A mass supercritical and Sobolev critical fractional Schrödinger system*

Mei-Qi Liu[1] & Quanqing Li[2]

1. Department of Mathematical Sciences, Tsinghua University,

Beijing 100084, China

2. Department of Mathematics, Honghe University,

Mengzi, Yunnan, 661100, P.R. China.

e-mail: liumq20@mails.tsinghua.edu.cn, shili06171987@126.com

Abstract

We study the following coupled fractional Schrödinger system:

$$\begin{cases} (-\Delta)^s u = \lambda_1 u + \mu_1 |u|^{p-2}u + \beta r_1 |u|^{r_1-2}u|v|^{r_2} & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = \lambda_2 v + \mu_2 |v|^{q-2}v + \beta r_2 |u|^{r_1}|v|^{r_2-2}v & \text{in } \mathbb{R}^N, \end{cases}$$

with prescribed mass

$$\int_{\mathbb{R}^N} u^2 = a \quad \text{and} \quad \int_{\mathbb{R}^N} v^2 = b.$$

Here, $a, b > 0$ are prescribed, $N > 2s$, $s > \frac{1}{2}$, $2 + \frac{4s}{N} < p, q, r_1 + r_2 \leq 2_s^* = \frac{2N}{N-2s}$, and μ_1, μ_2, β are all positive constants. We first show that if $\beta > 0$ sufficiently large, a mountain pass-type normalized solution exists provided $2 \leq N \leq 4s$ and $2 + \frac{4s}{N} < p, q, r_1 + r_2 < 2_s^*$. Then we also prove that if $2 \leq N \leq 4s, p = q = r_1 + r_2 = 2_s^*$ the nonexistence of positive solution to the system.

Key words: Fractional Laplacian; Schrödinger system; Normalized solution.

Mathematics Subject Classification: 35R11, 35B09, 35B33.

*This work is supported by NSFC(11771234)

1 Introduction

In this paper, we consider the following nonlinear equations involving the fractional Laplace operator:

$$\begin{cases} i\frac{\partial \Psi_1}{\partial t} = (-\Delta)^s \Psi_1 - \mu_1 |\Psi_1|^{p-2}\Psi_1 - \beta r_1 |\Psi_1|^{r_1-2}|\Psi_2|^{r_2}\Psi_1, \\ i\frac{\partial \Psi_2}{\partial t} = (-\Delta)^s \Psi_2 - \mu_2 |\Psi_2|^{q-2}\Psi_2 - \beta r_2 |\Psi_1|^{r_1}|\Psi_2|^{r_2-2}\Psi_2, \\ \Psi_j = \Psi_j(x,t) \in \mathbb{R}, j = 1,2, \ (x,t) \in \mathbb{R}^N \times \mathbb{R}, \\ \Psi_j(x,t) \to 0, \ \text{ as } \ |x| \to +\infty, \ \ j = 1,2, \end{cases} \tag{1.1}$$

where $0 < s < 1$, $N > 2s$, i is the imaginary unite and $\beta > 0$ is a coupling constant. The differential equations involving fractional Laplace operator appear in many fields such as physics and mathematical finances (see [1, 2, 6]), and it can be construed as the infinitesimal generators of Lévy stable diffusion processes. System (1.1) with $s = 1$ possesses numerous physical motivations, such as it appears as models in the research of Bose-Einstein condensation or the incoherent solitons in nonlinear optics (see e.g. [5, 15]). To obtain solitary wave solutions of the system (1.1), one makes the ansatz

$$\Psi_1(x,t) = e^{-i\lambda_1 t}u(x) \ \text{ and } \ \Psi_2(x,t) = e^{-i\lambda_2 t}v(x),$$

where $\lambda_1, \lambda_2 \in \mathbb{R}$ and and $u, v \in H^s(\mathbb{R}^N)$ are time-independent real valued functions. Note that a couple (Ψ_1, Ψ_2) is a solution of (1.1) iff a couple of (u, v) is a solution of (1.2)

$$\begin{cases} (-\Delta)^s u = \lambda_1 u + \mu_1 |u|^{p-2}u + \beta r_1 |u|^{r_1-2}u|v|^{r_2} & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = \lambda_2 v + \mu_2 |v|^{q-2}v + \beta r_2 |u|^{r_1}|v|^{r_2-2}v & \text{in } \mathbb{R}^N, \\ u, v \in H^s(\mathbb{R}^N), \end{cases} \tag{1.2}$$

where the fractional Laplacian operator $(-\Delta)^s$ is defined as

$$(-\Delta)^s u(x) = -\frac{1}{2}\int_{\mathbb{R}^N} \frac{u(x+y) + u(x-y) - 2u(x)}{|y|^{N+2s}} dy, \ \ \forall \, x \in \ \mathbb{R}^N.$$

The studies on nonlinear coupled system with $s = 1$, mainly on the fixed frequency case, i.e., $-\lambda_1, -\lambda_2 > 0$ are prescribed, see e.g. [3, 13, 27] and references therein. In recent years, the problem involving the fractional Laplace operator have been extensively studied, considerable attentions have been paid to search for solutions of (1.2) having prescribed mass; and in that case $(\lambda_1, \lambda_2) \in \mathbb{R}^2$ is a part of the unknown quantities appearing as Lagrange multipliers. In the literature such solutions are called normalized solutions. As we observe that system (1.1) is conservation of masses: $\int_{\mathbb{R}^N} |\Phi_1(t,x)|^2 dx = \int_{\mathbb{R}^N} |u|^2 dx$ and $\int_{\mathbb{R}^N} |\Phi_2(t,x)|^2 dx = \int_{\mathbb{R}^N} |v|^2 dx$ for all $t > 0$. From a physics point of view, prescribed mass represents the law of conservation of mass. Hence, it has particularly meaningful for finding normalized solutions. Many works about the normalized solutions of Schrödinger equations can be further referred to [7, 20, 21, 28, 29]. Soave in [28] well dealed with the existence of normalized ground states for the scalar nonlinear Schrödinger equation with combined power nonlinearities. More specifically, Soave obtained the constraint Palais-Smale sequence of I satisfying the additional condition which is the key ingredient to obtain the boundedness of Palais-Smale sequence, and gave a fine classification about the nonexistence and existence of normalized solutions. Luo and Zhang in [26] extended it to the scalar fractional Schrödinger equation and proved some existence and nonexistence results about normalized solutions. When $s = 1$, T. Gou and L. Jeanjean in [17] solved the normalized solutions of Sobolev subcritical cases. Li and Zou in [23] considered the system with critical and subcritical nonlinearities when the dimension $N = 3, 4$, and obtained the existence of positive normalized solutions. Schrödinger system can also be referred to [8, 9, 10, 11, 18]. We know that the fractional Laplace problem is non-local, which is more challenging than the Laplace problem. Here, we have to emphasize that the system (1.2) considered by Liu and Zou in [24] when the nonlinear coupling terms are replaced by βv and βu respectively. In this paper, we study the existence of normalized solutions of the nonlinear coupling terms system (1.2).

A natural method to obtain solutions of (1.2) satisfying the normalized conditions

$$\int_{\mathbb{R}^N} u^2 = a \quad \text{and} \int_{\mathbb{R}^N} v^2 = b, \tag{1.3}$$

consists in searching for critical points $(u, v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$ of the C^1 energy functional

$$I(u,v) = \int_{\mathbb{R}^N} \frac{1}{2}(|(-\Delta)^{\frac{s}{2}}u|^2 + |(-\Delta)^{\frac{s}{2}}v|^2) - \frac{\mu_1}{p}|u|^p - \frac{\mu_2}{q}|v|^q - \beta|u|^{r_1}|v|^{r_2}, \tag{1.4}$$

under the constraint

$$S_a \times S_b := \Big\{(u,v) \in H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N) : \int_{\mathbb{R}^N} |u|^2 = a, \int_{\mathbb{R}^N} |v|^2 = b\Big\}, \tag{1.5}$$

where

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 = \iint_{\mathbb{R}^{2N}} \frac{(u(x)-u(y))^2}{|x-y|^{N+2s}} dxdy, \tag{1.6}$$

and

$$H^s(\mathbb{R}^N) := \Big\{u \in L^2(\mathbb{R}^N) : \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}u|^2 < +\infty\Big\}$$

is a Hilbert space endowed with the norm $||u||^2 = \int_{\mathbb{R}^N} [|(-\Delta)^{\frac{s}{2}}u|^2 + u^2]$. Particularly, the parameters λ_1, λ_2 appear as the Lagrange multipliers. We would like to point out that many difficulties will be encountered in this process: in the mass-supercritical case, it is difficult to obtain the existence and boundedness of the Palais-Smale sequence, the weak limit of the Palais-Smale sequence might not be on $S_a \times S_b$ (even in the radial space), due to the fact of the embedding $H^s(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ and $H^s_r(\mathbb{R}^N) \hookrightarrow L^2(\mathbb{R}^N)$ are not compact. Consequently, it becomes much more sophisticated to find the normalized solutions of (1.2) comparing with the study of (1.2) of fixed frequency $(\lambda_1, \lambda_2) \in \mathbb{R}^2$.

Since the scalar setting will of course be relevant when solving systems, it is necessary to recall a few results of the scalar fractional Schrödinger equation. Fixed $a > 0$, $\mu > 0$, solving the problem

$$\begin{cases}(-\Delta)^s u = \lambda u + \mu |u|^{p-2} u \quad \text{in } \mathbb{R}^N, \\ u > 0, \\ \int_{\mathbb{R}^N} |u|^2 = a, \end{cases} \tag{1.7}$$

is equivalent to searching for the critical points of the functional $I_{p,\mu} : H^s(\mathbb{R}^N) \to \mathbb{R}$

$$I_{p,\mu}(u) := \frac{1}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 - \frac{\mu}{p} \int_{\mathbb{R}^N} |u|^p, \tag{1.8}$$

constrained on S_a. In this direction, the L^2-critical exponent for fractional NLS equation is denoted by

$$\bar{p} := 2 + \frac{4s}{N}.$$

Observe that $I_{p,\mu}$ is not bounded from below on S_a if $\bar{p} < p < 2_s^*$, which makes one to introduce the L^2-Pohozaev manifold

$$\mathcal{P}_{\mu,a} := \Big\{ u \in S_a : \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 = \frac{\mu \gamma_p}{s} |u|_p^p \Big\}, \tag{1.9}$$

where $\gamma_p = \frac{N(p-2)}{2p}$, $S_a = \{u \in H^s(\mathbb{R}^N), \int_{\mathbb{R}^N} |u|^2 = a\}$. Let $m_{p,\mu}(a) = \inf\limits_{\mathcal{P}_{\mu,a}} I_{p,\mu}(u)$. It is well known from the Pohozaev identity (see [14, Appendix]) of (1.7) that $\mathcal{P}_{\mu,a}$ contains all nontrivial solutions of (1.7). We thus have the following definition of ground states of (1.7).

Definition 1.1. *We say u_0 is a ground state of* (1.7) *on S_a if u_0 is a critical point of $I_{p,\mu}|_{S_a}(u)$ with $I_{p,\mu}|_{S_a}(u_0) = m_{p,\mu}(a)$.*

We are now in the position to present our main results.

Theorem 1.1. *Assume that $\frac{1}{2} \le s < 1$, $2 \le N \le 4s$, $r_1, r_2 > 1$, $\bar{p} < p, q, r_1 + r_2 < 2_s^* = \frac{2N}{N-2s}$ and $\mu_1, \mu_2, \beta > 0$, then there exists $\beta_0 > 0$ sufficiently large, such that for any $\beta \ge \beta_0$,* (1.2)-(1.3) *has a positive radially symmetric normalized solution (u_0, u_0) with $\lambda_1 < 0, \lambda_2 < 0$.*

Remark 1.1. *We deduce from Remark 3.1 and Proposition 3.4 that the critical point of mountain pass-type (u_0, v_0) is also a ground state of* (1.2)-(1.3) *in the sense of*

$$I|_{\mathcal{S}}'(u_0, v_0) = 0 \quad \textit{and} \quad I(u_0, v_0) = \inf_{(u,v) \in \mathcal{P}_{a,b}} I(u, v),$$

where $\mathcal{P}_{a,b}$ is given by (3.2).

Remark 1.2. *The nonlinear coupling terms of system* (1.2) *are replaced by* βv *and* βu *respectively, under assumptions of Theorem 1.1, [24, Theorem 1.3] has proved the existence of positive radially symmetric normalized solution.*

Theorem 1.2. *(Nonexistence). Assume that* $\frac{1}{2} \leq s < 1$, $2 \leq N \leq 4s$, $\mu_1, \mu_2, \beta > 0$, $r_1, r_2 > 1$ *and* $p = q = r_1 + r_2 = 2_s^*$, *then* (1.2)-(1.3) *has no positive solution.*

Notation. The usual norm in the Lebesgue space $L^p := L^p(\mathbb{R}^N)$ is denoted by $|u|_p$. $D_1, D_2 \cdots$ will represent positive constants (possibly different). We write H_r^s instead of $H_{rad}^s(\mathbb{R}^N)$, that is the space of radially symmetric functions in $H^s(\mathbb{R}^N)$. For convenience, we denote $H = H^s(\mathbb{R}^N) \times H^s(\mathbb{R}^N)$, $H_r = H_r^s \times H_r^s$, $S_{a,r} = S_a \cap H_r^s(\mathbb{R}^N)$, $\mathcal{S} = S_a \times S_b$, $\mathcal{S}_r = S_{a,r} \times S_{b,r}$. $||\cdot||$ denotes the norm of H or H^s. u^* denotes the symmetric decreasing rearrangement of $u \in H^s$, Recall that (see [25]) for $1 \leq p, q < +\infty$,

$$|u^*|_p = |u|_p \quad \text{and} \quad \int_{\mathbb{R}^N} |u|^p |v|^q \leq \int_{\mathbb{R}^N} (|u|^*)^p (|v|^*)^q.$$

2 Preliminary results

In this section, we shall present some results for the proof of our main Theorems 1.1-1.2. Let $u \in H^s(\mathbb{R}^N)$ and $2 < p < 2_s^*$, the fractional Gagliardo-Nirenberg-Sobolev (GNS) inequality([16]) is

$$\int_{\mathbb{R}^N} |u|^p \leq C_{N,p,s} (\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2)^{\frac{N(p-2)}{4s}} (\int_{\mathbb{R}^N} |u|^2)^{\frac{p}{2} - \frac{N(p-2)}{4s}}. \tag{2.1}$$

For notational convenience, we set $\gamma_p := \frac{N(p-2)}{2p}$, and then

$$p\gamma_p \begin{cases} > 2s & \text{if } \overline{p} < p < 2_s^* \\ = 2_s^* s & \text{if } p = 2_s^*. \end{cases}$$

By (2.1) and Hölder inequality, we have that, for any $(u, v) \in \mathcal{S}$,

$$\begin{aligned} \frac{\mu_1}{p} |u|_p^p \leq C_1 (\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + (\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2))^{\frac{p\gamma_p}{2s}}, \\ \frac{\mu_2}{q} |v|_q^q \leq C_2 (\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + (\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2))^{\frac{q\gamma_q}{2s}}, \end{aligned} \tag{2.2}$$

and

$$\begin{aligned} \beta \int_{\mathbb{R}^N} |u|^{r_1} |v|^{r_2} \leq & \beta |u|_r^{r_1} |v|_r^{r_2} \\ \leq & C_3 \beta (\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + (\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v|^2))^{\frac{r\gamma_r}{2s}}, \end{aligned} \tag{2.3}$$

where $C_1 = C(N, s, p, a, \mu_1)$, $C_2 = C(N, s, q, b, \mu_2)$ and $C_3 = C(N, s, r_1, r_2, a, b)$. Recall that (see [4, Section 9]), for any $u \in H^s(\mathbb{R}^N)$,

$$\iint_{\mathbb{R}^{2N}} \frac{(|u|^*(x) - |u|^*(y))^2}{|x-y|^{N+2s}} dxdy \leq \iint_{\mathbb{R}^{2N}} \frac{(u(x) - u(y))^2}{|x-y|^{N+2s}} dxdy. \tag{2.4}$$

Lemma 2.1. *Assume that $N > 2s$, $0 < s < 1$ and $\bar{p} < p < 2_s^*$, $\mu > 0$, then for any $a > 0$, problem* (1.7) *admits a unique positive solution $u_{p,\mu,a} \in \mathcal{P}_{\mu,a}$. Moreover,*

$$m_{p,\mu}(a) = \inf_{u \in S_a} \max_{\tau \in \mathbb{R}} I_{p,\mu}(\tau \star u) = \inf_{\mathcal{P}_{\mu,a}} I(u) = I_{p,\mu}(u_{p,\mu,a}) > 0,$$

and $m_{p,\mu}(a)$ is strictly decreasing with respect to $a > 0$.

Proof. Since the Lemma can be proved following closely the method of [26, Theorem 1.2] (also see [31, Lemma 2.4]), we only provide the summary of the proof. It follows from [16, Theorem 3.4] that $Q_{N,p}$ is the unique positive radial ground state solution of (2.5)

$$(-\Delta)^s u + u = |u|^{p-2}u \quad \text{in } \mathbb{R}^N. \tag{2.5}$$

We obtain that the unique positive solution of (1.7) formulated as follows,

$$u_{p,\mu,a} = \mu^{-\frac{1}{p-2}} \kappa^{-1} Q_{N,p}(\frac{1}{\epsilon}x),$$

with κ, ϵ satisfying

$$\begin{cases} -\lambda\epsilon^{2s} = 1, \\ \kappa^{2-p}\epsilon^{2s} = 1, \\ \mu^{-\frac{2}{p-2}} \kappa^{-2} \epsilon^N |Q_{N,p}|_2^2 = a, \end{cases} \tag{2.6}$$

and then

$$\begin{aligned} m_{p,\mu}(a) =& I_{p,\mu}(u_{p,\mu,a}) \\ =& (\frac{1}{2} - \frac{s}{p\gamma_p}) \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_{p,\mu,a}|^2 \\ =& (\frac{1}{2} - \frac{s}{p\gamma_p}) \Big[\frac{s}{\mu\gamma_p C_{N,p,s}} a^{\frac{p\gamma_p - ps}{2s}} \Big]^{\frac{2s}{p\gamma_p - 2s}}. \end{aligned}$$

The last equality means that $m_{p,\mu}(a) > 0$ is strictly decreasing with respect to $a > 0$. $\square$

Lemma 2.2. *Let $2 \leq N \leq 4s$, $\frac{1}{2} \leq s < 1$ and suppose $(u, v) \in H$ is a nonnegative solution of* (1.2) *with $2 < p, q, r \leq 2_s^*$, then $u \gneqq 0$ reads $\lambda_1 < 0$; $v \gneqq 0$ reads $\lambda_2 < 0$.*

Proof. Since $u \gneqq 0$ satisfies

$$(-\Delta)^s u = \lambda_1 u + \mu_1 u^{p-1} + \beta r_1 u^{r_1 - 1} v^{r_2},$$

it yields that the right hand side is nonnegative if $\lambda_1 \geq 0$. We claim have that $u \equiv 0$. Indeed, it is a standard method to combine [22, Proposition 3.1] and the Kelvin transform $u(x)$ centered at 0, it is not difficult to check that this contradicts to $u \in L^2(\mathbb{R}^N)$. Therefore,we infet that $\lambda_1 < 0$. The proof of the other part is identical. $\square$

We will need the following version of Brézis-Lieb Lemmas (see [12]) in the working space H and $H^s(\mathbb{R}^N)$ respectively. Since their proofs are standard, we would like to drop them.

Lemma 2.3. *If $\{(u_n, v_n)\} \subset H$ is a bounded sequence, up to a subsequence, $(u_n, v_n) \to (u, v)$ a.e. in $\mathbb{R}^N$, then for $2 \leq p \leq 2_s^*$ and $r_1, r_2 > 1,\ 2 \leq r_1 + r_2 \leq 2_s^*$, we have*

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^p - |u|^p - |u_n - u|^p = 0,$$

and

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} |u_n|^{r_1}|v_n|^{r_2} - |u_n - u|^{r_1}|v_n - v|^{r_2} - |u|^{r_1}|v|^{r_2} = 0.$$

Lemma 2.4. *If $\{u_n\} \subset H^s(\mathbb{R}^N)$ is a bounded sequence, up to a subsequence, $u_n \to u$ a.e. in $\mathbb{R}^N$, then*

$$\lim_{n\to+\infty} \Big(||u_n||_{H^s}^2 - ||u||_{H^s}^2 - ||u_n - u||_{H^s}^2 \Big) = 0,$$
$$\lim_{n\to+\infty} \Big(||u_n||_{L^2}^2 - ||u||_{L^2}^2 - ||u_n - u||_{L^2}^2 \Big) = 0.$$

Furthermore, we also have that

$$\lim_{n\to+\infty} \int_{\mathbb{R}^N} [|(-\Delta)^{\frac{s}{2}} u_n|^2 - |(-\Delta)^{\frac{s}{2}} u|^2 - |(-\Delta)^{\frac{s}{2}} (u_n - u)|^2] = 0.$$

3 Proof of Theorem 1.1

In this section, we deal with the purely mass supercritical case $\overline{p} < p, q, r < 2_s^*$, $\mu_1, \mu_2, \beta > 0$ and prove Theorem 1.1. We denote by Φ the fiber map

$$\begin{aligned} \Phi(\tau, u, v) :=& \Phi_{(u,v)}(\tau) := I(\mathcal{H}(\tau, u, v)) \\ =& \frac{e^{2s\tau}}{2} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2 - \frac{e^{\tau p \gamma_p}}{p} \mu_1 \int_{\mathbb{R}^N} |u|^p \\ &- \frac{e^{\tau q \gamma_q}}{q} \mu_2 \int_{\mathbb{R}^N} |v|^q - e^{\tau r \gamma_r} \beta \int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2}, \end{aligned} \tag{3.1}$$

where $(u, v) \in \mathcal{S}$ and $\tau \in \mathbb{R}$ and

$$\begin{aligned} \tau \star u(x) :&= e^{\frac{N\tau}{2}} u(e^\tau x), \text{ for a.e. } x \in \mathbb{R}^N, \\ \mathcal{H}(\tau, u, v) :&= \tau \star (u, v) = (\tau \star u, \tau \star v). \end{aligned}$$

Obviously, $\Phi(\tau, u, v)$ is a C^1 functional. In additional, setting the L^2-Pohozaev manifold

$$\mathcal{P}_{a,b} = \big\{(u, v) \in \mathcal{S} \mid P(u, v) = 0\big\}, \tag{3.2}$$

with

$$\begin{aligned} P(u, v) = \int_{\mathbb{R}^N} & |(-\Delta)^{\frac{s}{2}} u|^2 + (-\Delta)^{\frac{s}{2}} v|^2 - \frac{\mu_1 \gamma_p}{s} |u|^p \\ & - \frac{\mu_2 \gamma_q}{s} |v|^q - \frac{1}{s} r \gamma_r \beta |u|^{r_1}|v|^{r_2}. \end{aligned} \tag{3.3}$$

Observe that $\Phi'_{(u,v)}(\tau) = sP(\tau \star (u, v))$.

Lemma 3.1. *Let $(u,v)\in\mathcal{S}_r$ be arbitrary but fixed. Then the following hold.*

(1) $\frac{e^{2s\tau}}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2+|(-\Delta)^{\frac{s}{2}}v|^2\to 0^+$, $I(\mathcal{H}(\tau,u,v))\to 0^+$ *as* $\tau\to-\infty$*;*

(2) $\frac{e^{2s\tau}}{2}\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2+|(-\Delta)^{\frac{s}{2}}v|^2\to+\infty$, $I(\mathcal{H}(\tau,u,v))\to-\infty$ *as* $\tau\to+\infty$.

Proof. The first limit of (1), (2) are obviously holds. The second limit of (1), (2) are the consequence of (3.1) and $p\gamma_p,q\gamma_q,r\gamma_r>2s$. $\square$

Lemma 3.2. *There exists $K(a,b)>0$ sufficiently small such that*

$$0<\sup_{(u,v)\in\mathcal{A}_1}I(u,v)<\inf_{(u,v)\in\mathcal{A}_2}I(u,v)$$

with

$$\begin{cases}\mathcal{A}_1:=\{(u,v)\in\mathcal{S}_r:\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2+|(-\Delta)^{\frac{s}{2}}v|^2<K(a,b)\},\\ \mathcal{A}_2:=\{(u,v)\in\mathcal{S}_r:\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2+|(-\Delta)^{\frac{s}{2}}v|^2=2K(a,b)\}.\end{cases}$$

Proof. Let $K>0$ be arbitrary but fixed, and for any $(u,v)\in\mathcal{S}_r$ satisfying $l:=\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u|^2+|(-\Delta)^{\frac{s}{2}}v|^2<K$, we see that, for $K>0$ sufficiently small,

$$I(u,v)\geq\frac{1}{2}l-C_1l^{\frac{p\gamma_p}{2s}}-C_2l^{\frac{q\gamma_p}{2s}}-C_3\beta l^{\frac{r\gamma_r}{2s}}>0 \tag{3.4}$$

is a consequence of (2.1) and $p\gamma_p,q\gamma_q,r\gamma_r>2s$. For another thing, $(u_1,v_1),(u_2,v_2)\in\mathcal{S}_r$ such that $l_1=2K$ and $l_2<K$, here $l_1=\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_1|^2+|(-\Delta)^{\frac{s}{2}}v_1|^2$ and $l_2=\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_2|^2+|(-\Delta)^{\frac{s}{2}}v_2|^2$. It yields from (2.3)-(2.4) that

$$\begin{aligned}I(u_1,v_1)-I(u_2,v_2)\geq&\frac{1}{2}(l_1-l_2)-C_1l_1^{\frac{p\gamma_p}{2s}}-C_2l_1^{\frac{q\gamma_q}{2s}}-C_3\beta l_1^{\frac{r\gamma_r}{2s}}\\ \geq&\frac{1}{4}K,\end{aligned}$$

for $K>0$ small enough. In summary, we can obtain the desired results by taking a suitable sufficiently small positive number $K(a,b)$. $\square$

Remark 3.1. *By Lemmas 3.1, 3.2, we know that there exists $(\hat{u},\hat{v}),(\tilde{u},\tilde{v})\in\mathcal{S}_r$ such that $(\hat{u},\hat{v})\in\bar{\mathcal{A}}_1$ and $\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}\tilde{u}|^2+|(-\Delta)^{\frac{s}{2}}\tilde{v}|^2>2K(a,b)$ with the property that $I(\hat{u},\hat{v})>0\geq I(\tilde{u},\tilde{v})$. Let us define*

$$\hat{\Gamma}:=\left\{\hat{h}=(\alpha,\psi_1,\psi_2)\in([0,1],\mathbb{R}\times\mathcal{S}_r):\hat{h}(0)\in\{0\}\times\bar{\mathcal{A}}_1,\hat{h}(1)\in\{0\}\times I^0\right\},$$

where $I^0:=\{(u,v)\in\mathcal{S}_r:I(u,v)\leq 0\}$, and

$$\hat{c}_\beta(a,b):=\inf_{\hat{h}\in\hat{\Gamma}}\sup_{t\in[0,1]}\Phi(\hat{h}(t)).$$

Let
$$\Gamma := \left\{h \in C([0,1], \mathcal{S}_r) : h(0) \in \bar{\mathcal{A}}_1,\ h(1) \in I^0\right\},$$
and
$$c_\beta(a,b) := \inf_{h\in\Gamma} \max_{t\in[0,1]} I(h(t)).$$
We claim that $c_\beta(a,b) = \hat{c}_\beta(a,b)$*. Indeed, since* $\Gamma \subset \hat{\Gamma}$*, we firstly have that* $c_\beta(a,b) \geq \hat{c}_\beta(a,b)$*. For any* $\hat{h} \in \hat{\Gamma}$*, we can write it into* $\hat{h}(t) = (\hat{h}_1(t), \hat{h}_2(t)) \in \mathbb{R} \times \mathcal{S}_r$*. Since* $\hat{h}_1(t) \star \hat{h}_2(t) \in \Gamma$*, the opposite inequality is also holds. It follows from Lemma 3.2 and* (2.3)-(2.4) *that, gonging if necessary we can choose a smaller* $K = K(a,b)$*, for any* $(u,v) \in \mathcal{A}_1$*,*
$$P(u,v) \geq l - l^{\frac{p\gamma_p}{2s}} - C_2 l^{\frac{q\gamma_q}{2s}} - C_3\beta l^{\frac{r\gamma_r}{2s}} > 0, \tag{3.5}$$
and
$$0 < I(u,v) < c_\beta(a,b),$$
where $l = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u|^2 + |(-\Delta)^{\frac{s}{2}} v|^2$.

Lemma 3.3. *For every* $(u,v) \in \mathcal{S}$*, the functional* $\Phi_{(u,v)}(\tau)$ *has a unique critical point* $\tau_{(u,v)}$*, which is a strict maximum point at the positive level and* $\tau_{(u,v)} \star (u,v) \in \mathcal{P}_{a,b}$*. Moreover,*

(1) $\Phi_{(u,v)}(\tau)$ *is strictly increasing in* $(-\infty, \tau_{(u,v)})$.

(2) *The map* $(u,v) \in \mathcal{S} \mapsto \tau_{(u,v)} \in \mathbb{R}$ *is of class* C^1.

(3) $P(u,v) < 0$ *iff* $\tau_{(u,v)} < 0$.

Proof. Let $\mathcal{P}^0_{a,b} := \left\{(u,v) \in S_a \times S_b : \Phi''_{(u,v)}(0) = 0\right\}$, where $\Phi_{(u,v)}(\tau)$ is given by (3.1). We firstly claim that $\mathcal{P}^0_{a,b} = \varnothing$. By negation, we assume that $(u,v) \in \mathcal{P}^0_{a,b}$, then
$$(p\gamma_p - 2s)\gamma_p\mu_1|u|_p^p + (q\gamma_q - 2s)\gamma_q\mu_2|v|_q^q + (r\gamma_r - 2s)r\gamma_r\beta\int_{\mathbb{R}^N} |u|^{r_1}|v|^{r_2} = 0.$$
Since $p\gamma_p, q\gamma_q, r\gamma_r > 2s$, this implies $(u,v) = (0,0)$, we get a contradiction. Similarly to [26, Lemma 6.5], we can also infer that $\mathcal{P}_{a,b}$ is a smooth of codimension 3 in H. The rest proof can argue in the same way as that of [26, Lemma 6.14], so we drop the details.

$\square$

Proposition 3.4. *The following properties of* $c_\beta(a,b)$ *hold.*

(1) $\lim\limits_{\beta\to+\infty} c_\beta(a,b) = 0$.

(2) $c_\beta(a,b) = \inf\limits_{(u,v)\in\mathcal{P}_{a,b}} I(u,v)$.

Proof. (1) We apply Remark 3.1, $r\gamma_r > 2s$ and let $h_0(t) := [(1-t)\tau_1 + t\tau_2] \star (\hat{u}, \hat{v})$, yielding that $h_0 \in \Gamma$ and

$$\begin{aligned}
c_\beta(a,b) \le & \max_{t\in[0,1]} I(h_0(t)) \\
\le & \max_{\tau\ge 0} \Big[\frac{\tau^{2s}}{2}\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}}\hat{u}|^2 + |(-\Delta)^{\frac{s}{2}}\hat{v}|^2 - \beta\tau^{r\gamma_r}\int_{\mathbb{R}^N} |\hat{u}|^{r_1}|\hat{v}|^{r_2}\Big] \\
\le & D(\frac{1}{\beta})^{\frac{2s}{r\gamma_r - 2s}} \to 0 \qquad \text{as} \quad \beta \to +\infty.
\end{aligned}$$

(2) Define

$$P_{\hat{h}} : t \in [0,1] \to P(\alpha(t) \star (\psi_1(t), \psi_2(t))) \in \mathbb{R} \tag{3.6}$$

for all $\hat{h} = (\alpha, \psi_1, \psi_2) \in \hat{\Gamma}$, it then yields from (3.5) that $P_{\hat{h}}(0) > 0$. We claim that $P_{\hat{h}}(1) < 0$. If the claim is false, then $P_{\hat{h}}(1) \ge 0$, this implies from Lemma 3.3(1), (3) that $\tau_{\hat{h}(1)} \ge 0$ and

$$I(\psi_1(1), \psi_2(1)) = \Phi_{(\psi_1(1),\psi_2(1))}(0) > \Phi_{(\psi_1(1),\psi_2(1))}(-\infty) = 0^+,$$

we get a contradiction. We then deduce from the continuity of $P_{\hat{h}}$ that there exists $\tau_{\hat{h}} \in (0,1)$ such that $P_{\hat{h}}(\tau_{\hat{h}}) = 0$, i.e., $\hat{h}(\tau_{\hat{h}}) \in \mathcal{P}_{a,b}$. Then

$$\max_{t\in[0,1]} \Phi(\hat{h}(t)) \ge I(\hat{h}(\tau_{\hat{h}})) \ge \inf_{(u,v)\in\mathcal{P}_{a,b}} I(u,v),$$

this implies that $c_\beta(a,b) \ge \inf\limits_{(u,v)\in\mathcal{P}_{a,b}} I(u,v)$. In summary, we conclude that, for any $\hat{h} \in \hat{\Gamma}$ and $t \in (0,1)$,

$$\hat{h}(t) \cap \mathcal{P}_{a,b} \ne \emptyset. \tag{3.7}$$

On the other hand, for all $(u,v) \in \mathcal{P}_{a,b}$, we have that $(|u|^*, |v|^*) \in \mathcal{S}_r$ and $P(|u|^*, |v|^*) \le P(u,v) = 0$. Then $\tau_* := \tau_{(|u|^*,|v|^*)} \le 0 = \tau_{(u,v)}$ and $I(u,v) \ge I(\tau_* \star (u,v)) \ge I(\tau_* \star (|u|^*, |v|^*)) = \max\limits_{\tau\in\mathbb{R}} I(\tau \star (|u|^*, |v|^*))$ is a consequence of Lemma 3.3. For $\tau_0 \ll -1$ and $\tau_1 \gg 1$, we know that $\hat{\gamma}_{(|u|^*,|v|^*)}(t) := (0, [(1-t)\tau_0 + t\tau_1] \star (|u|^*, |v|^*))$ is a path in $\hat{\Gamma}$ with

$$I(u,v) \ge \max_{\tau\in\mathbb{R}} I(\tau \star (|u|^*, |v|^*)) \ge \max_{t\in[0,1]} I(\gamma_{(|u|^*,|v|^*)}(t)) \ge c_\beta(a,b).$$

Hence the reverse inequality follows. Furthermore, by Lemma 3.2,

$$c_\beta(a,b) = \inf_{\mathcal{P}_{a,b}} I > \sup_{\bar{A}_1 \cup I^0} I. \tag{3.8}$$

□

Lemma 3.5. *There exists a Palais-Smale sequence $\{(u_n, v_n)\} \subset \mathcal{S}_r$ for $I|_{\mathcal{S}}$ at the level $c_\beta(a,b)$ with the properties of $u_n^-, v_n^- \to 0$ and $P(u_n, v_n) \to 0$.*

Proof. Set

$$\mathcal{F} := \hat{\Gamma},\ A := \{\hat{h}([0,1]) : \hat{h} \in \hat{\Gamma}\},\ F := \mathcal{P}_{a,b} \text{ and } B = \{0\} \times \bar{\mathcal{A}}_1 \cup \{0\} \times I^0.$$

We firstly claim that $\mathcal{F}$ is homotopy-stable family with extended boundary B. Indeed, for any $\hat{h}(t) \in A$ and $\eta \in C([0,1] \times \mathcal{S}_r, \mathcal{S}_r)$, $\eta(1, \hat{h}(t)) \in \mathcal{F}$ is a consequence of $\eta(1, \hat{h}(t)) \in C([0,1], \mathcal{S}_r)$, $\eta(1, \hat{h}(0)) = \hat{h}(0) \in \{0\} \times \mathcal{A}_1$ and $\eta(1, \hat{h}(1)) = \hat{h}(1) \in \{0\} \times I^0$. It yields from (3.5), (3.7) and (3.8) that

$$A \cap F \backslash B = A \cap (F \backslash B) = A \cap F \neq \emptyset.$$

Note that, for any $\hat{h} \in \hat{\Gamma}$, we have that $|\hat{h}| \in \hat{\Gamma}$ and $\inf\limits_{\mathcal{P}_{a,b}} I \leq \max\limits_{t\in[0,1]} I(|\hat{h}(t)|) \leq \max\limits_{t\in[0,1]} I(\hat{h}(t))$, the first inequality is a consequence of Proposition 3.4(2). So, we can take a minimizing sequence $\{\hat{h}_n([0,1]), \hat{h}_n = (\alpha_n, \psi_1^n, \psi_2^n)\}$ for Φ at the level $c_\beta(a,b)$ with the properties that $\alpha_n(t) = 0, \psi_1^n \geq 0, \psi_2^n \geq 0$ for all $t \in [0,1]$. We thus derive from [19, Theorem 3.2] that there exists a sequence $\{(\tilde{\tau}_n, \tilde{u}_n, \tilde{v}_n)\} \subset \mathbb{R} \times \mathcal{S}_r$ such that

$$\begin{aligned} &\Phi(\tilde{\tau}_n, \tilde{u}_n, \tilde{v}_n) \to c_\beta(a,b), \\ \partial_\tau \Phi(\tilde{\tau}_n, \tilde{u}_n, \tilde{v}_n) \to 0,\ \ &||\partial_{(u,v)}\Phi(\tilde{\tau}_n, \tilde{u}_n, \tilde{v}_n)||_{T_{\tilde{u}_n}S_{a,r} \times T_{\tilde{v}_n}S_{b,r}} \to 0, \end{aligned} \tag{3.9}$$

$$|\tilde{\tau}_n| + \text{dist}((\tilde{u}_n, \tilde{v}_n), (\psi_1^n([0,1]), \psi_2^n([0,1]))) \to 0, \tag{3.10}$$

as $n \to \infty$. So, from now on let $(u_n, v_n) = \tilde{\tau}_n \star (\tilde{u}_n, \tilde{v}_n)$, we then see from (3.10) that $u_n^-, v_n^- \to 0$ a.e. in $\mathbb{R}^N$. For any $(\varphi_1, \varphi_2) \in T_{u_n}S_{a,r} \times T_{v_n}S_{b,r}$, we then see that $((-\tilde{\tau}_n) \star \varphi_1, (-\tilde{\tau}_n) \star \varphi_2) \in T_{\tilde{u}_n}S_{a,r} \times T_{\tilde{v}_n}S_{b,r}$ follows from

$$\begin{aligned} 0 = \int_{\mathbb{R}^N} u_n \varphi_1 dx = \int_{\mathbb{R}^N} \tilde{u}_n e^{\frac{-\tilde{\tau}_n}{2}} \varphi_1(e^{-\tilde{\tau}} x) dx, \\ 0 = \int_{\mathbb{R}^N} v_n \varphi_2 dx = \int_{\mathbb{R}^N} \tilde{v}_n e^{\frac{-\tilde{\tau}_n}{2}} \varphi_2(e^{-\tilde{\tau}} x) dx. \end{aligned}$$

This together with (3.9)-(3.10) implies that

$$\begin{aligned}
&I'(u_n,v_n)[\varphi_1,\varphi_2]\\
=&e^{2s\tilde{\tau}_n}\iint_{\mathbb{R}^{2N}}\frac{(\tilde{u}_n(x)-\tilde{u}_n(y))(e^{\frac{-N\tilde{\tau}_n}{2}}\varphi_1(e^{-\tilde{\tau}_n}x)-e^{\frac{-N\tilde{\tau}_n}{2}}\varphi_1(e^{-\tilde{\tau}_n}y))}{|x-y|^{N+2s}}dxdy\\
&+e^{2s\tilde{\tau}_n}\iint_{\mathbb{R}^{2N}}\frac{(\tilde{v}_n(x)-\tilde{v}_n(y))(e^{\frac{-N\tilde{\tau}_n}{2}}\varphi_2(e^{-\tilde{\tau}_n}x)-e^{\frac{-N\tilde{\tau}_n}{2}}\varphi_2(e^{-\tilde{\tau}_n}y))}{|x-y|^{N+2s}}dxdy\\
&-e^{\tilde{\tau}_n p\gamma_p}\mu_1\int_{\mathbb{R}^N}|\tilde{u}_n|^{p-2}\tilde{u}_n e^{\frac{-N\tilde{\tau}_n}{2}}\varphi_1(e^{-\tilde{\tau}_n}x)dx\\
&-e^{\tilde{\tau}_n q\gamma_q}\mu_2\int_{\mathbb{R}^N}|\tilde{v}_n|^{q-2}\tilde{v}_n e^{\frac{-N\tilde{\tau}_n}{2}}\varphi_2(e^{-\tilde{\tau}_n}x)dx\\
&-e^{\tilde{\tau}_n r\gamma_r}r_1\int_{\mathbb{R}^N}|\tilde{u}_n|^{r_1-2}\tilde{u}_n e^{\frac{-N\tilde{\tau}_n}{2}}\varphi_1(e^{-\tilde{\tau}_n}x)|\tilde{v}_n|^{r_2}dx\\
&-e^{\tilde{\tau}_n r\gamma_r}r_2\int_{\mathbb{R}^N}|\tilde{u}_n|^{r_1}|\tilde{v}_n|^{r_2-2}\tilde{v}_n e^{\frac{-N\tilde{\tau}_n}{2}}\varphi_2(e^{-\tilde{\tau}_n}x)dx\\
=&\partial_{(\tilde{u}_n,\tilde{v}_n)}\Phi(\tilde{\tau}_n,\tilde{u}_n,\tilde{v}_n)[(-\tilde{\tau}_n)\star\varphi_1,(-\tilde{\tau}_n)\star\varphi_2]\\
=&o_n(1)||((-\tilde{\tau}_n)\star\varphi_1,(-\tilde{\tau}_n)\star\varphi_2)||_H=o_n(1),
\end{aligned}$$

and

$$P(u_n,v_n)=s\partial_{\tilde{\tau}_n}\Phi(\tilde{\tau}_n,\tilde{u}_n,\tilde{v}_n)=o_n(1).$$

where $o_n(1)\to 0$ as $n\to\infty$. Hence, $\{(u_n,v_n)\}$ gives our desired results.

□

Lemma 3.6. *The Palais-Smale sequence $\{(u_n,v_n)\}\subset\mathcal{S}_r$ in Lemma 3.5 is bounded in H_r.*

Proof. Let $\rho_n:=\int_{\mathbb{R}^N}|(-\Delta)^{\frac{s}{2}}u_n|^2+|(-\Delta)^{\frac{s}{2}}v_n|^2$. It follows from $P(u_n,v_n)\to 0$ that

$$\rho_n=\frac{\gamma_p}{s}\mu_1|u_n|_p^p+\frac{\gamma_q}{s}\mu_2|v_n|_q^q+\frac{r\gamma_r}{s}\beta\int_{\mathbb{R}^N}|u_n|^{r_1}|v_n|^{r_2}+o_n(1). \tag{3.11}$$

Since $I(u_n,v_n)\to c_\beta(a,b)$, there exists a constant $D>0$ such that

$$\begin{aligned}
D\geq&c(a,b)+o_n(1)\\
=&\frac{p\gamma_p-2s}{2sp}\mu_1|u_n|_p^p+\frac{q\gamma_q-2s}{2sq}\mu_2|v_n|_q^q+\frac{r\gamma_r-2s}{2s}\beta\int_{\mathbb{R}^N}|u_n|^{r_1}|v_n|^{r_2}.
\end{aligned}$$

Three coefficients on the right side are positive number, it implies the boundedness of $|u_n|_p^p$, $|v_n|_q^p$ and $\int_{\mathbb{R}^N}|u_n|^{r_1}|v_n|^{r_2}$. This together with (3.11) implies that $\{(u_n,v_n)\}$ is bounded in H. □

Remark 3.2. *We derive from Lemma 3.6 that there exists a nonnegative* $(u_0, v_0) \in H_r$ *such that, up to a subsequence,*

$$
\begin{aligned}
(u_n, v_n) &\rightharpoonup (u_0, v_0) \quad \text{in } H_r, \\
(u_n, v_n) &\to (u_0, v_0) \quad \text{in } L^p \times L^q, \ \text{ and } \ L^r \times L^r, \text{ and a.e. in } \mathbb{R}^N.
\end{aligned} \tag{3.12}
$$

It then follows from the Lagrange multipliers rule and the fact that $I|'_{\mathcal{S}}(u_n, v_n) \to 0$ *that there exists a sequence* $\{(\lambda_1^n, \lambda_2^n)\} \subset \mathbb{R} \times \mathbb{R}$ *with*

$$
I'(u_n, v_n) - \lambda_1^n (u_n, 0) - \lambda_2^n (0, v_n) \to 0 \qquad \text{in} \quad H^*, \tag{3.13}
$$

Now, take $(u_n, 0)$ *and* $(0, v_n)$ *as test functions in* (3.13)*, we conclude the boundedness of* $(\lambda_1^n \, \lambda_2^n)$*. Hence, there exists* $(\lambda_1, \lambda_2) \in \mathbb{R} \times \mathbb{R}$ *such that, going if necessary to a subsequence,* $(\lambda_1^n, \lambda_2^n) \to (\lambda_1, \lambda_2)$*.*

Setting $c_0(a, 0) := m_{p,\mu_1}(a)$ and $c_0(0, b) := m_{q,\mu_2}(b)$.

Remark 3.3. *For any* $a, b > 0$*, it yields from Proposition 3.4(1) that there exists* $\beta_0 > 0$ *sufficiently large such that* $c_\beta(a, b) < \min\{c_0(a, 0), c_0(0, b)\}$ *for any* $\beta \geq \beta_0$*.*

Lemma 3.7. *Assume that* $\beta \geq \beta_0 \gg 1$*, then* $(u_n, v_n) \to (u_0, v_0)$ *in* H_r*. Moreover,* (u_0, v_0) *is a positive solution of system* (1.2) *with* $\lambda_1 < 0, \lambda_2 < 0$*.*

Proof. First we claim $u_0, v_0 \neq 0$. Arguing by contradiction, suppose $(u_0, v_0) = (0, 0)$. It then follows from (3.12) that $\lim\limits_{n \to \infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 + |(-\Delta)^{\frac{s}{2}} v_n|^2 = 0$, and $c_\beta(a, b) = \lim\limits_{n \to \infty} I(u_n, v_n) = 0$. This clearly contradicts to $c_\beta(a, b) > 0$. Therefore, in what follows, we split two cases to show that $u_0 > 0$ and $v_0 > 0$.

Case 1. If $u_0 \gneqq 0$, $v_0 \equiv 0$. We derive from the maximum principle (see [30, Proposition 2.17]) that $u_0 > 0$. Observe that $u_0 > 0$ is a radially symmetric solution of (1.7) with parameters p, μ_1, a_1, where $0 < a_1 = |u_0|_2^2 \leq a$ and $c_0(a, 0) \leq c_0(a_0, 0) = I(u_0, 0)$. It follows from Lemmas 2.3-2.4 and Remark 3.3 that

$$
\begin{aligned}
c_\beta(a, b) &= \lim_{n \to \infty} I(u_n, v_n) = \lim_{n \to \infty} I(u_n - u_0, v_n) + I(u_0, 0) \\
&\geq \frac{1}{2} \lim_{n \to \infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} (u_n - u_0)|^2 + |(-\Delta)^{\frac{s}{2}} v_n|^2 + c_0(a, 0) \\
&\geq c_0(a, 0)
\end{aligned}
$$

a contradiction.

Case 2. If $u_0 \equiv 0$, $v_0 \gneqq 0$. Analogously as case 1, $v_0 > 0$ is a radially symmetric solution of (1.7) with parameters q, μ_2, b_1, where $b_1 = |v_0|_2^2 \leq b$ and $c_0(0, b) \leq c_0(0, b_1) = I(0, v_0)$. It also yields that

$$
\begin{aligned}
c_\beta(a, b) &= \lim_{n \to \infty} I(u_n, v_n) = \lim_{n \to \infty} I(u_n, v_n - v_0) + I(0, v_0) \\
&\geq c_0(0, b)
\end{aligned}
$$

a contradiction.

Therefore, we derive that $u_0, v_0 > 0$. It then follows from Remark 3.2 that (u_0, v_0) is a positive solution of systems (1.2). This together with $P(u_0, v_0) = 0$ implies that

$$\begin{aligned}\lambda_1|u_0|_2^2 + \lambda_2|v_0|_2^2 =&(\frac{\gamma_p}{s} - 1)\mu_1|u_0|_p^p + (\frac{\gamma_q}{s} - 1)\mu_2|v_0|_q^q \\ &+ (\frac{\gamma_r}{s} - 1)r\beta \int_{\mathbb{R}^N} |u_0|^{r_1}|v_0|^{r_2}. \end{aligned} \tag{3.14}$$

Using $P(u_n, v_n) \to 0$ and Remark 3.2-(3.12),(3.13), we see that

$$\begin{aligned}\lambda_1 a + \lambda_2 b =& \lim_{n\to+\infty} \lambda_1^n|u_n|_2^2 + \lambda_2^n|v_n|_2^2 \\ =& \lim_{n\to+\infty} \Big(\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 + |(-\Delta)^{\frac{s}{2}} v_n|^2 - \int_{\mathbb{R}^N} \beta r|u_n|^{r_1}|v_n|^{r_2} \\ &- \mu_1|u_n|_p^p - \mu_2|v_n|_q^q \Big) \\ =&(\frac{\gamma_p}{s} - 1)\mu_1|u_0|_p^p + (\frac{\gamma_q}{s} - 1)\mu_2|v_0|_q^q + (\frac{\gamma_r}{s} - 1)r\beta \int_{\mathbb{R}^N} |u_0|^{r_1}|v_0|^{r_2}.\end{aligned}$$

This together with (3.14), yielding that

$$\lambda_1(|u_0|_2^2 - a) + \lambda_2(|v_0|_2^2 - b) = 0. \tag{3.15}$$

Then $\lambda_1, \lambda_2 < 0$ follows from Lemma 2.2, we thus have that $(u_n, v_n) \to (u_0, v_0)$ in $L^2(\mathbb{R}^N) \times L^2(\mathbb{R}^N)$. We apply (3.12)-(3.13), yielding that

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_n|^2 - \lambda_1|u_n|^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_0|^2 - \lambda_1|u_0|^2,$$

and

$$\lim_{n\to\infty} \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_n|^2 - \lambda_2|v_n|^2 = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} v_0|^2 - \lambda_2|v_0|^2.$$

It is easy to see that $(u_n, v_n) \to (u_0, v_0)$ in H_r. □

Proof of Theorem 1.1. Theorem 1.1 follows from Lemmas 3.5-3.7, then we finish the proof. □

4 Proof of Theorem 1.2

Lemma 4.1. *Let* $2 \le N \le 4s$, $\frac{1}{2} \le s < 1$ *and suppose* $\mu_1, \mu_2, \beta > 0, r_1 + r_2 = 2_s^*$, *then*

$$\begin{cases} (-\Delta)^s u = \lambda_1 u + \mu_1|u|^{2_s^*-2}u + \beta r_1|u|^{r_1-2}u|v|^{r_2} & \text{in } \mathbb{R}^N, \\ (-\Delta)^s v = \lambda_2 v + \mu_2|v|^{2_s^*-2}v + \beta r_2|u|^{r_1}|v|^{r_2-2}v & \text{in } \mathbb{R}^N, \\ u, v \in H^s(\mathbb{R}^N), \end{cases} \tag{4.1}$$

has no positive solution.

Proof. Arguing by contradiction, we may assume without loss of generality that (u_1, v_1) be a positive solution of the system (4.1) and satisfy $|u_1|_2^2 = a, |v_1|_2^2 = b$. We have by Lemma 2.2 that $\lambda_1 < 0$ and $\lambda_2 < 0$. Observe that

$$0 = P(u_1, v_1) = \int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_1|^2 + |(-\Delta)^{\frac{s}{2}} v_1|^2 - \mu_1 \int_{\mathbb{R}^N} |u_1|^{2_s^*} - \mu_2 \int_{\mathbb{R}^N} |v_1|^{2_s^*} - \beta 2_s^* \int_{\mathbb{R}^N} |u_1|^{r_1} |v_1|^{r_2} \tag{4.2}$$

and

$$\int_{\mathbb{R}^N} |(-\Delta)^{\frac{s}{2}} u_1|^2 + |(-\Delta)^{\frac{s}{2}} v_1|^2 = \lambda_1 \int_{\mathbb{R}^N} |u_1|^2 + \lambda_2 \int_{\mathbb{R}^N} |v_1|^2 + \mu_1 \int_{\mathbb{R}^N} |u_1|^{2_s^*} + \mu_2 \int_{\mathbb{R}^N} |v_1|^{2_s^*} + \beta 2_s^* \int_{\mathbb{R}^N} |u_1|^{r_1} |v_1|^{r_2}, \tag{4.3}$$

the last equality is a consequence of the weak solution definition. It then yields from (4.2)-(4.3) that

$$0 > \lambda_1 \int_{\mathbb{R}^N} |u_1|^2 + \lambda_2 \int_{\mathbb{R}^N} |v_1|^2 = \lambda_1 a + \lambda_2 b = 0$$

a contradiction. □

Proof of Theorem 1.2. By Lemma 4.1, we finish the proof of Theorem 1.2. □

References

[1] D. Applebaum, Lévy processes-from probability to finance and quantum groups. *Notices Am. Math. Soc.* **51**(11), 1336–1347, 2004.

[2] D. Applebaum, Lévy processes and stochastic calculus.In: Cambridge Studies in Advanced Mathematics (vol. 116, 2nd edn). *Cambridge University Press*, 2009.

[3] A. Ambrosetti and E. Colorado, Standing waves of some coupled nonlinear Schrödinger equations. *J. Lond. Math. Soc.*, **2**(1), 67–82, 2007.

[4] F. J. Almgren and E. H. Lieb, Symmetric Decreasing Rearrangement Is Sometimes Continuous. *J. Amer. Math. Soc.*, **2**(4), 683–773, 1989.

[5] V. S. Bagnato, D. J. Frantzeskakis, P. G. Kevrekidis, B. A. Malomed and D. Mihalache., Bose-Einstein condensation: twenty years after. *Roman. Rep. Phys.*, **67**, 5–50, 2015.

[6] J. P. Bouchaud and A. Georges, Anomalous diffusion in disordered media: Statistical mechanisms, models and physical applications. *Physics Reports*, **195**, 127–293, 1990.

[7] T. Bartsch and de Valeriola, S., Normalized solutions of nonlinear Schrödinger equations. *Arch. Math. (Basel)*, **100**(1), 75–83, 2013.

[8] T. Bartsch and L. Jeanjean, Normalized solutions for nonlinear Schrödinger systems. *Proc. Roy. Soc. Edinburgh Sect. A*, **148**(2), 225–242, 2018.

[9] T. Bartsch, L. Jeanjean, and N. Soave, Normalized solutions for a system of coupled cubic Schrödinger equations on $\mathbb{R}^3$. *J. Math. Pures Appl.*, **106**(4), 583–614, 2016.

[10] T. Bartsch and N. Soave, Multiple normalized solutions for a competing system of Schrödinger equations. *Calc. Var. Partial Differential Equations*, **58**(1), 2019.

[11] T. Bartsch and N. Soave, A natural constraint approach to normalized solutions of nonlinear Schrödinger equations and systems. *J. Funct. Anal.*, **272**(12), 4998–5037, 2017.

[12] H. Brézis and E. H. Lieb, A relation between pointwise convergence of functions and convergence of functionals. *Proc. Amer. Math. Soc.*, **88**(3), 486–490, 1983.

[13] Z. Chen and W. Zou, An optimal constant for the existence of least energy solutions of a coupled Schrödinger system. *Calc. Var. Partial Differential Equations*, **48**, 695–711, 2013.

[14] X. Chang and Z.-Q. Wang, Ground state of scalar field equations involving a fractional Laplacian with general nonlinearity. *Nonlinearity*, **26**(2), 479–494, 2013.

[15] B.D. Esry, C. H. Greene, J.P. Burke Jr and J. L. Bohn, Hartree-Fock theory for double condensates. *Phys. Rev. Lett.*, **78**, 3594–3597, 1997.

[16] R. L. Frank, E. Lenzmann and L. Silvestre, Uniqueness of radial solutions for the fractional Laplacian. *Commun. Pure Appl. Math.*, **69**(9), 1671–1726, 2016.

[17] T. Gou and L. Jeanjean, Multiple positive normalized solutions for nonlinear Schrödinger systems. *Nonlinearity*, **31**(2), 2319–2345, 2018.

[18] T. Gou and L. Jeanjean, Existence and orbital stability of standing waves for nonlinear Schrödinger equations systems. *Nonlinear Anal.*, **144**, 10–22, 2016.

[19] N. Ghoussoub, Duality and perturbation methods in critical point theory. 1993.

[20] N. Ikoma and K. Tanaka, A note on deformation argument for $L^2(\mathbb{R}^N)$ normalized solutions of nonlinear Schrödinger equations and systems. *Adv. Differ. Equ.*, **24**, 609–646, 2019.

[21] L. Jeanjean, Existence of solutions with prescribed norm for semilinear elliptic equations. *Nonlinear Anal.*, **28**(10), 1633–1659, 1997.

[22] T. Jin, Y. Y. Li, and J. Xiong, On a fractional Nirenberg problem, part I: blow up analysis and compactness of solutions. *J. Eur. Math. Soc.* (JEMS), **16**(6), 1111–1171, 2014.

[23] H. Li and W. Zou, Normalized ground states for semilinear elliptic systems with critical and subcritical nonlinearities. *arXiv:2006.14387*.

[24] M. Liu and W. Zou, Normalized Solutions for a System of Fractional Schrödinger Equations with Linear Coupling. to appear.

[25] E. H. Lieb and M. Loss., Analysis. Graduate studies in Mathematics, vol. 14. *Amercian Mathematical Society, Provience*, 2001.

[26] H. Luo and Z. Zhang, Normalized solutions to the fractional Schrödinger equations with combined nonlinearities. *Calc. Var. Partial Differential Equations*, **59**(4), 143, 2020.

[27] L. A. Maia, E. Montefusco, and B. Pellacci, Positive solutions for a weakly coupled nonlinear Schrödinger system. *J. Differential Equations*, **229**(2), 743–767, 2006.

[28] N. Soave, Normalized ground states for the NLS equation with combined nonlinearities. *J. Differ. Equ.*, **269**(9), 6941–6987, 2020.

[29] N. Soave, Normalized ground states for the NLS equation with combined nonlinearities: The Sobolev critical case. *J. Funct. Anal.*, **279**(6), 2020.

[30] L. Silvestre, Regularity of the obstacle problem for a fractional power of the Laplace operator. *Comm. Pure Appl. Math.*, **60**(1), 67–112, 2007.

[31] M. D. Zhen, B. L. Zhang and Vicentiu D. Rădulescu, Normalized solutions for nonlinear coupled fractional systems: Low and high perturbations in the attractive case. *Discrete Contin. Dyn. Syst.*, **41**(6), 2653–2676, 2021.