Global well-posedness for the three-dimensional incompressible viscous non-resistive MHD equations in an infinite slab

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Abstract

In this paper, we investigate the global well-posedness of the system of incompressible viscous non-resistive MHD fluids in a three-dimensional horizontally infinite slab with finite height. We reformulate our analysis to Lagrangian coordinates, and then develop a new mathematical approach to establish global well-posedness of the MHD system, which requires no nonlinear compatibility conditions on the initial data.

Keywords: MHD fluids; incompressible fluids; global well-posedness.

1. Introduction

1.1. Formulation in Eulerian coordinates

The dynamics of electrically conducting fluids interacting with magnetic field can be described by the equations of magnetohydrodynamics (MHD). It is widely applied in astrophysics, geophysics, cosmology and engineering, among may others. In this paper, we shall investigate the following three-dimensional incompressible viscous non-resistive MHD equations:

$$\begin{cases}\rho v_t+\rho v\cdot\nabla v-\mu\Delta v+\nabla\left(p+\lambda|M|^2/2\right)=\lambda M\cdot\nabla M,\\ M_t+v\cdot\nabla M=M\cdot\nabla v-M\mathrm{div}v,\\ \mathrm{div}v=\mathrm{div}M=0\end{cases}\tag{1.1}$$

in a horizontally infinite slab with finite height Ω:

$$\Omega:=\{x:=(x_{\mathrm{h}},x_3)^{\mathrm{T}}\in\mathbb{R}^3\mid x_{\mathrm{h}}\in\mathbb{R}^2,\ x_3\in(0,1)\}.\tag{1.2}$$

Here the unknowns $v:=v(x,t)$, $M:=M(x,t)$ and $p:=p(x,t)$ denote the velocity, magnetic field and the kinetic pressures of MHD fluids, resp., the three positive (physical) parameters ρ, μ and λ stand for the density, shear viscosity coefficient and permeability of vacuum dividing by 4π, resp. The equations (1.1) is supplemented with the following initial-boundary value conditions

$$(v,M)|_{t=0}=(v^0,M^0),\tag{1.3}$$

$$v|_{\partial\Omega}=0.\tag{1.4}$$

Here $\partial\Omega$ represents the boundary of Ω, meaning $\partial\Omega:=\mathbb{R}^2\times\{0,1\}$.

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1.2. Background

Below we review some previous results. The study for the MHD equations has been attracting many mathematicians over past many years, and have been achieved a lot of important results. It is classical that the viscous and resistive MHD equations has a unique global classical solution for the small initial data [25]. Kawashima [19] proved global well-posedness for compressible viscous resistive MHD equations when the initial data are close to a non-vacuum equilibrium state. Ducomet and Feireisl [4] proved the global existence of weak solutions to 3D full compressible viscous resistive MHD equations based on the compressible Navier-Stokes system [5, 21], see also Hu and Wang [10] for similar result. On the other hand, it has been physically conjectured that in MHD fluids, the energy is dissipated at a rate that is independent of the resistivity [3]. Hence, one can easily conclude that a non-resistive MHD fluid may still be dissipative. In fact, Jiang and Jiang [14] revealed that a viscous non-resistive MHD fluid strains when stretched and will quickly return to its original rest state by the magnetic tension once the stress is removed. This means that the motion of a non-resistive MHD fluid, which exhibits elastic characteristics, has stabilizing effects. However, it turns out that the problem for viscous non-resistive MHD equations becomes quite delicate in mathematically due to the dissipation of a non-resistive MHD fluid only along the direction of impressive magnetic field (i.e., partial dissipations), and thus received more attention in recent years.

For compressible viscous non-resistive MHD equations, Jiang and Zhang [18] proved the existence and uniqueness of global strong solution to the 1D compressible viscous non-resistive MHD equations with large initial data. Wu and Wu [29] presented a systematic approach to the small data global well-posedness and stability of the 2D compressible viscous non-resistive MHD equations. Jiang and Jiang [13] investigated the Rayleigh-Taylor stability/instability problem of stratified compressible case. Let us come to the incompressible MHD equations, meaning $\mathrm{div}v = 0$. For the existence of global solution for system (1.1) with small initial data, please see [20, 23] for the Cauchy problems, and see [24, 26] for the initial-boundary value problems. For the existence of global solutions for system (1.1) with large initial data under strong impressive magnetic fields, please see [30] for the Cauchy problem, and see [16] for the initial-boundary value problem very recently. In [26], Tan and Wang proved global well-posedness for 3D incompressible viscous non-resistive MHD equations based on the multi-tier energy method. Such method is quite effective in studying the compressible/incompressible MHD system, please refer to a series of work by Jiang and Jiang [11–13, 15] and Wang [28] for more results in this research. Motivated by those works, in this paper, we shall give an alternative version of multi-tier energy method by developing a new mathematical approach, which requires lower normal regularity, and thus requires no compatibility conditions on the time derivatives of the velocity.

1.3. Reformulation in Lagrangian coordinates

Similar to [11, 17, 26], it is more convenient and effective to work with Lagrangian coordinates. To this end, we first assume that there exists an invertible mapping $\zeta^0 := \zeta^0(y) : \Omega \to \Omega$, such that

$$\partial\Omega = \zeta^0(\partial\Omega) \;\text{ and }\; \det(\nabla\zeta^0(y)) = 1 \text{ for any } y \in \bar{\Omega}.$$

The flow mapping ζ is then defined through

$$\begin{cases} \partial_t\zeta(y,t) = v(\zeta(y,t),t) & \text{in } \Omega, \\ \zeta(y,0) = \zeta^0 & \text{in } \Omega. \end{cases}$$

We denote the Eulerian coordinates by (x, t) with $x = \zeta(y, t)$, whereas $(y, t) \in \Omega \times \mathbb{R}^+$ stand for the Lagrangian coordinates. In order to switch back and forth from Lagrangian to Eulerian coordinates, we assume that $\zeta(\cdot, t)$ is invertible and $\Omega = \zeta(\Omega, t)$, which can be achieved when the flow mapping ζ is a small perturbation around the identity mapping Id. In addition, since v is divergence-free,

$$\mathrm{det}\nabla\zeta = 1. \tag{1.5}$$

Define the Lagrangian unknowns by

$$(u, B, q)(y, t) = (v, M, p + \lambda|M|^2/2)(\zeta(y, t), t), \quad (y, t) \in \Omega \times \mathbb{R}^+.$$

Then the initial-boundary value problem (1.1), (1.3)–(1.4) are thus reformulated as follows:

$$\begin{cases} \zeta_t = u & \text{in } \Omega, \\ \rho u_t - \mu\Delta_{\mathcal{A}} u + \nabla_{\mathcal{A}} q = \lambda B \cdot \nabla_{\mathcal{A}} B & \text{in } \Omega, \\ B_t - B \cdot \nabla_{\mathcal{A}} u = 0 & \text{in } \Omega, \\ \mathrm{div} u = \mathrm{div} B = 0 & \text{in } \Omega, \\ (\zeta, u) = (y, 0) & \text{on } \partial\Omega, \\ (u, B)|_{t=0} = (u^0, B^0), \end{cases} \tag{1.6}$$

where $(u^0, B^0) := (v^0(\zeta^0), M^0(\zeta^0))$. Here and in what follows, the notations f^0 (or f_0) denotes the initial data of function $f(\cdot, t)$.

Next we further introduce the notations involving $\mathcal{A}$. The matrix $\mathcal{A} := (\mathcal{A}_{ij})_{3\times3}$ is defined via $\mathcal{A}^T = (\nabla\zeta)^{-1} := (\partial_j\zeta_i)^{-1}_{3\times3}$, and the differential operators $\nabla_{\mathcal{A}}$, $\mathrm{div}_{\mathcal{A}}$ and $\Delta_{\mathcal{A}}$ are defined by

$$\begin{aligned} &\nabla_{\mathcal{A}} w := (\nabla_{\mathcal{A}} w_1, \nabla_{\mathcal{A}} w_2, \nabla_{\mathcal{A}} w_3)^{\mathrm{T}}, \quad \nabla_{\mathcal{A}} w_i := (\mathcal{A}_{1k}\partial_k w_i, \mathcal{A}_{2k}\partial_k w_i, \mathcal{A}_{3k}\partial_k w_i)^{\mathrm{T}}, \\ &\mathrm{div}_{\mathcal{A}}(f_1, f_2, f_3)^{\mathrm{T}} = (\mathrm{div}_{\mathcal{A}} f_1, \mathrm{div}_{\mathcal{A}} f_2, \mathrm{div}_{\mathcal{A}} f_3)^{\mathrm{T}}, \quad \mathrm{div}_{\mathcal{A}} f_i := \mathcal{A}_{lk}\partial_k f_{il}, \\ &\Delta_{\mathcal{A}} w := (\Delta_{\mathcal{A}} w_1, \Delta_{\mathcal{A}} w_2, \Delta_{\mathcal{A}} w_3)^{\mathrm{T}} \quad \text{and} \quad \Delta_{\mathcal{A}} w_i := \mathrm{div}_{\mathcal{A}}\nabla_{\mathcal{A}} w_i \end{aligned} \tag{1.7}$$

for vector functions $w := (w_1, w_2, w_3)^{\mathrm{T}}$ and $f_i := (f_{i1}, f_{i2}, f_{i3})^{\mathrm{T}}$, where we have used the Einstein convention of summation over repeated indices and $\partial_k := \partial_{y_k}$. Additionally, thanks to (1.5), we have the following so called geometric identity:

$$\partial_l\mathcal{A}_{kl} = 0. \tag{1.8}$$

We turn to study the equivalently initial-boundary value problem (1.6) in Lagrangian coordinates, and will prove global well-posedness of the initial-boundary value problem (1.6) around the equilibrium state $(u, B) = (0, \bar{M})$, where the constant vector $\bar{M} := (0, 0, m)^{\mathrm{T}}$ with $m \neq 0$. Our result shows that the problem (1.6) admits a global unique strong solution for the small initial perturbation; and the solution converges to the equilibrium state at an algebraic rate as time tends to infinity. Compare with the previous result in [26], the novelty of this paper comes from a new mathematical approach to establish the global well-posedness of the problem, which requires no nonlinear compatibility conditions on the initial data.

We shall find out the conserved quantities for the transformed problem (1.6). These quantities will help us reformulate the system in a proper way, and the reformulation will be more suitable. Indeed, from $(1.6)_3$ we can derive the differential version of magnetic flux conservation [14]:

$$\mathcal{A}_{jl}B_j = \mathcal{A}^0_{jl}B^0_j,$$

which implies that

$$B = \bar{M} \cdot \nabla\zeta \tag{1.9}$$

provided the initial data (ζ^0, B^0) satisfies

$$B^0 = \bar{M} \cdot \nabla\zeta^0. \tag{1.10}$$

Here we should point out that B given by (1.9) automatically satisfies $(1.6)_3$ and $(1.6)_4$. For full details please refer to [14, 26].

Let

$$\eta := \zeta - y, \quad \text{i.e.,} \quad \zeta = \eta + y.$$

Based on (1.9), on can directly calculate that

$$B \cdot \nabla_{\mathcal{A}} B = (\bar{M} \cdot \nabla)^2\zeta = m^2\partial_3^2\eta. \tag{1.11}$$

Consequently, under assumption (1.10), we use (1.11) to transform (1.6) into

$$\begin{cases} \eta_t = u & \text{in } \Omega, \\ \rho u_t + \nabla_{\mathcal{A}} q - \mu\Delta_{\mathcal{A}} u = \lambda m^2\partial_3^2\eta & \text{in } \Omega, \\ \mathrm{div}_{\mathcal{A}} u = 0 & \text{in } \Omega, \\ (\eta, u) = (0, 0) & \text{on } \partial\Omega, \\ (\eta, u)|_{t=0} = (\eta^0, u^0). \end{cases} \tag{1.12}$$

The rest of this paper is devoted to showing the global well-posedness for (1.12) around the equilibrium state and that the solution converges at an algebraic rate.

2. Main result

2.1. Notations

Before stating our result, we shall introduce some simplified notations:

(1) Basic notations: Let $\int := \int_\Omega$, $\nabla_{\mathrm{h}} := (\partial_1, \partial_2)^{\mathrm{T}}$, $a \lesssim b$ means that $a \leqslant cb$ for some constant $c > 0$, where the constant c may depend on the domain Ω, and other known physical parameters (or functions) such as μ and λ, and may vary from line to line. For any differential operator ∂^β, we define the standard commutators

$$[\partial^\beta, f]g = \partial^\beta(fg) - f\partial^\beta g, \quad f[\partial^\beta, g] = \partial^\beta(fg) - \partial^\beta fg,$$
$$[\partial^\beta, f, g] = \partial^\beta(fg) - \partial^\beta fg - f\partial^\beta g.$$

(2) Simplified Banach spaces, norms and semi-norms:

$$\begin{aligned}
&L^p := L^p(\Omega) := W^{0,p}(\Omega), \qquad H^i := W^{i,2} = W^{i,2}(\Omega),\\
&\|\cdot\|_{L^p} := \|\cdot\|_{L^p(\Omega)}, \qquad\qquad \|\cdot\|_i := \|\cdot\|_{W^{i,2}(\Omega)},\\
&\|\cdot\|_{i,k}^2 := \sum_{|\alpha|=i} \|\partial_{\mathrm{h}}^{\alpha}\cdot\|_k^2, \qquad \|\cdot\|_{\underline{i},k}^2 := \sum_{j=0}^{i} \|\cdot\|_{j,k}^2,
\end{aligned}$$

where $1 \leqslant p \leqslant \infty$, i, j, k are nonnegative integers, and $\partial_{\mathrm{h}}^{\alpha}$ denotes $\partial_1^{\alpha_1}\partial_2^{\alpha_2}$ for some multi-index of order $\alpha := (\alpha_1, \alpha_2)$ with $|\alpha| := \alpha_1 + \alpha_2$.

(3) Simplified functional classes: for nonnegative integers $i \geqslant 1$, $j \geqslant 0$,

$$\begin{aligned}
&H^{j,i} := \{f \in H^i \mid \partial_{\mathrm{h}}^{\alpha} f \in H^i \text{ for } 0 \leqslant |\alpha| \leqslant j\},\\
&H_*^{j,2} := \{f \in H^{j,2} \mid \zeta := f(y,t) + y : \Omega \to \Omega \text{ is a } C^0\text{-diffeomorphism mapping}\},\\
&H_1^{j+2,2} := \{f \in H^{j+2,2} \mid \det(\nabla f + I) = 1\},\\
&H_0^{j,i} := \{f \in H^{j,i} \mid f|_{\partial\Omega} = 0\}, \quad H_{*,0}^{j,2} := H_*^{j,2} \cap H_0^{j,1}.
\end{aligned}$$

(4) Define the following energy functionals $\mathcal{E}_L$ and $\mathcal{E}_H$:

$$\mathcal{E}_L := \|\eta\|_{\underline{2},2}^2 + \|\nabla\eta\|_{\underline{4},0}^2 + \|u\|_{\underline{4},0}^2 + \|\nabla u\|_{\underline{2},0}^2,$$

$$\mathcal{E}_H := \|\eta\|_{\underline{4},2}^2 + \|\nabla\eta\|_{\underline{6},0}^2 + \|u\|_{\underline{6},0}^2 + \|\nabla u\|_{\underline{4},0}^2,$$

and the dissipative functionals $\mathcal{D}_L$ and $\mathcal{D}_H$ are defined as follows:

$$\mathcal{D}_L := \|(\eta,u)\|_{\underline{2},2}^2 + \|\nabla q\|_{\underline{2},0}^2 + \|\partial_3\eta\|_{\underline{4},0}^2 + \|\nabla u\|_{\underline{4},0}^2 + \|u_t\|_{\underline{2},0}^2,$$

$$\mathcal{D}_H := \|(\eta,u)\|_{\underline{4},2}^2 + \|\nabla q\|_{\underline{4},0}^2 + \|\partial_3\eta\|_{\underline{6},0}^2 + \|\nabla u\|_{\underline{6},0}^2 + \|u_t\|_{\underline{4},0}^2.$$

(5) $\mathcal{G}_i(t)$ for $1 \leq i \leq 4$ are the final objects of the *a priori* estimate process in the proof, and are defined by

$$\begin{aligned}
&\mathcal{G}_1(t) = \sup_{0\leq\tau<t} \|\eta(\tau)\|_{\underline{5},2}^2, \qquad \mathcal{G}_2(t) = \int_0^t \frac{\|(\eta,u)(\tau)\|_{\underline{5},2}^2 + \|\nabla q(\tau)\|_{\underline{5},0}^2}{(1+\tau)^{3/2}}\mathrm{d}\tau,\\
&\mathcal{G}_3(t) = \sup_{0\leq\tau<t} \mathcal{E}_H(\tau) + \int_0^t \mathcal{D}_H(\tau)\mathrm{d}\tau,\\
&\mathcal{G}_4(t) = \sup_{0\leq\tau<t} (1+\tau)^2\mathcal{E}_L(\tau) + \int_0^t (1+\tau)^{3/2}\mathcal{D}_L(\tau)\mathrm{d}\tau.
\end{aligned}$$

Now we state the global well-posedness result:

Theorem 2.1. *Let the initial data $(\eta^0, u^0) \in (H_{*,0}^{5,2} \cap H_1^{5,2}) \times H_0^{5,1}$. There exists a sufficiently small constant $\delta > 0$ such that, if the initial data (η^0, u^0) satisfies*

$$\|\eta^0\|_{\underline{5},2}^2 + \|u^0\|_{\underline{5},1}^2 \leqslant \delta. \tag{2.1}$$

Then the initial-boundary value problem (1.12) *admits a unique solution (η, u, q) on $[0,\infty)$. Moreover, the solution enjoys the uniform estimate*

$$\mathcal{G}(\infty) := \sum_{i=1}^{4} \mathcal{G}_i(\infty) \lesssim \|\eta^0\|_{\underline{5},2}^2 + \|u^0\|_{\underline{5},1}^2. \tag{2.2}$$

Remark 2.1. By anisotropic type Sobolev inequality (3.7), we see that

$$\|\nabla\eta\|_{L^\infty} \lesssim \|\nabla\eta\|_{\underline{2},0}^{1/2}\|\nabla\eta\|_{\underline{2},1}^{1/2} \lesssim \|\nabla\eta\|_{\underline{2},1},$$

which implies that the flow map $\zeta := \eta + y \in H^{5,2}$ is diffeomorphism for suitable small δ. As such, we may change coordinates to $x \in \Omega$ to produce global-in-time, decaying solutions to the original incompressible MHD problem.

Remark 2.2. We notice from the uniform estimate (2.2) that $\mathcal{E}_L(t) \lesssim (1+t)^{-2}$; moreover, it should be noted here that with the help of interpolation inequality (3.8) between low-order derivatives and bounded high-order derivatives, the higher horizontal derivatives regularity of $\mathcal{E}_H(t)$ is, the faster time-decay rate of $\mathcal{E}_L(t)$ will be.

Now we briefly sketch the proof idea of Theorem 2.1. Since the local well-posedness of the problem (1.12) can follow exactly in the same way as that of the previous works, such as [9], the key step is to derive the *a priori* estimate (2.2). The proof of the *a priori* estimate (2.2) is based on a multi-tier energy method motivated by the surface wave problem [7, 8]. In fact, below we introduce a new version of multi-tier energy method by developing a new approach, which is different from the previous works. More precisely, note that the coefficients of these perturbed nonlinear terms in the problem (1.12) depend on the entries of $\mathcal{A}-I$, we must get the uniformly in time estimates of them, which requires that $\|\mathcal{A}-I\|_{L^\infty(\Omega)} < 1$, and thus requires that $\|\mathcal{A}-I\|_{H^2(\Omega)} < 1$ at least in the previous works. Nevertheless, thanks to the anisotropic type Sobolev inequality (3.7), it only requires that $\|\mathcal{A}-I\|_{L^\infty(\Omega)} \lesssim \|\mathcal{A}-I\|_{\underline{2},1}$ in this paper, meaning $\|\nabla\eta\|_{L^\infty(\Omega)} \lesssim \|\nabla\eta\|_{\underline{2},1}$, which tells us how the size of η can control the variation of the deformation $\nabla\eta$ (the normal regularity of η is no more than two-order). Notice also that the horizontal derivatives naturally preserve the homogeneous boundary conditions; whence it is convenient to integrate by parts and apply the Poincaré inequality.

When we use the standard energy method to obtain the *a priori* estimate (2.2) to the problem (1.12), the difficulty appears in the differential form of the energy which may be understood in the basic momentum equations

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\|u\|_0^2 + \|\nabla\eta\|_0^2\right) + \|\nabla u\|_0^2 + \|\partial_3\eta\|_0^2 \lesssim \mathcal{N},$$

where $\mathcal{N}$ represents the higher-order terms. This shows that the dissipation provides no direct control of $\nabla\eta$ in the energy. To overcome such difficulty, the celebrated two-tier energy method of Guo and Tice [7, 8] is introduced to consider separately the low-order energy and the bounded high-order energy. It should be noted that in previous works, the two-tier energy method couples the decay of low-order energy and the boundedness of high-order (both spatial and time derivatives) energy, where higher regularity and more compatibility conditions, thus the global well-posedness was established for the initial data which have high normal regularity, and also some compatibility conditions in terms of the time derivatives of the velocity on the initial data are needed. However, it is not easy to verify the validity of the compatibility condition in terms of the time derivatives of the velocity, especially for the higher-order time derivatives of the velocity. In this paper, we establish the global well-posedness of the problem by developing a new mathematical analysis, which requires lower normal regularity for (η, u, q), and thus requires no compatibility conditions on the time derivatives of the velocity. More precisely, similar to the case in [11], we shall establish three differential form energy inequalities (cf. Proposition 5.1), i.e., the lower energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_L + \mathcal{D}_L \leq 0, \tag{2.3}$$

the higher energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_H + \mathcal{D}_H \lesssim (\mathcal{Q} + \|\nabla\eta\|_{\underline{2},1}\|\eta\|_{\underline{5},2})\left(\|(\eta, u)\|_{\underline{5},2} + \|\nabla q\|_{\underline{5},0}\right), \tag{2.4}$$

and the highest-order energy inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\|\eta\|}^2_{\underline{5},2} + \|(\eta, u)\|^2_{\underline{5},2} + \|\nabla q\|^2_{\underline{5},0} \lesssim \mathcal{E}_H,$$

where the energy functionals $\tilde{\mathcal{E}}_L$, $\tilde{\mathcal{E}}_H$ and $\overline{\|\eta\|}^2_{\underline{5},2}$ are equivalent to $\mathcal{E}_L$, $\mathcal{E}_H$ and $\|\eta\|^2_{\underline{5},2} + \|u\|^2_{\underline{5},1}$, resp. Note that the functional $\mathcal{Q}$ in (2.4) is involving the terms of velocity u, cannot be controlled by our original basic goal $\sqrt{\mathcal{E}_L}\|\eta\|_{\underline{5},2}$, which is quite different from the case in [11, 26]. However we observe that the functional $\mathcal{Q}$ can be controlled by $\sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}$, please see (4.11) as an example. Hence the higher energy inequality (2.4) reduces to

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_H + \mathcal{D}_H \lesssim \sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\left(\|(\eta, u)\|_{\underline{5},2} + \|\nabla q\|_{\underline{5},0}\right). \tag{2.5}$$

In the spirit of [6], we expect to seek a suitable integrable decay for the dissipation $\mathcal{D}_L$ so as to close the the higher energy inequality. Instead of proving the inequality $C\tilde{\mathcal{E}}_L \leq \mathcal{D}_L$, but we fortunately can prove $C\tilde{\mathcal{E}}_L^{3/2} \leq \mathcal{D}_L$ by using the interpolation between low-order derivatives and bounded high-order derivatives. From this, we may derive the differential inequality

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_L + C\tilde{\mathcal{E}}_L^{3/2} \leq 0,$$

which implies

$$\tilde{\mathcal{E}}_L(t) \lesssim \mathcal{E}_L(0)(1+t)^{-2}. \tag{2.6}$$

Furthermore, combining (2.3) with (2.6) gives an integrable decay of the dissipation $\mathcal{D}_L$ in the following sense:

$$\sup_{0\leq\tau<t}(1+\tau)^{3/2}\mathcal{E}_L(\tau) + \int_0^t (1+\tau)^{3/2}\mathcal{D}_L(\tau)\mathrm{d}\tau \lesssim \mathcal{E}_L(0).$$

Full details will present in Section 5. Consequently, by the multi-tier energy method, we can deduce the *a priori* estimate (2.2) which, together with the local well-posedness result, yields Theorem 2.1.

Remark 2.3. It should be noted that the term $\|\nabla\eta\|_{\underline{2},1}$ in the right-hand side of (2.4) can be bounded by both $\sqrt{\mathcal{D}_L}$ and $\sqrt{\mathcal{E}_L}$. The reason why we choose the bound $\|\nabla\eta\|_{\underline{2},1} \lesssim \sqrt{\mathcal{D}_L}$ (i.e., (2.5)) lies in an integrable decay of $\mathcal{D}_L$ only requires the time-decay rate of $\mathcal{E}_L$ as $(1+t)^{-(1+s)}$ with $s > 0$ (cf. [6, (1.34)–(1.35)]). However, the case $\|\nabla\eta\|_{\underline{2},1} \lesssim \sqrt{\mathcal{E}_L}$ will result in

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_H + \mathcal{D}_H \lesssim \left(\sqrt{\mathcal{D}_L} + \sqrt{\mathcal{E}_L}\right)\|\eta\|_{\underline{5},2}\left(\|(\eta, u)\|_{\underline{5},2} + \|\nabla q\|_{\underline{5},0}\right), \tag{2.7}$$

which in turn requires the time-decay rate of $\mathcal{E}_L$ as $(1+t)^{-(2+s)}$ with $s > 0$ (cf. [11, (4.2)–(4.3)]. Compare with (2.7), the inequality (2.5) requires fewer regularities on the initial data.

The rest of the paper is organized as follows. In section 3, we first derive the inhomogeneous forms of (1.12), and then introduce some preliminaries estimates and nonlinear perturbed estimates. In Section 4, we derive the tangential estimates of the the solution (η, u), meaning tangential energy evolution; and Stokes regularity estimates. As a consequence, the total energy estimates are stated in Section 5. Thus the proof of Theorem 2.1 is completed.

3. Preliminaries

Let (η, u, q) be the solution of the problem (1.12) such that

$$\sqrt{\sup_{0\leq\tau\leq T}\left(\|\eta(\tau)\|_{\underline{5},2}^2+\|u(\tau)\|_{\underline{5},1}^2\right)}\leq\delta\in(0,1)\quad\text{for some } T>0, \tag{3.1}$$

where δ is sufficiently small. Moreover, we also assume that the solution (η, u, q) possesses proper regularity, so that the procedure of formal calculations makes sense.

3.1. Inhomogeneous forms

In this subsection, we further deduce the inhomogeneous forms of (1.12), which are very useful to establish a priori estimates for the transformed problem. To do this, we let $\tilde{\mathcal{A}} := \mathcal{A} - I$, and let

$$\tilde{\mathcal{A}}^L := \begin{pmatrix} \partial_2\eta_2+\partial_3\eta_3 & -\partial_1\eta_2 & -\partial_1\eta_3 \\ -\partial_2\eta_1 & \partial_1\eta_1+\partial_3\eta_3 & -\partial_2\eta_3 \\ -\partial_3\eta_1 & -\partial_3\eta_2 & \partial_1\eta_1+\partial_2\eta_2 \end{pmatrix}$$

and

$$\tilde{\mathcal{A}}^N := \begin{pmatrix} \partial_2\eta_2\partial_3\eta_3-\partial_2\eta_3\partial_3\eta_2 & \partial_1\eta_3\partial_3\eta_2-\partial_1\eta_2\partial_3\eta_3 & \partial_1\eta_2\partial_2\eta_3-\partial_1\eta_3\partial_2\eta_2 \\ \partial_2\eta_3\partial_3\eta_1-\partial_2\eta_1\partial_3\eta_3 & \partial_1\eta_1\partial_3\eta_3-\partial_1\eta_3\partial_3\eta_1 & \partial_1\eta_3\partial_2\eta_1-\partial_2\eta_3\partial_1\eta_1 \\ \partial_2\eta_1\partial_3\eta_2-\partial_2\eta_2\partial_3\eta_1 & \partial_1\eta_2\partial_3\eta_1-\partial_1\eta_1\partial_3\eta_2 & \partial_1\eta_1\partial_2\eta_2-\partial_1\eta_2\partial_2\eta_1 \end{pmatrix}.$$

Then

$$\tilde{\mathcal{A}}=\tilde{\mathcal{A}}^L+\tilde{\mathcal{A}}^N\quad\text{and}\quad\mathcal{A}=I+\tilde{\mathcal{A}}^L+\tilde{\mathcal{A}}^N.$$

Thus we have

$$\nabla_{\mathcal{A}}u=\nabla u+\tilde{\mathcal{A}}\nabla u^{\mathrm{T}}:=\nabla u+\tilde{\mathcal{N}}_u, \tag{3.2}$$

$$\nabla_{\mathcal{A}}q=\nabla q+\tilde{\mathcal{A}}\nabla q:=\nabla q+\mathcal{N}_q,$$

$$\mathrm{div}_{\mathcal{A}}u=\mathrm{div}u+\tilde{\mathcal{A}}:\nabla u:=\mathrm{div}u+\mathrm{div}_{\tilde{\mathcal{A}}}u. \tag{3.3}$$

In view of (3.2), we further deduce that

$$\Delta_{\mathcal{A}}u=\Delta u+\mathrm{div}\left(\tilde{\mathcal{A}}^{\mathrm{T}}\nabla u+(\tilde{\mathcal{A}}^{\mathrm{T}}+I)\tilde{\mathcal{N}}_u\right):=\Delta u+\mathcal{N}_u.$$

Summing up the relations above, we get

$$\begin{cases} \eta_t=u & \text{in }\Omega,\\ \rho u_t+\nabla q-\mu\Delta u=\lambda m^2\partial_3^2\eta-\mathcal{N}_q+\mu\mathcal{N}_u & \text{in }\Omega,\\ \mathrm{div}u=-\mathrm{div}_{\tilde{\mathcal{A}}}u & \text{in }\Omega,\\ (\eta,u)=(0,0) & \text{on }\partial\Omega,\\ (\eta,u)|_{t=0}=(\eta^0,u^0). \end{cases} \tag{3.4}$$

3.2. Preliminary estimates

This subsection is devoted to the introduction of preliminary estimates for the proof of Theorem 2.1. First, we state some basic estimates and inequalities, which will be repeatedly used throughout this paper. And then, we further derive some estimates involving $\mathcal{A}$, $\mathrm{div}\eta$, $\mathrm{div}u$ and the estimates of the nonlinear terms.

Lemma 3.1. *(1) Embedding inequality (see [1, Theorem 4.12]):*

$$\|f\|_{L^p} \lesssim \|f\|_1 \quad \text{for } 2 \leqslant p \leqslant 6. \tag{3.5}$$

(2) Poincaré's inequality (see [28, Lemma A.4]):

$$\|f\|_0^2 \lesssim \|\partial_3 f\|_0^2 \quad \text{for } f \in H_0^1. \tag{3.6}$$

Lemma 3.2. *There hold that*

$$\|f\|_{L^\infty} \lesssim \|f\|_{\underline{2},0}^{1/2} \left(\|f\|_{\underline{2},0}^{1/2} + \|\partial_3 f\|_{\underline{2},0}^{1/2} \right), \tag{3.7}$$

$$\|f\|_{\underline{j},0} \lesssim \|f\|_{\underline{i},0}^{(k-j)/(k-i)} \|f\|_{\underline{k},0}^{(j-i)/(k-i)} \quad \text{for } 0 \leqslant i < j \leqslant k, \tag{3.8}$$

$$\left\| \prod_{j=1}^k \varphi_j \right\|_{\underline{i},0} \lesssim \sum_{j=1}^k \left(\prod_{l=1, l\neq j}^k \|\varphi_l\|_{\underline{2},1}^{1/2} \|\varphi_l\|_{\underline{2},0}^{1/2} \right) \|\varphi_j\|_{\underline{i},0} \quad \text{for } i \geqslant 2, \tag{3.9}$$

$$\|\varphi\phi\|_{\underline{i},0} \lesssim \|\varphi\|_{\underline{i},1}^{1/2} \|\varphi\|_{\underline{i},0}^{1/2} \|\phi\|_{\underline{i},0} \quad \text{for } i \geqslant 2. \tag{3.10}$$

Proof. In view of the interpolation inequalities (see [22, Theorem]):

$$\begin{aligned} &\|\omega\|_{L^\infty(0,1)} \lesssim \|\omega\|_{L^2(0,1)}^{1/2} \|\omega'\|_{L^2(0,1)}^{1/2} + \|\omega\|_{L^2(0,1)}, \\ &\|\nabla_{\mathrm{h}}^j \omega\|_{L^2(\mathbb{R}^2)} \lesssim \|\nabla_{\mathrm{h}}^i \omega\|_{L^2(\mathbb{R}^2)}^{(k-j)/(k-i)} \|\nabla_{\mathrm{h}}^k \omega\|_{L^2(\mathbb{R}^2)}^{(j-i)/(k-i)}, \end{aligned} \tag{3.11}$$

and the following embedding inequality in $H^s(\mathbb{R}^2)$ (see [2, Theorem 1.66]):

$$\|\omega\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\omega\|_{H^s(\mathbb{R}^2)}, \ \ \forall\, s > 1. \tag{3.12}$$

We easily get estimates (3.7)–(3.8).

Similarly, we can obtain (3.9)–(3.10) by further using (3.12) and the following product estimate in $H^s(\mathbb{R}^2)$ (see [2, Corollary 2.86]):

$$\|\omega\varpi\|_{H^s(\mathbb{R}^2)} \lesssim \|\omega\|_{L^\infty(\mathbb{R}^2)} \|\varpi\|_{H^s(\mathbb{R}^2)} + \|\omega\|_{H^s(\mathbb{R}^2)} \|\varpi\|_{L^\infty(\mathbb{R}^2)}, \ \ \forall\, s > 0.$$

For full details please refer to [16, Lemma 3.1]. The proof is completed. □

Lemma 3.3. *Under assumption* (3.1)*, there hold that*

(1) Estimates involving $\mathcal{A}$:

$$\|\tilde{\mathcal{A}}\|_{\underline{i},k} \lesssim \|\nabla \eta\|_{\underline{i},k}, \tag{3.13}$$

$$\|\mathcal{A}_t\|_{\underline{i},k} \lesssim \|\nabla u\|_{\underline{i},k}. \tag{3.14}$$

(2) Estimates involving $\mathrm{div}\eta$ *and* $\mathrm{div}u$*:*

$$\|\mathrm{div}\eta\|_{\underline{i},k} \lesssim \|\nabla\eta\|_{\underline{i},1}\|\nabla\eta\|_{\underline{i},k}, \tag{3.15}$$

$$\|\mathrm{div}u\|_{\underline{i},k} \lesssim \|\nabla\eta\|_{\underline{i},1}\|\nabla u\|_{\underline{i},k}, \tag{3.16}$$

where $2 \leqslant i \leqslant 5$ *and* $0 \leqslant k \leqslant 1$.

Proof. (1) Let the indices $1 \leqslant l, m, n, w, z \leqslant 3$. By (3.10), we have for $2 \leqslant i \leqslant 5$,

$$\|\partial_l\eta_m\partial_n\eta_w\|_{\underline{i},0} \lesssim \|\partial_l\eta_m\|_{\underline{i},0}^{1/2}\|\partial_l\eta_m\|_{\underline{i},1}^{1/2}\|\partial_n\eta_w\|_{\underline{i},0} \lesssim \|\nabla\eta\|_{\underline{i},0}, \tag{3.17}$$

and

$$\begin{aligned}\|\partial_z\left(\partial_l\eta_m\partial_n\eta_w\right)\|_{\underline{i},0} &\lesssim \|\partial_z\partial_l\eta_m\partial_n\eta_w\|_{\underline{i},0} + \|\partial_l\eta_m\partial_z\partial_n\eta_w\|_{\underline{i},0}\\ &\lesssim \|\partial_z\partial_l\eta_m\|_{\underline{i},0}\|\partial_n\eta_w\|_{\underline{i},0}^{1/2}\|\partial_n\eta_w\|_{\underline{i},1}^{1/2} + \|\partial_l\eta_m\|_{\underline{i},0}^{1/2}\|\partial_l\eta_m\|_{\underline{i},1}^{1/2}\|\partial_z\partial_n\eta_w\|_{\underline{i},0}\\ &\lesssim \|\nabla\eta\|_{\underline{i},1}.\end{aligned} \tag{3.18}$$

Using that (3.17)–(3.18) and the fact $\tilde{\mathcal{A}} := \tilde{\mathcal{A}}^L + \tilde{\mathcal{A}}^N$, we get (3.13) immediately.

Moreover, observe that by $(3.4)_1$,

$$\mathcal{A}_t = \tilde{\mathcal{A}}_t^L + \tilde{\mathcal{A}}_t^N \sim \nabla u + \nabla\eta\nabla u.$$

Exploiting (3.10) and (3.1), it is easy to see that

$$\|\mathcal{A}_t\|_{\underline{j},0} \lesssim \|\nabla u\|_{\underline{j},0} + \|\nabla\eta\|_{\underline{j},0}^{1/2}\|\nabla\eta\|_{\underline{j},1}^{1/2}\|\nabla u\|_{\underline{2},0} \lesssim \|\nabla u\|_{\underline{j},0},$$

$$\|\nabla\mathcal{A}_t\|_{\underline{j},0} \lesssim \|\nabla u\|_{\underline{j},1} + \|\nabla\eta\|_{\underline{j},0}^{1/2}\|\nabla\eta\|_{\underline{j},1}^{1/2}\|\nabla u\|_{\underline{j},1} + \|\nabla u\|_{\underline{j},0}^{1/2}\|\nabla u\|_{\underline{j},1}^{1/2}\|\nabla\eta\|_{\underline{j},1} \lesssim \|\nabla u\|_{\underline{j},1}.$$

Hence (3.14) holds.

(2) Making use of determinant expansion theorem and recalling (1.5), we see that

$$1 = \det(\nabla\eta + I) = 1 + \mathrm{div}\eta + r_\eta,$$

where $r_\eta := r_2^\eta + r_3^\eta$ with

$$r_2^\eta := -\partial_2\eta_3\partial_2\eta_3 - \partial_3\eta_1\partial_1\eta_3 - \partial_2\eta_1\partial_1\eta_2 + \partial_2\eta_2\partial_3\eta_3 + \partial_1\eta_1\partial_3\eta_3 + \partial_1\eta_1\partial_2\eta_2,$$

$$r_3^\eta := \partial_1\eta_1(\partial_2\eta_2\partial_3\eta_3 - \partial_2\eta_3\partial_3\eta_2) - \partial_2\eta_1(\partial_1\eta_2\partial_3\eta_3 - \partial_1\eta_3\partial_3\eta_2) + \partial_3\eta_1(\partial_1\eta_2\partial_2\eta_3 - \partial_1\eta_3\partial_2\eta_2).$$

Consequently,

$$\mathrm{div}\eta = -r_\eta. \tag{3.19}$$

Similar to (3.13), we can estimate that for $2 \leqslant i \leqslant 5$, $0 \leqslant k \leqslant 1$,

$$\|r_\eta\|_{\underline{i},k} \lesssim \|r_2^\eta\|_{\underline{i},k} + \|r_3^\eta\|_{\underline{i},k} \lesssim \|\nabla\eta\|_{\underline{i},0}^{1/2}\|\nabla\eta\|_{\underline{i},1}^{1/2}\|\nabla\eta\|_{\underline{i},k}.$$

Hence we obtain (3.15) by the relation (3.19).

On the other hand, by $(3.4)_3$ and the definition of $\mathrm{div}_{\tilde{\mathcal{A}}}u$ in (3.3), we use (3.10) to get

$$\|\mathrm{div}u\|_{\underline{i},0} \lesssim \|\tilde{\mathcal{A}}\|_{\underline{i},0}^{1/2}\|\tilde{\mathcal{A}}\|_{\underline{i},1}^{1/2}\|\nabla u\|_{\underline{i},0},$$

$$\|\nabla\mathrm{div}u\|_{\underline{i},0} \lesssim \|\tilde{\mathcal{A}}\|_{\underline{i},0}^{1/2}\|\tilde{\mathcal{A}}\|_{\underline{i},1}^{1/2}\|\nabla^2 u\|_{\underline{i},0} + \|\nabla\tilde{\mathcal{A}}\|_{\underline{i},0}\|\nabla u\|_{\underline{i},0}^{1/2}\|\nabla u\|_{\underline{i},1}^{1/2}.$$

These two estimates together with (3.13) give (3.16). The proof is completed. □

Now we record the estimates for nonlinear terms.

Lemma 3.4. *Under assumption* (3.1)*, there hold that*

$$\|\mathcal{N}_q\|_{\underline{j},0} \lesssim \|\nabla\eta\|_{\underline{j},1}\|\nabla q\|_{\underline{j},0} \quad \textit{for } 2 \leqslant j \leqslant 5, \tag{3.20}$$

$$\|\mathcal{N}_u\|_{\underline{j},0} \lesssim \|\nabla\eta\|_{\underline{j},1}\|u\|_{\underline{j},2}, \quad \textit{for } 2 \leqslant j \leqslant 5. \tag{3.21}$$

Proof. Let $\varphi, \psi \in \{\tilde{\mathcal{A}}_{mn}\}_{1\leqslant m,n\leqslant 3}$, and

$$f_1 := \varphi\psi, \quad f_2 := \varphi.$$

Base on the definition of $\mathcal{N}_q$ resp., $\mathcal{N}_u$ and its entries in (3.4), one can identify the principal terms in $\mathcal{N}_q$ resp., $\mathcal{N}_u$ as

$$\mathcal{N}_q \sim f_2\partial_n q \quad \text{and} \quad \mathcal{N}_u \sim \sum_{i=1}^{2} \partial_l \left(f_i \partial_n u_m\right) \quad 1 \leqslant k, l, m, n \leqslant 3.$$

Making use of (3.10), we can estimate that

$$\|\mathcal{N}_q\|_{\underline{j},0} \lesssim \|f_2\|_{\underline{j},0}^{1/2}\|f_2\|_{\underline{j},1}^{1/2}\|\nabla q\|_{\underline{j},0},$$

$$\|\mathcal{N}_u\|_{\underline{j},0} \lesssim \sum_{i=1}^{2}\left(\|\nabla f_i \nabla u\|_{\underline{j},0} + \|f_i\nabla^2 u\|_{\underline{j},0}\right) \lesssim \sum_{i=1}^{5}\|f_i\|_{\underline{j},1}\|u\|_{\underline{j},2}.$$

In addition, using (3.10) again, we easily estimate that

$$\|f_1\|_{\underline{j},1} \lesssim \|\varphi\|_{\underline{j},1}\|\psi\|_{\underline{j},1}.$$

Consequently, combining with above three estimates and then using (3.13) with $k = 1$, we obtain (3.20)–(3.21) for sufficiently small δ. This completes the proof of Lemma 3.4. □

4. Energy evolution

4.1. Energy evolution of tangential derivatives

In this subsection, we shall derive the tangential estimates of the solution (η, u), that is, we shall derive both the horizontal spatial derivatives and the times derivatives of the solution (η, u) in the nonhomogeneous form.

Lemma 4.1. *For any given α satisfying $0 \leqslant |\alpha| \leqslant 6$, there hold that*

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(2\rho\int \partial_{\mathrm{h}}^{\alpha}u\cdot\partial_{\mathrm{h}}^{\alpha}\eta\mathrm{d}y + \mu\|\nabla\partial_{\mathrm{h}}^{\alpha}\eta\|_0^2\right) + \lambda m^2\|\partial_3\partial_{\mathrm{h}}^{\alpha}\eta\|_0^2 \\ &\lesssim \|\sqrt{\rho}\partial_{\mathrm{h}}^{\alpha}u\|_0^2 + \begin{cases} \sqrt{\mathcal{E}_H}\mathcal{D}_L, & \textit{for } |\alpha| \leqslant 4; \\ \sqrt{\mathcal{E}_H}\mathcal{D}_H + \sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\left(\|(\eta,u)\|_{\underline{5},2} + \|\nabla q\|_{\underline{5},0}\right), & \textit{for } |\alpha| \leqslant 6, \end{cases} \end{aligned} \tag{4.1}$$

$$\begin{aligned} &\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\sqrt{\rho}\partial_{\mathrm{h}}^{\alpha}u\|_0^2 + \lambda m^2\|\partial_3\partial_{\mathrm{h}}^{\alpha}\eta\|_0^2\right) + \mu\|\partial_{\mathrm{h}}^{\alpha}\nabla u\|_0^2 \\ &\lesssim \begin{cases} \sqrt{\mathcal{E}_H}\mathcal{D}_L, & \textit{for } |\alpha| \leqslant 4; \\ \|\nabla\eta\|_{\underline{2},1}\|\nabla u\|_{\underline{6},0}^2 + \sqrt{\mathcal{E}_H}\mathcal{D}_H + \sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\left(\|(\eta,u)\|_{\underline{5},2} + \|\nabla q\|_{\underline{5},0}\right), & \textit{for } |\alpha| \leqslant 6. \end{cases} \end{aligned} \tag{4.2}$$

Proof. Applying $\partial_{\mathrm{h}}^{\alpha}$ to (3.4) yields

$$\begin{cases}\partial_{\mathrm{h}}^{\alpha}\eta_t=\partial_{\mathrm{h}}^{\alpha}u & \text{in } \Omega,\\ \rho\partial_{\mathrm{h}}^{\alpha}u_t+\nabla\partial_{\mathrm{h}}^{\alpha}q-\mu\Delta\partial_{\mathrm{h}}^{\alpha}u=\lambda m^2\partial_{\mathrm{h}}^{\alpha}\partial_3^2\eta-\partial_{\mathrm{h}}^{\alpha}\left(\mathcal{N}_q-\mu\mathcal{N}_u\right) & \text{in } \Omega,\\ \mathrm{div}\partial_{\mathrm{h}}^{\alpha}u=\mathrm{div}\partial_{\mathrm{h}}^{\alpha}u & \text{in } \Omega,\\ (\partial_{\mathrm{h}}^{\alpha}\eta,\partial_{\mathrm{h}}^{\alpha}u)=(0,0) & \text{on } \partial\Omega.\end{cases} \tag{4.3}$$

Multiplying $(4.3)_2$ by $\partial_{\mathrm{h}}^{i}\eta$ in L^2, and then integrating by parts over Ω, using $(4.3)_4$, we have

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\int\rho\partial_{\mathrm{h}}^{\alpha}u\cdot\partial_{\mathrm{h}}^{\alpha}\eta\mathrm{d}y+\mu\int\nabla\partial_{\mathrm{h}}^{\alpha}u:\partial_{\mathrm{h}}^{\alpha}\eta\mathrm{d}y+\lambda m^2\int|\partial_3\partial_{\mathrm{h}}^{\alpha}\eta|^2\mathrm{d}y\\ &=\int\rho|\partial_{\mathrm{h}}^{\alpha}u|^2\mathrm{d}y-\int\partial_{\mathrm{h}}^{\alpha}\left(\mathcal{N}_q-\mu\mathcal{N}_u\right)\cdot\partial_{\mathrm{h}}^{\alpha}\eta\mathrm{d}y+\int\partial_{\mathrm{h}}^{\alpha}q\mathrm{div}\partial_{\mathrm{h}}^{\alpha}\eta\mathrm{d}y\\ &=:\int\rho|\partial_{\mathrm{h}}^{\alpha}u|^2\mathrm{d}y+\sum_{i=1}^{2}K_i.\end{aligned}$$

By $(4.3)_1$, we have

$$\mu\int\nabla\partial_{\mathrm{h}}^{\alpha}u:\partial_{\mathrm{h}}^{\alpha}\eta\mathrm{d}y=\frac{\mu}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\nabla\partial_{\mathrm{h}}^{\alpha}\eta|^2\mathrm{d}y.$$

Putting it into above identity then gives

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(2\rho\int\partial_{\mathrm{h}}^{\alpha}u\cdot\partial_{\mathrm{h}}^{\alpha}\eta\mathrm{d}y+\mu\|\nabla\partial_{\mathrm{h}}^{\alpha}\eta\|_0^2\right)+\lambda m^2\|\partial_3\partial_{\mathrm{h}}^{\alpha}\eta\|_0^2=\|\sqrt{\rho}\partial_{\mathrm{h}}^{\alpha}u\|_0^2+\sum_{i=1}^{2}K_i. \tag{4.4}$$

In the same manner, multiplying $(4.3)_2$ by $\partial_{\mathrm{h}}^{i}u$ in L^2, we get

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int\rho|\partial_{\mathrm{h}}^{\alpha}u|^2\mathrm{d}y+\mu\int|\nabla\partial_{\mathrm{h}}^{\alpha}u|^2\mathrm{d}y+\lambda m^2\int\partial_3\partial_{\mathrm{h}}^{\alpha}\eta\cdot\partial_3\partial_{\mathrm{h}}^{\alpha}u\mathrm{d}y\\ &=-\int\partial_{\mathrm{h}}^{\alpha}\left(\mathcal{N}_q-\mu\mathcal{N}_u\right)\cdot\partial_{\mathrm{h}}^{\alpha}u\mathrm{d}y+\int\partial_{\mathrm{h}}^{\alpha}q\mathrm{div}\partial_{\mathrm{h}}^{\alpha}u\mathrm{d}y=:\sum_{i=3}^{4}K_i.\end{aligned}$$

Using $(4.3)_1$ again, we have

$$\lambda m^2\int\partial_3\partial_{\mathrm{h}}^{\alpha}\eta\cdot\partial_3\partial_{\mathrm{h}}^{\alpha}u\mathrm{d}y=\frac{\lambda m^2}{2}\frac{\mathrm{d}}{\mathrm{d}t}\int|\partial_3\partial_{\mathrm{h}}^{\alpha}\eta|^2\mathrm{d}y.$$

Plugging it into above identity then yields

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\|\sqrt{\rho}\partial_{\mathrm{h}}^{\alpha}u\|_0^2+\lambda m^2\|\partial_3\partial_{\mathrm{h}}^{\alpha}\eta\|_0^2\right)+\mu\|\partial_{\mathrm{h}}^{\alpha}\nabla u\|_0^2=\sum_{i=3}^{4}K_i. \tag{4.5}$$

Next we estimate nonlinear terms $K_1, ..., K_4$ in sequence. Indeed, an integration by parts and nonlinear estimates (3.20)–(3.21) yields that

$$|K_1|+|K_3|\lesssim\begin{cases}\|(\mathcal{N}_u,\mathcal{N}_q)\|_{\underline{2},0}\|(\eta,u)\|_{\underline{6},0}\lesssim\sqrt{\mathcal{E}_H}\mathcal{D}_L & \text{for } 0\leqslant|\alpha|\leqslant 4,\\ \|(\mathcal{N}_u,\mathcal{N}_q)\|_{\underline{4},0}\|(\eta,u)\|_{\underline{6},0}\lesssim\sqrt{\mathcal{E}_H}\mathcal{D}_H & \text{for } 0\leqslant|\alpha|\leqslant 5.\end{cases} \tag{4.6}$$

Thanks to (3.15)–(3.16),

$$|K_2|+|K_4|\lesssim\begin{cases}\|\nabla q\|_{\underline{2},0}\|(\mathrm{div}\eta,\mathrm{div}u)\|_{\underline{5},0}\lesssim\sqrt{\mathcal{E}_H}\mathcal{D}_L & \text{for } 1\leqslant|\alpha|\leqslant 4,\\ \|\nabla q\|_{\underline{4},0}\|(\mathrm{div}\eta,\mathrm{div}u)\|_{\underline{5},0}\lesssim\sqrt{\mathcal{E}_H}\mathcal{D}_H & \text{for } 1\leqslant|\alpha|\leqslant 5.\end{cases}\tag{4.7}$$

Moreover, it is known that (see [26, (4.45)-(4.46)])

$$\int q\mathrm{div}\eta\mathrm{d}y=-\int\psi\cdot\nabla q\mathrm{d}y\lesssim\|\eta\|_2\|\eta\|_1\|\nabla q\|_0,\tag{4.8}$$

where

$$\psi:=-\begin{pmatrix}\eta_1(\partial_2\eta_2+\partial_3\eta_3)-\eta_1(\partial_2\eta_3\partial_3\eta_2-\partial_2\eta_2\partial_3\eta_3)\\ \eta_2\partial_3\eta_3-\eta_1\partial_1\eta_2-\eta_1(\partial_1\eta_2\partial_3\eta_3-\partial_1\eta_3\partial_3\eta_2)\\ -\eta_1\partial_1\eta_3-\eta_2\partial_2\eta_3-\eta_1(\partial_1\eta_3\partial_2\eta_2-\partial_1\eta_2\partial_2\eta_3)\end{pmatrix}\quad\text{and }\ \mathrm{div}\eta=\mathrm{div}\psi.$$

By (1.8) and $(3.4)_3$, we integrate by parts over Ω to get

$$\int q\mathrm{div}u\mathrm{d}y=-\int q\mathrm{div}_{\tilde{\mathcal{A}}}u\mathrm{d}y=\int\nabla q\cdot(\tilde{\mathcal{A}}^{\mathrm{T}}u)\mathrm{d}y\lesssim\|\eta\|_2\|u\|_1\|\nabla q\|_0.\tag{4.9}$$

By collecting (4.7), (4.8) as well as (4.9), we conclude that

$$|K_2|+|K_4|\lesssim\begin{cases}\|\nabla q\|_{\underline{2},0}\|(\mathrm{div}\eta,\mathrm{div}u)\|_{\underline{5},0}\lesssim\sqrt{\mathcal{E}_H}\mathcal{D}_L & \text{for } 0\leqslant|\alpha|\leqslant 4,\\ \|\nabla q\|_{\underline{4},0}\|(\mathrm{div}\eta,\mathrm{div}u)\|_{\underline{5},0}\lesssim\sqrt{\mathcal{E}_H}\mathcal{D}_H & \text{for } 0\leqslant|\alpha|\leqslant 5.\end{cases}\tag{4.10}$$

While for the case of $|\alpha|=6$, it seems to be more subtle. First of all, recall the definition of $\mathcal{N}_u$ in (3.4), we integrate by parts over Ω to see that

$$\begin{aligned}\int\partial_{\mathrm{h}}^6\mathcal{N}_u\cdot\partial_{\mathrm{h}}^6\eta\mathrm{d}y&\lesssim\left(\|\tilde{\mathcal{A}}\|_{\underline{2},1}\|\nabla u\|_{\underline{6},0}+\|\nabla u\|_{\underline{2},1}\|\tilde{\mathcal{A}}\|_{\underline{6},0}\right)\|\nabla\eta\|_{6,0}\\&\lesssim\|(\nabla\eta,\nabla u)\|_{\underline{2},1}\|\eta\|_{\underline{5},2}\|(\eta,u)\|_{\underline{5},2}\lesssim\sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\|(\eta,u)\|_{\underline{5},2}.\end{aligned}\tag{4.11}$$

In view of the relation (3.19) and product estimate (3.9), we can estimate that

$$\|\mathrm{div}\eta\|_{6,0}\lesssim\|\nabla\eta\|_{\underline{2},1}(1+\|\nabla\eta\|_{\underline{2},1})\|\nabla\eta\|_{\underline{6},0}\lesssim\|\nabla\eta\|_{\underline{2},1}\|\nabla\eta\|_{\underline{6},0},$$

which implies that

$$\int\partial_{\mathrm{h}}^6 q\mathrm{div}\partial_{\mathrm{h}}^6\eta\mathrm{d}y\lesssim\|q\|_{6,0}\|\mathrm{div}\eta\|_{6,0}\lesssim\|\nabla\eta\|_{\underline{2},1}\|\nabla\eta\|_{\underline{6},0}\|\nabla q\|_{5,0}.\tag{4.12}$$

In addition, we directly use (1.8) to compute that

$$\int\partial_{\mathrm{h}}^6\mathcal{N}_q\cdot\partial_{\mathrm{h}}^6\eta\mathrm{d}y=\int\partial_{\mathrm{h}}^6\tilde{\mathcal{A}}^{\mathrm{T}}\nabla q\cdot\partial_{\mathrm{h}}^6\eta\mathrm{d}y+\int\tilde{\mathcal{A}}^{\mathrm{T}}[\partial_{\mathrm{h}}^6,\nabla q]\cdot\partial_{\mathrm{h}}^6\eta\mathrm{d}y.\tag{4.13}$$

By (3.11) and (3.12), we can estimate that

$$\begin{aligned}\int\partial_{\mathrm{h}}^6\tilde{\mathcal{A}}^{\mathrm{T}}\nabla q\cdot\partial_{\mathrm{h}}^6\eta\mathrm{d}y&\lesssim\|\partial_{\mathrm{h}}^6\tilde{\mathcal{A}}\|_0\|\nabla q\|_{L^\infty(\mathbb{R}^2)L^2(0,1)}\|\partial_{\mathrm{h}}^6\eta\|_{L^2(\mathbb{R}^2)L^\infty(0,1)}\\&\lesssim\|\tilde{\mathcal{A}}\|_{6,0}\|\nabla q\|_{\underline{2},0}\|\eta\|_{6,1}\lesssim\sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}^2.\end{aligned}$$

While making full use of (1.8), (3.5), (3.7) and (3.13), we claim that

$$\int \tilde{\mathcal{A}}^{\mathrm{T}}[\partial_{\mathrm{h}}^6, \nabla q] \cdot \partial_{\mathrm{h}}^6 \eta \mathrm{d}y \lesssim \sqrt{\mathcal{E}_H}\mathcal{D}_H + \sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\left(\|\eta\|_{\underline{5},2} + \|\nabla q\|_{\underline{5},0}\right).$$

Putting above two estimates into (4.13) then follows

$$\int \partial_{\mathrm{h}}^6 \mathcal{N}_q \cdot \partial_{\mathrm{h}}^6 \eta \mathrm{d}y \lesssim \sqrt{\mathcal{E}_H}\mathcal{D}_H + \sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\left(\|\eta\|_{\underline{5},2} + \|\nabla q\|_{\underline{5},0}\right). \tag{4.14}$$

Consequently, it follows from (4.11), (4.12) and (4.14) that

$$\sum_{|\alpha|=6}\sum_{i=1}^{2} K_i \lesssim \sqrt{\mathcal{E}_H}\mathcal{D}_H + \sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\left(\|(\eta,u)\|_{\underline{5},2} + \|\nabla q\|_{\underline{5},0}\right). \tag{4.15}$$

On the other hand, similar to (4.11), we can estimate that

$$\begin{aligned}\int \partial_{\mathrm{h}}^6 \mathcal{N}_u \cdot \partial_{\mathrm{h}}^6 u \mathrm{d}y &\lesssim \left(\|\tilde{\mathcal{A}}\|_{\underline{2},1}\|\nabla u\|_{\underline{6},0} + \|\nabla u\|_{\underline{2},1}\|\tilde{\mathcal{A}}\|_{\underline{6},0}\right)\|\nabla u\|_{6,0} \\ &\lesssim \|\nabla \eta\|_{\underline{2},1}\|\nabla u\|_{\underline{6},0}^2 + \sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\|u\|_{\underline{5},2}.\end{aligned} \tag{4.16}$$

Now we turn to estimate the remaining terms $-\int \partial_{\mathrm{h}}^6 \mathcal{N}_q \cdot \partial_{\mathrm{h}}^\alpha u \mathrm{d}y$ and $\int \partial_{\mathrm{h}}^6 q \mathrm{div}\partial_{\mathrm{h}}^6 u \mathrm{d}y$. Analogously to (4.13), we have

$$\int \partial_{\mathrm{h}}^6 \mathcal{N}_q \cdot \partial_{\mathrm{h}}^6 u \mathrm{d}y = \int \tilde{\mathcal{A}}^{\mathrm{T}}\nabla\partial_{\mathrm{h}}^6 q \cdot \partial_{\mathrm{h}}^6 u \mathrm{d}y + \int \partial_{\mathrm{h}}^6 \tilde{\mathcal{A}}^{\mathrm{T}}\nabla q \cdot \partial_{\mathrm{h}}^6 u \mathrm{d}y + \int [\partial_{\mathrm{h}}^6, \tilde{\mathcal{A}}^{\mathrm{T}}, \nabla q] \cdot \partial_{\mathrm{h}}^6 u \mathrm{d}y.$$

In view of (3.3) and $(3.4)_3$, we use (1.8) to obtain

$$\begin{aligned}\int \tilde{\mathcal{A}}^{\mathrm{T}}\nabla\partial_{\mathrm{h}}^6 q \cdot \partial_{\mathrm{h}}^6 u \mathrm{d}y &= -\int \partial_{\mathrm{h}}^6 q \mathrm{div}_{\tilde{\mathcal{A}}}\partial_{\mathrm{h}}^6 u \mathrm{d}y \\ &= -\int \partial_{\mathrm{h}}^6 q \partial_{\mathrm{h}}^6\left(\mathrm{div}_{\tilde{\mathcal{A}}} u\right)\mathrm{d}y + \int \partial_{\mathrm{h}}^6 q \mathrm{div}_{[\partial_{\mathrm{h}}^6, \tilde{\mathcal{A}}]} u \mathrm{d}y \\ &= \int \partial_{\mathrm{h}}^6 q \mathrm{div}\partial_{\mathrm{h}}^6 u \mathrm{d}y + \int \partial_{\mathrm{h}}^6 q \mathrm{div}_{[\partial_{\mathrm{h}}^6, \tilde{\mathcal{A}}]} u \mathrm{d}y.\end{aligned}$$

Thus from above two identities we get that

$$\begin{aligned}&-\int \partial_{\mathrm{h}}^6 \mathcal{N}_q \cdot \partial_{\mathrm{h}}^6 u \mathrm{d}y + \int \partial_{\mathrm{h}}^6 q \mathrm{div}\partial_{\mathrm{h}}^6 u \mathrm{d}y \\ &= -\int \partial_{\mathrm{h}}^6 \tilde{\mathcal{A}}^{\mathrm{T}}\nabla q \cdot \partial_{\mathrm{h}}^6 u \mathrm{d}y - \int [\partial_{\mathrm{h}}^6, \tilde{\mathcal{A}}^{\mathrm{T}}, \nabla q] \cdot \partial_{\mathrm{h}}^6 u \mathrm{d}y - \int \partial_{\mathrm{h}}^6 q \mathrm{div}_{[\partial_{\mathrm{h}}^6, \tilde{\mathcal{A}}]} u \mathrm{d}y.\end{aligned} \tag{4.17}$$

Moreover, similar to the derivation of (4.14),

$$\int \partial_{\mathrm{h}}^6 \tilde{\mathcal{A}}^{\mathrm{T}}\nabla q \cdot \partial_{\mathrm{h}}^6 u \mathrm{d}y \lesssim \|\tilde{\mathcal{A}}\|_{6,0}\|\nabla q\|_{\underline{2},0}\|u\|_{6,1} \lesssim \sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\|u\|_{\underline{5},2},$$

$$\int \tilde{\mathcal{A}}^{\mathrm{T}}[\partial_{\mathrm{h}}^6, \nabla q] \cdot \partial_{\mathrm{h}}^6 u \mathrm{d}y \lesssim \sqrt{\mathcal{E}_H}\mathcal{D}_H + \sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\left(\|u\|_{\underline{5},2} + \|\nabla q\|_{\underline{5},0}\right),$$

$$\int \partial_{\mathrm{h}}^6 q \mathrm{div}_{[\partial_{\mathrm{h}}^6, \tilde{\mathcal{A}}]} u \mathrm{d}y \lesssim \sqrt{\mathcal{E}_H}\mathcal{D}_H + \sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\left(\|u\|_{\underline{5},2} + \|\nabla q\|_{\underline{5},0}\right).$$

Hence we can further deduce that

$$-\int \partial_{\mathrm{h}}^{6}\mathcal{N}_q\cdot\partial_{\mathrm{h}}^{6}u\mathrm{d}y+\int \partial_{\mathrm{h}}^{6}q\mathrm{div}\partial_{\mathrm{h}}^{6}u\mathrm{d}y\lesssim\sqrt{\mathcal{E}_H}\mathcal{D}_H+\sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\left(\|(\eta,u)\|_{\underline{5},2}+\|\nabla q\|_{\underline{5},0}\right). \tag{4.18}$$

Therefore, combine with (4.16) and (4.18), we get

$$\sum_{|\alpha|=6}\sum_{i=3}^{4}K_i\lesssim\|\nabla\eta\|_{\underline{2},1}\|\nabla u\|_{\underline{6},0}^2+\sqrt{\mathcal{E}_H}\mathcal{D}_H+\sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\left(\|(\eta,u)\|_{\underline{5},2}+\|\nabla q\|_{\underline{5},0}\right). \tag{4.19}$$

In the light of (4.6), (4.10), (4.15) as well as (4.19), we conclude that

$$\sum_{i=1}^{2}K_i\lesssim\begin{cases}\sqrt{\mathcal{E}_H}\mathcal{D}_L, & \text{for } |\alpha|\leqslant 4;\\ \sqrt{\mathcal{E}_H}\mathcal{D}_H+\sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\left(\|(\eta,u)\|_{\underline{5},2}+\|\nabla q\|_{\underline{5},0}\right), & \text{for } |\alpha|\leqslant 6,\end{cases}$$

$$\sum_{i=3}^{4}K_i\lesssim\begin{cases}\sqrt{\mathcal{E}_H}\mathcal{D}_L, & \text{for } |\alpha|\leqslant 4;\\ \|\nabla\eta\|_{\underline{2},1}\|\nabla u\|_{\underline{6},0}^2+\sqrt{\mathcal{E}_H}\mathcal{D}_H+\sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\left(\|(\eta,u)\|_{\underline{5},2}+\|\nabla q\|_{\underline{5},0}\right), & \text{for } |\alpha|\leqslant 6.\end{cases}$$

Finally, inserting them into (4.4) and (4.5), respectively, we obtain (4.1) and (4.2) immediately. The proof is completed. □

Lemma 4.2. *For any given α satisfying $0\leqslant|\alpha|\leqslant 5$, there holds that*

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda m^2\int\partial_3\partial_{\mathrm{h}}^{\alpha}\eta\cdot\partial_3\partial_{\mathrm{h}}^{\alpha}u\mathrm{d}y+\mu\|\nabla\partial_{\mathrm{h}}^{\alpha}u\|_0^2\right)+\|\sqrt{\rho}u_t\|_0^2\\ &\lesssim\|\partial_3\partial_{\mathrm{h}}^{\alpha}u\|_0^2+\begin{cases}\sqrt{\mathcal{E}_L}\mathcal{D}_L, & \textit{for } |\alpha|\leqslant 2;\\ \sqrt{\mathcal{E}_H}\mathcal{D}_H, & \textit{for } |\alpha|\leqslant 4;\\ \|(\nabla\eta,u)\|_{\underline{5},1}\left(\|(\eta,u)\|_{\underline{5},2}^2+\|\nabla q\|_{\underline{5},0}^2+\|u_t\|_{\underline{5},0}^2\right), & \textit{for } |\alpha|\leqslant 5.\end{cases}\end{aligned} \tag{4.20}$$

Proof. Multiplying $(4.3)_2$ by $\partial_{\mathrm{h}}^{\alpha}u_t$ in L^2, following the derivation of (4.1), we easily obtain

$$\begin{aligned}&\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda m^2\int\partial_3\partial_{\mathrm{h}}^{\alpha}\eta\cdot\partial_3\partial_{\mathrm{h}}^{\alpha}u\mathrm{d}y+\mu\|\nabla\partial_{\mathrm{h}}^{\alpha}u\|_0^2\right)+\|\sqrt{\rho}u_t\|_0^2\\ &=\lambda m^2\|\partial_3\partial_{\mathrm{h}}^{\alpha}u\|_0^2-\int\partial_{\mathrm{h}}^{\alpha}\left(\mathcal{N}_q-\mu\mathcal{N}_u\right)\cdot\partial_{\mathrm{h}}^{\alpha}u_t\mathrm{d}y+\int\partial_{\mathrm{h}}^{\alpha}q\partial_{\mathrm{h}}^{\alpha}\mathrm{div}u_t\mathrm{d}y.\end{aligned} \tag{4.21}$$

By $(1.12)_3$ and geometric identity (1.8), we easily verify that

$$\mathrm{div}u_t=-\mathrm{div}\left(\tilde{\mathcal{A}}_t^{\mathrm{T}}u+\tilde{\mathcal{A}}^{\mathrm{T}}u_t\right).$$

Substituting it into (4.21) and integrating by parts, we get

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}t}\left(\lambda m^2\int\partial_3\partial_{\mathrm{h}}^{\alpha}\eta\cdot\partial_3\partial_{\mathrm{h}}^{\alpha}u\mathrm{d}y+\mu\|\nabla\partial_{\mathrm{h}}^{\alpha}u\|_0^2\right)+\|\sqrt{\rho}u_t\|_0^2=\lambda m^2\|\partial_3\partial_{\mathrm{h}}^{\alpha}u\|_0^2+\sum_{i=5}^{6}K_i, \tag{4.22}$$

where

$$K_5 := -\int \partial_{\mathrm{h}}^{\alpha}\left(\mathcal{N}_q - \mu \mathcal{N}_u\right) \cdot \partial_{\mathrm{h}}^{\alpha} u_t \mathrm{d}y, \quad K_6 := \int \nabla \partial_{\mathrm{h}}^{\alpha} q \cdot \partial_{\mathrm{h}}^{\alpha}\left(\tilde{\mathcal{A}}_t^{\mathrm{T}} u + \tilde{\mathcal{A}}^{\mathrm{T}} u_t\right) \mathrm{d}y.$$

Exploiting (3.20)–(3.21), we have that

$$K_5 \lesssim \begin{cases} \|(\mathcal{N}_q, \mathcal{N}_u)\|_{\underline{2},0} \|u_t\|_{\underline{2},0} \lesssim \sqrt{\mathcal{E}_L} \mathcal{D}_L, & \text{for } |\alpha| \leqslant 2; \\ \|(\mathcal{N}_q, \mathcal{N}_u)\|_{\underline{4},0} \|u_t\|_{\underline{4},0} \lesssim \sqrt{\mathcal{E}_H} \mathcal{D}_H, & \text{for } |\alpha| \leqslant 4; \\ \|(\mathcal{N}_u, \mathcal{N}_q)\|_{\underline{5},0} \|u_t\|_{\underline{5},0} \lesssim \|\nabla \eta\|_{\underline{5},1}\left(\|u\|_{\underline{5},2} + \|\nabla q\|_{\underline{5},0}\right) \|u_t\|_{\underline{5},0}, & \text{for } |\alpha| \leqslant 5. \end{cases}$$

Utilizing (3.10) and (3.14), we can estimate that

$$K_6 \lesssim \begin{cases} \|\nabla q\|_{\underline{2},0}\left(\|\tilde{\mathcal{A}}\|_{\underline{2},1} \|u_t\|_{\underline{2},0} + \|u\|_{\underline{2},1} \|\tilde{\mathcal{A}}_t\|_{\underline{2},0}\right) \lesssim \sqrt{\mathcal{E}_L} \mathcal{D}_L, & \text{for } |\alpha| \leqslant 2; \\ \|\nabla q\|_{\underline{4},0}\left(\|\tilde{\mathcal{A}}\|_{\underline{4},1} \|u_t\|_{\underline{4},0} + \|u\|_{\underline{4},1} \|\tilde{\mathcal{A}}_t\|_{\underline{4},0}\right) \lesssim \sqrt{\mathcal{E}_H} \mathcal{D}_H, & \text{for } |\alpha| \leqslant 4; \\ \|\nabla q\|_{\underline{5},0}\left(\|\tilde{\mathcal{A}}\|_{\underline{5},1} \|u_t\|_{\underline{5},0} + \|u\|_{\underline{5},1} \|\tilde{\mathcal{A}}_t\|_{\underline{5},0}\right) & \\ \quad \lesssim \|\nabla q\|_{\underline{5},0}\left(\|\nabla \eta\|_{\underline{5},1} \|u_t\|_{\underline{5},0} + \|u\|_{\underline{5},1} \|\nabla u\|_{\underline{5},0}\right), & \text{for } |\alpha| \leqslant 5. \end{cases}$$

Putting above two estimates into (4.22) then yields (4.20) immediately. This completes the proof of Lemma 4.2. □

With Lemmas 4.1–4.2 in hand, now we are ready to complete the tangential energy estimates. To this end, we shall first introduce the tangential energy functionals as

$$\begin{aligned} \bar{\mathcal{E}}_L &:= \|\nabla \eta\|_{\underline{4},0}^2 + \|u\|_{\underline{4},0}^2 + \|\nabla u\|_{\underline{2},0}^2, \\ \bar{\mathcal{E}}_H &:= \|\nabla \eta\|_{\underline{6},0}^2 + \|u\|_{\underline{6},0}^2 + \|\nabla u\|_{\underline{4},0}^2 \end{aligned}$$

and the the tangential dissipation functionals as

$$\begin{aligned} \bar{\mathcal{D}}_L &:= \|(\eta, \partial_3 \eta)\|_{\underline{4},0}^2 + \|\nabla u\|_{\underline{4},0}^2 + \|u_t\|_{\underline{2},0}^2, \\ \bar{\mathcal{D}}_H &:= \|(\eta, \partial_3 \eta)\|_{\underline{6},0}^2 + \|\nabla u\|_{\underline{6},0}^2 + \|u_t\|_{\underline{4},0}^2. \end{aligned}$$

Proposition 4.1. *Under assumption* (3.1) *with sufficiently small* δ*, then there hold*

$$\frac{\mathrm{d}}{\mathrm{d}t} \bar{\mathfrak{E}}_L + \bar{\mathcal{D}}_L \lesssim \sqrt{\mathcal{E}_H} \mathcal{D}_L, \tag{4.23}$$

$$\frac{\mathrm{d}}{\mathrm{d}t} \bar{\mathfrak{E}}_H + \bar{\mathcal{D}}_H \lesssim \sqrt{\mathcal{E}_H} \mathcal{D}_H + \sqrt{\mathcal{D}_L} \|\eta\|_{\underline{5},2}\left(\|(\eta, u)\|_{\underline{5},2} + \|\nabla q\|_{\underline{5},0}\right), \tag{4.24}$$

where $\bar{\mathfrak{E}}_L$ *and* $\bar{\mathfrak{E}}_H$ *are equivalent to* $\bar{\mathcal{E}}_L$ *and* $\bar{\mathcal{D}}_H$*, resp. In particular, we also have*

$$\frac{\mathrm{d}}{\mathrm{d}t} \bar{\mathfrak{E}}_{H+1} + \bar{\mathcal{D}}_{H+1} \lesssim \|(\nabla \eta, u)\|_{\underline{5},1}\left(\|(\eta, u)\|_{\underline{5},2}^2 + \|\nabla q\|_{\underline{5},0}^2\right), \tag{4.25}$$

where $\bar{\mathfrak{E}}_{H+1}$ *and* $\bar{\mathfrak{D}}_{H+1}$ *are equivalent to* $\bar{\mathcal{E}}_H + \|\nabla u\|_{\underline{5},0}^2$ *and* $\bar{\mathcal{D}}_H + \|u_t\|_{\underline{5},0}^2$*, resp.*

Proof. Let $0 \leqslant j \leqslant 4$. We first define

$$\bar{\mathcal{E}}_{j+2} := 2\rho \sum_{|\alpha|=0}^{j+2} \int \partial_{\mathrm{h}}^{\alpha} u \cdot \partial_{\mathrm{h}}^{\alpha} \eta \mathrm{d}y + \mu \|\nabla \eta\|_{\underline{j+2},0}^2 + c_1 \left(\|\sqrt{\rho} u\|_{\underline{j+2},0}^2 + \lambda m^2 \|\partial_3 \partial_{\mathrm{h}}^{\alpha} \eta\|_{\underline{j+2},0}^2 \right),$$

$$\bar{\mathcal{D}}_{j+2} := \lambda m^2 \|\partial_3 \partial_{\mathrm{h}}^{\alpha} \eta\|_{\underline{j+2},0}^2 + \mu c_1 \|\nabla u\|_{\underline{j+2},0}^2,$$

where the constant c_1 is suitable large.

By summing over α with $|\alpha| \leqslant j+2$, we easily derive from Lemma 4.1 that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \bar{\mathcal{E}}_{j+2} + c \bar{\mathcal{D}}_{j+2} \\
&\lesssim \begin{cases} \sqrt{\mathcal{E}_H} \mathcal{D}_L, & \text{for } \ j = 2; \\ \delta \|\nabla u\|_{\underline{6},0}^2 + \sqrt{\mathcal{E}_H} \mathcal{D}_H + \sqrt{\mathcal{D}_L} \|\eta\|_{\underline{5},2} \left(\|(\eta, u)\|_{\underline{5},2} + \|\nabla q\|_{\underline{5},0} \right), & \text{for } \ j = 4. \end{cases}
\end{aligned}$$

Choosing a suitable large c_1, we further get

$$\frac{\mathrm{d}}{\mathrm{d}t} \bar{\mathcal{E}}_{j+2} + c \bar{\mathcal{D}}_{j+2} \lesssim \begin{cases} \sqrt{\mathcal{E}_H} \mathcal{D}_L, & \text{for } \ j = 2; \\ \sqrt{\mathcal{E}_H} \mathcal{D}_H + \sqrt{\mathcal{D}_L} \|\eta\|_{\underline{5},2} \left(\|(\eta, u)\|_{\underline{5},2} + \|\nabla q\|_{\underline{5},0} \right), & \text{for } \ j = 4. \end{cases} \tag{4.26}$$

On the other hand, by (4.20) and sum over such α with $|\alpha| \leqslant j$, we have

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \bar{\mathfrak{E}}_j + \|\sqrt{\rho} u_t\|_{\underline{j},0}^2 \lesssim \|\partial_3 u\|_{\underline{j},0}^2 + \begin{cases} \sqrt{\mathcal{E}_L} \mathcal{D}_L, & \text{for } \ j = 2; \\ \sqrt{\mathcal{E}_H} \mathcal{D}_H, & \text{for } \ j = 4, \end{cases} \tag{4.27}$$

where we have defined that

$$\bar{\mathfrak{E}}_j := \sum_{|\alpha|=0}^{j} \left(\lambda m^2 \int \partial_3 \partial_{\mathrm{h}}^{\alpha} \eta \cdot \partial_3 \partial_{\mathrm{h}}^{\alpha} u \mathrm{d}y + \mu \|\nabla \partial_{\mathrm{h}}^{\alpha} u\|_0^2 \right).$$

Consequently, we further deduce from (4.27) and (4.26) that there exists a suitable large constant c_2, such that

$$\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t} \left(c_2 \bar{\mathcal{E}}_{j+2} + \bar{\mathfrak{E}}_j \right) + c \left(c_2 \bar{\mathcal{D}}_{j+2} + \|u_t\|_{\underline{j},0}^2 \right) \\
&\lesssim \begin{cases} \sqrt{\mathcal{E}_H} \mathcal{D}_L, & \text{for } \ j = 2; \\ \sqrt{\mathcal{E}_H} \mathcal{D}_H + \sqrt{\mathcal{D}_L} \|\eta\|_{\underline{5},2} \left(\|(\eta, u)\|_{\underline{5},2} + \|\nabla q\|_{\underline{5},0} \right), & \text{for } \ j = 4. \end{cases}
\end{aligned} \tag{4.28}$$

Applying Poincaré's inequality (3.6), then from (4.28) we easily see the desired (4.23)–(4.24) are obtained by redefining $\bar{\mathfrak{E}}_L := \left(c_2 \bar{\mathcal{E}}_4 + \bar{\mathfrak{E}}_2 \right)$, $\bar{\mathfrak{E}}_H := \left(c_2 \bar{\mathcal{E}}_6 + \bar{\mathfrak{E}}_4 \right)$ and $\bar{\mathcal{D}}_L := c \left(c_2 \bar{\mathcal{D}}_4 + \|u_t\|_{\underline{2},0}^2 \right)$, $\bar{\mathcal{D}}_H := c \left(c_2 \bar{\mathcal{D}}_6 + \|u_t\|_{\underline{4},0}^2 \right)$.

Now we turn to derive (4.25). By (4.22) and sum over such α with $|\alpha| \leqslant 5$, we have

$$\frac{1}{2} \frac{\mathrm{d}}{\mathrm{d}t} \bar{\mathfrak{E}}_5 + \|\sqrt{\rho} u_t\|_{\underline{5},0}^2 \lesssim \|\partial_3 u\|_{\underline{5},0}^2 + \|(\nabla \eta, u)\|_{\underline{5},1} \left(\|(\eta, u)\|_{\underline{5},2}^2 + \|(\nabla q, u_t)\|_{\underline{5},0}^2 \right), \tag{4.29}$$

where $\bar{\mathfrak{E}}_5 := \sum_{|\alpha|=0}^{5}\left(\lambda m^2\int\partial_3\partial_{\mathrm{h}}^{\alpha}\eta\cdot\partial_3\partial_{\mathrm{h}}^{\alpha}u\mathrm{d}y+\mu\|\nabla\partial_{\mathrm{h}}^{\alpha}u\|_0^2\right)$. Hence we further derive from (4.29) and (4.26) with $j=4$ that there exists a suitable large constant c_3, such that

$$
\begin{aligned}
&\frac{\mathrm{d}}{\mathrm{d}t}\left(c_3\bar{\mathcal{E}}_6+\bar{\mathfrak{E}}_5\right)+c\left(c_2\bar{\mathcal{D}}_6+\|u_t\|_{\underline{5},0}^2\right)\\
&\lesssim\sqrt{\mathcal{E}_H}\mathcal{D}_H+\sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\|(\eta,u)\|_{\underline{5},2}+\|(\nabla\eta,u)\|_{\underline{5},1}\left(\|u\|_{\underline{5},2}^2+\|(\nabla q,u_t)\|_{\underline{5},0}^2\right),\\
&\lesssim\|(\nabla\eta,u)\|_{\underline{5},1}\left(\|(\eta,u)\|_{\underline{5},2}^2+\|(\nabla q,u_t)\|_{\underline{5},0}^2\right),
\end{aligned}
$$

which implies for sufficiently small δ,

$$
\frac{\mathrm{d}}{\mathrm{d}t}\left(c_3\bar{\mathcal{E}}_6+\bar{\mathfrak{E}}_5\right)+c\left(c_2\bar{\mathcal{D}}_6+\|u_t\|_{\underline{5},0}^2\right)\lesssim\|(\nabla\eta,u)\|_{\underline{5},1}\left(\|(\eta,u)\|_{\underline{5},2}^2+\|\nabla q\|_{\underline{5},0}^2\right). \tag{4.30}
$$

Hence (4.25) follows by redefining $\bar{\mathfrak{E}}_{H+1}:=\left(c_3\bar{\mathcal{E}}_6+\bar{\mathfrak{E}}_5\right)$ and $\bar{\mathfrak{D}}_{H+1}:=c\left(c_2\bar{\mathcal{D}}_6+\|u_t\|_{\underline{5},0}^2\right)$. This completes the proof of Proposition 4.1. □

4.2. Energy evolution of normal derivatives

In this subsection, we use the regularity theory of the Stokes problem to derive more estimates of (η,u). To do this, we first rewrite $(4.3)_2$–$(4.3)_4$ as the following Stokes problem:

$$
\begin{cases}
-\mu\Delta\partial_{\mathrm{h}}^{\alpha}w+\nabla\partial_{\mathrm{h}}^{\alpha}q=\lambda m^2\Delta_{\mathrm{h}}\partial_{\mathrm{h}}^{\alpha}\eta-\rho\partial_{\mathrm{h}}^{\alpha}u_t-\partial_{\mathrm{h}}^{\alpha}\left(\mathcal{N}_q-\mu\mathcal{N}_u\right) & \text{in }\Omega,\\
\mathrm{div}\partial_{\mathrm{h}}^{\alpha}w=\mathrm{div}\partial_{\mathrm{h}}^{\alpha}\left(u+\lambda m^2\eta/\mu\right) & \text{in }\Omega,\\
\partial_{\mathrm{h}}^{\alpha}w=0 & \text{on }\partial\Omega,
\end{cases} \tag{4.31}
$$

where $|\alpha|:=j\leqslant 5$, and

$$
w:=u+\lambda m^2\eta/\mu,\quad \Delta_{\mathrm{h}}:=\partial_1^2+\partial_2^2.
$$

Lemma 4.3. *Under assumption* (3.1), *it holds that*

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_{\underline{j},2}^2+\|(\eta,u)\|_{\underline{j},2}^2+\|\nabla q\|_{\underline{j},0}^2\lesssim\|(\Delta_{\mathrm{h}}\eta,u_t)\|_{\underline{j},0}^2,\quad \textit{for}\ \ j=2,4,5. \tag{4.32}
$$

Proof. Applying the classical Stokes regularity theory as in [27, Proposition 2.3] to the above Stokes problem (4.31), for $0\leqslant i\leqslant j$ we have

$$
\|w\|_{i,2}^2+\|\nabla q\|_{i,0}^2\lesssim\|(\Delta_{\mathrm{h}}\eta,u_t)\|_{i,0}^2+\|(\mathcal{N}_q,\mathcal{N}_u)\|_{i,0}^2+\|(\mathrm{div}u,\mathrm{div}\eta)\|_{i,1}^2. \tag{4.33}
$$

Note that by $(4.3)_1$,

$$
\|w\|_{i,2}^2=\frac{\mathrm{d}}{\mathrm{d}t}\left\|\sqrt{\lambda m^2/\mu}\eta\right\|_{i,2}^2+\|\lambda m^2\eta/\mu\|_{j,2}^2+\|u\|_{i,2}^2.
$$

Then we further derive from (4.33) that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_{i,2}^2+\|(\eta,u)\|_{i,2}^2+\|\nabla q\|_{i,0}^2\lesssim\|(\Delta_{\mathrm{h}}\eta,u_t)\|_{i,0}^2+\|(\mathcal{N}_q,\mathcal{N}_u)\|_{i,0}^2+\|(\mathrm{div}u,\mathrm{div}\eta)\|_{i,1}^2.
$$

Summing over i with $0\leqslant i\leqslant j$, we further deduce that

$$
\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|_{\underline{j},2}^2+\|(\eta,u)\|_{\underline{j},2}^2+\|\nabla q\|_{\underline{j},0}^2\lesssim\|(\Delta_{\mathrm{h}}\eta,u_t)\|_{\underline{j},0}^2+\|(\mathcal{N}_q,\mathcal{N}_u)\|_{\underline{j},0}^2+\|(\mathrm{div}u,\mathrm{div}\eta)\|_{\underline{j},1}^2. \tag{4.34}
$$

Moreover, by (3.20)–(3.21) and (3.15)–(3.16), we have that

$$\begin{aligned}&\|(\mathcal{N}_q,\mathcal{N}_u)\|^2_{\underline{j},0}+\|(\mathrm{div}u,\mathrm{div}\eta)\|^2_{\underline{j},1}\\ &\lesssim\|\nabla\eta\|^2_{\underline{j},1}\left(\|(\eta,u)\|^2_{\underline{j},2}+\|\nabla q\|^2_{\underline{j},0}\right)\lesssim\delta\left(\|(\eta,u)\|^2_{\underline{j},2}+\|\nabla q\|^2_{\underline{j},0}\right).\end{aligned}\tag{4.35}$$

Putting it into (4.34), then the desired (4.32) follows for sufficiently small δ. This completes the proof of Lemma 4.3. □

5. Global energy estimates

In this section we shall glue all the energy evolution of tangential and normal estimates obtained in section 4 so as to prove the *a priori* estimate (2.2).

Proposition 5.1. *Under assumption* (3.1) *with sufficiently small* δ*, there are three energy functionals* $\tilde{\mathcal{E}}_L$*,* $\tilde{\mathcal{E}}_H$ *and* $\overline{\|\eta\|}^2_{\underline{5},2}$*, which are equivalent to* $\mathcal{E}_L$*,* $\mathcal{E}_H$ *and* $\|\eta\|^2_{\underline{5},2}+\|u\|^2_{\underline{5},1}$*, resp., such that*

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_L+\mathcal{D}_L\leqslant 0,\tag{5.1}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_H+\mathcal{D}_H\lesssim\sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\left(\|(\eta,u)\|_{\underline{5},2}+\|\nabla q\|_{\underline{5},0}\right),\tag{5.2}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\|\eta\|}^2_{\underline{5},2}+\|(\eta,u)\|^2_{\underline{5},2}+\|\nabla q\|^2_{\underline{5},0}\lesssim\mathcal{E}_H\quad\text{on }(0,T].\tag{5.3}$$

Proof. To begin with, it follows from (4.32) that

$$\frac{\mathrm{d}}{\mathrm{d}t}\|\eta\|^2_{\underline{j},2}+c\left(\|(\eta,u)\|^2_{\underline{j},2}+\|\nabla q\|^2_{\underline{j},0}\right)\lesssim\begin{cases}\bar{\mathcal{D}}_L & \text{for } j=2,\\ \bar{\mathcal{D}}_H & \text{for } j=4,\\ \bar{\mathcal{E}}_H+\|u_t\|^2_{\underline{5},0} & \text{for } j=5,\end{cases}\tag{5.4}$$

which together with (4.23)–(4.24) and (4.25) yields that, for a suitable large constant c_4,

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E}_L+c\mathcal{D}_L\leqslant\sqrt{\mathcal{E}_H}\mathcal{D}_L,\tag{5.5}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\mathfrak{E}_H+c\mathcal{D}_H\lesssim\sqrt{\mathcal{E}_H}\mathcal{D}_H+\sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\left(\|(\eta,u)\|_{\underline{5},2}+\|\nabla q\|_{\underline{5},0}\right),\tag{5.6}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}\overline{\|\eta\|}^2_{\underline{5},2}+c\left(\|(\eta,u)\|^2_{\underline{5},2}+\|(\nabla q,u_t)\|^2_{\underline{5},0}\right)\lesssim\bar{\mathcal{E}}_H+\|(\nabla\eta,u)\|_{\underline{5},1}\left(\|(\eta,u)\|^2_{\underline{5},2}+\|\nabla q\|^2_{\underline{5},0}\right),\tag{5.7}$$

where for some constant c_4,

$$\begin{aligned}\mathfrak{E}_L&:=\|\eta\|^2_{\underline{2},2}+c_4\bar{\mathfrak{E}}_L,\\ \mathfrak{E}_H&:=\|\eta\|^2_{\underline{4},2}+c_4\bar{\mathfrak{E}}_H,\\ \overline{\|\eta\|}^2_{\underline{5},2}&:=\mathcal{H}_{\underline{5}}+c_4\bar{\mathfrak{E}}_{H+1}.\end{aligned}$$

It's easily to check that the functionals $\mathfrak{E}_L$, $\mathfrak{E}_H$ and $\overline{\|\eta\|}^2_{\underline{5},2}$ are equivalent to $\mathcal{E}_L$, $\mathcal{E}_H$ and $\|\eta\|^2_{\underline{5},2}+\|u\|^2_{\underline{5},1}$, resp. Therefore, the inequalities (5.1)–(5.3) directly follows from (5.5)–(5.7) under assumption (3.1) with sufficiently small δ. The proof of Proposition 5.1 is completed. □

Now we are in position to establish the *a priori* estimate (2.2). Making use of (5.3), we find that

$$\begin{aligned}\|\eta\|_{\underline{5},2}^2 &\lesssim \|\eta^0\|_{\underline{5},2}^2 e^{-t} + \int_0^t e^{-(t-\tau)}\mathcal{E}_H(\tau)\mathrm{d}\tau \\ &\lesssim \|\eta^0\|_{\underline{5},2}^2 e^{-t} + \sup_{\tau\in[0,t]} \mathcal{E}_H(\tau)\int_0^t e^{-(t-\tau)}\mathrm{d}\tau \\ &\lesssim \|\eta^0\|_{\underline{5},2}^2 e^{-t} + \mathcal{G}_3(t),\end{aligned}$$

which yields

$$\mathcal{G}_1(t) \lesssim \|\eta^0\|_{\underline{5},2}^2 + \mathcal{G}_3(t). \tag{5.8}$$

Multiplying (5.3) by $(1+t)^{-3/2}$, we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\overline{\|\eta\|}_{\underline{5},2}^2}{(1+t)^{3/2}} + \frac{3}{2}\frac{\overline{\|\eta\|}_{\underline{5},2}^2}{(1+t)^{5/2}} + \frac{\|(\eta,u)\|_{\underline{5},2}^2 + \|\nabla q\|_{\underline{5},0}^2}{(1+t)^{3/2}} \lesssim \frac{\mathcal{E}_H}{(1+t)^{3/2}},$$

which implies that

$$\mathcal{G}_2(t) \lesssim \|\eta^0\|_{\underline{5},2}^2 + \|u^0\|_{\underline{5},1}^2 + \mathcal{G}_3(t). \tag{5.9}$$

Moreover, an integration of (5.2) with respect to time t gives

$$\begin{aligned}\mathcal{G}_3(t) &\lesssim \mathcal{E}_H(0) + \int_0^t \sqrt{\mathcal{D}_L}\|\eta\|_{\underline{5},2}\left(\|(\eta,u)\|_{\underline{5},2} + \|\nabla q\|_{\underline{5},0}\right)(\tau)\mathrm{d}\tau \\ &\lesssim \mathcal{E}_H(0) + \mathcal{G}_1^{1/2}(t)\mathcal{G}_2^{1/2}(t)\left(\int_0^t (1+\tau)^{3/2}\mathcal{D}_L\mathrm{d}\tau\right)^{1/2}.\end{aligned}$$

Let

$$\mathcal{G}_5(t) := \mathcal{G}_1(t) + \sup_{\tau\in[0,t]}\mathcal{E}_H(\tau) + \mathcal{G}_4(t).$$

From now on, we further assume $\sqrt{\mathcal{G}_5(T)} \leqslant \delta$, which is a stronger requirement than (3.1). Then we can use above inequality and Young's inequality to get

$$\begin{aligned}\mathcal{G}_3(t) &\lesssim \mathcal{E}_H(0) + \delta\left(\mathcal{G}_1(t) + \mathcal{G}_2(t)\right) \\ &\lesssim \|\eta^0\|_{\underline{5},2}^2 + \|u^0\|_{\underline{5},1}^2 + \delta\left(\mathcal{G}_1(t) + \mathcal{G}_2(t)\right).\end{aligned} \tag{5.10}$$

Hence it follows from (5.8)–(5.10) that

$$\sum_{i=1}^3 \mathcal{G}_i(t) \lesssim \|\eta^0\|_{\underline{5},2}^2 + \|u^0\|_{\underline{5},1}^2 := \mathcal{I}^0. \tag{5.11}$$

Finally, we show the time decay behavior of $\mathcal{G}_4(t)$. Note that $\mathcal{E}_L$ can be controlled by $\mathcal{D}_L$ except for the term $\|\eta\|_{\underline{5},0}^2$ from $\|\nabla\eta\|_{\underline{4},0}^2$ in $\mathcal{E}_L$. However, we can use interpolation inequality (3.8) to get

$$\|\eta\|_{\underline{5},0} \lesssim \|\eta\|_{\underline{4},0}^{2/3}\|\eta\|_{\underline{7},0}^{1/3} \lesssim \|\eta\|_{\underline{4},0}^{2/3}\|\nabla\eta\|_{\underline{6},0}^{1/3}.$$

Thus we further conclude that

$$\mathcal{E}_L \lesssim (\mathcal{D}_L)^{2/3}(\mathcal{E}_H)^{1/3} \lesssim (\mathcal{D}_L)^{2/3}(\mathcal{I}^0)^{1/3}.$$

Plugging the above estimate into (5.5), we obtain

$$\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_L + c\frac{(\tilde{\mathcal{E}}_L)^{3/2}}{(\mathcal{I}^0)^{1/2}} \leqslant 0,$$

which yields

$$\mathcal{E}_L \lesssim \tilde{\mathcal{E}}_L \lesssim \frac{\mathcal{E}_L(0)}{((\mathcal{I}^0/\mathcal{E}_L(0))^{1/2}+t/2)^2} \lesssim \frac{\mathcal{E}_L(0)}{(1+t)^2}.$$

Hence we get

$$\sup_{0\le\tau<t}(1+\tau)^2\mathcal{E}_L(\tau) \lesssim \mathcal{E}_L(0). \tag{5.12}$$

On the other hand, since

$$\begin{aligned}&\frac{\mathrm{d}}{\mathrm{d}t}\left((1+t)^{\frac{3}{2}}\tilde{\mathcal{E}}_L\right)+(1+t)^{\frac{3}{2}}\tilde{\mathcal{D}}_L\\&=(1+t)^{\frac{3}{2}}\frac{\mathrm{d}}{\mathrm{d}t}\tilde{\mathcal{E}}_L+(1+t)^{\frac{3}{2}}\tilde{\mathcal{D}}_L+\frac{3}{2}\tilde{\mathcal{E}}_L(1+t)^{\frac{1}{2}},\end{aligned}$$

from (5.1), we get

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((1+t)^{\frac{3}{2}}\tilde{\mathcal{E}}_L\right)+(1+t)^{\frac{3}{2}}\tilde{\mathcal{D}}_L \leqslant \frac{3}{2}\tilde{\mathcal{E}}_L(1+t)^{\frac{1}{2}},$$

so by (5.12) we can see

$$\frac{\mathrm{d}}{\mathrm{d}t}\left((1+t)^{\frac{3}{2}}\tilde{\mathcal{E}}_L\right)+(1+t)^{\frac{3}{2}}\tilde{\mathcal{D}}_L \lesssim \tilde{\mathcal{E}}_L(0)(1+t)^{-\frac{3}{2}}.$$

Hence we arrive at

$$\sup_{0\le\tau<t}(1+\tau)^{3/2}\mathcal{E}_L(\tau)+\int_0^t(1+\tau)^{3/2}\mathcal{D}_L\mathrm{d}\tau \lesssim \mathcal{E}_L(0). \tag{5.13}$$

Combine (5.12) with (5.13), we obtain

$$\mathcal{G}_4(t) \lesssim \mathcal{E}_L(0). \tag{5.14}$$

Now we sum up (5.11) and (5.14) to conclude that

$$\mathcal{G}(t) := \sum_{i=1}^{4}\mathcal{G}_i \lesssim \left(\|\eta^0\|^2_{\underline{5},2}+\|u^0\|^2_{\underline{5},1}\right). \tag{5.15}$$

Consequently, by virtue of (5.15), we have proved the following estimate, which combine with the well-posedness result and a continuity argument yields Theorem 2.1.

Proposition 5.2. *Let (η,u,q) be a solution of the initial-boundary value problem* (1.12). *Then there is a sufficiently small δ, such that (η,u,q) enjoys the following uniform estimate*

$$\mathcal{G}(T) \lesssim \|\eta^0\|^2_{\underline{5},2}+\|u^0\|^2_{\underline{5},1},$$

provided that $\sqrt{\mathcal{G}_5(T)} \leqslant \delta$ for some T.

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