EXISTENCE AND LONG-TIME BEHAVIOR OF SOLUTIONS TO THE VELOCITY-VORTICITY-VOIGT MODEL OF THE 3D NAVIER-STOKES EQUATIONS WITH DAMPING AND MEMORY

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ABSTRACT. In this paper, we study the long-time dynamical behavior of the non-autonomous velocity-vorticity-Voigt model of the 3D Navier-Stokes equations with damping and memory. We first investigate the existence and uniqueness of weak solutions to the initial boundary value problem for above-mentioned model. Next, we prove the existence of uniform attractor of this problem, where the time-dependent forcing term $f \in L^2_b(\mathbb{R}; H^{-1}(\Omega))$ is only translation bounded instead of translation compact. The results in this paper will extend and improve some results in Yue, Wang (Comput. Math. Appl., 2020) in the case of non-autonomous and contain memory kernels which have not been studied before.

1. INTRODUCTION

In this paper, we study the long-time dynamical behavior of the solutions for the following velocity-vorticity-Voigt system with memory:

$$\begin{cases} u_t - \alpha^2 \Delta u_t - \vartheta \Delta u - \int_0^\infty \kappa(s) \Delta u(t-s) ds + w \times u + \nabla p = f, & x \in \Omega, t \geq \tau, \\ w_t - \vartheta \Delta w - \int_0^\infty l(s) \Delta w(t-s) ds + (u \cdot \nabla) w - (w \cdot \nabla) u + \lambda w = \nabla \times f, & x \in \Omega, t \geq \tau, \\ \nabla \cdot u = 0, \, \nabla \cdot w = 0, & x \in \Omega, t \geq \tau, \\ u(x,t) = 0, \, w(x,t) = 0, & x \in \partial\Omega, t \geq \tau, \\ u(x,\tau) = u_\tau(x), w(x,\tau) = w_\tau(x), & x \in \Omega, \\ u(x, \tau - s) = q_\tau(x,s), w(x, \tau - s) = p_\tau(x), & x \in \Omega, s > 0, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^3$ with smooth boundary $\partial\Omega$; the vector field $u = (u_1, u_2, u_3)$ is averaged velocity of the fluid; $w = (w_1, w_2, w_3)$ represents vorticity but we do not assume $w = \nabla \times u$; $f = f(x,t)$ is an external forcing term; $p = p(x,t)$ is the pressure; $\vartheta > 0$ is the kinematic viscosity and the term λw (λ is a positive constant determined later) is damping term, which parameterizes the extra dissipation occurring in the planetary boundary layer (see [16]), α is a length scale parameter characterizing the elasticity of the fluid (see [7]). The second equation of system (1.1) is called vorticity equation with memory.

In 2019, Larios et al. [8] pointed out that the Voigt-regularization and the velocity-vorticity formulation have not been able to overcome all the analytical and computational difficulty inherent in the 3D Navier-Stokes equations of incompressible fluid flow. Therefore, they combine these two approaches and contructed the new system which will retain the best qualities of both systems and have solutions that are closer to the actual physics of fluids, while still having enough regularization that the equations are better behaved from the standpoints of mathematical analysis, numerical stability, and computational efficiency. The new system is a new regularization of the 3D Navier-Stokes equations, which is called the 3D velocity-vorticity-Voigt (VVV) model, with a Voigt regularization term added to momentum equation in velocity-vorticity form, but with no regularizing term in the vorticity equation. In [8], the authors only proved the global well-posedness by Galerkin approximation and convergence properties of the system.

In 2020, G.Yue and J. Wang [15] also considered VVV model as in [8], but added the damping term λw to the second equation, which parameterizes the extra dissipation occurring in the planetary boundary layer

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(see [17]), such as

$$\begin{cases} u_t - \alpha^2 \Delta u_t - \vartheta \Delta u + w \times u + \nabla p = f, \\ w_t - \vartheta \Delta w + (u \cdot \nabla) w - (w \cdot \nabla) u + \lambda w = \nabla \times f, \\ \nabla \cdot u = 0, \, \nabla \cdot w = 0. \end{cases} \tag{1.2}$$

The authors proved the existence of global and exponential attractors of the three-dimensional VVV system.

In the last few years, in the case of $\lambda \equiv 0$, $\alpha \equiv 0$ and $w = \nabla \times u$, the system (1.2) becomes the model studied by the authors in [11]. They investigated well-posedness of a velocityvorticity formulation of the 3D Navier-Stokes (NS) equations with no-slip boundary conditions.

Now if we consider the system (1.2) in the case of incorporating hereditary effects, we add a fading memory term to (1.2), the system can be turned into the VVV system with memory that we will study. The speed of energy dissipation for (1.1) is faster than for the usual VVV system. The conduction of energy is not only affected by present external forces but also by historic external forces. The system (1.1) appears as an extension of the usual VVV system in the realm of viscoelastic incompressible fluid models.

In this paper, we will prove the existence of weak solutions, the existence of uniform attractor for VVV system, while the time-dependent forcing term is only translation bounded instead of translation compact (see (F) below) and the memory kernel satisfies general assumption (see (M) below). To study the problem (1.1), we assume that the external force and the memory kernel satisfy the following conditions:

(M) The convolution (or memory) kernel κ and l are nonnegative summable functions having the explicit form

$$\kappa(s) = \int_s^\infty \mu(r) dr, \quad l(s) = \int_s^\infty \nu(r) dr,$$

where $\mu, \nu \in L^1(\mathbb{R}^+)$ are decreasing (hence nonnegative) piecewise absolutely continuous in each interval $[0, T]$ with $T > 0$ and satisfy $\int_0^\infty \mu(s) ds = 1$, $\int_0^\infty \nu(s) ds = 1$. In particular, μ, ν are allowed to exhibit (infinitely many) jumps. Moreover, we require that

$$\kappa(s) \le \theta_1 \mu(s), \qquad l(s) \le \theta_2 \nu(s), \tag{1.3}$$

for some $\theta_1, \theta_2 > 0$ and every $s > 0$. As shown in Gatti et al [5], this is completely equivalent to the requirement that

$$\mu(r+s) \le M e^{-\delta r} \mu(s), \qquad \nu(r+s) \le M e^{-\delta r} \nu(s), \tag{1.4}$$

for some $M \ge 1, \delta > 0$, every $r \ge 0$ and almost every $s > 0$.

(F) The external force $f \in L^2_b(\mathbb{R}; H^{-1}(\Omega))$, the space of translation bounded functions in $L^2_{\rm loc}(\mathbb{R}; H^{-1}(\Omega))$, that is, $f \in L^2_{\rm loc}(\mathbb{R}; H^{-1}(\Omega))$ satisfies

$$\|f\|^2_{L^2_b} = \sup_{t \in \mathbb{R}} \int_t^{t+1} \|f(s)\|^2_{H^{-1}} ds < +\infty.$$

For $f \in L^2_b\left(\mathbb{R}; H^{-1}(\Omega)\right)$, we denote by $\mathcal{H}_w(f)$ the closure of the set $\{f(\cdot + h) | h \in \mathbb{R}\}$ in $L^2_b(\mathbb{R}; H^{-1}(\Omega))$ with the weak topology. Noting that, as in [3, Chapter 5, Proposition 4.2], we have: $\mathcal{H}_w(f)$ is weakly compact and for all $\sigma \in \mathcal{H}_w(f)$,

$$\|\sigma\|^2_{L^2_b} \le \|f\|^2_{L^2_b}.$$

The paper is organized as follows. In Section 2, we introduce some notations, functions spaces, and recall some basic inequalities that will be used frequently in this paper. In Section 3, we prove the existence and uniqueness of weak solutions by using the Faedo-Galerkin method. Finally, in Section 4, we show the existence of uniform attractor for the continuous semigroup generated by the weak solutions.

2. Notations and preliminaries

In this section, we recall some notations about function spaces and preliminary results. We can find it, for example, in [6, 13, 15].

For $1 \leqslant p \leqslant \infty$ and $k \in \mathbb{N}$, spaces $\mathbf{L}^p(\Omega) = (L^p(\Omega))^3$, $\mathbf{H}^k(\Omega) = (H^k(\Omega))^3$, and $\mathbf{H}^k_0(\Omega) = (H^k_0(\Omega))^3$ will denote the Lebesgue and Sobolev spaces of vector-valued functions on Ω as usual, where $H^k = W^{k,2}$ is a Hilbert space. We also denote by $\|.\|$ and $\langle \cdot, \cdot \rangle$ the normal and scalar product in $\mathbf{L}^2(\Omega)$, respectively. Let

$C_w(I;X)$ be the space of the weakly continous functions u that take values in X for almost every $t \in I$, such that the $\mathbf{L}^p$ norm of $\|u\|_X$ is finite. For $u=(u_1,u_2,u_3)$ and $v=(v_1,v_2,v_3)$, we denote by

$$\langle u,v\rangle=\sum_{i=1}^{3}\langle u_i,v_i\rangle_{L^2(\Omega)},\ \|u\|^2=\sum_{i=1}^{3}\|u_i\|^2,\ \|\nabla v\|^2=\sum_{i,j=1}^{3}\left\|\frac{\partial v_j}{\partial x_i}\right\|^2.$$

We set

$$\mathcal{V}=\{u\in (C_0^\infty(\Omega))^3:\nabla\cdot u=0\},$$

H and V are the closures of $\mathcal{V}$ in $\mathbf{L}^2(\Omega)$ and $\mathbf{H}^1(\Omega)$. Besides, V^{-1} denote the dual space of V. For convenience, we use $\|u\|$ and $\|u\|_1$ to denote the norms of u in H and V.
Let

$$P:L^2(\Omega)=H\oplus H^\perp\to H$$

be the Helmholtz-Leray orthogonal projection. The Stokes operator is defined as

$$A=-P\triangle, D(A)=H^2(\Omega)\cap V.$$

It is known to us all that the operator A is a self-adjoint positively definite operator in H, and its inverse A^{-1} is a compact operator from H into H. Moreover $D(A^{\frac{1}{2}})=V$ and

$$\|u\|_V=\|\nabla u\|=\|A^{\frac{1}{2}}u\|,\quad \forall u\in V.$$

Denote the family of Hilbert spaces $V_s=D(A^{\frac{s}{2}}),\quad (s\in\mathbb{R},\alpha\in(0,1])$, endowed with inner product

$$\langle u,v\rangle_{V_s}=\langle A^{\frac{s-1}{2}}u,A^{\frac{s-1}{2}}v\rangle+\alpha^2\langle A^{\frac{s}{2}}u,A^{\frac{s}{2}}v\rangle,$$

and norm

$$\|u\|_{V_s}^2=\|A^{\frac{1}{2}}u\|^2+\alpha^2\|A^{\frac{s}{2}}u\|^2.$$

So we have the following results

$$\begin{aligned}
\|u\|_{V_1}^2&=\|u\|^2+\alpha^2\|A^{\frac{1}{2}}u\|^2=\|u\|^2+\alpha^2\|\nabla u\|^2,\\
\|u\|_{V_2}^2&=\|A^{\frac{1}{2}}u\|^2+\alpha^2\|Au\|^2=\|\nabla u\|^2+\alpha^2\|Au\|^2,\\
\|\nabla u\|^2&=\frac{1}{\alpha^2}\cdot\alpha^2\|\nabla u\|^2\leqslant\frac{1}{\alpha^2}\left(\|u\|^2+\alpha^2\|\nabla u\|^2\right)=\frac{1}{\alpha^2}\|u\|_{V_1}^2,\\
\|Au\|^2&=\frac{1}{\alpha^2}\cdot\alpha^2\|Au\|^2\leqslant\frac{1}{\alpha^2}\left(\|\nabla u\|^2+\alpha^2\|Au\|^2\right)=\frac{1}{\alpha^2}\|u\|_{V_2}^2.
\end{aligned}$$

Then we introduce the standard bilinear and the trilinear forms

$$B(u,v)=P((u\cdot\nabla)v),$$

$$b(u,v,w)=\langle B(u,v),w\rangle.$$

It is clear that the bilinear form $B(\cdot,\cdot)$ can be extended to a continuous map $B:V\times V\to V^{-1}$, where V^{-1} is the dual space of V. And for smooth functions $u,v,w\in\nu$, we have

$$\langle B(u,v),w\rangle=\int(u\cdot\nabla)v\cdot wdx.$$

And then the trilinear form $b(\cdot,\cdot,\cdot)$ satisfies the following equalities and inequalities (see e.g. [10]):
For every $u,v,w\in V$, we have

$$b(u,v,v)=0,\quad b(u,v,w)=-b(u,w,v), \tag{2.1}$$

$$|b(u,v,w)|\leqslant C\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}}\|\nabla v\|\|\nabla w\|, \tag{2.2}$$

$$|b(u,v,w)|\leqslant C\|v\|^{\frac{1}{2}}\|\nabla v\|^{\frac{1}{2}}\|\nabla u\|\|\nabla w\|, \tag{2.3}$$

$$|b(u,v,w)|\leqslant C\|w\|^{\frac{1}{2}}\|\nabla w\|^{\frac{1}{2}}\|\nabla u\|\|\nabla v\|. \tag{2.4}$$

Next we give more estimates that we will often use later: For any $u\in V$, the following inequalities hold

$$\|u\|_{L^3}\leqslant C\|u\|^{\frac{1}{2}}\|\nabla u\|^{\frac{1}{2}},\quad \|u\|_{L^4}\leqslant C\|u\|^{\frac{1}{4}}\|\nabla u\|^{\frac{3}{4}},\quad \|u\|_{L^6}\leqslant C\|\nabla u\|. \tag{2.5}$$

Next, in order to cary out our analysis, we introduce some new variables which reflect the past history of (1.1), given by

$$\eta^t(x,s)=\eta(x,t,s)=\int_0^s u(x,t-r)dr,\quad \zeta^t(x,s)=\zeta(x,t,s)=\int_0^s w(x,t-r)dr,\ (x,t,s)\in\Omega\times[0,\infty)\times\mathbb{R}^+.$$

We can check that

$$\partial_t\eta^t(x,s)=u(x,t)-\partial_s\eta^t(x,s),\quad \partial_t\zeta^t(x,s)=w(x,t)-\partial_s\zeta^t(x,s),\ (x,t,s)\in\Omega\times[0,\infty)\times\mathbb{R}^+,$$

and

$$\eta^t(x,0):=\lim_{s\to 0}\eta^t(x,s)=0,\quad \zeta^t(x,0):=\lim_{s\to 0}\zeta^t(x,s)=0,\ (x,t)\in\Omega\times[0,\infty),$$

$$\eta^\tau(x,s)=\eta_\tau(x,s)=\int_0^s u(r)dr,\quad \zeta^\tau(x,s)=\zeta_\tau(x,s)=\int_0^s w(r)dr,\ (x,s)\in\Omega\times\mathbb{R}^+.$$

Setting $\mu(s)=-\kappa'(s)$ and $\nu(s)=-k'(s)$, problem (1.1) can be transformed into the following system

$$\begin{cases} u_t-\alpha^2\Delta u_t-\vartheta\Delta u-\int_0^\infty\mu(s)\Delta\eta^t(s)ds+w\times u+\nabla p=f, & (x,t)\in\Omega\times[\tau,\infty),\\ w_t-\vartheta\Delta w-\int_0^\infty\nu(s)\Delta\zeta^t(s)ds+(u\cdot\nabla)w-(w\cdot\nabla)u+\lambda w=\nabla\times f, & (x,t)\in\Omega\times[\tau,\infty),\\ \eta^t_t+\eta^t_s=u, & (x,t,s)\in\Omega\times[\tau,\infty)\times\mathbb{R}^+,\\ \zeta^t_t+\zeta^t_s=w, & (x,t,s)\in\Omega\times[\tau,\infty)\times\mathbb{R}^+,\\ \nabla\cdot u=0,\ \nabla\cdot w=0, & (x,t)\in\Omega\times[\tau,\infty),\\ u(x,t)=0,\ w(x,t)=0, & x\in\partial\Omega, t\geq\tau,\\ u(x,\tau)=u_\tau(x), w(x,\tau)=w_\tau(x), & x\in\Omega,\\ \eta^\tau(x,s)=\eta_\tau(x,s)=\int_0^s q_\tau(x,r)dr,\quad \zeta^\tau(x,s)=\zeta_\tau(x,s)=\int_0^s p_\tau(x,r)dr, & x\in\Omega, s>0,\end{cases}\tag{2.6}$$

Besides, to simplify the notation, we take the viscosity $\vartheta=1$ and then we apply the Helmholtz-Leray projector P to the system (2.6) to obtain the following equivalent functional differential equation

$$\begin{cases} u_t+\alpha^2Au_t+Au+\int_0^\infty\mu(s)A\eta^t(s)ds+P(w\times u)=Pf,\\ w_t+Aw+\int_0^\infty\nu(s)A\zeta^t(s)ds+B(u,w)-B(w,u)+\lambda w=\nabla\times(Pf),\\ \eta^t_t+\eta^t_s=u,\\ \zeta^t_t+\zeta^t_s=w,\end{cases}\tag{2.7}$$

with boundary conditions

$$u(x,t)=w(x,t)=0,\quad \eta^t(x,0)=\zeta^t(x,0)=0 \text{ on } (x,t)\in\partial\Omega\times\mathbb{R}^+,$$

and initial conditions

$$u(x,\tau)=u_\tau(x),\ w(x,\tau)=w_\tau(x),\quad \eta^\tau(x,s)=\eta_\tau(x,s),\ \zeta^\tau(x,s)=\zeta_\tau(x,s)\quad (x,s)\in\Omega\times\mathbb{R}^+.\tag{2.8}$$

Next, we define the memory space. Let $L^2_\mu(\mathbb{R}^+,H)$ be the Hilbert space of functions $\varphi\colon\mathbb{R}^+\to L^2(\Omega)$ endowed with the inner product

$$\langle\varphi_1,\varphi_2\rangle_\mu=\int_0^\infty\mu(s)\langle\varphi_1(s),\varphi_2(s)\rangle ds,$$

and let $\|\varphi\|_\mu$ denote the corresponding norm. In a similar manner we introduce the inner product $\langle\cdot,\cdot\rangle_{1,\mu}$, $\langle\cdot,\cdot\rangle_{2,\mu}$ and relative norms $\|\cdot\|_{1,\mu}$, $\|\cdot\|_{2,\mu}$ on $L^2_\mu(\mathbb{R}^+,V)$ and $L^2_\mu(\mathbb{R}^+,D(A))$ as

$$\langle\cdot,\cdot\rangle_{1,\mu}=\langle\cdot,\cdot\rangle_{V,\mu};\langle\cdot,\cdot\rangle_{2,\mu}=\langle\cdot,\cdot\rangle_{D(A),\mu};\quad \|\varphi\|^2_{1,\mu}=\int_0^\infty\mu(s)\|\nabla\varphi(s)\|^2ds,\ \|\varphi\|^2_{2,\mu}=\int_0^\infty\mu(s)\|\Delta\varphi(s)\|^2ds.$$

We also introduce the Hilbert spaces

$$\mathcal{H}_1=V_1\times H\times L^2_\mu(\mathbb{R}^+,V)\times L^2_\nu(\mathbb{R}^+,V),$$
$$\mathcal{H}_2=V_2\times V_1\times L^2_\mu(\mathbb{R}^+,D(A))\times L^2_\nu(\mathbb{R}^+,D(A)).$$

The norm induced on $\mathcal{H}_i$ for $i=1,2$, are

$$\|(u,w,\eta,\zeta)\|^2_{\mathcal{H}_1}=\|u\|^2_{V_1}+\|w\|^2_H+\int_0^\infty\mu(s)\|\nabla\eta(s)\|^2ds+\int_0^\infty\nu(s)\|\nabla\zeta(s)\|^2ds,$$

and
$$\|(u,w,\eta,\zeta)\|^2_{\mathcal{H}_2}=\|u\|^2_{V_2}+\|w\|^2_{V_1}+\int_0^\infty \mu(s)\|\Delta\eta(s)\|^2ds+\int_0^\infty \nu(s)\|\Delta\zeta(s)\|^2ds.$$

In addition, we recall some useful inequalities (see e.g. [13]) which will be used throughout this paper.

• Poincaré inequality: $\|u\|_{V_{s+1}}\leqslant \frac{1}{\sqrt{\lambda_1}}\|u\|_{V_{s+1}},\quad \forall u\in V_{s+1}$.

Therefore, taking advantage of Poincaré inequality, we have the following two estimates
$$\begin{aligned}\|u\|^2_{V_1}&=\|u\|^2+\alpha^2\|\nabla u\|^2\leqslant\frac{1}{\lambda_1}\|\nabla u\|^2+\alpha^2\|\nabla u\|^2=\frac{1}{k_\alpha}\|\nabla u\|^2,\\ \|u\|^2_{V_2}&=\|\nabla u\|^2+\alpha^2\|Au\|^2\leqslant\frac{1}{\lambda_1}\|Au\|^2+\alpha^2\|Au\|^2=\frac{1}{k_\alpha}\|Au\|^2,\end{aligned}\tag{2.9}$$
where $k_\alpha=\frac{\lambda_1}{1+\lambda_1\alpha^2}$.

• Agmon inequalities in 3D: For any $u\in D(A)$, we have
$$\|u\|_{L^\infty}\leqslant C\|\nabla u\|^{\frac12}\|Au\|^{\frac12},\qquad \|u\|_{L^\infty}\leqslant C\|u\|^{\frac14}\|Au\|^{\frac34}.$$

• Gronwall's inequality: Let $\varphi(t)\in\mathbb{R}$ satisfy the differential inequality
$$\varphi_t\leqslant g(t)\varphi+h(t).$$
Then
$$\varphi(t)\leqslant\varphi(0)e^{G(t)}+\int_0^t e^{G(t)-G(s)}h(s)ds,$$
where $G(t)=\int_0^t g(r)dr$. In particular, if $\varphi_t\leqslant b\varphi+\gamma$, where b and γ are constants, then
$$\varphi(t)\leqslant\varphi_0e^{bt}+\frac{\gamma}{\beta}\left(e^{bt}-1\right).$$

Finally, we will provide a auxiliary lemma to serve later sections.

Lemma 2.1. *Assume that hypotheses **(M)** hold. Then for any $u,w\in V_1$ and $\eta^t\in L^2_\mu(\mathbb{R}^+,V),\zeta^t\in L^2_\nu(\mathbb{R}^+,V)$, the following inequalities hold*
$$\int_0^\infty\kappa(s)\|\nabla\eta^t(s)\|^2ds\le\theta_1\|\eta^t\|^2_{1,\mu}\le\theta_1(\|u\|^2_{V_1}+\|\eta^t\|^2_{1,\mu});\tag{2.10}$$
$$\int_0^\infty l(s)\|\nabla\zeta^t(s)\|^2ds\le\theta_2\int_0^\infty\nu(s)\|\nabla\zeta^t(s)\|^2ds\le\theta_2\left(\|\nabla w\|^2+\int_0^\infty\nu(s)\|\nabla\zeta^t(s)\|^2ds\right);\tag{2.11}$$
$$\frac{d}{dt}\left(\int_0^\infty\kappa(s)\|\nabla\eta^t(s)\|^2ds\right)\le-\frac12\int_0^\infty\mu(s)\|\nabla\eta^t(s)\|^2ds+2\theta_1^2\kappa(0)\|\nabla u\|^2;\tag{2.12}$$
$$\frac{d}{dt}\left(\int_0^\infty l(s)\|\nabla\zeta^t(s)\|^2ds\right)\le-\frac12\int_0^\infty\nu(s)\|\nabla\zeta^t(s)\|^2ds+2\theta_2^2l(0)\|\nabla w\|^2.\tag{2.13}$$

Proof. By hypotheses (1.3), we immediately obtain (2.10) and (2.11).
Besides, using the third equation of (2.7) and exploiting again (1.4), we have
$$\begin{aligned}&\frac{d}{dt}\left(\int_0^\infty\kappa(s)\|\nabla\eta^t(s)\|^2ds\right)\\ =&-2\int_0^\infty\kappa(s)\int_\Omega\nabla\eta^t_s\nabla\eta^tdxds+2\int_0^\infty\kappa(s)\langle\eta^t(s),u\rangle_Vds\\ =&-\int_0^\infty\kappa(s)\frac{d}{ds}\|\nabla\eta^t(s)\|^2ds+2\int_0^\infty\kappa(s)\langle\eta^t(s),u\rangle_Vds\\ =&-\kappa(s)\|\nabla\eta^t(s)\|^2\Big|_{s=0}^{s=\infty}+\int_0^\infty\kappa'(s)\|\nabla\eta^t(s)\|^2ds+2\int_0^\infty\kappa(s)\langle\eta^t(s),u\rangle_Vds\\ \le&-\int_0^\infty\mu(s)\|\nabla\eta^t(s)\|^2ds+2\theta_1\int_0^\infty\mu(s)\int_\Omega\nabla\eta^t\cdot\nabla udxds\end{aligned}$$

$$
\begin{aligned}
&\leq -\int_0^\infty \mu(s)\|\nabla\eta^t(s)\|^2 ds + 2\theta_1\Big(\int_0^\infty \mu(s)\|\nabla\eta^t\|^2 ds\Big)^{1/2}\Big(\int_0^\infty \mu(s)\|\nabla u\|^2 ds\Big)^{1/2}\\
&\leq -\frac{1}{2}\int_0^\infty \mu(s)\|\nabla\eta^t(s)\|^2 ds + 2\theta_1^2\|\nabla u\|^2\int_0^\infty \kappa'(s)ds\\
&\leq -\frac{1}{2}\int_0^\infty \mu(s)\|\nabla\eta^t(s)\|^2 ds + 2\theta_1^2\kappa(0)\|\nabla u\|^2.
\end{aligned}
$$

Similarly, we also get (2.13).

□

3. Existence and uniqueness of weak solutions

Definition 3.1. *A function $z = (u, w, \eta^t, \zeta^t)$ is called a weak solution on the time interval $[\tau, T]$ of problem (2.7)-(2.8) with the initial datum $z(\tau) = z_\tau \in \mathcal{H}_1$ and external force $f \in L^2(\tau, T; V^{-1})$ if*

$$
\begin{aligned}
&u \in C([\tau,T];V_1), \quad w \in C_w([\tau,T];H)\cap L^2(\tau,T;V),\\
&u_t \in L^2(\tau,T;V), \quad w_t \in L^2(\tau,T;V^{-1}),\\
&\eta^t \in C([\tau,T];L^2_\mu(\mathbb{R}^+,V)), \quad \zeta^t \in C([\tau,T];L^2_\nu(\mathbb{R}^+,V)),\\
&\partial_t\eta^t + \partial_s\eta^t \in L^\infty(\tau,T;L^2_\mu(\mathbb{R}^+,H))\cap L^2(\tau,T;L^2_\mu(\mathbb{R}^+,V_1)),\\
&\partial_t\zeta^t + \partial_s\zeta^t \in L^\infty(\tau,T;L^2_\nu(\mathbb{R}^+,H))\cap L^2(\tau,T;L^2_\nu(\mathbb{R}^+,V_1)),
\end{aligned}
$$

and

$$
\begin{cases}
\langle u_t,\varphi\rangle + \alpha^2\langle u_t,\varphi\rangle_V + \langle u,\varphi\rangle_V + \langle \eta^t,\varphi\rangle_{1,\mu} + \langle w\times u,\varphi\rangle = \langle f,\varphi\rangle_{V^{-1},V},\\
\langle \eta^t_t + \eta^t_s, \xi^t\rangle_{1,\mu} = \langle u,\xi^t\rangle_{1,\mu},\\
\langle w_t,\varphi\rangle + \langle w,\varphi\rangle_V + \langle\zeta^t,\varphi\rangle_{1,\nu} - \langle B(u,\varphi),w\rangle - \langle B(w,u),\varphi\rangle = -\langle f,\nabla\times\varphi\rangle_{V^{-1},V},\\
\langle \zeta^t_t + \zeta^t_s,\psi\rangle_{1,\nu} = \langle w,\psi\rangle_{1,\nu},
\end{cases}
$$

for every test functions $\varphi \in L^2(\tau,T;V\cap L^\infty(\Omega))$ and $\xi^t \in L^2_\mu(\mathbb{R}^+,V)$, $\psi^t \in L^2_\nu(\mathbb{R}^+,V)$.

The following result on the existence and uniqueness of weak solutions to the model (1.1) (also (2.7)-(2.8)) was proved by a Faedo-Garlerkin.

Theorem 3.1. *Assume that hypotheses* **(F), (M)** *hold. Then for any $z_\tau = (u_\tau, w_\tau, \eta^\tau, \zeta^\tau) \in \mathcal{H}_1$, any $\sigma \in \mathcal{H}_w(f)$ and any $T > \tau$, $\tau \in \mathbb{R}$ given, problem (2.7)-(2.8) (with σ in place of f) has a unique weak solution $z = (u, w, \eta^t, \zeta^t)$ on the interval $[\tau, T]$ satisfying*

$$z \in C([\tau,T];\mathcal{H}_1).$$

Moreover, the weak solution depends continuously on the initial data on $\mathcal{H}_1$.

Proof. i) Existence.

Consider the approximate solution $z_n(t) = (u_n(t), w_n(t), \eta^t_n, \zeta^t_n)$ in the form

$$u_n(t) = \sum_{j=1}^n u_{nj}(t)\varphi_j, \quad w_n(t) = \sum_{j=1}^n w_{nj}(t)\varphi_j \quad \text{and} \quad \eta^t_n(s) = \sum_{j=1}^n \eta_{nj}(t)\xi_j(s), \quad \zeta^t_n(s) = \sum_{j=1}^n \zeta_{nj}(t)\phi_j(s)$$

satisfying

$$
\begin{aligned}
&\big\langle(\partial_t u_n + \alpha^2 A\partial_t u_n, \partial_t\eta^t_n),(\varphi_k,\xi_j)\big\rangle_{H\times L^2_\mu(\mathbb{R}^+,V)}\\
&\quad= \Big\langle\Big(-Au_n - \int_0^\infty \mu(s)A\eta^t_n(s)ds - P(w_n\times u_n) + P\sigma,\ u_n - \partial_s\eta^t_n\Big),(\varphi_k,\xi_j)\Big\rangle_{H\times L^2_\mu(\mathbb{R}^+,V)},\\
&\big\langle(\partial_t w_n, \partial_t\zeta^t_n),(\varphi_k,\phi_j)\big\rangle_{H\times L^2_\mu(\mathbb{R}^+,V)}\\
&\quad= \Big\langle\Big(-Aw_n - \int_0^\infty \mu(s)A\zeta^t_n(s)ds - B(u_n,w_n) + B(w_n,u_n) + \nabla\times P\sigma,\ w_n - \partial_s\zeta^t_n\Big),(\varphi_k,\phi_j)\Big\rangle_{H\times L^2_\mu(\mathbb{R}^+,V)},\\
&(u_n,\eta^t_n)\big|_{t=\tau} = (P_n u_\tau, Q_n\eta_\tau),\\
&(w_n,\zeta^t_n)\big|_{t=\tau} = (P_n w_\tau, Q_n\zeta_\tau),
\end{aligned}
\tag{3.1}
$$

for a.e. $t \le T$, for every $k, j = 0, \ldots, n$, where φ_0, ξ_0 and ϕ_0 are the zero vectors in the respective spaces. Taking (φ_k, ξ_0), (φ_k, ϕ_0) and (φ_0, ξ_k), (φ_0, ϕ_k) in (3.1), and applying the divergence theorem to the term

$$\left\langle \int_0^\infty \mu(s) A\eta_n^t(s)ds, \varphi_k \right\rangle \quad \text{and} \quad \left\langle \int_0^\infty \nu(s) A\zeta_n^t(s)ds, \varphi_k \right\rangle$$

we get a system of ODE in the variable $z_k(t)$ of the form

$$\begin{aligned}
&\frac{d}{dt}(1+\alpha^2 a_k)u_{nk} = -a_k u_{nk} - \sum_{j=1}^n \eta_{nj} \left\langle \xi_j, \varphi_k \right\rangle_{1,\mu} + \left\langle P\sigma, \varphi_k \right\rangle, \\
&\frac{d}{dt} w_{nk} = -a_k w_{nk} - \sum_{j=1}^n \zeta_{nj} \left\langle \xi_j, \varphi_k \right\rangle_{1,\nu} + \langle B(w_{nk}, u_{nk}), \varphi_k\rangle + \langle \nabla \times P\sigma, \varphi_k \rangle, \\
&\frac{d}{dt}\eta_{nk} = \sum_{j=1}^n u_{nj} \left\langle \varphi_j, \xi_k \right\rangle_{1,\mu} - \sum_{j=1}^n \eta_{nj} \left\langle \xi_j', \xi_k \right\rangle_{1,\mu}, \\
&\frac{d}{dt}\zeta_{nk} = \sum_{j=1}^n w_{nj} \left\langle \varphi_j, \phi_k \right\rangle_{1,\nu} - \sum_{j=1}^n \zeta_{nj} \left\langle \xi_j', \phi_k \right\rangle_{1,\nu},
\end{aligned} \tag{3.2}$$

subject to the initial conditions

$$\begin{aligned}
u_{nk}(\tau) &= \langle u_\tau, \varphi_k \rangle_V, \\
w_{nk}(\tau) &= \langle w_\tau, \varphi_k \rangle_V, \\
\eta_{nk}^\tau &= \langle \eta_\tau, \xi_k \rangle_{1,\mu}, \\
\zeta_{nk}^\tau &= \langle \zeta_\tau, \phi_k \rangle_{1,\nu}.
\end{aligned} \tag{3.3}$$

According to the Picard-Lindelöf Theorem, we know that there exists a solution of (3.2)-(3.3) on some interval (τ, T_n). The *a priori* estimates below imply that in fact $T_n = +\infty$.

Multiplying the first equation of(3.1) by the function (u_{nk}, η_{nk}), then summing from $k = 1$ to n, we have

$$\begin{aligned}
\frac{1}{2}\frac{d}{dt}\left(\|u_n\|^2 + \alpha^2\|\nabla u_n\|^2\right) + \|\nabla u_n\|^2 + \int_0^\infty \mu(s)\langle \nabla \eta_n^t(s), \nabla u_n\rangle ds = \langle P\sigma, u_n\rangle_{V^{-1},V}, \\
u_n = \partial_t \eta_n^t + \partial_s \eta_n^t.
\end{aligned} \tag{3.4}$$

Integrating by parts, we have

$$\begin{aligned}
\int_0^\infty \mu(s)\langle \nabla \eta^t(s), \nabla u_n\rangle ds &= \int_0^\infty \mu(s)\langle \nabla \eta_n^t(s), \partial_t \nabla \eta_n^t(s)\rangle ds + \int_0^\infty \mu(s)\langle \nabla \eta_n^t(s), \partial_s \nabla \eta^t(s)\rangle ds \\
&= \frac{1}{2}\frac{d}{dt}\left(\int_0^\infty \mu(s)\|\nabla \eta_n^t(s)\|^2 ds\right) - \int_0^\infty \mu'(s)\|\nabla \eta_n^t(s)\|^2 ds.
\end{aligned}$$

Besides, from conditions (**M**) and the Cauchy inequality, we can see that

$$-2\int_0^\infty \mu'(s)\|\nabla \eta_n^t(s)\|^2 ds \ge 0 \text{ and } \quad 2\langle P\sigma, u_n\rangle_{V^{-1},V} \le \|P\sigma\|_{V^{-1}}^2 + \|\nabla u_n\|^2.$$

Combining all the above estimates, we get

$$\frac{d}{dt}\left(\|u_n\|^2 + \alpha^2\|\nabla u_n\|^2 + \int_0^\infty \mu(s)\|\nabla \eta_n^t(s)\|^2 ds\right) + \|\nabla u_n\|^2 \le \|P\sigma\|_{V^{-1}}^2. \tag{3.5}$$

Integrating on (τ, t), $t \in (\tau, T)$, leads to

$$y(t) + \int_\tau^t \|\nabla u_n(r)\|^2 dr \le y(\tau) + C\int_\tau^t \|P\sigma\|_{V^{-1}}^2 dr,$$

where $y(t) = \|u_n(t)\|^2 + \alpha^2\|\nabla u_n(t)\|^2 + \int_0^\infty \mu(s)\|\nabla \eta_n^t(s)\|^2 ds$.
This inequality implies that

$$\begin{aligned}
\{u_n\} &\quad \text{is bounded in} \quad L^\infty(\tau, T; V_1), \\
\{\eta_n^t\} &\quad \text{is bounded in} \quad L^\infty(\tau, T; L_\mu^2(\mathbb{R}^+, V)).
\end{aligned} \tag{3.6}$$

Therefore, by the Banach-Alaoglu Theorem, there exists functions u and η^t such that

$$\begin{aligned} u_n \rightharpoonup u \quad &\text{weakly-star in} \quad L^\infty(\tau, T; V_1), \\ \eta_n^t \rightharpoonup \eta^t \quad &\text{weakly-star in} \quad L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+, V)), \end{aligned} \tag{3.7}$$

and

$$\begin{aligned} Au_n \rightharpoonup Au \quad &\text{weakly-star in} \quad L^\infty(\tau, T; V^{-1}), \\ A\eta_n^t \rightharpoonup A\eta^t \quad &\text{weakly-star in} \quad L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+, V^{-1})), \end{aligned} \tag{3.8}$$

up to a subsequence.

Multiplying the second equation of (3.1) by the function (w_{nk}, ζ_{nk}), then summing from $k = 1$ to n, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}\|w_n\|^2 + \|\nabla w_n\|^2 + \lambda\|w_n\|^2 + \int_0^\infty \mu(s)\langle \nabla\zeta_n^t(s), \nabla w_n\rangle ds &= b(w_n, u_n, w_n) + \langle \nabla \times P\sigma, w_n\rangle, \\ w_n &= \partial_t \zeta_n^t + \partial_s \zeta_n^t. \end{aligned} \tag{3.9}$$

Integrating by parts, we have

$$\begin{aligned} \int_0^\infty \nu(s)\langle \nabla\zeta^t(s), \nabla w_n\rangle ds &= \int_0^\infty \nu(s)\langle \nabla\zeta_n^t(s), \partial_t\nabla\zeta_n^t(s)\rangle ds + \int_0^\infty \nu(s)\langle \nabla\zeta_n^t(s), \partial_s\nabla\zeta^t(s)\rangle ds \\ &= \frac{1}{2}\frac{d}{dt}\left(\int_0^\infty \nu(s)\|\nabla\zeta_n^t(s)\|^2 ds\right) - \int_0^\infty \nu'(s)\|\nabla\zeta_n^t(s)\|^2 ds. \end{aligned}$$

On the other hand, by conditions (**M**) and the Cauchy inequality, we obtain

$$-2\int_0^\infty \nu'(s)\|\nabla\zeta_n^t(s)\|^2 ds \geq 0 \quad \text{and} \quad 2\langle \nabla\times P\sigma, w_n\rangle \leq 2\|\nabla\times P\sigma\|^2 + \frac{1}{2}\|\nabla w_n\|^2.$$

Using the (2.12) and the Cauchy inequality once again, we have

$$\begin{aligned} 2|b(w_n, u_n, w_n)| &\leq C\|\nabla u_n\|\|w_n\|^{\frac{1}{2}}\|\nabla w_n\|^{\frac{3}{2}} \\ &\leq C\|\nabla u_n\|^4\|w_n\|^2 + \frac{1}{2}\|\nabla w_n\|^2. \end{aligned}$$

Combining all the above estimates, we have

$$\frac{d}{dt}\left(\|w_n\|^2 + \int_0^\infty \nu(s)\|\nabla\zeta_n^t(s)\|^2 ds\right) + \|\nabla w_n\|^2 \leq C\|\nabla u_n\|^4\|w_n\|^2 + 2\|\nabla\times P\sigma\|^2.$$

Therefore,

$$\frac{d}{dt}y(t) + \|\nabla w_n\|^2 \leq Cy(t) + 2\|\nabla\times P\sigma\|^2. \tag{3.10}$$

where $y(t) = \|w_n\|^2 + \int_0^\infty \nu(s)\|\nabla\zeta_n^t(s)\|^2 ds$. Applying the Gronwall lemma, we get

$$\|w_n\|^2 + \int_0^\infty \nu(s)\|\nabla\zeta_n^t(s)\|^2 ds \leq y(\tau)e^{C(T-\tau)} + 2\int_\tau^T e^{C(t-s)}\|\nabla\times P\sigma\|^2 ds \leq \infty.$$

This inequality implies that

$$\begin{aligned} \{w_n\} \quad &\text{is bounded in} \quad L^\infty(\tau, T; H), \\ \{\zeta_n^t\} \quad &\text{is bounded in} \quad L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+, H_0^1(\Omega))). \end{aligned} \tag{3.11}$$

Besides, integrating (3.10) from τ to T, leads to

$$\{w_n\} \quad \text{is bounded in} \quad L^2(\tau, T; H_0^1(\Omega)), \tag{3.12}$$

Now, multiplying the first equation of(3.1) by the function (w_{nk}, ζ_{nk}), and summing form $k = 1$ to n, then taking the divergence of this equation, and denoting $v_n := \nabla\cdot w_n$, $\hat{\zeta}_n^t := \nabla\cdot w_n$, we get

$$\frac{d}{dt}v_n + Av_n + B(u_n, v_n) + \int_0^\infty \nu A\hat{\zeta}_n^t(s)ds = 0. \tag{3.13}$$

Next we take inner-products with the above equation by v_n, we get

$$\frac{d}{dt}\left(\|v_n\|^2 + \int_0^\infty \nu\|\nabla\hat{\zeta}_n^t(s)\|^2 ds\right) + 2\|\nabla v_n\|^2 \le 0, \tag{3.14}$$

$$\frac{d}{dt}y(t) \le y(t),$$

where $y(t) = \|v_n\|^2 + \int_0^\infty \nu\|\nabla\hat{\zeta}_n^t(s)\|^2 ds$. Using the Gronwall lemma, we have

$$y(t) \le y(\tau)e^{(T-\tau)}.$$

If the initial data (w_τ, ζ_τ) is in $H \times L^2_\nu(\mathbb{R}^+, V)$, we have $(v_n(\tau), \hat{\zeta}_n^t) = (0,0)$, which implies $(\nabla \times w_n, \nabla \times \zeta_n^t) = (v_n, \hat{\zeta}_n^t) = (0,0)$ in $L^2([\tau,T]; L^2(\Omega)) \times L^2([\tau,T]; L^2_\nu(\mathbb{R}^+, H^1_0(\Omega))$. Since $L^2([\tau,T]; H) \times L^2([\tau,T]; L^2_\nu(\mathbb{R}^+, V)$ is closed in $L^2([\tau,T]; L^2(\Omega)) \times L^2([\tau,T]; L^2_\nu(\mathbb{R}^+, H^1_0(\Omega))$, this implies $(\nabla \cdot w, \nabla \cdot \zeta^t) = (0,0)$, so long as $(\nabla \cdot w_\tau, \nabla \cdot \zeta_\tau) = (0,0)$. Namely, we have $(w, \zeta^t) \in L^\infty(\tau, T; H) \cap L^2(\tau, T; V) \times L^2(\tau, T; L^2_\nu(\mathbb{R}^+, V)$. Besides, from (3.14), we also obtain

$$\|v_n\|^2 + \int_0^\infty \nu\|\nabla\hat{\zeta}_n^t(s)\|^2 ds + 2\int_\tau^T \|\nabla v_n\|^2 dt \le \|v_n(\tau)\|^2 + \int_0^\infty \nu\|\nabla\hat{\zeta}_n^\tau(s)\|^2 ds,$$

this implies that $(v_n, \hat{\zeta}_n^t)$ is uniformly bounded in $L^2(\tau, T; H^1_0(\Omega)) \times L^2(\tau, T; L^2_\nu(\mathbb{R}^+, H^1_0(\Omega))$. Therefore, by similar arguments as above for u_n and η_n^t, we get

$$\begin{aligned} &w_n \rightharpoonup w \quad \text{weakly in} \quad L^2(\tau, T; V), \\ &\zeta_n^t \rightharpoonup \zeta^t \quad \text{weakly-star in} \quad L^\infty(\tau, T; L^2_\nu(\mathbb{R}^+, V)), \end{aligned} \tag{3.15}$$

and

$$\begin{aligned} &Aw_n \rightharpoonup Aw \quad \text{weakly-star in} \quad L^\infty(\tau, T; V^{-1}), \\ &A\zeta_n^t \rightharpoonup A\zeta^t \quad \text{weakly-star in} \quad L^\infty(\tau, T; L^2_\mu(\mathbb{R}^+, V^{-1})), \end{aligned} \tag{3.16}$$

up to a subsequence.

Next, we estimate $\partial_t u_n, \partial_t w_n$. Multiplying the first equation of(3.1) by the function (u'_{nk}, η'_{nk}), then summing from $k = 1$ to n, we have

$$\frac{1}{2}\frac{d}{dt}\|\nabla u_n\|^2 + \|\partial_t u_n\|^2 + \alpha^2\|\partial_t \nabla u_n\|^2 = -\int_0^\infty \mu(s)\langle \nabla\eta_n^t(s), \partial_t \nabla u_n\rangle ds - \langle P(w_n \times u_n), \partial_t u_n\rangle + \langle P\sigma, \partial_t u_n\rangle_{V^{-1},V}, \tag{3.17}$$

Using the Holder and Young inequalities and (2.12), (3.6), (3.11), we get

$$-\int_0^\infty \mu(s)\langle \nabla\eta_n^t(s), \partial_t \nabla u_n\rangle ds \le C(\varepsilon)\int_0^\infty \nu(s)\|\nabla\eta_n^t(s)\|^2 ds + \varepsilon\|\mu\|_{L^1(\mathbb{R}^+)}\|\partial_t \nabla u_n\|^2,$$

$$\langle P\sigma, \partial_t u_n\rangle_{V^{-1},V} \le \frac{1}{\alpha^2}\|P\sigma\|^2_{V^{-1}} + \frac{\alpha^2}{4}\|\partial_t \nabla u_n\|^2,$$

$$\begin{aligned} -\langle P(w_n \times u_n), \partial_t u_n\rangle &\le \|w_n\|_{L^3}\|u_n\|\|\partial_t u_n\|_{L^6} \\ &\le C\|w_n\|^{\frac{1}{2}}\|\nabla w_n\|^{\frac{1}{2}}\|u_n\|\|\partial_t \nabla u_n\| \\ &\le C(\alpha, \|u_n\|)\|\nabla w_n\|\|\partial_t \nabla u_n\| \\ &\le C(\alpha, \|u_n\|)\|\nabla w_n\|^2 + \frac{\alpha^2}{4}\|\partial_t \nabla u_n\|^2. \end{aligned}$$

Combining all the above estimates, we get

$$\frac{d}{dt}\|\nabla u_n\|^2 + 2\|\partial_t u_n\|^2 + (\alpha^2 - 2\varepsilon\|\mu\|_{L^1(\mathbb{R}^+)})\|\partial_t \nabla u_n\|^2 \le C(\alpha, \|u_n\|)\|\nabla w_n\|^2 + \|P\sigma\|^2_{V^{-1}} + C(\varepsilon, \|\eta_n^t\|_{1,\mu}).$$

Choosing $\varepsilon > 0$ small enough such that $\alpha^2 - 4\varepsilon\|\mu\|_{L^1(\mathbb{R}^+)} \ge 0$ and then integrating on (τ, t), $t \in (\tau, T)$, leads to

$$\|\nabla u_n\|^2 + \int_\tau^t \left(\|\partial_t u_n(r)\|^2 + (\alpha^2 - 2\varepsilon\|\mu\|_{L^1(\mathbb{R}^+)})\|\partial_t \nabla u_n(r)\|^2\right) dr$$

$$\leq C(\alpha, \|u_n\|) \int_\tau^T \|\nabla w_n(t)\|^2 dt + \int_\tau^T \|P\sigma\|^2_{V^{-1}} dt + C(\varepsilon, \|\eta_n^t\|_{1,\mu})T.$$

This inequality implies that

$$\{\partial_t u_n\} \quad \text{is bounded in} \quad L^2(\tau, T; V_1) \tag{3.18}$$

thus

$$\partial_t u_n \rightharpoonup u_t \quad \text{weakly in} \quad L^2(\tau, T; V_1) \tag{3.19}$$

up to a subsequence.

Now, we consider the nonlinear terms in the second equation of (3.1). By (2.12), we get

$$|\langle (u_n \cdot \nabla) w_n, \phi \rangle| \leq \|u_n\|^{\frac{1}{2}} \|\nabla u_n\|^{\frac{1}{2}} \|\nabla w_n\| \|\nabla \phi\|$$

for all test functions $\phi \in V$. Thus, $-B(u_n, w_n)$ is uniformly bounded in $L^2(\tau, T; V^{-1})$. Similar estimates show that $B(w_n, u_n)$ is also uniformly bounded in $L^2(\tau, T; V^{-1})$. Therefore, from the second equation of (3.1) and by the bounds we obtain above, we have

$$\{\partial_t w_n\} \quad \text{is bounded in} \quad L^2(\tau, T; V^{-1}), \tag{3.20}$$

thus

$$\partial_t w_n \rightharpoonup w_t \quad \text{weakly in} \quad L^2(\tau, T; V^{-1}) \tag{3.21}$$

up to a subsequence. Applying the AubinLions lemma in [9], we can conclude that $(u_n, w_n) \to (u, w)$ strongly in $L^2(\tau, T; H) \times L^2(\tau, T; H)$.

Now, choosing some test function $(\psi, \xi) \in C_c^1([\tau, T]; V) \times \mathcal{D}(([\tau, T]; \mathcal{D}(\mathbb{R}^+, V))$, then (3.1) holds with (ψ, ξ) in place of (φ_k, ξ_j). Integrating the resulting equation over (τ, T) and integrating by parts, we get

$$\begin{aligned}
&- \int_\tau^T \langle u_n, \psi_t \rangle dt + \langle u_n(\cdot, T), \psi_t(\cdot, T) \rangle - \langle u_n(\cdot, \tau), \psi_t(\cdot, \tau) \rangle \\
&\qquad - \alpha^2 \int_\tau^T \langle \nabla u_n, \nabla \psi_t \rangle dt - \alpha^2 \langle \nabla u_n(\cdot, T), \nabla \psi_t(\cdot, T) \rangle + \alpha^2 \langle \nabla u_n(\cdot, \tau), \nabla \psi_t(\cdot, \tau) \rangle \\
&\qquad + \int_\tau^T \langle \nabla u_n, \nabla \psi \rangle dt + \int_\tau^T \langle w_n \times u_n, P_n \psi \rangle dt + \int_\tau^T \int_0^\infty \mu(s) \langle \nabla \eta_n^t, \nabla \psi \rangle = \int_\tau^T \langle Pf, \psi \rangle_{V^{-1}, V} dt, \\
&- \int_\tau^T \langle w_n, \psi_t \rangle dt + \langle w_n(\cdot, T), \psi_t(\cdot, T) \rangle - \langle w_n(\cdot, \tau), \psi_t(\cdot, \tau) \rangle \\
&\qquad + \int_\tau^T \langle \nabla u_n, \nabla \psi \rangle dt + \int_\tau^T \langle B(u_n, w_n), P_n \psi \rangle dt - \int_\tau^T \langle B(w_n, u_n), P_n \psi \rangle dt \\
&\qquad + \int_\tau^T \int_0^\infty \nu(s) \langle \nabla \zeta_n^t, \nabla \psi \rangle = \int_\tau^T \langle P\sigma, \nabla \times \psi \rangle dt.
\end{aligned} \tag{3.22}$$

By similar arguments as in [8], we have

$$\left| \int_\tau^T \langle w_n \times u_n, P_n \psi \rangle dt - \int_\tau^T \langle w \times u, \psi \rangle dt \right| \to 0 \quad \text{as} \quad n \to \infty,$$

$$\left| \int_\tau^T \langle B(u_n, w_n), P_n \psi dt - \int_\tau^T \langle B(u, w), \psi \rangle dt \right| \to 0 \quad \text{as} \quad n \to \infty,$$

$$\left| \int_\tau^T \langle B(w_n, u_n), P_n \psi dt - \int_\tau^T \langle B(w, u), \psi \rangle dt \right| \to 0 \quad \text{as} \quad n \to \infty.$$

Besides, the pressure term p can be recovered using a corollary of a deep result of G. de Rham. The corollary states that, for any distribution g, the equality $g = \nabla p$ holds for some distribution p if and only if $\langle g, w \rangle = 0$ for all $w \in V$. See [14] for an elementary proof of the corollary. Moreover, by standard arguments, we can check that z satisfies the initial condition $z(\tau) = z_\tau$. This implies that $z = (u, \eta^t, w, \zeta^t)$ is a weak solution of problem (2.7).

ii) Uniqueness and continuous dependence. We assume that $z_1=(u_1,\eta_1^t,w_1,\zeta_1^t)$ and $z_2=(u_2,\eta_2^t,w_2,\zeta_2^t)$ are two solutions subject to initial data $z_1(\tau)$ and $z_2(\tau)$, respectively. Denote $(U,\bar\eta^t,W,\bar\zeta^t)=(u_1-u_2,\eta_1^t-\eta_2^t,w_1-w_2,\zeta_1^t-\zeta_2^t)$, we have

$$\begin{cases} U_t+\alpha^2AU_t+AU+\int_0^\infty\mu(s)A\bar\eta^t(s)ds+P(w_1\times U)+P(W\times u_2)=0,\\ W_t+AW+\int_0^\infty\nu(s)A\bar\zeta^t(s)ds+B(u_1,W)-B(w_2,U)-B(W,u_1)+\lambda W=0,\\ \bar\eta_t^t+\bar\eta_s^t=U,\\ \bar\zeta_t^t+\bar\zeta_s^t=W.\end{cases}\tag{3.23}$$

Taking the inner product of the first equation of (4.1) in $L^2(\Omega)$ by U and the second one by W, we obtain

$$\frac12\frac{d}{dt}\left(\|U\|^2+\alpha^2\|\nabla U\|^2+\int_0^\infty\mu(s)\|\nabla\bar\eta^t(s)\|^2ds\right)+\|\nabla U\|^2-\int_0^\infty\mu'(s)\|\nabla\bar\eta^t(s)\|^2ds=-\langle P(W\times u_2),U\rangle,$$
$$\frac12\frac{d}{dt}\left(\|W\|^2+\int_0^\infty\nu(s)\|\nabla\bar\zeta^t(s)\|^2ds\right)+\|\nabla W\|^2+\lambda\|W\|^2-\int_0^\infty\nu'(s)\|\nabla\bar\zeta^t(s)\|^2ds$$
$$=b(W,u_1,W)+b(w_2,U,W)-b(U,w_2,W),$$

where $\langle P(w_1\times U),U\rangle=0$ and $U=\bar\eta_t^t+\bar\eta_s^t,\quad W=\bar\zeta_t^t+\bar\zeta_s^t$.
From the boundedness of $\|u_2\|$ and $\|\nabla w_2\|$, we deduce that there exists a constant C_1, such that $\|\nabla u_1\|,\|u_2\|,\|\nabla w_2\|\le C_1$. Therefore, using (2.5), (2.4) and Holder's inequality, Young's inequality, we get

$$\begin{aligned}|\langle P(W\times u_2),U\rangle|&\le\int_\Omega|W||u_2||U|dx\le\|W\|_{L^3}\|u_2\|\|U\|_{L^6}\\&\le C_1\|W\|^{\frac12}\|\nabla W\|^{\frac12}\|\nabla U\|\\&\le C_1\|W\|\|\nabla W\|+\frac{C_1}{4}\|\nabla U\|^2\\&\le C_1\|W\|^2+\frac{C_1}{4}\|\nabla W\|^2+\frac{C_1}{4}\|\nabla U\|^2,\end{aligned}$$

and

$$\begin{aligned}|b(w_2,U,W)-b(U,w_2,W)|&\le C\|\nabla U\|\|\nabla w_2\|\|W\|^{\frac12}\|\nabla W\|^{\frac12}\\&\le C_1\|\nabla U\|\|W\|^{\frac{C_1}{2}}\|\nabla W\|^{\frac12}\\&\le C_1\|W\|\|\nabla W\|+\frac{C_1}{4}\|\nabla U\|^2\\&\le C_1\|W\|^2+\frac{C_1}{4}\|\nabla W\|^2+\frac{C_1}{4}\|\nabla U\|^2,\end{aligned}$$

and

$$\begin{aligned}|b(W,u_1,W)|&\le C\|\nabla u_1\|\|W\|^{\frac12}\|\nabla W\|^{\frac32}\\&\le C_1\|W\|^{\frac12}\|\nabla W\|^{\frac32}\\&\le C_1\|W\|^2+\frac{C_1}{4}\|\nabla W\|^2.\end{aligned}$$

Combining the above three estimates, it yields

$$\frac12\frac{d}{dt}\left(\|U\|^2+\alpha^2\|\nabla U\|^2+\int_0^\infty\mu(s)\|\nabla\bar\eta^t(s)\|^2ds\right)+\|\nabla U\|^2\le C_1\|W\|^2+\frac{C_1}{4}\|\nabla W\|^2+\frac{C_1}{4}\|\nabla U\|^2,\tag{3.24}$$

$$\frac12\frac{d}{dt}\left(\|W\|^2+\int_0^\infty\nu(s)\|\nabla\bar\zeta^t(s)\|^2ds\right)+\|\nabla W\|^2+\lambda\|W\|^2-\int_0^\infty\nu'(s)\|\nabla\bar\zeta^t(s)\|^2ds$$
$$\le C_1\|W\|^2+\frac{C_1}{2}\|\nabla W\|^2+\frac{C_1}{4}\|\nabla U\|^2,\tag{3.25}$$

where $-\int_0^\infty \mu'(s)\|\nabla\bar{\eta}^t(s)\|^2 ds \geq 0$ and $-\int_0^\infty \nu'(s)\|\nabla\bar{\zeta}^t(s)\|^2 ds \geq 0$.
Adding (4.2) and (3.25) gives

$$\frac{1}{2}\frac{d}{dt}\left(\|U\|_{V_1}^2 + \|W\|^2 + \int_0^\infty \mu(s)\|\nabla\bar{\eta}^t(s)\|^2 ds + \int_0^\infty \nu(s)\|\nabla\bar{\zeta}^t(s)\|^2 ds\right) \leqslant 2C_1\left(\|W\|^2 + \|\nabla W\|^2 + \|\nabla U\|^2\right).$$

Thus,

$$\frac{d}{dt}y(t) \leqslant 4C_1 y(t),$$

where $y(t) = \|U\|_{V_1}^2 + \|W\|^2 + \int_0^\infty \mu(s)\|\nabla\bar{\eta}^t(s)\|^2 ds + \int_0^\infty \nu(s)\|\nabla\bar{\zeta}^t(s)\|^2 ds$.
By applying the Gronwall inequality, we obtain

$$y(t) \leq e^{4C_1 T} y(\tau).$$

Therefore,

$$\begin{aligned}
&\|U\|_{V_1}^2 + \|W\|^2 + \int_0^\infty \mu(s)\|\nabla\bar{\eta}^t(s)\|^2 ds + \int_0^\infty \nu(s)\|\nabla\bar{\zeta}^t(s)\|^2 ds \\
&\leqslant e^{2C_1 T}\left(\|U(\tau)\|_{V_1}^2 + \|W(\tau)\|^2 + \int_0^\infty \mu(s)\|\nabla\bar{\eta}^\tau(s)\|^2 ds + \int_0^\infty \nu(s)\|\nabla\bar{\zeta}^\tau(s)\|^2 ds\right) \\
&\leqslant \frac{1}{\alpha^2} e^{2C_1 T}\left(\|U(\tau)\|_{V_1}^2 + \|W(\tau)\|^2 + \|\bar{\eta}^\tau(s)\|_{1,\mu}^2 + \|\bar{\zeta}^\tau(s)\|_{1,\nu}^2\right).
\end{aligned}$$

This proves the uniqueness (when $z_1(\tau) = z_2(\tau)$) and the continous dependence on the initial data of the weak solution. This completes the proof.

□

4. Existence of a uniform attractor

Theorem 3.1 allows us to define a family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(f)}$ as follows

$$U_\sigma(t,\tau): \mathcal{H}_1 \to \mathcal{H}_1,$$

where $U_\sigma(t,\tau)z_\tau$ is the unique weak solution of (1.1) (with σ in place of f) at the time t with the initial datum z_τ at τ.

The aim of this section is to prove the following theorem.

Theorem 4.1. *Assume that conditions (F) and (M) hold. Then the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(f)}$, corresponding to problem (2.6) in $\mathcal{H}_1$ has a compact uniform (w.r.t $\sigma \in \mathcal{H}_w(f)$) attractor $\mathcal{A}^\varepsilon$ in the space $\mathcal{H}_1$. Moreover,*

$$\mathcal{A} = \bigcup_{\sigma\in\mathcal{H}_w(f)} \mathcal{K}_\sigma(s), \quad \forall s \in \mathbb{R},$$

where $\mathcal{K}_\sigma(s)$ is the kernel section at time s of the process $U_\sigma(t,\tau)$.

For proving this theorem, we need to show that the processes $U_\sigma(t,\tau)$ has a uniform absorbing set B_0 in $\mathcal{H}_1$ and $U_\sigma(t,\tau)$ is uniform asymptotically compact in $\mathcal{H}_1$.

4.1. **Existence of a uniform absorbing set.** Now, we prove the existence of a uniform absorbing set for the family of processes $\{U_\sigma(t,\tau)\}_{\sigma\in\mathcal{H}_w(f)}$.

Lemma 4.2 (Bounded uniformly absobing set)**.** *Let (F) and (M) hold. Then there exists a positive constant C_5 which only depends on $\|f\|_{L_b^2}$ and the constant α, such that for any bounded subset B in $\mathcal{H}_1$, there is a $t_3^* = t(\|B\|_{\mathcal{H}_1})$,*

$$\|U_\sigma(t,\tau)z_\tau\|_{\mathcal{H}_1}^2 \leq C_5, \quad \textit{for } t-\tau \geq t_3^*, z_\tau \in B, \sigma \in \mathcal{H}_w(f).$$

Moreover, there exists a positive constant $C_6 > 0$ which depends on $\|f\|_{L_b^2}$, such that the following inequality holds

$$\int_t^{t+1} \|\nabla w\|^2 \leq C_6.$$

Proof. Multiplying the first equation of (2.7) with u and integrating by parts over Ω, we can obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|u\|^2+\alpha^2\|\nabla u\|^2+\int_0^\infty \mu(s)\|\nabla\eta^t(s)\|^2ds\right)+\|\nabla u\|^2-\int_0^\infty \mu'(s)\|\nabla\eta^t(s)\|^2ds=\langle P\sigma,u\rangle_{V^{-1},V}.$$

Thus,

$$\frac{d}{dt}\left(\|u\|^2+\alpha^2\|\nabla u\|^2+\int_0^\infty \mu(s)\|\nabla\eta^t(s)\|^2ds\right)+\|\nabla u\|^2\leq\frac{1}{\lambda_1}\|P\sigma\|^2_{V^{-1}} \tag{4.1}$$

where $-\int_0^\infty \mu'(s)\|\nabla\eta^t(s)\|^2ds\geq 0$ and

$$|\langle P\sigma,u\rangle|\leqslant\|P\sigma\|_{V^{-1}}\|\nabla u\|\leqslant\frac{1}{2}\|\nabla u\|^2+\frac{1}{2}\|P\sigma\|^2_{V^{-1}}.$$

Now, for $\gamma_1>0$ to be fixed later, we define the functional

$$\Lambda_1(t)=E_1+4\gamma_1\int_0^\infty \kappa(s)\|\nabla\eta^t(s)\|^2ds,$$

where $E_1(t)=\|u\|^2+\alpha^2\|\nabla u\|^2+\int_0^\infty \mu(s)\|\nabla\eta^t(s)\|^2ds$.

From (4.1) and (4.2), we get

$$\frac{d}{dt}\Lambda_1(t)+\left(1-8\gamma_1\theta_1^2\kappa(0)\right)\|\nabla u\|^2+2\gamma_1\int_0^\infty \mu(s)\|\nabla\eta^t(s)\|^2ds\leq\|P\sigma\|^2_{V^{-1}}.$$

Choosing $\gamma_1>0$ small enough such that

$$2\gamma_1\left(\frac{1}{\lambda_1}+\alpha^2\right)\leq 1-8\gamma_1\theta_1^2\kappa(0),$$

we have

$$\frac{d}{dt}\Lambda_1(t)+2\gamma_1E_1(t)\leq\frac{1}{\lambda_1}\|P\sigma\|^2.$$

Using (2.10),(2.12) and choosing γ_1 small enough, we have

$$E_1(t)\leq\Lambda_2(t)\leq 2E_1 \tag{4.2}$$

thus

$$\frac{d}{dt}\Lambda_1(t)+\gamma_1\Lambda_1(t)\leq\|P\sigma\|^2_{V^{-1}}.$$

Applying the Gronwall inequality, we obtain

$$\Lambda_1(t)\leq\Lambda_1(\tau)e^{-\gamma_1(t-\tau)}+\int_\tau^t e^{-\gamma_1(t-r)}\|P\sigma(r)\|^2_{V^{-1}}dr. \tag{4.3}$$

Besides,

$$\begin{aligned}\int_\tau^t e^{-\gamma_1(t-r)}\|P\sigma(r)\|^2_{V^{-1}}dr&\leq\left(\int_{t-1}^t e^{-\gamma_1(t-r)}\|P\sigma(r)\|^2_{V^{-1}}ds+\int_{t-2}^{t-1} e^{-\gamma_1(t-r)}\|P\sigma(r)\|^2_{V^{-1}}dr+\dots\right)\\&\leq\left(1+e^{-\gamma_1}+e^{-2\gamma_1}+\dots\right)\|P\sigma\|^2_{L^2_b}\leq\frac{1}{1-e^{-\gamma_1}}\|Pf\|^2_{L^2_b},\end{aligned} \tag{4.4}$$

where we have used the fact that $\|P\sigma\|^2_{L^2_b}\leq\|Pf\|^2_{L^2_b}$ for all $P\sigma\in\mathcal{H}_w(Pf)$.
Combining (4.2), (4.3) and (4.18), we get

$$\|u\|^2_{V_1}+\|\eta^t(s)\|^2_{1,\mu}\leq\Lambda_1(t)\leq 2\left(\|u_\tau\|^2_{V_1}+\|\eta_\tau(s)\|^2_{1,\mu}\right)e^{-\gamma_1(t-\tau)}+\frac{1}{\lambda_1\left(1-e^{-\gamma_1}\right)}\|Pf\|^2_{L^2_b}. \tag{4.5}$$

Obviously, $\lim\limits_{t\to+\infty}e^{-\gamma_1(t-\tau)}=0$, there exists a $t_1^*=t_1^*(\|u_0\|_{V_1},\|\eta_\tau(s)\|_{1,\mu})>0$ such that

$$\|u\|^2_{V_1}+\|\eta^t\|^2_{1,\mu}\leq C\|Pf\|^2_{L^2_b}=C_2(\|f\|,\alpha),\ \forall t\geqslant t_1^*. \tag{4.6}$$

Now, we give the proof of the boundedness of $\|w\|$ and $\|\zeta^t\|_{1,\nu}$.
Multiplying the second equation of (2.7) with w and integrate by parts over Ω to obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|^2+\int_0^\infty \nu(s)\|\nabla\zeta^t(s)\|^2ds\right)+\|\nabla w\|^2+\lambda\|w\|^2-\int_0^\infty \nu'(s)\|\nabla\zeta^t(s)\|^2ds$$

$$= b(w,u,w) + \langle \nabla \times (P\sigma), w\rangle. \tag{4.7}$$

When $t \geqslant t_1^*$, by using the Holder and Young inequalities, and (2.4), (4.6), we get

$$|\langle \nabla \times (P\sigma), w\rangle| \leqslant \|P\sigma\|_{V^{-1}}\|\nabla w\| \leqslant 2\|P\sigma\|_{V^{-1}}^2 + \frac{1}{8}\|\nabla w\|^2,$$

and

$$\begin{aligned}|b(w,u,w)| &\leqslant C\|\nabla u\|\|\nabla w\|^{\frac{3}{2}}\|w\|^{\frac{1}{2}} \leqslant \frac{3}{4}\|\nabla w\|^2 + \frac{C^4\|\nabla u\|^4}{4}\|w\|^2\\ &\leqslant \frac{3}{4}\|\nabla w\|^2 + \frac{C^4\|u\|_{V_1}^4}{4\alpha^4}\|w\|^2 \leqslant \frac{3}{4}\|\nabla w\|^2 + C_3\|w\|^2,\end{aligned}$$

where $C, C_3 = \dfrac{C^4C_2^2}{4\alpha^4}$ are positive constants.
Besides, since the term $-\int_0^\infty \nu'(s)\|\nabla\zeta^t(s)\|^2 ds$ in (4.7) is positive, it can be neglected.
Combining (4.7) and the above inequalities, we obtain

$$\frac{1}{2}\frac{d}{dt}\left(\|w\|^2 + \int_0^\infty \nu(s)\|\nabla\zeta^t(s)\|^2 ds\right) + \frac{1}{4}\|\nabla w\|^2 + C_4\|w\|^2 \leqslant 4\|P\sigma\|_{V^{-1}}^2,\ \forall t \geqslant t_1^*, \tag{4.8}$$

where $C_4 = 2(\lambda - C_3) > 0$. For $\gamma_2 > 0$ to be fixed later, we define the functional

$$\Lambda_2(t) = \|w\|^2 + \|\zeta^t\|_{1,\nu}^2 + 4\gamma_2\int_0^\infty l(s)\|\nabla\zeta^t(s)\|^2 ds.$$

Using (2.11),(2.13) and choosing γ_2 small enough, we have

$$\|w\|^2 + \|\zeta^t\|_{1,\nu}^2 \leq \Lambda_2(t) \leq 2\left(\|w\|^2 + \|\zeta^t\|_{1,\nu}^2\right). \tag{4.9}$$

From (4.8) and (4.9), we get

$$\frac{d}{dt}\Lambda_2(t) + \left(\frac{1}{2} - 8\gamma_2\theta_2^2 l(0)\right)\|\nabla w\|^2 + C_4\|w\|^2 + 2\gamma_2\|\zeta^t\|_{1,\nu}^2 \leq 4\|P\sigma\|_{V^{-1}}^2. \tag{4.10}$$

Choosing $\gamma_2 > 0$ small enough such that $\frac{1}{2} - 8\gamma_2\theta_2^2 l(0) > 0$ and putting $\gamma_3 = \min\{\gamma_2, C_3\}$, we have

$$\frac{d}{dt}\Lambda_2(t) + \gamma_3\Lambda_2(t) \leq 4\|Pf\|_{V^{-1}}^2.$$

Applying the Gronwall inequality and using (4.18), (4.9), we obtain

$$\begin{aligned}\|w\|^2 + \|\zeta^t\|_{1,\nu}^2 &\leqslant 2\left(\|w_\tau\|^2 + \|\zeta^t\|_{1,\nu}^2\right)e^{-\gamma_3(t-\tau)} + 4\int_\tau^t e^{-\gamma_1(t-r)}\|P\sigma(r)\|_{V^{-1}}^2 dr\\ &\leqslant 2\left(\|w_\tau\|^2 + \|\zeta_\tau\|_{1,\nu}^2\right)e^{-\gamma_3(t-\tau)} + \frac{4}{1-e^{-\gamma_3}}\|Pf\|_{L_b^2}^2.\end{aligned}$$

Therefore, there exists a $t_2^* = t_2^*\left(\|(u_\tau, w_\tau, \eta_\tau, \zeta_\tau)\|_{\mathcal{H}_1}\right) > t_1^*$, such that

$$\|w\|^2 + \|\zeta^t\|_{1,\nu}^2 \leq \frac{8}{1-e^{-\gamma_3}}\|Pf\|_{L_b^2}^2 = C_4,\ \forall t \geqslant t_2^*. \tag{4.11}$$

Combining (4.6) and (4.11), we immediately deduce that

$$\|u(t)\|_{V_1}^2 + \|\eta^t\|_{1,\mu}^2 + \|w(t)\|^2 + \|\zeta^t\|_{1,\nu}^2 \leq C_2 + C_4,$$

or

$$\|z(t)\|_{\mathcal{H}_1}^2 \leq C_5, \quad C_5 = C_2 + C_4, \tag{4.12}$$

for all $z_\tau \in B, \sigma \in \mathcal{H}_w(f)$ and for all $t \geq t_2^*$, where B is an arbitrary bounded subset of $\mathcal{H}_1$.

On the other hand, integrating (4.8) between t and $t+1$, we obtain

$$\int_t^{t+1}\|\nabla w\|^2 \leq \frac{C_4}{\frac{1}{2} - 8\gamma_2\theta_2^2 l(0)} = C_6. \tag{4.13}$$

This completes the proof $\square$

4.2. **Asymptotic compactness.** The main difficulty of the problem is, of course, that the embeddings are no longer compact. In order to get the asymptotic compactness of the solution, we are going to take advantage of the method of the semigroup decomposition.

Recall that in this paper we only assume the external force $f \in V^{-1}$. However, we know that for any $f \in V^{-1}$ and $\varepsilon > 0$ given, there is a $f^\varepsilon \in H$, which depends on f and ε, such that

$$\|f - f^\varepsilon\|_{V^{-1}} < \varepsilon. \tag{4.14}$$

4.2.1. *Decomposition of the equation.* It is convenient to make asymptotic estimates when we decompose the solution $U_\sigma(t,\tau)z_\tau = z(t)$ of problem (4.2) (with σ in place of f) as

$$U_\sigma(t,\tau)z_\tau = D(t,\tau)z_\tau + K_\sigma(t,\tau)z_\tau,$$

where $D(t,\tau)z_\tau = z_1(t)$ and $K_\sigma(t,\tau)z_\tau = z_2(t)$, that is, $z = (u,\eta^t,w,\zeta^t) = z_1 + z_2$, $z_1 = (v^\varepsilon, \eta^{t\varepsilon}, r^\varepsilon, \zeta_1^{t\varepsilon})$, $z_2 = (\bar{v}^\varepsilon, \eta_2^{t\varepsilon}, \bar{r}^\varepsilon, \zeta_2^{t\varepsilon})$. For convenience, we still denote $z_1 = (v, \eta_1^t, r, \zeta_1^t)$, $z_2 = (\bar{v}, \eta_2^t, \bar{r}, \zeta_2^t)$. Besides, $D(t,\tau)z_\tau$ solves the following equation with initial data $z_1(\tau) = z_\tau$,

$$\begin{cases} v_t + \alpha^2 A v_t + Av + \int_0^\infty \mu(s) A\eta_1^t(s)ds + P(w \times v) & = P\sigma - P\sigma^\varepsilon, \\ r_t + Ar + B(u,r) - B(w,v) + \lambda r + \int_0^\infty \nu(s) A\zeta_1^t(s)ds & = 0, \\ \partial_t \eta_1^t + \partial_s \eta_1^t = v, \quad \partial_t \zeta_1^t + \partial_s \zeta_1^t = r \end{cases} \tag{4.15}$$

and $K_\sigma(t)z_\tau$ solves the following equation with initial data $z_2(\tau) = 0$

$$\begin{cases} \bar{v}_t + \alpha^2 A \bar{v}_t + A\bar{v} + \int_0^\infty \mu(s) A\eta_2^t(s)ds + P(w \times \bar{v}) & = P\sigma^\varepsilon, \\ \bar{r}_t + A\bar{r} + B(u,\bar{r}) - B(w,\bar{v}) + \lambda \bar{r} + \int_0^\infty \nu(s) A\zeta_2^t(s)ds & = \nabla \times (P\sigma), \\ \partial_t \eta_2^t + \partial_s \eta_2^t = \bar{v}, \quad \partial_t \zeta_2^t + \partial_s \zeta_2^t = \bar{r}. \end{cases} \tag{4.16}$$

By the standard Galerkin method, one can prove the existence and uniqueness of solutions to problems (4.15) and (4.16). Besides, for problem (4.16), because the external force $f^\varepsilon \in H$ and the initial data are zero (so belong to $\mathcal{H}_2$), we can show that the solution $z_2 = (\bar{v}, \eta_2^t, \bar{r}, \zeta_2^t)$ is in fact a strong solution. In particular, we will have $z_2 \in \mathcal{H}_2$ for any $T > 0$ and this will be used in the proof of Lemma 4.4 below.

We now prove some estimates for solutions of these two systems.

Lemma 4.3. *Assume that hypotheses* **(M)-(F)** *hold. Then the solutions of equation* (4.15) *satisfy the following estimate: there is a constant $\gamma_6 > 0$ and there exist $t_3^* > t_2^* > \tau$ large enough, such that*

$$\|D(t,\tau)z_\tau\|_{\mathcal{H}_1}^2 \le C\|z_(\tau)\|_{\mathcal{H}_1}^2 e^{-\gamma_6(t-\tau)} + \varepsilon, \text{ for all } t \ge t_3^*.$$

Proof. Multiplying the first equation of (4.15) by v, we have

$$\begin{aligned} \frac{1}{2}\frac{d}{dt}(\|v\|^2 + \alpha^2\|\nabla v\|^2 + \|\eta_1^t\|_{1,\mu}^2) + \|\nabla v(t)\|^2 - \int_0^\infty \mu'(s)\|\nabla \eta_1^t(s)\|^2 ds &= \langle P\sigma - P\sigma^\varepsilon, v\rangle_{V^{-1},V} \\ &\le \|P\sigma - P\sigma^\varepsilon\|_{V^{-1}}^2 + \frac{1}{4}\|\nabla v(t)\|^2. \end{aligned}$$

Now, for $\gamma_3 > 0$ to be fixed later, we define the functional

$$\Lambda_3(t) = E_3 + 4\gamma_4 \int_0^\infty \kappa(s)\|\nabla \eta_1^t(s)\|^2 ds, \quad E_3(t) = \|v\|^2 + \alpha^2\|\nabla v\|^2 + \|\eta_1^t\|_{1,\mu}^2.$$

Using (2.10),(2.12) and choosing γ_4 small enough, we have

$$E_3(t) \le \Lambda_3(t) \le 2E_3.$$

From (4.1) and (4.2), we get

$$\frac{d}{dt}\Lambda_3(t) + \left(1 - 8\gamma_4\theta_1^2\kappa(0)\right)\|\nabla v\|^2 + 2\gamma_4 \int_0^\infty \mu(s)\|\nabla \eta_1^t(s)\|^2 ds \le \|P\sigma - P\sigma^\varepsilon\|_{V^{-1}}^2.$$

Choosing $\gamma_4 > 0$ small enough such that

$$2\gamma_4\left(\frac{1}{\lambda_1} + \alpha^2\right) \le 1 - 8\gamma_4\theta_1^2\kappa(0),$$

we have

$$\frac{d}{dt}\Lambda_3(t) + 2\gamma_4 E_3(t) \leq \|P\sigma - P\sigma^\varepsilon\|^2_{V^{-1}},$$

thus

$$\frac{d}{dt}\Lambda_3(t) + \gamma_4 \Lambda_3(t) \leq \|P\sigma - P\sigma^\varepsilon\|^2_{V^{-1}},$$

And then, we can get the following estimate by Gronwall inequality

$$\|v\|^2 + \alpha^2\|\nabla v\|^2 + \|\eta_1^t\|^2_{1,\mu} \leq \Lambda_3(t) \leq C\left(\|u_\tau\|^2_{V_1} + \|\eta_\tau\|^2_{1,\mu}\right)e^{-\gamma_4(t-\tau)} + C\varepsilon^2. \tag{4.17}$$

Now we consider the $\|r(t)\|^2 + \|\zeta_1^t\|^2_{1,\nu}$. Taking the inner product of the second equation of (4.15) with r and then using (2.3) and (4.17), we have

$$\frac{1}{2}\frac{d}{dt}(\|r\|^2 + \|\zeta_1^t\|^2_{1,\nu}) + \|\nabla r\|^2 + \lambda\|r\|^2 - \int_0^\infty \mu'(s)\|\nabla\eta_1^t(s)\|^2 ds = b(w, v, r).$$

Using (2.3), (4.17) and Young inequality, we have

$$\begin{aligned} b(w,v,r) &\leq C\|v\|^{\frac{1}{2}}\|\nabla v\|^{\frac{1}{2}}\|\nabla w\|\|\nabla r\| \\ &\leq C\|\nabla v\|^2\|\nabla w\|^2 + \frac{1}{2}\|\nabla r\|^2 \\ &\leq \left(C\left(\|u_\tau\|^2_{V_1} + \|\eta_\tau\|^2_{1,\mu}\right)e^{-\gamma_4(t-\tau)} + C\varepsilon^2\right)\|\nabla w\|^2 + \frac{1}{2}\|\nabla r\|^2. \end{aligned}$$

Therefore,

$$\frac{d}{dt}(\|r\|^2 + \|\zeta_1^t\|^2_{1,\nu}) + \|\nabla r\|^2 + 2\lambda\|r\|^2 \leq \left(C\left(\|u_\tau\|^2_{V_1} + \|\eta_\tau\|^2_{1,\mu}\right)e^{-\gamma_4(t-\tau)} + C\varepsilon^2\right)\|\nabla w\|^2.$$

Now, for $\gamma_5 > 0$ to be fixed later, we define the functional

$$\Lambda_4(t) = \|r\|^2 + \|\zeta_1^t\|^2_{1,\nu} + 4\gamma_5\int_0^\infty l(s)\|\nabla\zeta_1^t(s)\|^2 ds.$$

Using (2.10),(2.12) and choosing γ_5 small enough, we have

$$\|r\|^2 + \|\zeta_1^t\|^2_{1,\nu} \leq \Lambda_3(t) \leq 2\left(\|r\|^2 + \|\zeta_1^t\|^2_{1,\nu}\right).$$

Thus, we get

$$\begin{aligned} \frac{d}{dt}\Lambda_4(t) + \left(1 - 8\gamma_5\theta_2^2 l(0)\right)\|\nabla r\|^2 + 2\lambda\|r\|^2 + 2\gamma_5\int_0^\infty \mu(s)\|\nabla\eta_1^t(s)\|^2 ds \\ \leq \left(C\|z_\tau\|^2_{\mathcal{H}_1}e^{-\gamma_4(t-\tau)} + C\varepsilon^2\right)\|\nabla w\|^2. \end{aligned}$$

Choosing $\gamma_5 > 0$ small enough such that

$$1 - 8\gamma_5\theta_2^2 l(0) > 0 \quad \text{and} \quad \gamma_5 \leq \lambda,$$

we have

$$\frac{d}{dt}\Lambda_4(t) + 2\gamma_5\left(\|r\|^2 + \|\zeta_1^t\|^2_{1,\nu}\right) \leq \left(C\|z_\tau\|^2_{\mathcal{H}_1}e^{-\gamma_4(t-\tau)} + C\varepsilon^2\right)\|\nabla w\|^2,$$

thus

$$\frac{d}{dt}\Lambda_4(t) + \gamma_5\Lambda_4(t) \leq \left(C\|z_\tau\|^2_{\mathcal{H}_1}e^{-\gamma_4(t-\tau)} + C\varepsilon^2\right)\|\nabla w\|^2,$$

Applying the Gronwall inequality, we deduce that

$$\begin{aligned} \|r\|^2 + \|\zeta_1^t\|^2_{1,\nu} \leq \Lambda_4(t) \leq C\|z_\tau\|^2_{\mathcal{H}_1}e^{-\gamma_5(t-\tau)} + C\|z_\tau\|^2_{\mathcal{H}_1}\int_\tau^t e^{-\gamma_5(t-s)}e^{-\gamma_4(s-\tau)}\|\nabla w(s)\|^2 ds \\ + C\varepsilon^2\int_\tau^t e^{-\gamma_5(t-r)}\|\nabla w(r)\|^2 dr. \end{aligned}$$

On the other hand, using (4.13), we have

$$\int_\tau^t e^{-\gamma_5(t-s)}e^{-\gamma_4(s-\tau)}\|\nabla w(s)\|^2 ds$$

$$
\begin{aligned}
&\le e^{-\gamma_4(t-\tau)}\int_\tau^t e^{-\gamma_5(t-s)}\|\nabla w(s)\|^2ds\\
&\le e^{-\gamma_4(t-\tau)}\left(\int_{t-1}^t e^{-\gamma_5(t-s)}\|\nabla w(s)\|^2ds+\int_{t-2}^{t-1} e^{-\gamma_5(t-s)}\|\nabla w(s)\|^2ds+\dots\right)\\
&\le e^{-\gamma_4(t-\tau)}\left(1+e^{-\gamma_5}+e^{-2\gamma_5}+\dots\right)C_6\\
&\le \frac{C_6}{1-e^{-\gamma_5}}e^{-\gamma_4(t-\tau)},
\end{aligned}
$$

and

$$
\int_\tau^t e^{-\gamma_5(t-r)}C\varepsilon^2\|\nabla w(r)\|^2dr\le\frac{CC_6\varepsilon^2}{1-e^{-\gamma_1}}. \tag{4.18}
$$

Therefore,

$$
\|r\|^2+\|\zeta_1^t\|_{1,\nu}^2\le C\|z_\tau\|_{\mathcal{H}_1}^2e^{-\gamma_5(t-\tau)}+\frac{CC_6\|z_\tau\|_{\mathcal{H}_1}^2}{1-e^{-\gamma_5}}e^{-\gamma_4(t-\tau)}+C\varepsilon^2. \tag{4.19}
$$

Combining (4.17) and (4.19) and taking $2C\varepsilon^2\le\varepsilon$, we can conclude that

$$
\|D(t,\tau)z_\tau\|_{\mathcal{H}_1}^2\le C\|z_\tau\|_{\mathcal{H}_1}^2e^{-\gamma_6(t-\tau)}+\varepsilon.
$$

Next we concern the solution operator $K(t)$. We will show that, for every fixed time, as (u_0,w_0) belongs to the absorbing set $\mathbb{B}_\beta$, the component related to $K(t)u_0$ belongs to a compact subset of V_1. $\square$

Lemma 4.4. *Let (F) and (M) hold. Then, for any $\varepsilon>0$, $\alpha\in(0,1]$, $z_\tau\in\mathcal{H}_1$, there exists $M>0, t_3^*>\tau$ large enough, which depends on $\|z_\tau\|_{\mathcal{H}_1}^2, \|Pf^\varepsilon\|_{L_b^2}, \|\nabla\times Pf\|_{L_b^2}$, such that*

$$
\|K_\sigma(t,\tau)z_\tau\|_{\mathcal{H}_2}^2\le M \quad \text{for all } t\ge t_3^*. \tag{4.20}
$$

Proof. Taking the second equation of (4.16) by $A\bar v$ and integrating by parts over ω to obtain

$$
\begin{aligned}
&\frac12\frac{d}{dt}\left(\|\nabla\bar v\|^2+\alpha^2\|A\bar v\|^2+\int_0^\infty\mu(s)\|A\eta_2^t(s)\|^2ds\right)\\
&\qquad+\|A\bar v\|^2-\int_0^\infty\mu'(s)\|A\eta_2^t(s)\|^2ds+\langle P(w\times\bar v),A\bar v\rangle=\langle P\sigma^\varepsilon,A\bar v\rangle.
\end{aligned}
$$

Using Holder and Young inequalities, we get

$$
\langle P\sigma^\varepsilon,A\bar v\rangle\le\|P\sigma^\varepsilon\|^2+\frac14\|A\bar v\|^2.
$$

By the boundedness of $\|\nabla u\|$ and $\|\nabla v\|$, we can deduce that $\|\nabla\bar v\|$ is bounded. Using Holder, Young and Agmon inequalities, as well as the boundedness of $\|\nabla\bar v\|$ and $\|w\|$, we get

$$
\begin{aligned}
|\langle P(w\times\bar v),A\bar v\rangle|&\le\|w\|\|\bar v\|_{L^\infty}\|A\bar v\|\\
&\le C\|w\|\|\nabla\bar v\|^{\frac12}\|A\bar v\|^{\frac32}\\
&\le C\|\nabla\bar v\|^{\frac12}\|A\bar v\|^{\frac32}\\
&\le C\|\nabla\bar v\|^2+\frac14\|A\bar v\|^2\\
&\le C+\frac14\|A\bar v\|^2 \text{ for all } t\ge t_2^*.
\end{aligned}
$$

Combining the above inequalities and $-\int_0^\infty\mu'(s)\|A\eta_2^t(s)\|^2ds\ge0$, we have

$$
\frac{d}{dt}\left(\|\nabla\bar v\|^2+\alpha^2\|A\bar v\|^2+\int_0^\infty\mu(s)\|A\eta_2^t(s)\|^2ds\right)+\|A\bar v\|^2\le2\|P\sigma^\varepsilon\|^2+C.
$$

Now, for $\gamma_6>0$ to be fixed later, we also define the functional

$$
\Lambda_5(t)=E_5(t)+4\gamma_6\int_0^\infty\kappa(s)\|A\eta_2^t(s)\|^2ds,\quad E_5(t)=\|\bar v\|_{V_2}^2+\|\eta_2^t\|_{2,\mu}^2.
$$

Thus, we obtain

$$\frac{d}{dt}\Lambda_5(t)+\left(1-8\gamma_6\theta_1^2\kappa(0)\right)\|A\bar v\|^2+2\gamma_6\int_0^\infty\mu(s)\|\nabla\eta^t(s)\|^2ds\le 2\|P\sigma^\varepsilon\|^2+C.$$

Choosing γ_5 small enough such that $\frac{2\gamma_6}{k_\alpha}\le 1-8\gamma_6\theta_1^2\kappa(0)$ and $E_5\le\Lambda_5(t)\le 2E_5$, then using (2.9), we can deduce that

$$\frac{d}{dt}\Lambda_5(t)+2\gamma_6E_5(t)\le 2\|P\sigma^\varepsilon\|^2+C,$$

thus

$$\frac{d}{dt}\Lambda_5(t)+\gamma_6\Lambda_5(t)\le 2\|P\sigma^\varepsilon\|^2+C. \tag{4.21}$$

By the Gronwall inequality, we obtain

$$\Lambda_5(t)\le\frac{1}{(1-e^{-\gamma_6})}\|Pf^\varepsilon\|^2_{L^2_b}+C,$$

thus

$$\|\bar v\|^2_{V_2}+\|\eta_2^t\|^2_{2,\mu}\le M_1,\quad \text{where } M_1=\frac{1}{(1-e^{-\gamma_6})}\|Pf^\varepsilon\|^2_{L^2_b}+C \text{ and } t\ge t_2^*. \tag{4.22}$$

Next, we will show the boundedness of $\|\nabla\bar r\|$ and $\|\zeta_2^t\|_{2,\nu}$. Multiplying the second equation of (4.16) by $A\bar r$ and integrating by parts over ω, we obtain

$$\begin{aligned}\frac12\frac{d}{dt}\left(\|\nabla\bar r\|^2+\int_0^\infty\nu(s)\|A\zeta_2^t(s)\|^2ds\right)+\|A\bar r\|^2+\lambda\|\nabla\bar r\|^2-\int_0^\infty\mu'(s)\|A\eta_2^t(s)\|^2ds\\ =\langle\nabla\times(P\sigma),A\bar r\rangle+b(w,\bar v,A\bar r)-b(u,\bar r,A\bar r).\end{aligned}$$

Now we deal with the three terms on the right hand side of above equality as follows

$$|\langle\nabla\times(P\sigma),A\bar r\rangle|\le\|\nabla\times(P\sigma)\|^2+\frac14\|A\bar r\|^2;$$

$$\begin{aligned}|b(w,\bar v,A\bar r)|&\le\|w\|_{L^6}\|\nabla\bar v\|_{L^3}\|A\bar r\|\\ &\le C\|\nabla w\|\|A\bar v\|\|A\bar r\|\\ &\le CM_1\|\nabla w\|\|A\bar r\|\\ &\le M_2\|\nabla w\|^2+\frac14\|A\bar r\|^2,\end{aligned}$$

where we have used the Holder and Young inequalities and (4.22);

$$\begin{aligned}|b(u,\bar r,A\bar r)|&\le\|u\|_{L^6}\|\nabla\bar r\|_{L^3}\|A\bar r\|\\ &\le C\|\nabla u\|\|\nabla\bar r\|^{\frac12}\|A\bar r\|^{\frac32}\\ &\le CC_5\|\nabla\bar r\|^{\frac12}\|A\bar r\|^{\frac32}\\ &\le M_3\|\nabla\bar r\|^2+\frac14\|A\bar r\|^2,\end{aligned}$$

where we have also used the Holder and Young inequalities, $\|\nabla\bar r\|_{L^3}\le C\|\nabla\bar r\|^{\frac12}\|A\bar r\|^{\frac12}$, and (4.12). Combining the above inequalities and the term $-\int_0^\infty\mu'(s)\|A\eta_2^t(s)\|^2ds\ge 0$ can be neglected, we have

$$\frac{d}{dt}\left(\|\nabla\bar r\|^2+\|\zeta_2^t\|^2_{2,\nu}\right)+\|A\bar r\|^2+M_4\|\nabla\bar r\|^2\le 2M_2\|\nabla w\|^2+2\|\nabla\times(P\sigma)\|^2,\quad\forall t\ge t_2^*,$$

where $M_4=2(\lambda-M_3)>0$. Similarly to the proof of (4.21) we obtain

$$\frac{d}{dt}\Lambda_6(t)+\gamma_7\Lambda_6(t)\le 2M_2\|\nabla w\|^2+2\|\nabla\times(P\sigma)\|^2.$$

where $\Lambda_6(t) = \|\nabla \bar{r}\|^2 + \|\zeta_2^t\|^2_{2,\nu} + 4\gamma_6 \int_0^\infty l(s)\|A\zeta_2^t(s)\|^2 ds$ and $\gamma_7 > 0$ is small enough. Then, we can get the following estimate by Gronwall inequality, (F) and (4.13)

$$\|\nabla \bar{r}\|^2 + \|\zeta_2^t\|^2_{2,\nu} \leq \frac{C}{(1-e^{-\gamma_7})}\left(M_2 + \|\nabla \times f\|^2_{L^2_b}\right) = M_5 \text{ for all } t \geq t_2^*. \tag{4.23}$$

Combining (4.22) and (4.23), we end up with

$$\|K_\sigma(t,\tau)z_\tau\|^2_{\mathcal{H}_2} \leq M \quad \text{ for all } t \geq t_3^* > t_2^*.$$

□

In addition, for any $(\eta_{2\tau}, \zeta_{2\tau}) \in L^2_\mu(\mathbb{R}^+, V) \times L^2_\nu(\mathbb{R}^+, V)$, the Cauchy problem (see e.g. [2, 12])

$$\begin{cases} \partial_t(\eta_2^t, \zeta_2^t) = -\partial_s(\eta_2^t, \zeta_2^t) + (\bar{v}, \bar{r}), & t > \tau, \\ (\eta_2^\tau, \zeta_2^\tau) = (\eta_{2\tau}, \zeta_{2\tau}), \end{cases}$$

has a unique solution $(\eta_2^t, \zeta_2^t) \in C((\tau, +\infty); L^2_\mu(\mathbb{R}^+, V) \times L^2_\nu(\mathbb{R}^+, V))$, and

$$\eta_2^t(s) = \begin{cases} \int_0^s \bar{v}(t-r)dr, & \tau < s \leq t, \\ \eta_{2\tau}(\tau + s - t) - \eta_{2\tau}(\tau) + \int_0^t \bar{v}(t-y)dy, & s > t. \end{cases} \tag{4.24}$$

$$\zeta_2^t(s) = \begin{cases} \int_0^s \bar{r}(t-r)dr, & \tau < s \leq t, \\ \zeta_{2\tau}(\tau + s - t) - \zeta_{2\tau}(\tau) + \int_0^t \bar{r}(t-y)dy, & s > t. \end{cases} \tag{4.25}$$

So, for the equations (4.24) and (4.25), thanks to $(\eta_2^\tau(x,s), \zeta_2^\tau(x,s)) = (0,0)$, we have

$$\eta_2^t(s) = \begin{cases} \int_0^s \bar{v}(t-y)dy, & \tau < s \leq t, \\ \int_0^t \bar{v}(t-y)dy, & s > t. \end{cases} \tag{4.26}$$

$$\zeta_2^t(s) = \begin{cases} \int_0^s \bar{r}(t-y)dy, & \tau < s \leq t, \\ \int_0^t \bar{r}(t-y)dy, & s > t. \end{cases} \tag{4.27}$$

Let B_0 be the bounded uniformly absorbing set obtained from Lemma 4.2; then

Lemma 4.5. *Assume that the external force f satisfies (F), $\sigma \in \mathcal{H}_w(f)$ and (M) hold. Setting*

$$\mathcal{K}_T = PK_\sigma(T,\tau)B_0,$$

for $T > \tau$ large enough, where $\{K_\sigma(t,\tau)\}_{t\geq\tau}$ is the solution process of (4.16), $P : V_1 \times H \times L^2_\mu(\mathbb{R}^+, V) \times L^2_\nu(\mathbb{R}^+, V) \to L^2_\mu(\mathbb{R}^+, V) \times L^2_\nu(\mathbb{R}^+, V)$ is the projection operator. Then there is a positive constant $N_1 = N_1(\|B_0\|_{\mathcal{H}_1})$ such that

(i) *$\mathcal{K}_T$ is bounded in $L^2_\mu(\mathbb{R}^+, D(A)) \cap H^1_\mu(\mathbb{R}^+; V) \times L^2_\nu(\mathbb{R}^+, D(A)) \cap H^1_\nu(\mathbb{R}^+; V)$,*

(ii) $\sup_{(\eta_2^t, \zeta_2^t)\in\mathcal{K}_T} \|(\eta_2^t(s), \zeta_2^t(s))\|^2_{V\times V} \leq N_1$.

Moreover, $\mathcal{K}_T$ is relatively compact in $L^2_\mu(\mathbb{R}^+, V) \times L^2_\nu(\mathbb{R}^+, V)$.

Proof. Due to the explicit expression (4.26), we have

$$\partial_s\eta_2^t(s) = \begin{cases} \bar{v}(t-s), & 0 < s \leq t, \\ 0, & s > t, \end{cases} \quad \partial_s\zeta_2^t(s) = \begin{cases} \bar{r}(t-s), & 0 < s \leq t, \\ 0, & s > t. \end{cases}$$

Combining with Lemma 4.4, this implies that (i) holds.
After that, using (4.26) once again, we can deduce that

$$\|\eta_2^T(s)\|^2_V \leq \begin{cases} \int_0^s \|\bar{v}(T-y)\|^2_V dr \leq \int_0^T \|\bar{v}(T-y)\|^2_V dy, & 0 < s \leq T, \\ \int_0^T \|\bar{v}(T-y)\|^2_V dy, & s > T; \end{cases}$$

$$\|\zeta_2^T(s)\|^2_V \leq \begin{cases} \int_0^s \|\bar{r}(T-y)\|^2_V dr \leq \int_0^T \|\bar{r}(T-y)\|^2_V dy, & 0 < s \leq T, \\ \int_0^T \|\bar{r}(T-y)\|^2_V dy, & s > T; \end{cases}$$

By virtue of (4.20), we know that (ii) holds. Because $D(A) \hookrightarrow V$ compactly, we conclude that $\mathcal{K}_T$ is relatively compact in $L^2_\mu(\mathbb{R}^+, V) \times L^2_\nu(\mathbb{R}^+, V)$ thanks to the following lemma:

Lemma 4.6. [12] *Assume that* $\mu \in C^1(\mathbb{R}^+) \cap L^1(\mathbb{R}^+)$ *is a nonnegative function and satisfies the condition: if there exists* $s_0 \in \mathbb{R}^+$ *such that* $\mu(s_0) = 0$*, then* $\mu(s) = 0$ *for all* $s \geq s_0$*. Moreover, let* X_0, X_1, X_2 *be Banach spaces, here* X_0, X_2 *are reflexive and satisfy*

$$X_0 \hookrightarrow X_1 \hookrightarrow X_2,$$

where the embedding $X_0 \hookrightarrow X_1$ *is compact. Let* $\mathcal{C} \subset L^2_\mu(\mathbb{R}^+, X_1)$ *satisfy*

(i) $\mathcal{C}$ *is a subset in* $L^2_\mu(\mathbb{R}^+, X_0) \cap H^1_\mu(\mathbb{R}^+, X_2)$;
(ii) $\sup_{\eta \in \mathcal{C}} \|\eta(s)\|^2_{X_1} \leq h(x, s), \forall s \in \mathbb{R}^+,$ *where* $h \in L^1_\mu(\mathbb{R}^+)$.

Then $\mathcal{C}$ *is relatively compact in* $L^2_\mu(\mathbb{R}^+, X_1)$.

□

4.3. **Proof of Theorem 4.1.**

Proof. By Lemma 4.2, the family of processes $U_\sigma(t, \tau)$ has a bounded absorbing B_0 in $\mathcal{H}_1$. Moreover, $U_\sigma(t, \tau)$ is uniform asymptotically compact in $\mathcal{H}_1$ due to Lemmas 4.3 and 4.5. Therefore, the family of process $U_\sigma(t, \tau)$ has the uniform attractor $\mathcal{A}$ in $\mathcal{H}_1$.

□

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