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# Depth of almost strictly sign regular matrices (CMMSE-2021)

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**The concept of depth of an almost strictly sign regular matrix is introduced and used to simplify some algorithmic characterizations of these matrices. Copyright © 2021 John Wiley & Sons, Ltd.**

**Keywords:** Depth; Almost Strictly Sign Regular matrices; Neville elimination; characterization

## 1. Introduction

Matrices with all their minors nonnegative are called Totally Positive (TP) and matrices whose minors of the same order have the same sign are called Sign Regular (SR). They arise in approximation theory, differential equations, statistics, combinatorics, mechanics, computer-aided geometric design and economics, among other subjects (see [1], [2], [3] and [4]). An important subclass of TP matrices are the Almost Strictly Totally Positive (ASTP) matrices, matrices whose minors are positive if and only if all their diagonal entries are positive (see [5]). ASTP matrices contain Hurwitz matrices and B-splines collocation matrices. In [6], the class of Almost Strictly Sign Regular (ASSR) matrices was introduced and characterized by a reduced number of minors. ASSR matrices form a subclass of SR matrices including all ASTP matrices.

In [7], an algorithmic characterization of ASSR matrices is provided. It used the Neville elimination (NE) of a matrix, which is an elimination procedure alternative to Gaussian elimination. Roughly speaking, NE makes zeros in a column of a matrix by adding to each row an adequate multiple of the previous one (see [8] for more details), instead of using just a row with a fixed pivot as in Gaussian elimination. So, NE transforms a nonsingular matrix into an upper triangular matrix. The entries of each columns used to make zeros below them are called pivots of the NE. More results about NE and SR matrices can be seen in [9], [10], [11], [12], [13], [14] and [15].

In this paper, we introduce the concept of depth of an ASSR matrix. At the end of Section 4, we see that strictly M-banded ASSR matrices (see [16]) are included in the class of ASSR matrices with depth $n-M$. We also see in Section 4 that the use of the depth of an ASSR matrix allows us to simplify the algorithms of [7] and to reduce its computational cost. If the matrix has a high depth, then this reduction is considerable.

The paper is organized as follows. Section 2 contains basic notations and definitions concerning the zero pattern of a matrix. Section 3 recalls definitions and some fundamental results on ASSR matrices. In Section 4 we introduce the concept of depth of an ASSR matrix and prove that the depth of an ASSR matrix determines its initial signature. We also provide the mentioned simplified characterization and the corresponding algorithms. Section 5 include some numerical results and applications. Finally, Section 6 summarizes the main conclusions of this work.

## 2. Basic notations and definitions

For $k, n \in \mathbb{N}$, with $1 \leq k \leq n$, $Q_{k,n}$ denotes the set of all increasing sequences of $k$ natural numbers not greater than $n$. If $A$ is a real $n \times n$ matrix and $\alpha = (\alpha_1, \ldots, \alpha_k)$, $\beta = (\beta_1, \ldots, \beta_k) \in Q_{k,n}$, then $A[\alpha|\beta]$ is by definition the $k \times k$ submatrix of $A$ containing rows $\alpha_1, \ldots, \alpha_k$ and columns $\beta_1, \ldots, \beta_k$ of $A$. In particular, when $\alpha = \beta$, $A[\alpha] := A[\alpha|\alpha]$ is the corresponding

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principal submatrix. Besides, $Q^0_{k,n}$ denotes the set of increasing sequences of $k$ consecutive natural numbers not greater than $n$ and if $\alpha \in Q^0_{k,n}$, then $\det A[\alpha]$ is a principal minor.

From now on, it will be frequently used the backward identity matrix $n \times n$, $P_n$, whose element $(i,j)$ is defined as

$$\begin{cases} 1 & \text{if } i+j=n+1, \\ 0 & \text{otherwise.} \end{cases}$$

Throughout this document, we will work with matrices whose zero and nonzero entries are grouped in certain positions. Then we introduce the type-I and type-II staircase matrices.

**Definition 2.1** *A real matrix $A=(a_{ij})_{1\le i,j,\le n}$ is called type-I staircase matrix if it satisfies simultaneously the following conditions:*

*(a)* $a_{ii} \neq 0$, $\forall i \in \{1,\ldots,n\}$*;*

*(b)* $a_{ij}=0, i>j \Rightarrow a_{kl}=0$*, if* $l \le j$, $i \le k$*;*

*(c)* $a_{ij}=0, i<j \Rightarrow a_{kl}=0$*, if* $k \le i$, $j \le l$*.*

**Definition 2.2** *A matrix $A$ is called type-II staircase if $P_nA$ is a type-I staircase matrix.*

Conditions introduced in definitions 2.1 and 2.2 produce a staircase structure for the zero pattern, which is set through the following indices (see [5, 7]):

**Definition 2.3** *For a matrix $A=(a_{ij})_{1\le i,j\le n}$, type-I staircase, we define*

$$i_0=1, \qquad j_0=1, \tag{1}$$

*and for $k=1,\ldots,\ell$:*

$$i_k=\max\left\{i \;/\; a_{ij_{k-1}}\neq 0\right\}+1 \quad (\le n+1), \tag{2}$$

$$j_k=\max\left\{j\le i_k \;/\; a_{i_kj}=0\right\}+1 \quad (\le n+1), \tag{3}$$

*where $l$ is given in this recurrent definition by $j_\ell=n+1$.*

*Analogously we define*

$$\widehat{j}_0=1, \qquad \widehat{i}_0=1 \tag{4}$$

*and for $k=1,\ldots,r$:*

$$\widehat{j}_k=\max\left\{j \;/\; a_{\widehat{i}_{k-1}j}\neq 0\right\}+1 \; (\le n+1), \tag{5}$$

$$\widehat{i}_k=\max\left\{i\le \widehat{j}_k \;/\; a_{i\widehat{j}_k}=0\right\}+1 \; (\le n+1), \tag{6}$$

*where $\widehat{i}_r=n+1$.*

*Finally, we denote by $I$, $J$, $\widehat{I}$ and $\widehat{J}$ the following sets of indices*

$$\begin{array}{rclrcl} I &=& \{i_0,i_1,\ldots,i_\ell\}, & J &=& \{j_0,j_1,\ldots,j_\ell\}, \\ \widehat{I} &=& \left\{\widehat{i}_0,\widehat{i}_1,\ldots,\widehat{i}_r\right\}, & \widehat{J} &=& \left\{\widehat{j}_0,\widehat{j}_1,\ldots,\widehat{j}_r\right\}, \end{array}$$

*thereby defining the zero pattern in the matrix $A$.*

**Note 2.4** *Note that, if $card(I)=2$ then $a_{ij}\neq 0$ when $1\le j<i\le n$. In the same way, if $card(\widehat{I})=2$, then $a_{ij}\neq 0$ when $1\le i<j\le n$.*

So, if $A$ is a type-II staircase matrix the zero pattern of $A$ is the zero pattern of $P_nA$.

To mark the positions of the matrix in which a new echelon starts, we define the following indices.

**Definition 2.5** *Let $A$ be a real $n\times n$ matrix, type-I staircase, with zero pattern $I$, $J$, $\widehat{I}$ and $\widehat{J}$. Let be $1\le i,j\le n$. If $j\le i$ we define*

$$j_t=\max\left\{j_s \;/\; 0\le s\le k-1,\; j-j_s\le i-i_s\right\}, \tag{7}$$

*being $k$ the unique index satisfying that $j_{k-1}\le j<j_k$, and if $i<j$*

$$\widehat{i}_t=\max\left\{\widehat{i}_s \;/\; 0\le s\le k'-1,\; i-\widehat{i}_s\le j-\widehat{j}_s\right\}, \tag{8}$$

*being $k'$ the only index satisfying that $\widehat{i}_{k'-1}\le i<\widehat{i}_{k'}$.*

## 3. Almost strictly sign regular matrices

Next, we define ASSR matrices, wich are matrices whose nontrivial minors of the same order have all the same strict sign. To store the sign we are going to define the vector of signatures.

**Definition 3.1** *Given a vector $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \mathbb{R}^n$, we say that $\varepsilon$ is a signature sequence, or simply, is a signature, if $\varepsilon_i \in \{+1, -1\}$ for all $i \leq n$.*

ASSR matrices form a subclass of SR matrices. A matrix is SR if all its minors of the same order have the same sign. That is:

**Definition 3.2** *A real matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ is said to be SR, with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, if all its minors satisfy that*

$$\varepsilon_m \det A[\alpha|\beta] \geq 0, \quad \alpha, \beta \in Q_{m,n}, \qquad m \leq n. \tag{9}$$

In a staircase matrix there are some minors which are trivially zero due to the position of their zero entries. We are going to distinguish those minors from those that do not verify that condition.

**Definition 3.3** *Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a type-I (type-II) staircase matrix. A submatrix $A[\alpha|\beta]$, with $\alpha, \beta \in Q_{m,n}$, is said to be nontrivial if all its main diagonal (secondary diagonal) elements are nonzero.*

The minor associated to a nontrivial submatrix ($A[\alpha|\beta]$) is called nontrivial minor ($\det A[\alpha|\beta]$).
The nontrivial minors play an important role in ASSR matrices.

**Definition 3.4** *A real matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ is said to be ASSR, with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, if all its nontrivial minors $\det A[\alpha|\beta]$ satisfy that*

$$\varepsilon_m \det A[\alpha|\beta] > 0, \quad \alpha, \beta \in Q_{m,n}, \qquad m \leq n. \tag{10}$$

**Note 3.5** *Observe that an ASSR matrix is SR, since the trivial minors are zero and the nontrivial minors satisfy the strict inequality (9). Observe also that an ASSR matrix is nonsingular.*

Next, we present the characterization given in [6] for ASSR matrices (see Theorem 10).

**Theorem 3.6** *Let $A$ be a real $n \times n$ matrix and $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ be a signature. Then $A$ is ASSR with signature $\varepsilon$ if and only if $A$ is a type-I or type-II staircase matrix, and all its nontrivial minors with $\alpha, \beta \in Q^0_{m,n}$, $m \leq n$, satisfy*

$$\varepsilon_m \det A[\alpha|\beta] > 0. \tag{11}$$

A characterization of the ASSR matrices using the pivots of NE is given in [7]. This characterization uses the following results:

**Theorem 3.7** *Let $B = (b_{ij})_{1 \leq i,j \leq n}$ be a nonsingular type-I staircase matrix, with zero pattern defined by $I$, $J$, $\widehat{I}$ and $\widehat{J}$. If $B$ is ASSR with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, then the NE of $B$ can be performed without row exchanges and the pivots $p_{ij}$ satisfy, for $1 \leq j \leq i \leq n$,*

$$p_{ij} = 0 \quad \Leftrightarrow \quad b_{ij} = 0, \tag{12}$$

$$\varepsilon_{j-j_t}\varepsilon_{j-j_t+1} p_{ij} > 0 \quad \Leftrightarrow \quad b_{ij} \neq 0, \tag{13}$$

*where $\varepsilon_0 = 1$ and $j_t$ is defined in (7).*

**Theorem 3.8** *Let $B = (b_{ij})_{1 \leq i,j \leq n}$ be a nonsingular type-II staircase matrix, with zero pattern defined by $I$, $J$, $\widehat{I}$ and $\widehat{J}$. If $B$ is ASSR with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, then the NE of $B^T$ can be performed without row exchanges and the pivots $q_{ij}$ satisfy, for $1 \leq i < j \leq n$,*

$$q_{ij} = 0 \quad \Leftrightarrow \quad b_{ij} = 0, \tag{14}$$

$$\varepsilon_{i-\widehat{i}_t}\varepsilon_{i-\widehat{i}_t+1} p_{ij} > 0 \quad \Leftrightarrow \quad b_{ij} \neq 0, \tag{15}$$

*where $\varepsilon_0 = 1$ and $\widehat{i}_t$ is defined in (8).*

From now on, we will denote by $A_h = A[h, \ldots, n]$. Note that $A_1 = A$ and $A_n = (a_{nn})$.

**Note 3.9** *Let $A$ be a type-I (type-II) staircase matrix. Then, in relation to $A_h$ ($A_h^T$) matrices, the next results are verified:*

1. *$A_h$ and $A_h^T$ are also type-I (type-II) staircase matrix for all $h \in \{1, 2, \ldots, n\}$.*


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2. *If the zero pattern of $A$ is given by $I = \{i_0, \ldots, i_\ell\}$, $J = \{j_0, \ldots, j_\ell\}$, $\widehat{I} = \{\widehat{i}_0, \ldots, \widehat{i}_r\}$, $\widehat{J} = \{\widehat{j}_0, \ldots, \widehat{j}_r\}$, then the zero pattern of $A_h$ matrices is given by $I^h = \{1, i_a - h + 1, \ldots, i_l - h + 1\}$, $J^h = \{1, j_a - h + 1, \ldots, j_l - h + 1\}$, $\widehat{I}^h = \{1, \widehat{i}_b - h + 1, \ldots, \widehat{i}_r - h + 1\}$, $\widehat{J}^h = \{1, \widehat{j}_b - h + 1, \ldots, \widehat{j}_r - h + 1\}$,*

   *where $a = \min\{s \ / \ j_s - h + 1 \geq 2\}$ and $b = \min\{s \ / \ \widehat{i}_s - h + 1 \geq 2\}$.*

3. *If $A$ is ASSR matrix with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ then $A_h$ and $A_h^T$ are ASSR matrices with signature $\varepsilon' = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_{n-h+1})$.*

Finally, in the following result (Theorem 5 of [7]) a characterization of ASSR matrices, with $\varepsilon_2 = 1$, is presented:

**Theorem 3.10** *A nonsingular matrix $A = (a_{ij})_{1 \leq i,j \leq n}$ is ASSR with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$, with $\varepsilon_2 = 1$ if and only if for every $h = 1, \ldots, n-1$ the following properties hold simultaneously:*

*(i) $A$ is type-I staircase;*

*(ii) the NE of the matrices $A_h = A[h, \ldots, n]$ and $A_h^T$ can be performed without row exchanges;*

*(iii) the pivots $p_{ij}^h$ of the NE of $A_h$ satisfy conditions corresponding to (12), (13), and the pivots $q_{ij}^h$ of the NE $A_h^T$ satisfy (14) and (15);*

*(iv) for the positions $(i^h, j^h)$ of matrix $A_h$:*

- *if $i^h \geq j^h$ and $i^h - j^h = i_t^h - j_t^h$ then $\varepsilon_{j^h - j_t^h} \varepsilon_{j^h - j_t^h + 1} = \varepsilon_{j^h - 1} \varepsilon_{j^h}$,*
- *if $i^h < j^h$ and $i^h - j^h = \widehat{i}_t^h - \widehat{j}_t^h$ then $\varepsilon_{i^h - \widehat{i}_t^h} \varepsilon_{i^h - \widehat{i}_t^h + 1} = \varepsilon_{i^h - 1} \varepsilon_{i^h}$,*

*where indices $i_t^h, j_t^h, \widehat{i}_t^h, \widehat{j}_t^h$ are given by conditions corresponding to (7) and (8).*

## 4. Depth and characterization of ASSR matrices

In this section, the characterization of ASSR matrices obtained in Theorem 3.10 is simplified. To that end, following the Definition 2.3, we denote by $r = \mathrm{card}(I) - 1$ and by $s = \mathrm{card}(\widehat{I}) - 1$. Observe that if $r = 1$ and $s = 1$, then $A$ has not zero entries.

For further result, it is convenient to know the length of the longest diagonal with all elements nonzero and which is close to zero entry.

**Definition 4.1** *Let $A$ an $n \times n$ matrix, type-I staircase with zero pattern give by $I$, $J$, $\widehat{I}$, $\widehat{J}$. We define, $\theta_L$ ($\theta_U$) as the length of the shortest diagonal below (above) the main diagonal without zero entries, that is:*

$$\theta_L := \begin{cases} 1 & \text{if} \quad r = 1, \\ \max\limits_{k \in \{1, \ldots, r-1\}} \{n - (i_k - j_k)\} & \text{if} \quad r > 1, \end{cases}$$

*and*

$$\theta_U := \begin{cases} 1 & \text{if} \quad s = 1, \\ \max\limits_{k \in \{1, \ldots, s-1\}} \{n - (\widehat{j}_k - \widehat{i}_k)\} & \text{if} \quad s > 1. \end{cases}$$

To illustrate Definition 4.1 let us look at the following example.

**Example 4.2** *Given the matrix*

$$A = \begin{pmatrix} 1 & 2 & 10^{-4} & 0 & 0 & 0 \\ 2 & 6 & 6 & 8 & 0 & 0 \\ 0 & 6 & 21 & 30 & 9 & 0 \\ 0 & 8 & 30 & 48 & 42 & 28 \\ 0 & 10 & 39 & 82 & 172 & 176 \\ 0 & 0 & 0 & 28 & 176 & 259 \end{pmatrix},$$

*the zero pattern bellow the main diagonal is $I = \{1, 3, 6, 7\}$ and $J = \{1, 2, 4, 7\}$. We observe that $\theta_L = \max\{6 - (3 - 2), 6 - (6 - 4)\} = 5$ and 5 is the length of the shortest diagonal below the main diagonal without zero entries.*

*The zero pattern above the main diagonal is $\widehat{I} = \{1, 2, 3, 4, 7\}$ and $\widehat{J} = \{1, 4, 5, 6, 7\}$. We observe that $\theta_U = \max\{6 - (4 - 2), 6 - (5 - 3), 6 - (6 - 4)\} = 4$ and 4 is the length of the shortest diagonal above the main diagonal without zero entries.*

Considering the previous definition, the depth $\theta$ of $A$ is defined.

**Definition 4.3** *Let $A = (a_{ij})_{1 \leq i,j \leq n}$ be a real matrix. We define $\theta$ the depth of $A$ as,*

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- *if the matrix $A$ is type-I staircase, then $\theta = \max\{\theta_L, \theta_U\}$,*
- *if the matrix $A$ is type-II staircase, then $\theta$ is the depth of $P_nA$.*

**Example 4.4** *Let $A$ be the matrix given in Example 4.2. Then $\theta = \max\{5, 4\} = 5$ and 5 is the length of the longest diagonal close to a zero.*

The next result shows that the depth $\theta$ of an ASSR matrix determines the first $\theta$ components of its signature.

**Theorem 4.5** *Let $A = (a_{ij})_{1\le i,j\le n}$ be an ASSR type-I staircase matrix with depth $\theta$. If $A$ is nonnegative, then its signature is $\varepsilon = (1, 1, \dots, 1, \varepsilon_{\theta+1}, \dots, \varepsilon_n)$. If $A$ is nonpositive, then its signature is $\varepsilon = (-1, 1, \dots, (-1)^{\theta-1}, (-1)^{\theta}, \varepsilon_{\theta+1}, \dots, \varepsilon_n)$.*

*Proof:* Observe that it is enough to see that

$$\varepsilon_k = (\varepsilon_1)^k \quad \forall\, k \in \{1, 2, \dots, \theta\}.$$

Firstly we suppose that $\varepsilon_1 = 1$. If $\theta = 1$ the result holds.

We can suppose that $\theta = \theta_L$ because, otherwise, we can apply the same reasoning to $A^t$. Let $m$ be such that $\theta = \theta_L = n - (i_m - j_m)$. The proof is performed by induction on $k$.

If $k = 1$ then $\varepsilon_1 = 1$ by hypothesis.

We suppose that $\varepsilon_k = 1$ for all $k \le \ell - 1$ with $\ell \le \theta$, and we will prove that $\varepsilon_k = 1$ for $k = \ell$.

If $\ell \le n + 2 - i_m$ we consider the square matrix of order $m$

$$B = A[i_m - 1, \dots, i_m + \ell - 2 | j_m - 1, \dots, j_m + \ell - 2]$$

then

$$\det B = a_{i_m-1, j_m-1} \det A[i_m, \dots, i_m + \ell - 2 | j_m, \dots, j_m + \ell - 2].$$

Thus $\varepsilon_\ell = \varepsilon_1 \varepsilon_{\ell-1}$. By hypothesis $\varepsilon_1 = 1$ and, by the induction hypothesis, $\varepsilon_{\ell-1} = 1$, thus $\varepsilon_\ell = 1$.

If $\ell > n + 2 - i_m$ we consider the square matrix of order $m$

$$B = A[n-\ell+1, \dots, n | n-(i_m-j_m)-\ell+1, \dots, n-(i_m-j_m)]$$

then $\det B =$

$$\det A[n-\ell+1, \dots, i_m - 1 | n-(i_m-j_m)-\ell+1, \dots, j_m - 1] \det A[i_m, \dots, n | j_m, \dots, n-(i_m-j_m)].$$

Thus $\varepsilon_\ell = \varepsilon_{\ell+i_m-n-1}\varepsilon_{n-i_m+1}$. By the induction hypothesis $\varepsilon_{\ell+i_m-n-1} = 1$ and $\varepsilon_{n-i_m+1} = 1$, thus $\varepsilon_\ell = 1$.

Therefore, in any case, $\varepsilon_\ell = 1$, and the result holds, that is, $\varepsilon = (1, 1, \dots, 1, \varepsilon_{\theta+1}, \dots, \varepsilon_n)$.

Suppose now that $\varepsilon_1 = -1$, then we define $B = \varepsilon_1 A = -A$. $B$ is an ASSR matrix with signature $\varepsilon'$ with $\varepsilon'_k = (-1)^k \varepsilon_k$, and $\theta' = \theta$. Applying the previous reasoning to $B$, $\varepsilon'_k = 1$ for all $k \in \{1, \dots, \theta\}$ and

$$\varepsilon_k = (-1)^k \varepsilon'_k = (\varepsilon_1)^k.$$

So, the result holds, that is, $\varepsilon = (-1, 1, \dots, (-1)^{\theta-1}, (-1)^{\theta}, \varepsilon_{\theta+1}, \dots, \varepsilon_n)$. □

**Example 4.6** *Given the matrix*

$$A = \begin{pmatrix} -4 & -16 & -2 & 0 & 0 & 0 \\ -8 & -40 & -44 & -64 & 0 & 0 \\ -2 & -32 & -133 & -240 & -60 & 0 \\ 0 & -32 & -184 & -368 & -248 & -144 \\ 0 & 0 & -36 & -240 & -972 & -1344 \\ 0 & 0 & 0 & -144 & -1752 & -13300 \end{pmatrix}$$

*ASSR with signature $\varepsilon = (-1, 1, -1, 1, -1, 1)$ and $\theta = 4$. Then $\varepsilon_1 = -1$, $\varepsilon_2 = (-1)^2$, $\varepsilon_3 = (-1)^3$, $\varepsilon_4 = (-1)^4$.*

In [7] the authors establish the relationship between the signatures of $A$ and $P_nA$, that we are going to use.

**Proposition 4.7** *A matrix $A$ is ASSR if and only if $P_nA$ is also ASSR. Furthermore, if the signature of $A$ is $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$, then the signature of $P_nA$ is $\varepsilon' = (\varepsilon'_1, \varepsilon'_2, \dots, \varepsilon'_n)$, with $\varepsilon'_m = (-1)^{\frac{m(m-1)}{2}} \varepsilon_m$.*

Taking into account the previous result, it is possible to determine the first $\theta$ components of the signature vector for type-II staircase ASSR matrices.


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**Corollary 4.8** *Let $A$ be an $n\times n$ ASSR type-II staircase matrix with signature $\varepsilon$. It is verified that $\varepsilon_k=(-1)^{\frac{k(k+1)}{2}}(\varepsilon_1)^k$ for all $k\in\{1,\dots,\theta\}$ where $\theta$ is the depth of the matrix $A$.*

*Proof:* If we define $B=\varepsilon_1 P_n A$ and we call $\varepsilon'$ its signature, then we have that $\varepsilon'_1=1$ and $B$ is type-I. By Theorem 4.5, $\varepsilon'_k=1$ for all $k\in\{1,\dots,\theta\}$. Finally, using Proposition 4.7, we have $\varepsilon_k=(-1)^{\frac{k(k+1)}{2}}(\varepsilon_1)^k\varepsilon'_k=(-1)^{\frac{k(k+1)}{2}}(\varepsilon_1)^k$. □

In order to simplify the relationship between the elements of the signature vector collected in equations (13) and (15), the following auxiliary results are obtained.

**Lemma 4.9** *Let $B=(b_{ij})_{1\le i,j\le n}$ be a type-I staircase nonsingular matrix, with zero pattern $I$, $J$, $\widehat{I}$ and $\widehat{J}$ and depth $\theta$. Let $\varepsilon=(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n)$ be a signature vector with $\varepsilon_2=1$ and $\varepsilon_k=(\varepsilon_1)^k$ for all $k\in\{1,\dots,\theta\}$. Then for all the pairs $(i,j)$ such that $1\le j\le i\le n$ it holds that*

$$\varepsilon_{j-j_t}\varepsilon_{j-j_t+1}=\varepsilon_{j-1}\varepsilon_j.$$

*Proof:*

- If $j_t=1$ then it is direct, $\varepsilon_{j-j_t}\varepsilon_{j-j_t+1}=\varepsilon_{j-1}\varepsilon_j$.
- If $j_t>1$ then, by (7) there exists $s$ such that $j-j_s\le i-i_s$. So,

$$j\le i-i_s+j_s=i-(i_s-j_s)\le n-(i_s-j_s)\le\max_{k\in\{1,\dots,r-1\}}\{n-(i_k-j_k)\}=\theta_L\le\theta.$$

Thus,

$$\varepsilon_{j-j_t}\varepsilon_{j-j_t+1}=(\varepsilon_1)^{j-j_t}(\varepsilon_1)^{j-j_t+1}=(\varepsilon_1)^{2(j_t-1)}(\varepsilon_1)^{j-j_t}(\varepsilon_1)^{j-j_t+1}=$$
$$(\varepsilon_1)^{j_t-1}(\varepsilon_1)^{j-j_t}(\varepsilon_1)^{j_t-1}(\varepsilon_1)^{j-j_t+1}=(\varepsilon_1)^{j-1}(\varepsilon_1)^{j}=\varepsilon_{j-1}\varepsilon_j.$$

□

By applying Lemma 4.9 to matrix $B^T$, the next result is obtained.

**Lemma 4.10** *Let $B=(b_{ij})_{1\le i,j\le n}$ be a type-I staircase nonsingular matrix, with zero pattern $I$, $J$, $\widehat{I}$ and $\widehat{J}$ and depth $\theta$. Let $\varepsilon=(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n)$ be a signature vector with $\varepsilon_2=1$ and $\varepsilon_k=(\varepsilon_1)^k$ for all $k\in\{1,\dots,\theta\}$. Then for all the pairs $(i,j)$ such that $1\le i<j\le n$ it is verify that*

$$\varepsilon_{i-\widehat{i}_t}\varepsilon_{i-\widehat{i}_t+1}=\varepsilon_{i-1}\varepsilon_i.$$

The following result provides a simplification of the conditions given in Theorem 3.7.

**Proposition 4.11** *Let $B=(b_{ij})_{1\le i,j\le n}$ be a nonsingular matrix ASSR with signature $\varepsilon=(\varepsilon_1,\varepsilon_2,\dots,\varepsilon_n)$ and $\varepsilon_2=1$. Then the NE of $B$ can be performed without row exchanges and the pivots $p_{ij}$ satisfy, for any $1\le j\le i\le n$,*

$$p_{ij}=0 \quad\Leftrightarrow\quad b_{ij}=0, \tag{16}$$
$$\varepsilon_{j-1}\varepsilon_j p_{ij}>0 \quad\Leftrightarrow\quad b_{ij}\neq 0, \tag{17}$$

*where $\varepsilon_0=1$.*

*In addition, for $j\in\{1,\dots,\theta\}$ the condition (17) can be expressed as*

$$\varepsilon_1 p_{ij}>0\Leftrightarrow b_{ij}\neq 0.$$

*Proof:* As $B$ is an ASSR matrix with $\varepsilon_2=1$, by Theorem 2 in [7], the matrix $B$ is type-I staircase. We denote by $\epsilon=(\epsilon_1,\dots,\epsilon_n)$ the signature, the zero pattern $I$, $J$, $\widehat{I}$, $\widehat{J}$ and $\theta$ the depth of $B$.

By Theorem 3.7, we know that the NE of $B$ can be performed without row exchanges and the pivots $p_{ij}$ satisfy (12) and (13), for any $1\le j\le i\le n$, that is

$$p_{ij}=0 \quad\Leftrightarrow\quad b_{ij}=0,$$
$$\varepsilon_{j-j_t}\varepsilon_{j-j_t+1}p_{ij}>0 \quad\Leftrightarrow\quad b_{ij}\neq 0,$$

where $\varepsilon_0=1$ and $j_t$ is defined in (7).
So, (16) holds and we only have to prove that (13) and (17) are equivalent.

By Theorem 4.5, $\varepsilon_k=(\varepsilon_1)^k$ for $k\le\theta$ and so the hypothesis of Lemma 4.9 holds. Therefore we have

$$\varepsilon_{j-j_t}\varepsilon_{j-j_t+1}=\varepsilon_{j-1}\varepsilon_j.$$

Then, $\varepsilon_{j-1}\varepsilon_j p_{ij}=\varepsilon_{j-j_t}\varepsilon_{j-j_t+1}p_{ij}>0$, and (17) holds.

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Finally, if $1 \leq j \leq \theta$, then $j-1 < \theta$ and $\varepsilon_j = (\varepsilon_1)^j$ and $\varepsilon_{j-1} = (\varepsilon_1)^{j-1}$

$$\varepsilon_{j-1}\varepsilon_j = (\varepsilon_1)^{j-1}(\varepsilon_1)^j = (\varepsilon_1)^{2j-1} = \varepsilon_1.$$

Thus, the condition $\varepsilon_{j-1}\varepsilon_j p_{ij} > 0$ is simplified to $\varepsilon_1 p_{ij} > 0$ □

Analogously, by using Theorem 3.10, the conditions (14) and (15) can be simplified:

**Proposition 4.12** *Let $B = (b_{ij})_{1\leq i,j\leq n}$ be a nonsingular matrix ASSR with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ where $\varepsilon_2 = 1$. Then, the NE of $B^T$ can be performed without row exchanges and the pivots $q_{ij}$ satisfy, for any $1 \leq i < j \leq n$,*

$$q_{ij} = 0 \Leftrightarrow b_{ij} = 0, \tag{18}$$

$$\varepsilon_{i-1}\varepsilon_i q_{ij} > 0 \Leftrightarrow b_{ij} \neq 0, \tag{19}$$

*where $\varepsilon_0 = 1$.*

*In addition, for indices $i \in \{1, \ldots, \theta\}$ the condition (19) can be expressed as*

$$\varepsilon_1 q_{ij} > 0 \Leftrightarrow b_{ij} \neq 0.$$

**Note 4.13** *Let $A$ be an $n \times n$ matrix, type-I staircase with depth $\theta$. Then, the depth, $\theta^h$, of $A_h$ verifies, $\theta^h \leq \theta - h + 1$.*

The following theorem is a simplification of Theorem 3.10 and the depth ($\theta$) of the matrix will play a key role.

**Theorem 4.14** *A nonsingular matrix $A = (a_{ij})_{1\leq i,j\leq n}$ is ASSR with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ and $\varepsilon_2 = 1$ if and only if $A$ is type-I staircase with depth $\theta$, $\varepsilon_k = (\varepsilon_1)^k$ for all $k \in \{1, 2, \ldots, \theta\}$, and for every $h = 1, \ldots, n-\theta+1$ the following properties hold simultaneously:*

*(i) the NE of the matrices $A_h$ and $A_h^T$ can be performed without row exchanges;*

*(ii) the pivots $p_{ij}^h$ of the NE of $A_h$ satisfy conditions corresponding to (16), (17), and the pivots $q_{ij}^h$ of the NE of $A_h^T$ satisfy (18) and (19).*

*Proof:* Let us start by assuming that the matrix $A$ is ASSR with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n)$ with $\varepsilon_2 = 1$.

By Theorem 2 in [7], as $\varepsilon_2 = 1$, $A$ is type-I staircase and, using Theorem 4.5, $\varepsilon_k = (\varepsilon_1)^k$ for all $k \in \{1, 2, \ldots, \theta\}$ holds.

For each $h \in \{1, \ldots, \theta\}$, we apply Theorems 4.11 and 4.12 to $B = A_h$, and conditions (i) and (ii) hold.

For the converse we assume that $A$ is type-I staircase with depth $\theta$, $\varepsilon_k = (\varepsilon_1)^k$ for all $k \in \{1, 2, \ldots, \theta\}$, and for every $h = 1, \ldots, n-\theta+1$, (i)-(ii) hold and we shall prove that the matrix is ASSR.

The proof is based on Theorem 3.10, so we have to prove that the conditions (i)-(iv) of this theorem are fullfilled.

- Condition (i) and (ii) of Theorem 3.10 are trivial by considering that $A$ is type-I staircase and (i).
- Using the hypothesis (ii), $\varepsilon_k = (\varepsilon_1)^k$ if $1 \leq k \leq \theta$ and Lemma 4.9, the condition (iii) of Theorem 3.10 is fulfilled.
- Finally, let us check the condition (iv) of the Theorem 3.10.
  Given the matrix $A_h$, its depth $\theta^h$ and the signature vector $\varepsilon^h = (\varepsilon_1, \ldots, \varepsilon_{n-h+1})$. It is clear that $\varepsilon_k^h = \varepsilon_k = (\varepsilon_1)^k = (\varepsilon_1^h)^k$, so we are under the hypothesis of Lemma 4.9 and Lemma 4.10 and it is verified, for the positions $(i^h, j^h)$ of matrix $A_h$, that:
  - if $i^h \geq j^h$ and $i^h - j^h = i_t^h - j_t^h$ then $\varepsilon_{j^h - j_t^h}\varepsilon_{j^h - j_t^h + 1} = \varepsilon_{j^h-1}\varepsilon_{j^h}$,
  - if $i^h < j^h$ and $i^h - j^h = \widehat{i_t^h} - \widehat{j_t^h}$ then $\varepsilon_{i^h - \widehat{i_t^h}}\varepsilon_{i^h - \widehat{i_t^h} + 1} = \varepsilon_{i^h-1}\varepsilon_{i^h}$,

  where indices $i_t^h, j_t^h, \widehat{i_t^h}, \widehat{j_t^h}$ are given by conditions corresponding to (7) and (8).

The last part of the proof is that the conditions must hold for the matrices $A_h$ with $h \in \{1, \ldots, \theta\}$. To prove this, we take the matrix $B = \varepsilon_1 A$, whose signature verifies that $\varepsilon_k = 1$ for all $k \in \{1, \ldots, \theta\}$. So the matrix $B_{n-\theta+1}$ with the signature vector $\varepsilon'^h = (1, \ldots, 1)$ verifies the hypothesis of Theorem 3.3 in [5] and the matrix $B_{n-\theta+1}$ it is ASTP. Thus the matrix $A_{n-\theta+1} = \varepsilon_1 B_{n-\theta+1}$ is ASSR and the conditions (i)-(iv) are fulfilled for it and all its submatrices $A_k$ with $k \in \{n-\theta+1, \ldots, n\}$. Thus the conditions (i)-(iv) of Theorem 3.10 are fulfilled and the matrix $A$ is ASSR with signature $\varepsilon$. □

**Note 4.15** *Notice that the previous result is a generalization of Theorem 6 of [16]. That result is applied to strictly M-banded matrices, while the new proposal allows us to characterize any ASSR matrices of type-I staircase, taking into account their depth. In fact, one can observe that a strictly M-banded type-I staircase matrix has depth $n - M$. On the other hand, we should take advantage of this moment to correct the last condition of Theorem 6 of [16], which should be written as $A_{M+1} = A[M+1, \ldots, n]$.*


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When the matrix $A$ is type-II staircase, a result similar to the Theorem 4.14 is presented. For that, we consider the matrix $B = P_n A$ and the submatrices $B_h = B[h, \dots, n]$ and $B_h^T$ with $h \in \{1, \dots, n-\theta+1\}$ being $\theta$ the depth of $A$. So, the following result is a consequence of Corollary 4.8 and applying Theorem 4.14 to $B$.

**Theorem 4.16** *A nonsingular matrix $A = (a_{ij})_{1 \le i,j \le n}$ is ASSR with signature $\varepsilon = (\varepsilon_1, \varepsilon_2, \dots, \varepsilon_n)$ and $\varepsilon_2 = -1$ if and only if $B = P_n A$ is type-I staircase with depth $\theta$, $\varepsilon_k = (-1)^{\frac{k(k-1)}{2}} (\varepsilon_1)^k$ for all $k \in \{1, 2, \dots, \theta\}$, and for every $h = 1, \dots, n-\theta+1$ the following properties hold simultaneously:*

*(i) the NE of the matrices $B_h$ and $B_h^T$ can be performed without row exchanges;*

*(ii) the pivots $p_{ij}^h$ of the NE of $B_h$ satisfy conditions corresponding to (16), (17), and the pivots $q_{ij}^h$ of the NE $B_h^T$ satisfy (18) and (19) for the signature vector $\bar{\varepsilon} = (\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_n)$ with $\bar{\varepsilon}_k = (-1)^{\frac{k(k-1)}{2}} \varepsilon_k$.*

Theorems 4.14 and 4.16 allow us to implement a fast algorithm to check if a matrix $A$ is ASSR by performing the NE algorithm to the submatrices $A_h$.

**Algorithm 1** NE characterization

**Input:** $A$: matrix of order $n$ and a signature vector $\varepsilon$
**Output:** If matrix $A$ is ASSR with signature $\varepsilon$, TRUE else FALSE

1: Check that $A$ is type-I staircase, calculate its depth $\theta$ and check that $\varepsilon_k = (\varepsilon_1)^k$ for $k \in \{2, \dots, \theta\}$
2: **for** $h = 1$ **to** $n - \theta + 1$ **do**
3: Apply NE to matrix $A_h$, checking that no row exchanges are needed and pivots $p_{ij}^h$ satisfy (16) and (17)
4: Apply NE to matrix $A_h^T$, checking that no row exchanges are needed and pivots $q_{ij}^h$ satisfy (18) and (19)
5: **end for**

Algorithm 1 allows us to test if a type-I staircase matrix is ASSR using the simplification obtained in Theorem 4.14. If we consider a type-II staircase matrix, then we apply Theorem 4.16 and the following changes in Algorithm 1 should be considered:

1. Check that $B = P_n A$ is type-I staircase, calculate its depth $\theta$ and check that $\varepsilon_k = (-1)^{\frac{k(k-1)}{2}} (\varepsilon_1)^k$ for $k \in \{2, \dots, \theta\}$
3. Apply NE to matrix $B_h$, checking that no row exchanges are needed and pivots $p_{ij}^h$ satisfy (16) and (17) for the signature vector $\bar{\varepsilon} = (\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_n)$ with $\bar{\varepsilon}_k = (-1)^{\frac{k(k-1)}{2}} \varepsilon_k$
4. Apply NE to matrix $B_h^T$, checking that no row exchanges are needed and pivots $q_{ij}^h$ satisfy (18) and (19) for the signature vector $\bar{\varepsilon} = (\bar{\varepsilon}_1, \dots, \bar{\varepsilon}_n)$ with $\bar{\varepsilon}_k = (-1)^{\frac{k(k-1)}{2}} \varepsilon_k$

It is possible to implement the algorithm so that the input argument is only the matrix and the output arguments are the signature, the zero pattern and the depth in case the matrix is ASSR.

We have done it in these terms and we have applied it to the following examples.

**Example 4.17** *Given the matrix*

$$A = \begin{pmatrix} -1 & -2 & 0 & 0 & 0 & 0 \\ -2 & -6 & -6 & -8 & 0 & 0 \\ 0 & 0 & -21 & -30 & -9 & 0 \\ 0 & 0 & -30 & -48 & -42 & -28 \\ 0 & 0 & -9 & -42 & -172 & -176 \\ 0 & 0 & 0 & -28 & -176 & -259 \end{pmatrix},$$

*the algorithm returns the following results*

$$\theta = 6, I = \{1, 3, 6, 7\}, J = \{1, 3, 4, 7\}, \hat{I} = \{1, 2, 3, 4, 7\}, \hat{J} = \{1, 3, 5, 6, 7\},$$

$$\varepsilon = (-1, 1, -1, 1, -1, 1).$$

*It should be noted that since the depth $\theta$ is 6, NE elimination must only be applied to the matrices $A_1 = A$ and $A_1^T = A^T$ while with the previous algorithms, it was necessary to apply the NE elimination to the matrices*

$$\{A_1, A_1^T, A_2, A_2^T, A_3, A_3^T, A_4, A_4^T, A_5, A_5^T\}.$$

**Example 4.18** *Given the matrix*

$$B = \begin{pmatrix} -1 & -2 & 0 & 0 & 0 & 0 \\ -2 & -6 & -6 & -8 & 0 & 0 \\ 0 & 0 & -21 & -30 & -9 & 0 \\ 0 & 0 & -30 & -48 & -42 & -45 \\ 0 & 0 & -9 & -42 & -172 & -176 \\ 0 & 0 & 0 & -28 & -176 & -259 \end{pmatrix},$$

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*the algorithm returns the following results*

$$\theta = 6,\ I = \{1,3,6,7\},\ J = \{1,3,4,7\},\ \hat{I} = \{1,2,3,4,7\},\ \hat{J} = \{1,3,5,6,7\},$$

$$\varepsilon = \text{“false”},$$

*because the matrix is not ASSR,* $\det(B[1,2|1,2]) = \det\begin{pmatrix} -1 & -2 \\ -2 & -6 \end{pmatrix} = 2$ *and* $\det(B[4,5|5,6]) = \det\begin{pmatrix} -42 & -45 \\ -172 & -176 \end{pmatrix} = -348$. *In this example the ASSR conditions fail when we apply the NE elimination to* $B^T$. *At the beginning of the fifth step, we get the matrix*

$$(B^T)^{(5)} = \begin{pmatrix} -1 & -2 & 0 & 0 & 0 & 0 \\ 0 & -2 & 0 & 0 & 0 & 0 \\ 0 & 0 & -21 & -30 & -9 & 0 \\ 0 & 0 & 0 & -5.1429 & -29.1430 & -28 \\ 0 & 0 & 0 & 0 & -3 & -17.3330 \\ 0 & 0 & 0 & 0 & 101.5000 & 116 \end{pmatrix},$$

*it can be seen that the pivots of the fifth column must all be negative, however the pivot* $q_{56} = (B^T)^{(5)}[6|5] = 101.5$ *is positive.*

## 5. Numerical experiments and applications

In this section we present a numerical experiment associated with a problem modeled through a differential equation. In this sense, it should be noted that classic problems as:

- an elastic cord held at both ends with unitary tension and subjected to a transverse load of intensity f(x)
- an elastic bar held at both ends and subjected to axial load of intensity f(x) or
- the conduction of heat in a bar subjected to a distributed heat source f(x) with constant temperature at the ends

can be established from a differential equation in one dimension and of the second order of the type:

$$\begin{aligned} &u''(t) + g(t)u(t) = f(t), \quad t \in [a,b], \\ &u(a) = 0, \\ &u(b) = 0. \end{aligned}$$

The finite element method consists of looking for the solution in a finite dimensional vector space and reducing the problem to calculating the coordinates of the solution with respect to a given base. This reduces the problem to a system of linear equations in which the matrix of coefficients, called the stiffness matrix of the system, is a structured matrix. When applying the method to certain cases, an ASSR matrix results. In these cases, high precision methods specifically designed for this type of matrices can be used and a highly accurate solution is obtained.

We perform several experiments and, here, we show one. We solve the next differential equation

$$\begin{aligned} &-u''(t) + 3000\,u(t) = t\,\sin(t + 2\pi/3), \qquad t \in [1,3], \\ &u(1) = 0, \\ &u(3) = 0. \end{aligned}$$

Using 21 nodes in the equation, we obtain a tridiagonal linear system, $A \cdot X = b$, where the matrix $A$ (stiffness matrix) is of order 19

$$A = \begin{pmatrix} 220 & 40 & 0 & \cdots & 0 \\ 40 & 220 & 40 & \cdots & 0 \\ 0 & 40 & 220 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \cdots & 220 \end{pmatrix},$$

and the matrix of constant terms is

$$b^t = \begin{pmatrix} -0.0060 & -0.0184 & -0.0326 & -0.0485 & -0.0657 & -0.0841 & -0.1033 & -0.1231 & -0.1431 & -0.1630 \\ -0.1824 & -0.2010 & -0.2183 & -0.2342 & -0.2481 & -0.2597 & -0.2689 & -0.2751 & -0.2783 \end{pmatrix}.$$

Thus, applying Algorithm 1, it is possible to verify that the obtained matrix (stiffness matrix) is an ASSR matrix with signature sequence $\varepsilon = (1,1,1,\ldots,1,1)$. Next, we will use Neville's method to solve the resulting system, and we will analyze the error made with respect to the exact solution of that system. It should be noted that the exact solution of the system has been obtained using symbolic calculus in Matlab.

To carry out the Neville process (see [8]), the following algorithm have been implemented using Matlab:


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```
function [A Piv Ak] = Neville(A,n)
% Function to apply the Neville Elimination method to a square matrix.
% Input arguments:
%     A  ........ a square matrix
%     n ......... the order of the matrix A (optional)
% Output arguments:
%     A ......... the matrix we obtain at the end of the process
%     Piv ....... Matrix with the pivots of the NE algorithm
%     Ak ........ A 3-dimensional array with all the matrices A_k
%                 of the procedure A_1=Ak(:,:,1), A_2=Ak(:,:,2),...
if nargin < 2; n = size(A,2); end
Piv = zeros(n,n);
for j = 1:n
    Ak(:,:,j) = A;
    [I,bnd] = Pivoting(A(j:end,j),j,n);
    A = A(I,:);
    Piv(j:end,j) = A(j:end,j);
    for i = bnd:-1:(j+1)
        A(i,:) = A(i,:)-A(i,j)/A(i-1,j)*A(i-1,:);
    end
end
```

In the next table, we compare the solution obtained by using the MatLab command (ML), this is $X = A \setminus b$, the Neville algorithm (NE) and the symbolic computation (SY) with the usual norms:

| | \|\|MC-SY\|\| | \|\|NE-SY\|\| |
|---|---|---|
| 1-norm | 1.3993e-18 | 7.0473e-19 |
| 2-norm | 5.1233e-19 | 3.0905e-19 |
| $\infty$-norm | 3.2526e-19 | 2.1684e-19 |

It can be seen that the combination between the presented algorithmic characterization shown and the Neville algorithm is efficient for this type of applications.

Finally, it should be noted that other efficient tools that use NE to work with SR matrices can be found, for instance, in [15] and [17]. In the first one, several algorithms (functions) have been implemented using Matlab to work with ASSR matrices. The second includes a software package that can perform virtually all matrix computations with nonsingular TP matrices to high relative accuracy (HRA), under certain conditions. HRA means that the relative errors of the computations are of the order of machine precision, independently of the size of the condition number. If the stiffness matrix (A) is a TP matrix, it is possible to use the function TNBD, to computes the bidiagonal decomposition of the matrix $A$ by performing NE, next, using function TNSolve, we can solve the linear system A X=b using backward substitution.

## 6. Conclusions

It is shown that the new concept of depth ($\theta$) of an $n \times n$ staircase matrix $A$ is a very useful tool to deal with ASSR matrices. On the one hand, it determines the initial $\theta$ components of the signature of $A$. In particular, if the depth is maximal, that is $\theta = n$ then the signature is completely determined. On the other hand, the depth can be used to simplify and reduce the computational cost of the algorithm to check if a given matrix is ASSR with a given signature. This reduction increases with the depth of the matrix.

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