are nonincreasing; for fixed $y, z \geq 0$, function $x \mapsto g(x, y, z)$ is nonincreasing; for fixed $x, y \geq 0$, function $z \mapsto f(x, y, z)$ is nonincreasing and function $z \mapsto g(x, y, z)$ is nondecreasing

Our next assumption relates to the ω -limit of solutions of (1) and is usually fulfilled by mathematical models in eco-epidemiology.

- S3) Each solution of (1) with positive initial condition is bounded and there is a bounded region \mathcal{R} that contains the ω -limit of all solutions of (1) with positive initial conditions.

Notice in particular that condition S3) implies that there is $L > 0$ such that

$$\limsup_{t \rightarrow +\infty} (S(t) + I(t) + P(t)) < L,$$

for all solutions $(S(t), I(t), P(t))$ of (1) with positive initial conditions.

To proceed, we need to consider two auxiliary equations and one auxiliary system. First, we consider the equation

$$s' = G(t, s), \tag{2}$$

that corresponds to the dynamics of uninfected preys in the absence of infected preys and predators (the first equation in system (1) with $I = 0$ and $P = 0$). We assume the following properties for the solutions of (2):

- S4) Each solution $s(t)$ of (2) with positive initial condition is bounded, bounded away from zero, and globally attractive on $]0, +\infty[$, that is $|s(t) - v(t)| \rightarrow 0$ as $t \rightarrow +\infty$ for each solution $v(t)$ of (2) with positive initial condition.

The second auxiliary equation we consider is the equation

$$y' = h(t, y)y, \quad (3)$$

that corresponds to the dynamics of predators in the absence of the considered preys (the third equation in system (1) with $I = 0$ and $S = 0$). We need the following property for the solutions of (3):

- S5) Each fixed solution $y(t)$ of (3) with positive initial condition is bounded and globally attractive on $[0, +\infty)$.

Finally, starting from the *uninfected subsystem*, that is, the system that describes the behavior of preys and predators in the absence of infected preys (the first and third equations of system (1) with $I = 0$), given by

$$\begin{cases} x' = G(t, x) - a(t)f(x, 0, z)z \\ z' = h(t, z)z + \gamma(t)a(t)f(x, 0, z)z \end{cases}, \quad (4)$$

we assume that we are able to construct families of auxiliary subsystems:

$$\begin{cases} x' = G_{1,\varepsilon}(t, x) - a(t)f(x, 0, 0)\hat{z}_\varepsilon(t) - v(\varepsilon)\rho(t)x \\ z' = h_{1,\varepsilon}(t, z)z + \gamma(t)a(t)f(x, v(\varepsilon)\rho^u, z)z \end{cases} \quad (5)$$

where $(\hat{x}_\varepsilon(t), \hat{z}_\varepsilon(t))$ is a solution of

$$\begin{cases} x' = G_{2,\varepsilon}(t, x) \\ z' = h_{2,\varepsilon}(t, z)z + \gamma(t)a(t)f(x, 0, z)z + v(\varepsilon)\rho(t)g(x, 0, z) \end{cases} \quad (6)$$

satisfying the following assumptions.

- S6) The following holds for systems (5) and (6):

- S6.1) for sufficiently small $\varepsilon > 0$, the functions $G_{i,\varepsilon}$ and $h_{i,\varepsilon}$, $i = 1, 2$, are continuous, the functionals $\varepsilon \mapsto G_{i,\varepsilon}$ and $\varepsilon \mapsto h_{i,\varepsilon}$, $i = 1, 2$, are continuous, $G_{1,0} = G_{2,0} = G$, $h_{1,0} = h_{2,0} = h$,

$$G_{1,\varepsilon}(t, x) \leq G(t, x) \leq G_{2,\varepsilon}(t, x)$$

and

$$h_{1,\varepsilon}(t, x) \leq h(t, x) \leq h_{2,\varepsilon}(t, x);$$

- S6.2) the real valued function $v : [0, +\infty[\rightarrow \mathbb{R}$ verifies $v(\varepsilon) > 0$ for $\varepsilon \in]0, +\infty[$, $v(0) = 0$ and is differentiable near $\varepsilon = 0$ with

$$A < v'(\varepsilon) < B,$$

for some $A, B > 0$ and sufficiently small $\varepsilon \geq 0$;

S6.3) there are constants ρ^u, ρ^ℓ such that, for all $t \geq 0$,

$$0 < \rho^\ell \leq \rho(t) \leq \rho^u;$$

S6.4) there is a family of nonnegative solutions, $\{(x_{1,\varepsilon}^*(t), z_{1,\varepsilon}^*(t))\}$ of system (5), one solution for each $\varepsilon \geq 0$ sufficiently small, depending on a solution $(x_{2,\varepsilon}^*(t), z_{2,\varepsilon}^*(t))$ of system (6), such that each solution in the family is globally asymptotically stable in a set containing the set $(\mathbb{R}^+)^2$ and the function

$$\varepsilon \mapsto (x_{1,\varepsilon}^*(t), z_{1,\varepsilon}^*(t)) \text{ is continuous;}$$

S6.5) the family of nonnegative solutions $\{(x_{2,\varepsilon}^*(t), z_{2,\varepsilon}^*(t))\}$ of system (6), one solution for each $\varepsilon \geq 0$ sufficiently small, is such that each solution in the family is globally asymptotically stable in a set containing the set $(\mathbb{R}^+)^2$ and the function

$$\varepsilon \mapsto (x_{2,\varepsilon}^*(t), z_{2,\varepsilon}^*(t)) \text{ is continuous.}$$

We write $x_{1,0}^* = x_1^*$, $x_{2,0}^* = x_2^*$, $z_{1,0}^* = z_1^*$ and $z_{2,0}^* = z_2^*$ for the components of the solutions in S6.4) and S6.5) corresponding to $\varepsilon = 0$. For the continuity of the functionals in S6.1), S6.4) and S6.5) we consider the usual supremum norm, $\|\cdot\|_0$ (notice that, by S3) the solutions are bounded). Note that we only aim to control two suitable families of perturbations of the uninfected subsystem, so that condition S6) is sufficiently flexible to adapt to a wide range of uninfected subsystems associated to the eco-epidemiological models.

We emphasize that our setting includes several of the most common functional responses for both functions f and g . Writing

$$f(S, I, P) = \frac{kS^\alpha}{h(S, I, P)} \text{ and } g(S, I, P) = \frac{kP^\alpha}{h(S, I, P)}$$

we may consider the following cases:

- Holling-type I: $h(S, I, P) = 1$, $\alpha = 1$
- Holling-type II : $h(S, I, P) = (1 + m(S + I))$, $\alpha = 1$
- Holling-type III: $h(S, I, P) = (1 + m(S + I))$, $\alpha > 0$
- Holling-type IV: $h(S, I, P) = (a + b(S + I) + c(S + I)^2)$, $\alpha = 1$
- Beddington-De Angelis: $h(S, I, P) = (a + b(S + I) + cP)$, $\alpha = 1$
- Crowley-Martin: $h(S, I, P) = (a + b(S + I) + cP + d(S + I)P)$, $\alpha = 1$,

where $k, m, a, b, c, d > 0$.

2.1. Main results. In this subsection, we will establish our results on the extinction and uniform strong persistence of the infective prey in system (1), assuming conditions S1) to S6). Given a function f we will use throughout the paper the notations $f^\ell = \inf_{t \geq 0} f(t)$, $f^u = \sup_{t \geq 0} f(t)$ and, for a ω -periodic function f we use the notation $\bar{f} = (1/\omega) \int_0^\omega f(s) ds$.

We define

$$\mathcal{R}^\ell(\lambda) = \liminf_{t \rightarrow +\infty} \int_t^{t+\lambda} \beta(s)x_1^*(s) - \eta(s)g(x_1^*(s), 0, z_2^*(s)) - c(s) ds \quad (7)$$

where we still denote by $x_1^*(t)$ and $z_2^*(t)$ the components of solutions in systems S6.4) and S6.5), with $\varepsilon = 0$, and

$$\mathcal{R}^u(\lambda) = \limsup_{t \rightarrow +\infty} \int_t^{t+\lambda} \beta(s)s^*(s) - \eta(s)g(s^*(s), 0, y^*(s)) - c(s) ds. \quad (8)$$

where $s^*(t)$ and $y^*(t)$ are particular solutions, respectively, of (2) and (3) with positive initial conditions.

As we will see in the following, using the global attractivity of solutions of (2) and (3) in $]0, +\infty[$ and the global attractivity of solutions given at S6.4) and S6.5) we can easily conclude that (7) is independent of the particular solutions considered in S6.4) and S6.5). Similarly, it is easy to conclude that (8) is independent of the particular solutions of (2) and (3) with positive initial conditions considered.

Proposition 1. *The numbers (7) and (8) are independent, respectively, of the particular solutions considered in S6.4) and S6.5) and of the particular solutions of (2) and (3) with positive initial conditions chosen.*

Proof. Let $(x_1^*(t), z_1^*(t))$, $(x_2^*(t), z_2^*(t))$ and $(\bar{x}_1^*(t), \bar{z}_1^*(t))$, $(\bar{x}_2^*(t), \bar{z}_2^*(t))$ be two distinct pairs of nonnegative solutions of (5) and (6) as in S6.4) and S6.5). Let $\delta > 0$. By S6), for $t \geq T_\delta$ sufficiently large, we have

$$x_1^*(t) - \delta \leq \bar{x}_1^*(t) \leq x_1^*(t) + \delta \quad \text{and} \quad z_2^*(t) - \delta \leq \bar{z}_2^*(t) \leq z_2^*(t) + \delta.$$

Additionally, by S1) and S2) we have, for every $t \geq T_\delta$,

$$\begin{aligned} & \left| \int_t^{t+\lambda} \beta(s)x_1^*(s) - \eta(s)g(x_1^*(s), 0, z_2^*(s)) - c(s) ds - \int_t^{t+\lambda} \beta(s)\bar{x}_1^*(s) - \eta(s)g(\bar{x}_1^*(s), 0, \bar{z}_2^*(s)) - c(s) ds \right| \\ & \leq \int_t^{t+\lambda} \beta(s)|x_1^*(s) - \bar{x}_1^*(s)| + \eta(s)|g(x_1^*(s), 0, z_2^*(s)) - g(\bar{x}_1^*(s), 0, \bar{z}_2^*(s))| ds \\ & \leq \lambda\beta^u\delta + 2\lambda\eta^u\varphi(\delta), \end{aligned}$$

with $\varphi(\delta) \rightarrow 0$ as $\delta \rightarrow 0$. We conclude that, for every $\delta > 0$,

$$\begin{aligned} & \liminf_{t \rightarrow +\infty} \int_t^{t+\lambda} \beta(s)x_1^*(s) - \eta(s)g(x_1^*(s), 0, z_2^*(s)) - c(s) ds - \lambda\beta^u\delta - 2\lambda\eta^u\varphi(\delta) \\ & \leq \liminf_{t \rightarrow +\infty} \int_t^{t+\lambda} \beta(s)\bar{x}_1^*(s) - \eta(s)g(\bar{x}_1^*(s), 0, \bar{z}_2^*(s)) - c(s) ds \\ & \leq \liminf_{t \rightarrow +\infty} \int_t^{t+\lambda} \beta(s)x_1^*(s) - \eta(s)g(x_1^*(s), 0, z_2^*(s)) - c(s) ds + \lambda\beta^u\delta + 2\lambda\eta^u\varphi(\delta), \end{aligned}$$

Thus $\mathcal{R}^\ell(\lambda)$ is independent of the chosen solution. Taking, respectively, \limsup , $s^*(t)$ and $y^*(t)$ instead of \liminf , $x_1^*(t)$ and $z_2^*(t)$ and using the same reasoning we can prove that $\mathcal{R}^u(\lambda)$ is also independent of the particular solutions chosen. The result follows. \square

Theorem 1. *Assume that conditions S1) to S5) hold. Assume further that either $G(t, S) = \Lambda(t) - \mu(t)S$ and $g(S + I, 0, P) \leq g(S, I, P)$ or g does not depend on I . If there is $\lambda > 0$ such that $\mathcal{R}^u(\lambda) < 0$, then the infectives in system (1) go to extinction.*

Proof. Assume that there is $\lambda > 0$ such that $\mathcal{R}^u(\lambda) < 0$ and let $s^*(t)$ and $y^*(t)$ be particular solutions, respectively, of (2) and (3) with positive initial conditions. Since functions β and η are bounded, there are $\kappa > 0$, $t_0 > 0$ and $\varepsilon_0 > 0$ such that, for $t \geq t_0$ and $\delta \in]0, \varepsilon_0]$, we have

$$\int_t^{t+\lambda} \beta(s)(s^*(s) + \delta) - \eta(s)g(s^*(s) + \delta, 0, y^*(s) - \delta) - c(s) ds \leq -\kappa < 0. \quad (9)$$

Let $(S(t), I(t), P(t))$ be a solution of (1) with positive initial conditions. We will prove first that

$$\liminf_{t \rightarrow +\infty} I(t) = 0. \quad (10)$$

Assume that (10) does not hold. Then, there is $\varepsilon > 0$ such that $I(t) > \varepsilon$ for all sufficiently large t . By the first equation of (1) we have

$$S' \leq G(t, S)$$

and thus $S(t) \leq s(t)$, where $s(t)$ is the solution of (2) with $s(t_0) = S(t_0)$. By condition S4), given $\varepsilon \in]0, \varepsilon_0]$, we have $S(t) \leq s^*(t) + \varepsilon$, for all sufficiently large t .

By the third equation of (1), we have

$$P' \geq h(t, P)P$$

and thus $P(t) \geq y(t)$, where $y(t)$ is the solution of (3) with $y(t_0) = P(t_0)$. By condition S5), given $\varepsilon \in]0, \varepsilon_0]$, we have $P(t) \geq y^*(t) - \varepsilon$, for all sufficiently large t .

When $G(t, S) = \Lambda(t) - \mu(t)S$,

$$(S + I)' \leq \Lambda(t) - \mu(t)S - c(t)I \leq \Lambda(t) - \mu(t)(S + I),$$

and consequently, for sufficiently large t

$$S(t) + I(t) \leq s^*(t) + \varepsilon.$$

Under this assumption on G , by the second equation of (1), since we assumed that $g(S + I, 0, P) \leq g(S, I, P)$, we have

$$I' \leq [\beta(t)(s^*(t) + \varepsilon) - \eta(t)g(s^*(t) + \varepsilon, 0, y^*(t) - \varepsilon) - c(t)] I, \quad (11)$$

for all sufficiently large t . Notice that, for a general G , if g does not depend on I we have $g(S, I, P) \geq g(s^*(t) + \varepsilon, 0, y^*(t) - \varepsilon)$ and we still obtain inequality (11).

Denoting by $\lfloor \alpha \rfloor$ the integer part of α and integrating the previous equation, we get

$$\begin{aligned} I(t) &\leq I(t_0) \exp \left\{ \int_{t_0}^t \beta(r)(s^*(r) + \varepsilon) - \eta(r)g(s^*(r) + \varepsilon, 0, y^*(r) - \varepsilon) - c(r) dr \right\} \\ &\leq I(t_0) e^{\lambda(\beta^u(s^*)^u + \varepsilon\beta^u)} \\ &\quad \times \exp \left\{ \int_{t_0}^{t_0 + \lfloor \frac{t-t_0}{\lambda} \rfloor \lambda} \beta(r)(s^*(r) + \varepsilon) - \eta(r)g(s^*(r) + \varepsilon, 0, y^*(r) - \varepsilon) - c(r) dr \right\} \\ &\leq I(t_0) e^{-\lfloor (t-t_0)/\lambda \rfloor \kappa} e^{\lambda(\beta^u(s^*)^u + \varepsilon\beta^u)}, \end{aligned}$$

for all sufficiently large t . Since $\lfloor (t - t_0)/\lambda \rfloor \kappa \rightarrow +\infty$ as $t \rightarrow +\infty$, we get a contradiction to the hypothesis that there is $\varepsilon > 0$ such that $I(t) > \varepsilon$ for sufficiently large t . We conclude that (10) holds.

Let $\varepsilon > 0$. Next we will prove that for sufficiently large t

$$I(t) \leq \varepsilon e^{h\lambda}, \quad (12)$$

where

$$h = \sup_{t \geq 0} |\beta(t)(s^*(t) + \varepsilon_0) - \eta(t)g(s^*(t) + \varepsilon_0, 0, y^*(t) - \varepsilon_0) - c(t)|.$$

By (10), there exists $t_1 \geq t_0$ such that $I(t_1) < \varepsilon$.

Assume, by contradiction that (12) does not hold. Then, there is $t_2 > t_1$ such that $I(t_2) > \varepsilon e^{h\lambda}$. Since $I(t_1) < \varepsilon$, there is $t_3 \in]t_1, t_2[$ such that $I(t_3) = \varepsilon$ and

$I(t) > \varepsilon$, for all $t \in]t_3, t_2[$. Integrating we get, by (9),

$$\begin{aligned} \varepsilon e^{h\lambda} &< I(t_2) \\ &\leq I(t_3) \exp \left\{ \int_{t_3}^{t_2} \beta(r)(s^*(r) + \varepsilon) - \eta(r)g(s^*(r) + \varepsilon, 0, y^*(r) - \varepsilon) - c(r) dr \right\} \\ &\leq \varepsilon \exp \left\{ \int_{t_3 + \lfloor (t_2 - t_3)/\lambda \rfloor \lambda}^{t_2} \beta(r)(s^*(r) + \varepsilon_0) - \eta(r)g(s^*(r) + \varepsilon_0, 0, y^*(r) - \varepsilon_0) - c(r) dr \right\} \\ &\leq \varepsilon e^{h\lambda}, \end{aligned}$$

which is a contradiction. Thus, we conclude that (12) holds and, since $\varepsilon \in]0, \varepsilon_0]$ is arbitrary, we conclude that $I(t) \rightarrow 0$ as $t \rightarrow 0$, as claimed. \square

Theorem 2. *Assume that conditions S1) to S3) and S6) hold. If there is $\lambda > 0$ such that $\mathcal{R}^\ell(\lambda) > 0$, then the infectives in system (1) are uniformly strong persistent.*

Proof. Assume that there is $\lambda > 0$ such that $\mathcal{R}^\ell(\lambda) > 0$ and let us fix particular families of solutions of systems (5) and (6), respectively $(x_{1,\varepsilon}^*(t), z_{1,\varepsilon}^*(t))$ and $(x_{2,\varepsilon}^*(t), z_{2,\varepsilon}^*(t))$, with positive initial conditions and satisfying N6.4) and N6.5). Then, we can choose $t_0 > 0$, $\kappa > 0$ and $\varepsilon_0 > 0$ such that, for $t \geq t_0$ and $\delta \in [0, \varepsilon_0]$ we have

$$\int_t^{t+\lambda} \beta(s)(x_1^*(s) - \delta) - \eta(s)g(x_1^*(s) - \delta, \delta, z_2^*(s) + \delta) - c(s) ds \geq \kappa > 0. \quad (13)$$

Let $(S(t), I(t), P(t))$ be a solution of (1) with positive initial conditions. We will prove first that there is $\varepsilon > 0$ such that

$$\limsup_{t \rightarrow +\infty} I(t) \geq \frac{v(\varepsilon)\rho^\ell}{(1 + \beta^u)(1 + \theta^u\eta^u)} > 0. \quad (14)$$

Assume that for all sufficiently small $\varepsilon > 0$

$$\limsup_{t \rightarrow +\infty} I(t) < \frac{v(\varepsilon)\rho^\ell}{(1 + \beta^u)(1 + \theta^u\eta^u)}.$$

Then, we conclude that there is $t_1 > t_0$, such that

$$I(t) < \frac{v(\varepsilon)\rho^\ell}{(1 + \beta^u)(1 + \theta^u\eta^u)} < v(\varepsilon)\rho(t), \quad (15)$$

for each $t \geq t_1$. By the first and third equations of (1) and the inequalities in S6.1) we have

$$\begin{cases} S' \leq G_{2,\varepsilon}(t, S) \\ P' \leq h_{2,\varepsilon}(t, P)P + \gamma(t)a(t)f(S, 0, P)P + v(\varepsilon)\rho(t)\theta(t)\eta(t)g(S, 0, P) \end{cases}.$$

Let $(\hat{x}_\varepsilon(t), \hat{z}_\varepsilon(t))$ be the solution of

$$\begin{cases} x' = G_{2,\varepsilon}(t, x) \\ z' = h_{2,\varepsilon}(t, z)z + \gamma(t)a(t)f(x, 0, z)z + v(\varepsilon)\rho(t)\theta(t)\eta(t)g(x, 0, z) \end{cases}$$

with $\hat{x}_\varepsilon(t_1) = S(t_1)$ and $\hat{z}_\varepsilon(t_1) = P(t_1)$. We have $S(t) \leq \hat{x}_\varepsilon(t)$ and $P(t) \leq \hat{z}_\varepsilon(t)$ for $t \geq t_1$. By the global stability assumption in N6.5), we have

$$|x_{2,\varepsilon}^*(t) - \hat{x}_\varepsilon(t)| \rightarrow 0 \quad \text{and} \quad |z_{2,\varepsilon}^*(t) - \hat{z}_\varepsilon(t)| \rightarrow 0, \quad \text{as } t \rightarrow +\infty$$

and, by continuity, again according to N6.5), we have for sufficiently large t

$$\begin{aligned} |x_2^*(t) - \hat{x}_\varepsilon(t)| &\leq |x_2^*(t) - x_{2,\varepsilon}^*(t)| + |x_{2,\varepsilon}^*(t) - \hat{x}_\varepsilon(t)| \\ &\leq \|x_2^* - x_{2,\varepsilon}^*\|_0 + |x_{2,\varepsilon}^*(t) - \hat{x}_\varepsilon(t)| \\ &\leq \varphi_1(\varepsilon), \end{aligned}$$

and

$$\begin{aligned} |z_2^*(t) - \hat{z}_\varepsilon(t)| &\leq |z_2^*(t) - z_{2,\varepsilon}^*(t)| + |z_{2,\varepsilon}^*(t) - \hat{z}_\varepsilon(t)| \\ &\leq \|z_2^* - z_{2,\varepsilon}^*\|_0 + |z_{2,\varepsilon}^*(t) - \hat{z}_\varepsilon(t)| \\ &\leq \varphi_2(\varepsilon), \end{aligned}$$

with $\varphi_1(\varepsilon), \varphi_2(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. In particular, for sufficiently large t ,

$$S(t) \leq \hat{s}_\varepsilon(t) \leq \varphi_1(\varepsilon) + x_2^*(t) \quad \text{and} \quad P(t) \leq \hat{z}_\varepsilon(t) \leq \varphi_2(\varepsilon) + z_2^*(t). \quad (16)$$

On the other hand, by (15) and the first and third equations of (1), we have

$$\begin{cases} S' \geq G_{1,\varepsilon}(t, S) - a(t)f(S, 0, 0)\hat{z}_\varepsilon(t) - v(\varepsilon)\rho(t)S \\ P' \geq h_{1,\varepsilon}(t, P)P + \gamma(t)a(t)f(S, v(\varepsilon)\rho^u, P)P \end{cases}$$

Letting $(\tilde{x}_\varepsilon(t), \tilde{z}_\varepsilon(t))$ be the solution of

$$\begin{cases} x' = G_{1,\varepsilon}(t, x) - a(t)f(x, 0, 0)\hat{z}_\varepsilon(t) - v(\varepsilon)\rho(t)x \\ z' = h_{1,\varepsilon}(t, z)z + \gamma(t)a(t)f(x, v(\varepsilon)\rho^u, z)z \end{cases}$$

with $\tilde{x}(t_1) = S(t_1)$ and $\tilde{z}(t_1) = P(t_1)$, we have $S(t) \geq \tilde{x}_\varepsilon(t)$ and $P(t) \geq \tilde{z}_\varepsilon(t)$, for all $t \geq t_1$. By the global stability assumption in N6.4), we have

$$|x_{1,\varepsilon}^*(t) - \tilde{x}_\varepsilon(t)| \rightarrow 0 \quad \text{and} \quad |z_{1,\varepsilon}^*(t) - \tilde{z}_\varepsilon(t)| \rightarrow 0, \quad \text{as } t \rightarrow +\infty.$$

and, by the continuity property in S6.4), for sufficiently large t , we have

$$\begin{aligned} |x_1^*(t) - \tilde{x}_\varepsilon(t)| &\leq |x_1^*(t) - x_{1,\varepsilon}^*(t)| + |x_{1,\varepsilon}^*(t) - \tilde{x}_\varepsilon(t)| \\ &\leq \|x_1^* - x_{1,\varepsilon}^*\|_0 + |x_{1,\varepsilon}^*(t) - \tilde{x}_\varepsilon(t)| \\ &\leq \psi_1(\varepsilon), \end{aligned}$$

and

$$\begin{aligned} |z_1^*(t) - \tilde{z}_\varepsilon(t)| &\leq |z_1^*(t) - z_{1,\varepsilon}^*(t)| + |z_{1,\varepsilon}^*(t) - \tilde{z}_\varepsilon(t)| \\ &\leq \|z_1^* - z_{1,\varepsilon}^*\|_0 + |z_{1,\varepsilon}^*(t) - \tilde{z}_\varepsilon(t)| \\ &\leq \psi_2(\varepsilon), \end{aligned}$$

with $\psi_1(\varepsilon), \psi_2(\varepsilon) \rightarrow 0$ as $\varepsilon \rightarrow 0$. In particular, for sufficiently large t ,

$$S(t) \geq \tilde{x}_\varepsilon(t) \geq x_1^*(t) - \psi_1(\varepsilon) \quad \text{and} \quad P(t) \geq \tilde{z}_\varepsilon(t) \geq z_1^*(t) - \psi_2(\varepsilon). \quad (17)$$

By the second equation in (1), (13), (16) and (17) we get, for $t \geq t_1$,

$$\begin{aligned} &\int_t^{t+\lambda} \beta(s)S(s) - \eta(s)g(S(s), I(s), P(s)) - c(s) ds \\ &\geq \int_t^{t+\lambda} \beta(s)(x_1^*(s) - \psi_1(\varepsilon)) - \eta(s)g(x_1^*(s) - \psi_1(\varepsilon), v(\varepsilon)\rho^u, z_2^*(s) + \varphi_2(\varepsilon)) - c(s) ds \geq \kappa. \end{aligned}$$

Thus, choosing $\varepsilon > 0$ such that $\max\{\varphi_2(\varepsilon), \psi_1(\varepsilon), v(\varepsilon)\rho^u\} < \varepsilon_0$, we have

$$\begin{aligned}
I(t) &= I(t_1) \exp \left\{ \int_{t_1}^t \beta(s) S(s) - \eta(s) g(S(s), I(s), P(s)) - c(s) ds \right\} \\
&\geq I(t_1) \exp \left\{ \int_{t_1}^t \beta(s) (x_1^*(s) - \psi_1(\varepsilon)) ds \right\} \\
&\quad \times \exp \left\{ \int_{t_1}^t -\eta(s) g(x_1^*(s) - \psi_1(\varepsilon), 0, z_2^*(s) + \varphi_2(\varepsilon)) - c(s) ds \right\} \\
&\geq I(t_1) e^{-\lambda(\beta^u \psi_1(\varepsilon) + \eta^u g((x_1^*)^u - \psi_1(\varepsilon), 0, (z_2^*)^\ell + \varphi_2(\varepsilon)) + c^u)} \\
&\quad \times \exp \left\{ \int_{t_1}^{t_1 + \lfloor (t-t_1)/\lambda \rfloor \lambda} \beta(s) (x_1^*(s) - \psi_1(\varepsilon)) ds \right\} \\
&\quad \times \exp \left\{ \int_{t_1}^{t_1 + \lfloor (t-t_1)/\lambda \rfloor \lambda} -\eta(s) g(x_1^*(s) - \psi_1(\varepsilon), 0, z_2^*(s) + \varphi_2(\varepsilon)) - c(s) ds \right\} \\
&\geq I(t_1) e^{\lfloor (t-t_1)/\lambda \rfloor \kappa} e^{-\lambda(\beta^u \psi_1(\varepsilon) + \eta^u g((x_1^*)^u - \psi_1(\varepsilon), 0, (x_1^*)^\ell + \varphi_2(\varepsilon)) + c^u)},
\end{aligned}$$

a contradiction to the fact that, according to S3), $I(t)$ is bounded. We conclude that (14) holds.

Next we will prove that there is $m_1 > 0$ such that for any solution $(S(t), I(t), P(t))$ with positive initial condition,

$$\liminf_{t \rightarrow +\infty} I(t) > m_1. \quad (18)$$

Assume that (18) does not hold. Then, given $\varepsilon \in]0, \varepsilon_0[$, there exists a sequence of initial values $(x_n)_{n \in \mathbb{N}}$, with $x_n = (S_n, I_n, P_n)$ and $S_n > 0$, $I_n > 0$ and $P_n > 0$ such that

$$\liminf_{t \rightarrow +\infty} I(t, x_n) < \frac{\rho^u v(\varepsilon/n^2)}{(1 + \theta^u \eta^u)(1 + \beta^u)},$$

where $(S(t, x_n), I(t, x_n), P(t, x_n))$ denotes the solution of (1) with initial conditions $S(0) = S_n$, $I(0) = I_n$, and $P(0) = P_n$. By (14), given $n \in \mathbb{N}$, there are two sequences $(t_{n,k})_{k \in \mathbb{N}}$ and $(s_{n,k})_{k \in \mathbb{N}}$ with

$$s_{n,1} < t_{n,1} < s_{n,2} < t_{n,2} < \dots < s_{n,k} < t_{n,k} < \dots$$

and $\lim_{k \rightarrow +\infty} s_{n,k} = +\infty$, such that

$$I(s_{n,k}, x_n) = \frac{\rho^\ell v(\varepsilon/n)}{(1 + \theta^u \eta^u)(1 + \beta^u)}, \quad I(t_{n,k}, x_n) = \frac{\rho^u v(\varepsilon/n^2)}{(1 + \theta^u \eta^u)(1 + \beta^u)} \quad (19)$$

and, for all $t \in]s_{n,k}, t_{n,k}[$,

$$\frac{\rho^u v(\varepsilon/n^2)}{(1 + \theta^u \eta^u)(1 + \beta^u)} < I(t, x_n) < \frac{\rho^\ell v(\varepsilon/n)}{(1 + \theta^u \eta^u)(1 + \beta^u)}. \quad (20)$$

By the second equation in (1) and S3), for sufficiently large t , we have

$$\begin{aligned}
I'(t, x_n) &= [\beta(t) S(t, x_n) - \eta(t) g(S(t, x_n), I(t, x_n), P(t, x_n)) - c(t)] I(t, x_n) \\
&\geq -(\eta^u g(L, 0, 0) + c^u) I(t, x_n).
\end{aligned}$$

Therefore we obtain

$$\int_{s_{n,k}}^{t_{n,k}} \frac{I'(r, x_n)}{I(r, x_n)} dr \geq -(\eta^u g(L, 0, 0) + c^u)(t_{n,k} - s_{n,k})$$

and thus $I(t_{n,k}, x_n) \geq I(s_{n,k}, x_n) e^{-(\eta^u g(L, 0, 0) + c^u)(t_{n,k} - s_{n,k})}$. By (19), and S6.3) we get

$$\frac{\rho^u v(\varepsilon/n^2)}{\rho^\ell v(\varepsilon/n)} \geq \frac{\rho(t_{n,k}) v(\varepsilon/n^2)}{\rho(s_{n,k}) v(\varepsilon/n)} \geq e^{-(\eta^u g(L, 0, 0) + c^u)(t_{n,k} - s_{n,k})}$$

and therefore we have

$$t_{n,k} - s_{n,k} \geq \frac{\log(\rho^\ell/\rho^u) + \log(v(\varepsilon/n)/v(\varepsilon/n^2))}{\eta^u g(L, 0, 0) + c^u} \rightarrow +\infty \quad (21)$$

as $n \rightarrow +\infty$, since, by S6.2) we have

$$\lim_{n \rightarrow +\infty} \frac{v(\varepsilon/n)}{v(\varepsilon/n^2)} = \lim_{n \rightarrow +\infty} \frac{n v'(\varepsilon/n)}{2 v'(\varepsilon/n^2)} \geq \lim_{n \rightarrow +\infty} \frac{An}{2B} = +\infty.$$

By the first and third equations of (1) and (20), we have, for $t \in]s_{n,k}, t_{n,k}[$,

$$\begin{cases} S' \leq G_{2,\varepsilon}(t, S(t, x_n)) \\ P' \leq h_{2,\varepsilon}(t, P(t, x_n))P(t, x_n) + \gamma(t)a(t)f(S(t, x_n), 0, P(t, x_n))P(t, x_n) \\ \quad + \rho(t)v(\varepsilon/n)\theta(t)\eta(t)g(S(t, x_n), 0, P(t, x_n)) \end{cases}.$$

Letting $(\hat{x}_{n,k}(t), \hat{z}_{n,k}(t))$ be the solution of

$$\begin{cases} x' = G_{2,\varepsilon}(t, x) \\ z' = h_{2,\varepsilon}(t, z)z + \gamma(t)a(t)f(x, 0, z)z + \rho(t)v(\varepsilon/n)\theta(t)\eta(t)g(x, 0, z) \end{cases}$$

with $\hat{x}_{n,k}(s_{n,k}) = S(s_{n,k})$ and $\hat{z}_{n,k}(s_{n,k}) = P(s_{n,k})$. We conclude that $S(t, x_n) \leq \hat{x}_{n,k}(t)$ and $P(t, x_n) \leq \hat{z}_{n,k}(t)$, for each $t \in]s_{n,k}, t_{n,k}[$. By N6.5), given $\delta > 0$, we have

$$|x_{2,\varepsilon/n}^*(t) - \hat{x}_{n,k}(t)| < \delta/2 \quad \text{and} \quad |z_{2,\varepsilon/n}^*(t) - \hat{z}_{n,k}(t)| < \delta/2,$$

for all sufficiently large k (that depends on n). By continuity, for sufficiently large n and all sufficiently large $k \geq K(n)$, we have

$$|x_2^*(t) - \hat{x}_{n,k}(t)| \leq |x_2^*(t) - x_{2,\varepsilon/n}^*(t)| + |x_{2,\varepsilon/n}^*(t) - \hat{x}_{n,k}(t)| \leq \delta.$$

and

$$|z_2^*(t) - \hat{z}_{n,k}(t)| \leq |z_2^*(t) - z_{2,\varepsilon/n}^*(t)| + |z_{2,\varepsilon/n}^*(t) - \hat{z}_{n,k}(t)| \leq \delta.$$

In particular, for sufficiently large n , all sufficiently large $k \geq K(n)$ and for $t \in]s_{n,k(n)}, t_{n,k(n)}[$, we have

$$S(t) \leq \hat{x}_{n,k}(t) \leq x_2^*(t) + \delta \quad \text{and} \quad P(t) \leq \hat{z}_{n,k}(t) \leq z_2^*(t) + \delta. \quad (22)$$

Similar computations show that, for sufficiently large n , all sufficiently large $k \geq K(n)$ and for $t \in]s_{n,k(n)}, t_{n,k(n)}[$, we obtain

$$S(t) \geq \tilde{x}_{n,k}(t) \geq x_1^*(t) - \delta \quad \text{and} \quad P(t) \geq \tilde{z}_{n,k}(t) \geq z_1^*(t) - \delta. \quad (23)$$

Notice that, for a given δ , eventually considering a larger n , we can take the same n and k in (22) and (23).

Given $l > 0$, by (21) we can choose $T > 0$ such that $t_{n,k} - s_{n,k} > l\lambda$ for all $n \geq T$. Therefore, by (19), (22) and (23), and by the second equation in (1), for $n \geq T$ and

$k \geq K(n)$ we get

$$\begin{aligned}
& \frac{\rho^u v(\varepsilon/n^2)}{(1 + \theta^u \eta^u)(1 + \beta^u)} \\
&= I(t_{n,k}, x_n) \\
&\geq I(s_{n,k}, x_n) \exp \left\{ \int_{s_{n,k}}^{t_{n,k}} \beta(r) S(r) - \eta(r) g(S(r), I(r), P(r)) - c(r) dr \right\} \\
&\geq I(s_{n,k}, x_n) \times \\
&\quad \times \exp \left\{ \kappa l + \int_{s_{n,k} + \lfloor (t_{n,k} - s_{n,k})/\lambda \rfloor}^{t_{n,k}} \beta(r)(x_1^*(r) - \delta) - \eta(r) g(x_1^*(r) - \delta, 0, z_2^*(r) + \delta) - c(r) dr \right\} \\
&\geq \frac{\rho^\ell v(\varepsilon/n)}{(1 + \theta^u \eta^u)(1 + \beta^u)} e^{\kappa l - \lambda(\beta^u \delta + \eta^u(g((x_1^*)^u - \delta, 0, (z_2^*)^u + \delta) + c^u))} \\
&> \frac{\rho^\ell v(\varepsilon/n)}{(1 + \theta^u \eta^u)(1 + \beta^u)},
\end{aligned}$$

for sufficiently large l (that requires that T is sufficiently large). We conclude that

$$\frac{\rho^u v(\varepsilon/n^2)}{\rho^\ell v(\varepsilon/n)} > 1$$

and this contradicts the fact that, by S6.2) and S6.3), we have

$$\lim_{n \rightarrow +\infty} \frac{\rho^u v(\varepsilon/n^2)}{\rho^\ell v(\varepsilon/n)} = \lim_{n \rightarrow +\infty} \frac{2\rho^u v'(\varepsilon/n^2)/n^3}{\rho^\ell v'(\varepsilon/n)/n^2} \leq \lim_{n \rightarrow +\infty} \frac{2\rho^u B}{n\rho^\ell A} = 0.$$

We conclude that there is $m_1 > 0$ such that $\liminf_{t \rightarrow +\infty} I(t) > m_1$ and the result follows from S2). \square

In [9], the authors obtain extinction and persistence results for eco-epidemiological model with Crowley-Martin functional response. In the extinction result the authors consider auxiliary equations different from (2) and (3) using some upper bound for S and some lower bound for P related to the dimension of some positive invariant region that contains the omega limit of all solutions. We will borrow and improve the idea of that paper in our context. To this purpose, we need to consider families of auxiliary equations. We begin by noticing that, by the proof of Theorem 1, for any solution $(S(t), I(t), P(t))$ of our problem with initial condition $(S(t_0), I(t_0), P(t_0)) = (S_0, I_0, P_0)$ we have $s^{1,\ell}(t) \leq S(t) \leq s^{1,u}(t)$ and $y^{1,\ell}(t) \leq P(t) \leq y^{1,u}(t)$, for all $t > 0$ sufficiently large, where $s^{1,\ell}(t) = 0$, $s^{1,u}(t)$ is the solution of (2) with initial condition $s^{1,u}(t_0) = S_0$, $y^{1,\ell}(t)$ is the solution of (3) with initial condition $y^{1,\ell}(t_0) = P_0$ and $y^{1,u}(t) = L$, where L is given in condition S3). Let $k \in \mathbb{N}$ be given and assume that $s^{k,u}(t)$, $s^{k,\ell}(t)$, $y^{k,u}(t)$ and $y^{k,\ell}(t)$ are defined. Consider the equations:

$$s' = G(t, s) - a(t)f(s, L, y^{k,u}(t))y^{k,\ell}(t), \quad (24)$$

and

$$s' = G(t, s) - a(t)f(s, 0, y^{k,\ell}(t))y^{k,u}(t) - \beta(t)sL, \quad (25)$$

and set $s^{k+1,u}(t)$ and $s^{k+1,\ell}(t)$ as particular solutions of (24) and (25) with positive initial condition, respectively. Consider also the equations

$$y' = h(t, y)y + \gamma(t)a(t)f(s^{k+1,u}(t), 0, y)y + \theta(t)\eta(t)g(0, 0, y)L, \quad (26)$$

and

$$y' = h(t, y)y + \gamma(t)a(t)f(s^{k+1,\ell}(t), L, y)y, \quad (27)$$

and set $y^{k+1,u}$ and $y^{k+1,\ell}$ as particular solutions of (26) and (27) with positive initial condition, respectively.

For each $k \in \mathbb{N}$ we consider the following conditions:

- S4_k) For each $i = 1, \dots, k$, $s^{i,u}(t)$ and $s^{i,\ell}(t)$ are bounded, bounded away from zero, and globally attractive on $]0, +\infty[$, that is $|s^{i,u}(t) - v(t)| \rightarrow 0$ as $t \rightarrow +\infty$ for each other solution $v(t)$ of (24) (respectively (25)), with $k = i$, with positive initial condition.
- S5_k) For each $i = 1, \dots, k$, $y^{i,u}(t)$ and $y^{i,\ell}(t)$ are bounded and globally attractive on $[0, +\infty)$.

We define the following number:

$$\mathcal{R}^{u,k}(\lambda) = \limsup_{t \rightarrow +\infty} \int_t^{t+\lambda} \beta(\tau) s^{k,u} - \eta(\tau) g(s^{k,u}, 0, y^{k,\ell}) - c(\tau) d\tau. \quad (28)$$

Notice that $\mathcal{R}^{u,1}(\lambda) = \mathcal{R}^u$ and, according to our assumptions, it is easy to prove, with similar arguments to the ones in Proposition 1, that for all $k \in \mathbb{N}$ the number $\mathcal{R}^{u,k}(\lambda)$ is independent of the particular solutions considered.

Theorem 3. *Assume that conditions S1) to S3), S4_k) and S5_k) hold for some $k \in \mathbb{N}$. Assume further $x \rightarrow f(x, y, z)$ in nondecreasing and that either $G(t, S) = \Lambda(t) - \mu(t)S$ and $g(S + I, 0, P) \leq g(S, I, P)$ or g does not depend on I . If there is $\lambda > 0$ such that $\mathcal{R}^{u,k}(\lambda) < 0$, then the infectives in system (1) go to extinction.*

Proof. The proof consists in repeating the steps in the proof of Theorem 1, with the changes that we will describe below. Let us consider $k \geq 2$ (otherwise the proof follows from Theorem 1). In the first place, instead of the bounds given by (2) and (3), we use bounds obtained in the following way: letting $y^{k-1,\ell}(t)$ and $y^{k-1,u}(t)$ be the solutions defined above, we know that

$$S' \leq G(t, S) - a(t)f(S, L, y^{k-1,u}(t))y^{k-1,\ell}(t) \quad (29)$$

and

$$S' \geq G(t, S) - a(t)f(S, 0, y^{k-1,\ell}(t))y^{k-1,u}(t) - \beta(t)SL. \quad (30)$$

Thus, using solutions $s^{k,\ell}(t)$ and $s^{k,u}(t)$, from the monotonicity properties of f we obtain

$$P' \geq h(t, P)P + \gamma(t)a(t)f(s^{k,\ell}(t), L, P)P \quad (31)$$

and

$$P' \leq h(t, P)P + \gamma(t)a(t)f(s^{k,u}(t), 0, P)P + \theta(t)\eta(t)g(0, 0, P)L.$$

The bounds in (29) and (31) allow us to conclude that, for sufficiently large $t > 0$, we have $S(t) \leq s^{k,u}(t)$ and $P(t) \geq y^{k,\ell}(t)$; using the number $\mathcal{R}^{u,k}(\lambda)$ in (28), similar arguments to the ones in Theorem 1 allow us to obtain the result. \square

We notice that Theorem 3 can be used to obtain a better understanding of the extinction threshold. For instance if $\mathcal{R}^{u,k} < \mathcal{R}^{u,m}$, for some $m < k$, condition $\mathcal{R}^{u,k} < 0$ is weaker than $\mathcal{R}^{u,m} < 0$, so we may improve the threshold for extinction. See Section 3.5.

3. EXAMPLES

In this section we will apply Theorem 1 and Theorem 2 to some particular cases of model (1). We finish with an example for Theorem 3.

3.1. Models with no predation on uninfected preys. In this section we will consider a family of models with no predation on uninfected preys by letting $f \equiv 0$ and $g(S, I, P) = P$. This family generalises the family of models in [10] by allowing a very general form for the vital dynamics of predators and preys. Thus, we consider in this subsection the following model:

$$\begin{cases} S' = G(t, S) - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)PI - c(t)I \\ P' = h(t, P)P + \theta(t)\eta(t)PI \end{cases}.$$

In this context, (7) and (8) become

$$\mathcal{R}_{np}^\ell(\lambda) = \liminf_{t \rightarrow +\infty} \int_t^{t+\lambda} \beta(s)s^*(s) - \eta(s)y^*(s) - c(s) ds$$

and

$$\mathcal{R}_{np}^u(\lambda) = \limsup_{t \rightarrow +\infty} \int_t^{t+\lambda} \beta(s)s^*(s) - \eta(s)y^*(s) - c(s) ds.$$

where $s^*(t)$ and $y^*(t)$ are particular solutions, respectively, of (2) and (3).

Under the hypotheses of Theorem 1, we obtain that if there is $\lambda > 0$ such that $\mathcal{R}_{np}^u(\lambda) < 0$ then the infectives in system (3.1) go to extinction, and under the hypotheses of Theorem 2, we conclude that if there is $\lambda > 0$ such that $\mathcal{R}_{np}^\ell(\lambda) > 0$ then the infectives in system (3.1) are uniform strong persistent.

As we already mentioned, model (3.1) includes the model discussed in [10] as the particular case where $G(t, S) = \Lambda(t) - \mu(t)S$ and $h(t, P) = b(t) - r(t)P$, with Λ , μ , r and b nonnegative, continuous and bounded functions satisfying:

$$\begin{aligned} \liminf_{t \rightarrow +\infty} \int_t^{t+\omega_1} \Lambda(s) ds &> 0, & \liminf_{t \rightarrow +\infty} \int_t^{t+\omega_2} \mu(s) ds &> 0, \\ \liminf_{t \rightarrow +\infty} \int_t^{t+\omega_3} r(s) ds &> 0 & \text{ and } & \liminf_{t \rightarrow +\infty} \int_t^{t+\omega_4} b(s) ds > 0, \end{aligned}$$

for some constants $w_i > 0$, $i = 1, \dots, 4$:

$$\begin{cases} S' = \Lambda(t) - \mu(t)S - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)PI - c(t)I \\ P' = (b(t) - r(t)P)P + \theta(t)\eta(t)PI \end{cases}. \quad (32)$$

Note that, for the model in (32), condition S1) is assumed, condition S2) is immediate from the particular forms of the functions g and h , conditions S4) and S5) follow from Lemmas 1 and 3 in [10] and condition S6) is a consequence of the fact that, in this setting, systems (5) and (6) are uncoupled and small perturbations of each of the equations in those systems is globally asymptotically stable by Lemmas 1 and 3 in [10]. Finally, condition S3) follows from Theorem 1 in [10]. We also note that $\mathcal{R}_{np}^u(\lambda)$ and $\mathcal{R}_{np}^\ell(\lambda)$ coincide with the corresponding numbers in [10].

Another possible choice for the functions h and G is $h(t, P) = -(\delta_1(t) + \delta_2(t)P)$, with δ_1 and δ_2 continuous and nonnegative functions and $G(t, S) = k(t, S)S$ with k a continuous and bounded function satisfying the conditions: $\partial k / \partial S(t, s) < 0$, for every $t, s \geq 0$; $k(t, 0) > 0$ for all $t \geq 0$; there is $S_1(t) > 0$ such that $k(t, S_1(t)) = 0$, for every $t \geq 0$. This choice makes the underlying predator-uninfected prey subsystem

in model (3.1) correspond to the model studied in section 3 of [3] with the function $f \equiv 0$. System (3.1) becomes in this case:

$$\begin{cases} S' = k(t, S)S - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)PI - c(t)I \\ P' = -(\delta_1(t) + \delta_2(t)P)P + \theta(t)\eta(t)PI \end{cases}.$$

Notice that the study of the function $k(t, S)$ in [3] allow us to conclude easily that conditions S1) to S5) are satisfied for this model. Condition S6) is a consequence of the fact that systems (5) and (6) are uncoupled and small perturbations of each of the equations in those systems is globally asymptotically stable (the global asymptotic stability of the first equation is consequence of Lemma 3.1 in [3] and the global asymptotic stability of the second equation is trivial).

To do some simulation, in this scenario we assumed that $G(t, S) = (0.7 - 0.6S)S$; $\beta(t) = \beta_0(1 + 0.7 \cos(2\pi t))$; $\eta(t) = 0.7(1 + 0.7 \cos(\pi + 2\pi t))$; $c(t) = 0.1$; $h(t, P) = -0.2 - 0.3P$; $\theta(t) = 0.9$. We obtain the model:

$$\begin{cases} S' = (0.7 - 0.6S)S - \beta_0(1 + 0.7 \cos(2\pi t))SI \\ I' = \beta_0(1 + 0.7 \cos(2\pi t))SI - 0.7(1 + 0.7 \cos(\pi + 2\pi t))PI - 0.1I \\ P' = (-0.2 - 0.3P)P + 0.63(1 + 0.7 \cos(\pi + 2\pi t))PI \end{cases},$$

When $\beta_0 = 0.01$ we obtain $\mathcal{R}^u = -0.15 < 0$ and we conclude that we have extinction (figure 1). When $\beta_0 = 0.3$ we obtain $\mathcal{R}^\ell = 1.3 > 0$ and we conclude that the infectives are uniform strong persistent (figure 2).

We considered the following initial conditions at $t = 0$: $(S_0, I_0, P_0) = (1, 0.5, 0.1)$, $(S_0, I_0, P_0) = (0.1, 0.2, 1)$ and $(S_0, I_0, P_0) = (0.5, 0.5, 0.5)$.

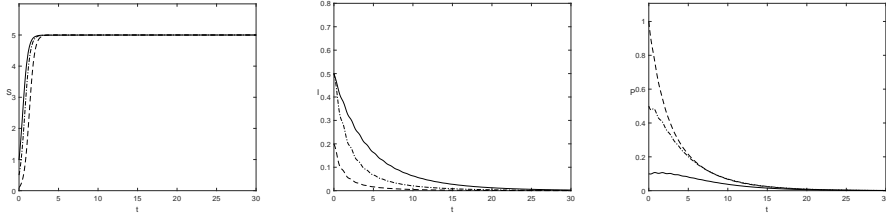


FIGURE 1. Extinction: $\beta_0 = 0.01$.

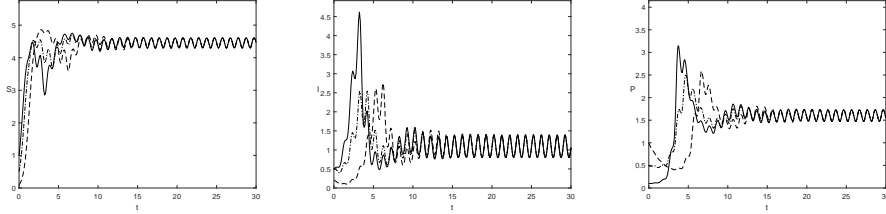


FIGURE 2. Uniform strong persistence: $\beta_0 = 0.3$.

3.2. Periodic coefficients. In this subsection we consider a family of models with periodic parameters and predation on uninfected preys that, in general, is not included in the general family of models considered in [9]. For periodic models, the thresholds become easier to deal with.

Assume that there is $\omega > 0$ such that all parameters in (1) are ω -periodic functions. In this case, (7) and (8) become, respectively,

$$\mathcal{R}^\ell(\omega) = \int_0^\omega \beta(s)x_1^*(s) - \eta(s)g(x_1^*(s), 0, z_2^*(s)) - c(s) ds,$$

and

$$\mathcal{R}^u(\omega) = \int_0^\omega \beta(s)s^*(s) - \eta(s)g(s^*(s), 0, y^*(s)) - c(s) ds.$$

Thus

$$\mathcal{R}^\ell(\omega) > 0 \quad \Leftrightarrow \quad \frac{\overline{\beta x_1^*}}{\overline{\eta g(x_1^*, 0, z_2^*)} + \bar{c}} > 1$$

and

$$\mathcal{R}^u(\omega) < 0 \quad \Leftrightarrow \quad \frac{\overline{\beta s^*}}{\overline{\eta g(s^*, 0, y^*)} + \bar{c}} < 1.$$

where $s^*(t)$ and $y^*(t)$ are particular solutions, respectively, of (2) and (3), and $x_1^*(t)$ and $z_2^*(t)$ still denote any particular solution of first and second equations in systems (5) and (6), respectively, with positive initial conditions. Define

$$\mathcal{R}_{per}^\ell = \frac{\overline{\beta x_1^*}}{\overline{\eta g(x_1^*, 0, z_2^*)} + \bar{c}} \quad \text{and} \quad \mathcal{R}_{per}^u = \frac{\overline{\beta s^*}}{\overline{\eta g(s^*, 0, y^*)} + \bar{c}}.$$

Under the hypotheses of Theorem 1, we have that if $\mathcal{R}_{per}^u < 1$ then the infectives in model (1) with periodic coefficients go to extinction, and under the hypotheses of Theorem 2, if $\mathcal{R}_{per}^\ell > 1$ then the infectives in model (1) with periodic coefficients are uniform strong persistent.

Note that the corollaries in [10], concerning the periodic case, are particular cases of the corollaries above. In fact, in [10] we have $f \equiv 0$ and in this case, as argued in the previous section, $(s^*(t), y^*(t))$ is a particular solution of (4), condition S1) is assumed, condition S2) is immediate, conditions S3) to S6) follow from results in [10]. Thus, when $f \equiv 0$, we get similar thresholds to the ones in the mentioned paper:

$$\mathcal{R}_{per}^\ell = \mathcal{R}_{per}^u = \frac{\overline{\beta s^*}}{\overline{\eta y^*} + \bar{c}}.$$

This threshold can also be obtained using the procedures in [11, 13].

We will focus now on a particular models with a function G that is different from the corresponding function in [9]. We consider the following setting: $G(t, S) = (\Lambda - \mu S)S$; $a(t) = a$; $f(S, I, P) = S$; $g(S, I, P) = P$; $h(t, P) = b - rP$; $\gamma(t) = \gamma$. We obtain the model:

$$\begin{cases} S' = (\Lambda - \mu S)S - aSP - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)PI - c(t)I \\ P' = (b - rP)P + \gamma aSP + \theta(t)\eta(t)PI \end{cases}, \quad (33)$$

For this model, condition S1) is assumed, condition S2) is immediate from the particular forms of the functions g and h , conditions S4) and S5) hold for our particular functions as already discussed in section 3.1. In this context, an endemic

equilibrium for (6) is $(\Lambda/\mu, \hat{z}_\varepsilon)$, with $\hat{z}_\varepsilon = (b\mu + a\gamma\Lambda + \varepsilon\mu)/\mu r$, and the endemic equilibrium for (5) exists if $\Lambda r > ab + a\gamma\Lambda/\mu$:

$$\left(\frac{\hat{\Lambda}}{\mu}, \frac{b\mu + a\gamma\hat{\Lambda}}{\mu r} \right),$$

with $\hat{\Lambda} = \Lambda - a\hat{z}_\varepsilon - \varepsilon$. These subsystems can be discussed using [4]. In fact, the global asymptotic stability result proved in section 3 of [4] implies that, if $\Lambda r > ab + a\gamma\Lambda/\mu$, condition S6) is satisfied. Finally, condition S3) is consequence of the following lemma:

Lemma 1. *There is a bounded region that contains the ω -limit of all orbits of (33).*

Proof. Let $\varepsilon > 0$. Since, by the first equation in (33), $S' \leq (\Lambda - \mu S)S$, we conclude that

$$S(t) \leq \frac{\Lambda}{\mu} + \varepsilon, \quad (34)$$

for all t sufficiently large. Additionally, we get

$$\sup_{S \in \mathbb{R}} (\Lambda - \mu S)S \leq \left(\Lambda - \frac{\mu\Lambda}{2\mu} \right) \frac{\Lambda}{2\mu} = \frac{\Lambda^2}{4\mu}. \quad (35)$$

Adding the first two equations in (33) and using (34) and (35) we have, for all t sufficiently large,

$$\begin{aligned} (S + I)' &= (\Lambda - \mu S)S - c(t)I \\ &\leq \frac{\Lambda^2}{4\mu} + c(t)S - c(t)(S + I) \\ &\leq \frac{\Lambda^2}{4\mu} + c^u \frac{\Lambda}{\mu} + c^u \varepsilon - c^\ell (S + I). \end{aligned} \quad (36)$$

Since $\varepsilon > 0$ is arbitrary, we conclude that

$$\limsup_{t \rightarrow +\infty} (S + I)(t) \leq \frac{1}{c^\ell} \left(\frac{\Lambda^2}{4\mu} + c^u \frac{\Lambda}{\mu} \right) =: A.$$

Finally, by the third equation in (33) and (36), given $\varepsilon > 0$, we get

$$\begin{aligned} P' &= (b - rP)P + \gamma aSP + \theta(t)\eta(t)PI \\ &\leq (b + \gamma aA + \theta^u \eta^u A - rP)P, \end{aligned} \quad (37)$$

for sufficiently large t . Thus,

$$\limsup_{t \rightarrow +\infty} P(t) \leq \frac{1}{r} (b + \gamma aA + \theta^u \eta^u A) =: B. \quad (38)$$

Equations (36) and (37) show that the region

$$\{(S, I, P) \in \mathbb{R}^3 : 0 \leq S + I \leq A \text{ and } 0 \leq P \leq B\}$$

contains the ω -limit of any orbit. \square

To do some simulation, in this scenario we assumed that $G(t, S) = (0.7 - 0.6S)S$; $a = 0.9$; $\beta(t) = \beta_0(1 + 0.7 \cos(2\pi t))$; $\eta(t) = 0.7(1 + 0.7 \cos(\pi + 2\pi t))$; $c(t) = 0.1$; $b = 0.2$; $r = 0.6$; $\gamma = 0.1$; $\theta(t) = 0.9$. We obtain the model:

$$\begin{cases} S' = (0.7 - 0.6S)S - 0.9SP - \beta_0(1 + 0.7 \cos(2\pi t))SI \\ I' = \beta_0(1 + 0.7 \cos(2\pi t))SI - 0.7(1 + 0.7 \cos(\pi + 2\pi t))PI - 0.1I \\ P' = (0.2 - 0.6P)P + 0.09SP + 0.63(1 + 0.7 \cos(\pi + 2\pi t))PI \end{cases}.$$

When $\beta_0 = 0.1$ we obtain $\mathcal{R}^u \approx -0.217 < 0$ ($\mathcal{R}_{per}^u \approx 0.35 < 1$) and we conclude that we have extinction (figure 3). When $\beta_0 = 0.8$ we obtain $\mathcal{R}^\ell \approx 0.167 > 0$ ($\mathcal{R}_{per}^\ell \approx 1.483 > 1$) and we conclude that the infectives are uniform strong persistent (figure 4).

We considered the following initial conditions at $t = 0$: $(S_0, I_0, P_0) = (1, 0.5, 0.1)$, $(S_0, I_0, P_0) = (0.1, 0.2, 1)$ and $(S_0, I_0, P_0) = (0.5, 0.5, 0.5)$.

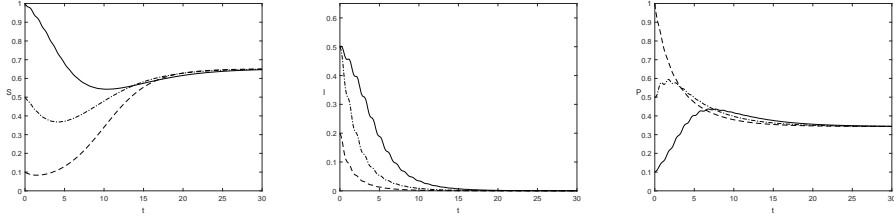


FIGURE 3. Extinction: $\beta_0 = 0.1$.

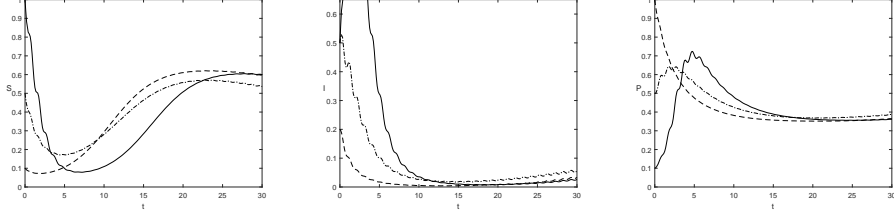


FIGURE 4. Uniform strong persistence: $\beta_0 = 0.8$.

3.3. Models with Gause-type uninfected subsystem. A model with Michaelis-Menten (or Holling-type I) functional response of predator to infected prey and a Holling-type II functional response of predator to susceptible prey is now considered. We consider the vital dynamics of uninfected prey as

$$G(t, S) = k(t, S)S,$$

where $k: \mathbb{R} \times [0, +\infty[\rightarrow \mathbb{R}$ is continuous, T -periodic ($T > 0$) in the t -variable, continuously differentiable in S and satisfying the following conditions:

$$\begin{cases} k(t, S) \text{ is bounded from above and } k(t, 0) > 0; \\ \text{for every } t \text{ there exists } S_1(t) > 0 \text{ such that } k(t, S_1(t)) = 0; \\ \frac{\partial k}{\partial S} < 0 \text{ for every } S \geq 0. \end{cases} \quad (39)$$

This type of vital dynamics was considered in [3] (see condition (k) in §3 in that reference). We notice that the general assumptions (39) are satisfied by a logistic growth of the prey population (with $S_1(t)$ equal to a constant). We also consider $a(t) = a > 0$; $f(S, I, P) = S/(m + S + I)$, with $m > 0$; $g(S, I, P) = P$; $h(t, P) = b - rP$, with $b, r > 0$, $\gamma(t) = \gamma > 0$. Moreover, we assume $\beta(t), \eta(t), c(t)$ and $\theta(t)$ to be bounded, nonnegative and continuous real valued functions. Henceforth we are considering the system

$$\begin{cases} S' = k(t, S)S - a\frac{SP}{m+S+I} - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)PI - c(t)I \\ P' = (b - rP)P + \gamma a\frac{SP}{m+S+I} + \theta(t)\eta(t)PI \end{cases}, \quad (40)$$

For model (40), conditions S1) and S2) follow immediate from hypothesis. The following lemma ensures condition S3).

Lemma 2. *There is a bounded region that contains the ω -limit of all orbits of (40).*

Proof. By the first equation in (40) we have $S' \leq k(t, S)S$. Notice that from (39) we have $k(t, S) > 0$ for $S < S_1(t)$ and $k(t, S) < 0$ for $S > S_1(t)$. Thus $[0, \max_{[0, T]} S_1(t)]$ is an attractor and so

$$\limsup_{t \rightarrow +\infty} S(t) \leq \max_{[0, T]} S_1(t) := S_1^u.$$

Additionally, writing $k^u = \sup_{(t, S)} k(t, S)$, we have

$$\begin{aligned} (S + I)' &\leq k(t, S)S - c(t)I \\ &\leq k^u S - c^\ell I \\ &\leq (k^u + c^\ell)(S_1^u + \delta) - c^\ell(S + I), \end{aligned}$$

for some $\delta > 0$ and all sufficiently large t , which lead us to

$$(S + I)(t) \leq \frac{(k^u + c^\ell)(S_1^u + \delta)}{c^\ell} =: C.$$

Finally, since for sufficiently large t

$$P' \leq (b - rP + \gamma a + \theta^u \eta^u C)P,$$

we have

$$\limsup_{t \rightarrow +\infty} P(t) \leq \frac{1}{r}(b + \gamma a + \theta^u \eta^u C) =: D.$$

Thus the region

$$\{(S, I, P) \in \mathbb{R}^3 : 0 \leq S + I \leq C \text{ and } 0 \leq P \leq D\}$$

contains the ω -limit of any orbit. \square

Notice that by [3, Lemma 3.1], equations $s' = k(t, s)s$ and $y' = (b - ry)y$, with positive initial condition, have a unique T -periodic solution which is bounded and globally asymptotically stable, hence globally attractive, on $]0, +\infty[$. This ensures conditions S4) and S5). For condition S6) we consider $G_{1,\varepsilon}(t, x) = G_{2,\varepsilon}(t, x) = k(t, x)x$, $h_{1,\varepsilon}(t, z) = h_{2,\varepsilon}(t, z) = (r - bz)z$, $v(\varepsilon) = \varepsilon$ and $\rho(t) = 1$. Clearly, conditions S6.1) to S6.3) hold. The auxiliary subsystem (6) becomes

$$\begin{cases} x' = k(t, x)x \\ z' = (b - rz)z + \gamma a\frac{xz}{m+x} + \varepsilon z. \end{cases} \quad (41)$$

By [3, Lemma 3.1] there exists a T -periodic globally asymptotic stable solution $x_{2,\varepsilon}^*(t)$ of the first equation of (41). Considering this solution in the second equation we get

$$z' = \left(b + \gamma a \frac{x_{2,\varepsilon}^*(t)}{m + x_{2,\varepsilon}^*(t)} + \varepsilon - rz \right) z,$$

which, again from [3, Lemma 3.1], has a T -periodic globally asymptotic stable solution $z_{2,\varepsilon}^*(t)$. Using this solutions $(x_{2,\varepsilon}^*(t), z_{2,\varepsilon}^*(t)^*)$, and writing

$$\tilde{k}(t, x) = k(t, x) - \frac{az_{2,\varepsilon}^*(t)}{m + x} - \varepsilon$$

the auxiliary subsystem (5) becomes

$$\begin{cases} x' = \tilde{k}(t, x)x \\ z' = (b - rz)z + \gamma a \frac{xz}{m + x + \varepsilon}. \end{cases} \quad (42)$$

It is straightforward to verify that $\tilde{k}(t, S)$ is bounded from above. Moreover, if

$$k(t, 0) - \frac{az_{2,\varepsilon}^*(t)}{m} - \varepsilon > 0 \quad (43)$$

for all sufficiently small $\varepsilon > 0$, then $\tilde{k}(t, 0) > 0$. Notice that we may find a bound for $z_{2,\varepsilon}^*(t)$ independent of ε . Moreover, if

$$\sup_{(t,x)} \left\{ \frac{\partial k}{\partial x}(t, x) + \frac{az_{2,\varepsilon}^*(t)}{(m + x)^2} \right\} < 0 \quad (44)$$

we have

$$\frac{\partial \tilde{k}}{\partial x}(t, x) = \frac{\partial k}{\partial x}(t, x) + \frac{az_{2,\varepsilon}^*(t)}{(m + x)^2} \leq \zeta < 0.$$

If (43) and (44) hold we have $\frac{\partial \tilde{k}}{\partial x}(t, x) \leq \zeta < 0$ and $\tilde{k}(t, 0) > 0$. Since $\lim_{x \rightarrow +\infty} \tilde{k}(t, x) = -\infty$ we conclude that for all t there exists $X_1(t)$ such that $\tilde{k}(t, X_1(t)) = 0$. Notice that having simultaneously (43) and (44) can be achieved, for instance, if a is small enough. Therefore, if (43) and (44) hold and $\varepsilon > 0$ is sufficiently small we are in conditions to apply [3, Lemma 3.1] to conclude that the first equation in (42) has a T -periodic solution $x_{1,\varepsilon}^*(t)$ which is globally asymptotically stable. Using this solution in the second equation gives

$$z' = (b - rz)z + \gamma a \frac{x_{1,\varepsilon}^*(t)}{m + x_{1,\varepsilon}^*(t) + \varepsilon} z = \left(b + \gamma a \frac{x_{1,\varepsilon}^*(t)}{m + x_{1,\varepsilon}^*(t) + \varepsilon} - rz \right) z,$$

which, proceeding as before, has a T -periodic solution $z_{1,\varepsilon}^*(t)$ that is globally asymptotically.

We have showed that conditions S6.4) and S6.5) also hold, so we may apply Theorem 1 and Theorem 2 to conclude that if $\mathcal{R}^u < 1$ then the infectives in model (40) go to extinction, and if $\mathcal{R}^\ell > 1$ then the infectives are uniform strong persistent.

Let us point out that in the particular case $k(t, x) = \Lambda - \mu x$, for some constants $\Lambda, \mu > 0$, we have the following globally asymptotically stable solution for (41)

$$\begin{cases} x_{2,\varepsilon}^* = \Lambda/\mu \\ z_{2,\varepsilon}^* = \left(b + \gamma a \frac{1}{m\mu+1} + \varepsilon \right) / r \end{cases}$$

and in this situation conditions (43) and (44) holds if

$$\frac{a}{mr} \left(b + \frac{\gamma a}{m\mu + 1} \right) < \min\{\Lambda, m\mu\}$$

and ε is sufficiently small.

To do some simulation, in this scenario we assumed that $G(t, S) = (0.7 - 0.6S)S$; $a = 0.9$; $\beta(t) = \beta_0(1 + 0.7 \cos(2\pi t))$; $\eta(t) = 0.7(1 + 0.7 \cos(\pi + 2\pi t))$; $c(t) = 0.1$; $b = 0.2$; $r = 0.6$; $m = 2$; $\gamma = 0.8$; $\theta(t) = 0.9$. We obtain the model:

$$\begin{cases} S' = (0.7 - 0.6S)S - 0.9 \frac{SP}{2+S+I} - \beta_0(1 + 0.7 \cos(2\pi t))SI \\ I' = \beta_0(1 + 0.7 \cos(2\pi t))SI - 0.7(1 + 0.7 \cos(\pi + 2\pi t))PI - 0.1I \\ P' = (0.2 - 0.6P)P + 0.9 \frac{SP}{2+S+I} + 0.6(1 + 0.7 \cos(\pi + 2\pi t))PI \end{cases}.$$

When $\beta_0 = 0.1$ we obtain $\mathcal{R}^u \approx -0.217 < 0$ and we conclude that we have extinction (figure 5). When $\beta_0 = 0.9$ we obtain $\mathcal{R}^\ell \approx 0.757 > 0$ and we conclude that the infectives are uniform strong persistent (figure 6).

We considered the following initial conditions at $t = 0$: $(S_0, I_0, P_0) = (1, 0.5, 0.1)$, $(S_0, I_0, P_0) = (0.1, 0.2, 1)$ and $(S_0, I_0, P_0) = (0.5, 0.5, 0.5)$.

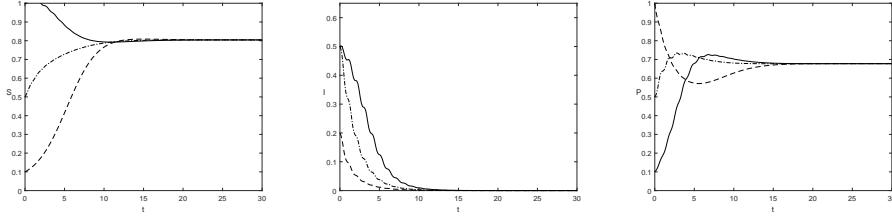


FIGURE 5. Extinction: $\beta_0 = 0.1$.

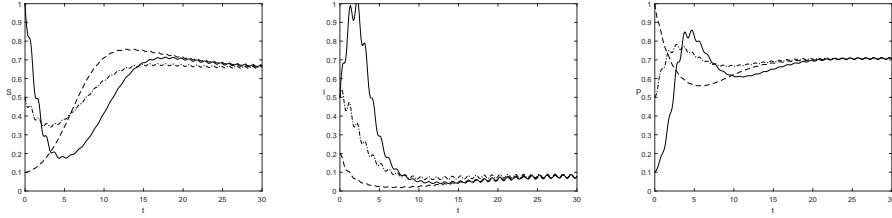


FIGURE 6. Uniform strong persistence: $\beta_0 = 0.9$.

3.4. Models with ratio-dependent uninfected subsystem. The functional response of predator to prey in the uninfected subsystem in the next example is ratio-dependent. Ratio-dependent functional responses were considered to overcome some paradoxes identified in Gause-type systems (see [6] and the references therein).

We consider now the vital dynamics of uninfected prey as

$$G(t, S) = \Lambda - \mu S,$$

for constants $\Lambda, \mu > 0$, and $a(t) = a > 0$, $f(S, I, P) = S/(Pm + S + I)$, with $m > 0$, $g(S, I, P) = P$; $h(t, P) = b - rP$, with $b, r > 0$, $\gamma(t) = \gamma > 0$, and $\beta(t), \eta(t), c(t)$ and $\theta(t)$ to be bounded, nonnegative and continuous real valued functions. With this assumptions system (1) becomes

$$\begin{cases} S' = \Lambda - \mu S - a \frac{SP}{mP + S + I} - \beta(t)SI \\ I' = \beta(t)SI - \eta(t)PI - c(t)I \\ P' = (b - rP)P + \gamma a \frac{SP}{mP + S + I} + \theta(t)\eta(t)PI \end{cases}. \quad (45)$$

For model (45), conditions S1) and S2) follow immediate from hypothesis and the particular forms of the functions g and h . Condition S3) is an immediate corollary of the following.

Lemma 3. *There is a bounded region that contains the ω -limit of all orbits of (40).*

Proof. By the first equation in (45) we have $S' \leq \Lambda - \mu S$ which implies

$$\limsup_{t \rightarrow +\infty} S(t) \leq \frac{\Lambda}{\mu}.$$

We have

$$\begin{aligned} (S + I)' &\leq \Lambda - \mu S - c(t)I \\ &\leq \Lambda - \min\{\mu, c^\ell\}(S + I) \end{aligned}$$

which implies

$$(S + I)(t) \leq \frac{\Lambda}{\min\{\mu, c^\ell\}} + \delta =: C.$$

for some $\delta > 0$ and all sufficiently large t . Finally, since for large t ,

$$P' \leq (b - rP + \gamma a + \theta^u \eta^u C)P$$

we have

$$\limsup_{t \rightarrow +\infty} P(t) \leq \frac{1}{r}(b + \gamma a + \theta^u \eta^u C) =: D.$$

Thus the region

$$\{(S, I, P) \in \mathbb{R}^3 : 0 \leq S + I \leq C \text{ and } 0 \leq P \leq D\}$$

contains the ω -limit of any orbit. \square

Similarly to the previous example, each solution of $s' = \Lambda - \mu s$ and of $y' = (b - ry)y$, with positive initial condition is bounded, bounded away from zero, and globally asymptotic stable (hence globally attractive) on $]0, +\infty[$, which ensures conditions S4) and S5).

For condition S6) we consider now $G_{1,\varepsilon}(t, x) = G_{2,\varepsilon}(t, x) = \Lambda - \mu x$, $h_{1,\varepsilon}(t, z) = h_{2,\varepsilon}(t, z) = (r - bz)z$, $v(\varepsilon) = \varepsilon$ and $\rho(t) = 1$. As before, conditions S6.1) to S6.3) are straightforward. The auxiliary subsystem (6) becomes

$$\begin{cases} x' = \Lambda - \mu x \\ z' = (b - rz)z + \gamma a \frac{xz}{mz + x} + \varepsilon z. \end{cases} \quad (46)$$

The first equation of (46) has solution $x_{2,\varepsilon}^*(t) = \Lambda/\mu$. Considering this solution in the second equation we get

$$z' = \left(b + \gamma a \frac{\Lambda/\mu}{mz + \Lambda/\mu} + \varepsilon - rz \right) z,$$

which has solution

$$z_{2,\varepsilon}^*(t)^* = \frac{A + \sqrt{A^2 + 4\Lambda r m \mu B}}{2r m \mu} := z_{2,\varepsilon},$$

where

$$A = -\Lambda r + b m \mu + \varepsilon m \mu \quad \text{and} \quad B = b + \varepsilon + a \gamma.$$

Proceeding as in previous example, from [3, Lemma 3.1] we conclude that $(x_{2,\varepsilon}^*(t), z_{2,\varepsilon}^*(t))$ is a globally asymptotically stable solution on $\{(x, y) \in \mathbb{R}^2 : x, y > 0\}$. Using this solution $(x_{2,\varepsilon}^*(t), z_{2,\varepsilon}^*(t))$, the auxiliary subsystem (5) becomes

$$\begin{cases} x' = \Lambda - \mu x - a z_{2,\varepsilon}^* - \varepsilon x \\ z' = (b - r z) z + \gamma a \frac{x z}{z m + x + \varepsilon}. \end{cases}$$

The first equation can be written as

$$x' = \Lambda - a z_{2,\varepsilon}^* - (\mu + \varepsilon)x,$$

which, if

$$\Lambda > a z_{2,\varepsilon}^* \tag{47}$$

it has solution

$$x_{1,\varepsilon}^*(t) = \frac{\Lambda - a z_{2,\varepsilon}^*}{\mu + \varepsilon} =: x_{1,\varepsilon}^*.$$

Using this solution in the second equation gives

$$z' = \left(b + \gamma a \frac{x_{1,\varepsilon}^*}{z m + x_{1,\varepsilon}^* + \varepsilon} - r z \right) z,$$

which in turn has solution

$$z_{1,\varepsilon}^*(t) = \frac{\tilde{A} + \sqrt{\tilde{A} + 4 m r \tilde{B}}}{2 m r}$$

where $\tilde{A} = b m - \varepsilon r - r x_{1,\varepsilon}^*$ and $\tilde{B} = b \varepsilon + b x_{1,\varepsilon}^* + a \gamma x_{1,\varepsilon}^*$.

$$z_{1,\varepsilon}^*(t) = \frac{\tilde{A} + \sqrt{\tilde{A} + 4 r m \tilde{B}}}{2 r m},$$

where $\tilde{A} = m b - r x_{1,\varepsilon}^*$ and $\tilde{B} = (b + \gamma a) x_{1,\varepsilon}^*$. Once more, proceeding as in the previous example, one can check that by [3, Lemma 3.1] we conclude that $(x_{1,\varepsilon}^*(t), z_{1,\varepsilon}^*(t))$ is a globally asymptotically stable solution on $\{(x, y) \in \mathbb{R}^2 : x, y > 0\}$.

We have showed that if (47) holds, which happens, for instance, for sufficiently small a , then conditions S6.4) and S6.5) also hold, and we may apply Theorem 1 and Theorem 2 to conclude that if $\mathcal{R}^u < 1$ then the infectives in model (45) go to extinction, and if $\mathcal{R}^\ell > 1$ then the infectives are uniform strong persistent.

To do some simulation, in this scenario we assumed that $G(t, S) = 3 - 0.6S$; $a = 0.9$; $\beta(t) = \beta_0(1 + 0.7 \cos(2\pi t))$; $\eta(t) = 0.7(1 + 0.7 \cos(\pi + 2\pi t))$; $c(t) = 0.1$; $b = 0.2$; $r = 0.6$; $m = 2$; $\gamma = 0.8$; $\theta(t) = 0.9$. We obtain the model:

$$\begin{cases} S' = 3 - 0.6S - 0.9 \frac{SP}{2P+S+I} - \beta_0(1 + 0.7 \cos(2\pi t))SI \\ I' = \beta_0(1 + 0.7 \cos(2\pi t))SI - 0.7(1 + 0.7 \cos(\pi + 2\pi t))PI - 0.1I \\ P' = (0.8 - 0.6P)P + 0.9 \frac{SP}{2P+S+I} + 0.6(1 + 0.7 \cos(\pi + 2\pi t))PI \end{cases}.$$

When $\beta_0 = 0.01$ we obtain $\mathcal{R}^u \approx -0.283 < 0$ and we conclude that we have extinction (figure 7). When $\beta_0 = 0.3$ we obtain $\mathcal{R}^\ell \approx 0.073 > 0$ and we conclude that the infectives are uniform strong persistent (figure 8).

We considered the following initial conditions at $t = 0$: $(S_0, I_0, P_0) = (1, 0.5, 0.1)$, $(S_0, I_0, P_0) = (0.1, 0.2, 1)$ and $(S_0, I_0, P_0) = (0.5, 0.5, 0.5)$.

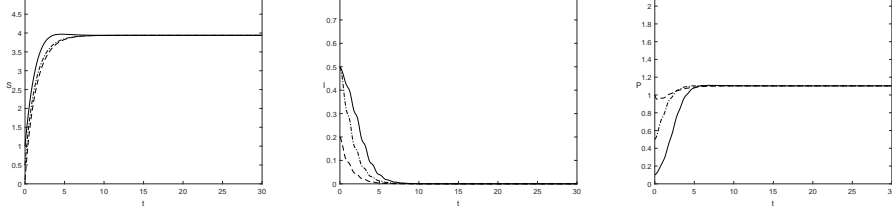


FIGURE 7. Extinction: $\beta_0 = 0.01$.

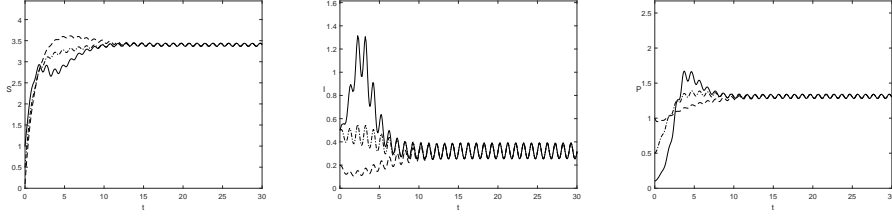


FIGURE 8. Uniform strong persistence: $\beta_0 = 0.3$.

3.5. Iterative scheme for models with Holling-type I functional response.

In this subsection we consider a simple example in order to illustrate the procedure in Section 2 which makes use of Theorem 3. This procedure can be extended for more complex models and can constitute a procedure in order to improve a threshold for extinction.

We will consider $G(t, S) = (\Lambda - \mu S)S$; $a(t) = a$; $\beta(t) = \beta$; $\eta(t) = \eta$; $\theta(t) = \theta$; $c(t) = c$; $f(S, I, P) = S$; $g(S, I, P) = P$; $h(t, P) = b - rP$; $\gamma(t) = \gamma$. We obtain the model:

$$\begin{cases} S' = (\Lambda - \mu S)S - aSP - \beta SI \\ I' = \beta SI - \eta PI - cI \\ P' = (b - rP)P + \gamma aSP + \theta \eta PI \end{cases}, \quad (48)$$

As discussed in section 3.1 conditions S1)-S2) hold for system (48). By Theorem 1, if

$$\mathcal{R}^u = \mathcal{R}^{u,0} = \beta \frac{\Lambda}{\mu} - \eta \frac{b}{r} - c < 0$$

the infectives in (48) go to extinction.

Set now $s^{1,\ell}(t) = 0$ and consider the solution $s^{1,u}(t) = \Lambda/\mu$ of equation (2). Recall B from (38) given by

$$B = \frac{1}{r} (b + \gamma a A + \theta \eta A),$$

where $A = \frac{\Lambda^2}{4\mu c} + \frac{\Lambda}{\mu}$. Set $y^{1,u}(t) = B$ and consider the solution $y^{1,\ell}(t) = b/r$ of equation (3). We define recursively the corresponding equations (24)-(26), respectively, by setting for $k \in \mathbb{N}$

$$\begin{aligned} (s^{k+1,u})' &= (\Lambda - ay^{k,\ell} - \mu s^{k+1,u})s^{k+1,u} \\ (s^{k+1,\ell})' &= (\Lambda - ay^{k,u} - \beta b - \mu s^{k+1,\ell})s^{k+1,\ell} \\ (y^{k+1,u})' &= (b + \gamma as^{k+1,u} + \theta\eta B - ry^{k+1,u})y^{k+1,u} \\ (y^{k+1,\ell})' &= (b + \gamma as^{k+1,\ell} - ry^{k+1,\ell})y^{k+1,\ell}, \end{aligned}$$

and consider the corresponding particular solutions

$$\begin{aligned} s^{k+1,u} &= (\Lambda - ay^{k,\ell})/\mu \\ s^{k+1,\ell} &= (\Lambda - ay^{k,u} - \beta B)/\mu \\ y^{k+1,u} &= (b + \gamma as^{k+1,u} + \theta\eta B)/r \\ y^{k+1,\ell} &= (b + \gamma as^{k+1,\ell})/r. \end{aligned}$$

It is easy to check that conditions $S4_k$) and $S5_k$) hold. We conclude that the infectives go to extinction if

$$\mathcal{R}^{u,k}(\lambda) = \beta s^{k,u} - \eta y^{k,\ell} - c < 0. \quad (49)$$

Condition (49) can be used to improve a threshold for extinction.

In order to undertake some simulation we fix $\mu = 0.6$; $a = 0.8$; $c = 0.1$; $b = 0.2$; $r = 0.9$; $\gamma = 0.1$ and $\theta = 0.9$. In the first situation we also fix $\eta = 0.5$ and analyse the persistence and extinction threshold in terms of variables Λ and β .

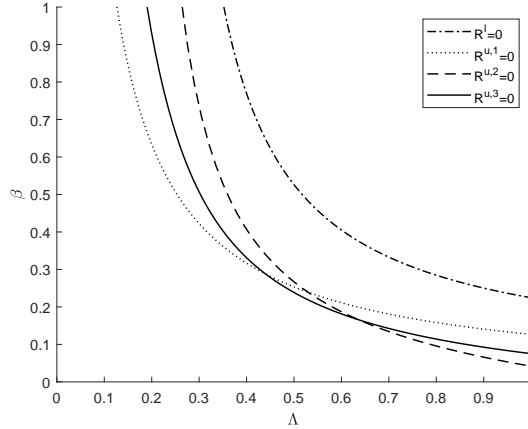


FIGURE 9. Persistence and extinction thresholds with respect to parameters β and Λ .

We can see the region of parameters corresponding to persistence above the line $\{(\Lambda, \beta) : \mathcal{R}^\ell = 0\}$. If, for each $k = 1, 2, 3$ we set the region for extinction $E_k = \{(\Lambda, \beta) : \Lambda \in [0, 1] \wedge \mathcal{R}^{k,u} < 0\}$, which can be easily seen in the picture as the region below the corresponding line, one can see that E_2 does not contain E_1 but we can improve the *known* region of extinction E_1 by considering $E_1 \cup E_2$ instead.

In general there is no guarantee that there is a real improvement in each iteration as we can see by the region E_3 which is contained in $E_1 \cup E_2$, and thus not adding new information on the set of parameters leading to extinction.

We fix now $\Lambda = 0.6$ and analyse the persistence and extinction as a function of parameters β and η .

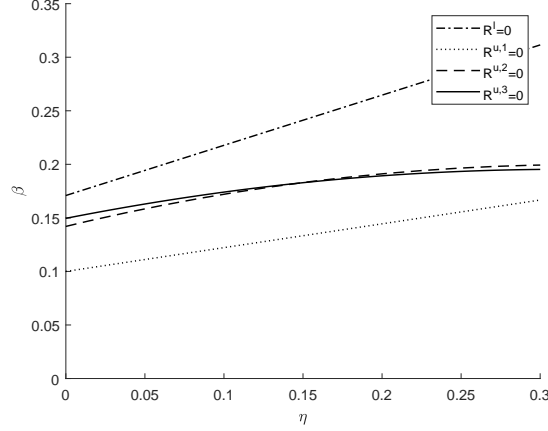


FIGURE 10. Persistence and extinction thresholds with respect to parameters β and η .

Set now the regions for extinction $F_k = \{(\eta, \beta) : \Lambda \in [0, 0.3] \wedge \mathcal{R}^{k,u} < 0\}$, $k = 1, 2, 3$. In this example we have an improvement in each of the first iterations, in the sense that $F_1 \subsetneq F_2 \subsetneq F_2 \cup F_3$.

This example intends to illustrate the potential of this approach and en raises several interesting questions.

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