Conservative compact difference scheme based on the scalar auxiliary variable method for the generalized Kawahara equation

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Abstract

In this paper, a conservative compact difference scheme for the generalized Kawahara equation is constructed based on the scalar auxiliary variable (SAV) approach. The discrete conservative laws of mass and Hamiltonian energy and boundedness estimates are studied in detail. The error estimates in discrete L^∞-norm and L^2-norm of the presented scheme are analyzed by using the discrete energy method. We give an efficiently algorithm of the presented scheme which only needs to solve two decoupled equations.

Key words: The generalized Kawahara equation, compact difference scheme, SAV method, conservation, convergence

1. Introduction

A dispersive model describing numerous wave phenomena, such as mogneto-acoustic waves in a cold plasma and the propagation of long wave in a shallow liquid beneath an ice sheet, was expressed by the following generalized Kawahara equation with periodic condition [1, 2, 3]:

$$u_t - \alpha u_{xxxxx} + \beta u_{xxx} + \gamma u_x + \mu(u^k)_x = 0, \quad (x,t) \in \mathbb{R} \times (0,T], \tag{1.1a}$$

$$u(x,0) = u_0(x), \quad x \in \mathbb{R}, \tag{1.1b}$$

$$u(x+L,t) = u(x,t), \quad (x,t) \in \mathbb{R} \times (0,T], \tag{1.1c}$$

where $\alpha > 0$, $\beta, \mu \neq 0$, $\gamma \geq 0$, $k \geq 2$ and $u_0(x)$ is known smooth periodic function with period L. This equation was proposed firstly by Kawahara in [1], and also referred as the singularly perturbed KdV equation or fifth-order KdV-type equation [4, 5]. It suffices to model only a single period $\Omega \in [0, L]$. We can be easily derive that the system (1.1) has important three conservative laws defined in the following

$$\frac{d}{dt}\mathcal{M} = \frac{d}{dt}\int_\Omega \frac{1}{2} u dx = 0, \tag{1.2}$$

$$\frac{d}{dt}\mathcal{E} = \frac{d}{dt}\int_\Omega \frac{1}{2} u^2 dx = 0, \tag{1.3}$$

$$\frac{d}{dt}\mathcal{H} = \frac{d}{dt}\int_\Omega \left(\frac{\alpha}{2} u_{xx}^2 + \frac{\beta}{2} u_x^2 - \frac{\gamma}{2} u^2 - \frac{\mu}{k+1} u^{k+1} \right) dx = 0, \tag{1.4}$$

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for $t > 0$, where $\mathcal{M}$, $\mathcal{E}$ and $\mathcal{H}$ are the mass, momentum and Hamiltonian energy of the system (1.1), respectively. For details of the physical description of Kawahara equation, the reader is advised to read [6, 7] and the references therein.

There are a great amount of works devoting to the traveling wave solution for the Kawahara equation. The nonsingular periodic-wave solutions of the Kawahara equation ($k = 2, 3$) were presented by using the elliptic Jacobi snoidal and cnoidal functions in [8]. Wazwaz [9] used the sine-cosine method, the tanh method, the extended tanh method and ansatze of hyperbolic functions for analytic treatment for the modified Kawahara equation ($k = 3$). The exact traveling wave solutions of the generalized Kawahara equation were presented by applying the modified extended direct algebraic method in [10]. Well-posedness and unique continuation property for the solutions to the generalized Kawahara equation below the energy space were studied in [11].

For the study of the numerical methods, Yuan *et al.* [12] proposed the numerical scheme of both the Kawahara equation and modified Kawahara equation, which consists of dual-Petrov-Galerkin method in space and Crank-Nicolson-leap-frog in time. The homotopy-analysis method was used to find the traveling wave solution of the Kawahara equation in [13, 14]. This method contains the auxiliary parameters to adjust and control the convergence region of solution. Bibi *et al.* [15] proposed an algorithm based on method of lines coupled with radial basis functions namely meshless method of lines for the Kawahara-type equation. Karakoc *et al.* [16] proposed a septic B-spline collocation method for the Kawahara equation, which is unconditionally stable proved by the von-Neumann stability analysis. Bashan [17, 18] developed the Crank-Nicolson-differential quadrature method based on modified cubic B-splines and fifth-order quintic B-spline, respectively. Gong *et al.* [19] derived a multi-symplectic Fourier pseudo-spectral scheme for the Kawahara equation with special attention to the relationship between the spectral differentiation matrix and discrete Fourier transform. Besides this, finite difference method for approximating the solution of the Kawahara equation has also been considered. Sepulveda *et al.* [20, 21] proposed implicit finite difference schemes for the Kawahara-type equation and proved its unconditionally stability. Koley *et al.* [22] developed the convergence of full discrete semi implicit and Crank-Nicolson implicit finite difference schemes. Wang and Cheng [23] derived the exact solutions of the 1D generalized Kawahara equation and proposed three-time level second order accuracy difference scheme for solving the 1D and 2D generalized Kawahara equation. To further improve the accuracy of the numerical solution, new compact fourth order, standard fourth order and standard second order finite difference schemes for the Kawahara equation were constructed by Chousuion *et al.* [24]. The standard fourth order and standard second order schemes can preserve both mass and energy.

Motivated by the above numerical methods for the Kawahara equation, we are interested in constructing a new high order accurate conservative compact difference scheme for solving the generalized Kawahara equation in (1.1). Note that the scalar auxiliary variable (SAV) approach is presented for solving a large class of gradient flow problems in [25]. The SAV approach is built upon the invariant energy quadratization (IEQ) approach in [26, 27, 28]. This method is not restricted to the nonlinear part of the free energy (see [28] and many works afterwards), and leads to conservative and unconditionally energy stable numerical schemes, which has been applied successfully to the gradient flows model (see [29, 30] and the references therein), the Camassa-Holm equation (see [31]), the Schrödinger equation (see [32, 33]), etc. Therefore, we shall apply the SAV approach to construct conservative scheme for solving the generalized Kawahara equation in (1.1) and prove the convergence order and energy stability for our scheme. The contributions of this work are concluded as follows:

- The presented compact difference scheme guarantees the discrete conservative laws of mass and energy.

- Under the conditions of $\int_\Omega(\frac{\gamma}{2}u^2+\frac{\mu}{k+1}u^{k+1})dx \geq -C_0$ for a positive constant C_0 and $h^4\tau^{-1}=o(1)$, the presented scheme achieves a convergence rate of $\mathcal{O}(\tau^2+h^4)$ in discrete L^∞-norm and L^2-norm. The boundedness of numerical solutions and convergence analysis are given by applying the matrix properties and the discrete energy method.
- An efficient algorithm for the presented scheme is given in detail. The first algorithm can be directly concluded by the presented scheme. This algorithm only needs to solve two decoupled equation by using a block-Gaussian elimination process. Numerical results verify the feasibility of the algorithm.

The rest of this paper are arranged as follows. In Section 2, some auxiliary notations and lemmas are introduced in detail. In Section 3, we construct high order conservative compact difference scheme based on SAV method for solving the generalized Kawahara equation. The discrete conservative laws and boundedness of the numerical solution are analyzed in Section 4. The convergence order $\mathcal{O}(\tau^2+h^4)$ in L^∞-norm and L^2-norm of the presented scheme are proved by the matrix properties and discrete energy method in Section 5. An efficient algorithm of the presented scheme is given in Section 6. Some numerical results are provided to verify our theoretical analysis in Section 7. Finally, some concluding remarks are given in Section 8.

2. Notations and lemmas

In this section, we introduce some useful notations and lemmas. We first divide the domain $[0,L]\times[0,T]$. Let $h=L/J$ and $\tau=T/N$ be the space-step and time-step, respectively, where J and N are given to be two positive integers. Defined $u=\{u_j^n|0\leq j\leq J,\ 0\leq n\leq N\}$ be a discrete grid function on $\Omega_{h,\tau}=\{(x_j,t_n)|x_j=jh,\ t_n=n\tau,\ 0\leq j\leq J,\ 0\leq n\leq N\}$, we denote the following notations

$$
\begin{aligned}
&\delta_t u_j^n=\frac{1}{\tau}(u_j^{n+1}-u_j^n),\quad u_j^{n+\frac{1}{2}}=\frac{1}{2}(u_j^{n+1}+u_j^n),\quad \delta_x u_j^n=\frac{1}{h}(u_j^{n+1}-u_j^n),\\
&\delta_{\bar{x}} u_j^n=\frac{1}{h}(u_j^n-u_{j-1}^n),\quad \delta_{\hat{x}} u_j^n=\frac{1}{2h}(u_{j+1}^n-u_{j-1}^n),\quad \delta_x^2 u_j^n=\delta_x\delta_{\bar{x}}u_j^n=\frac{1}{h^2}(u_{j+1}^n-2u_j^n+u_{j-1}^n),\\
&\delta_x^4 u_j^n=\delta_x^2\delta_x^2 u_j^n,\quad \mathcal{A}u_j^n=\frac{1}{12}(u_{j+1}^n+10u_j^n+u_{j-1}^n),\quad \mathcal{B}u_j^n=\frac{1}{6}(u_{j+1}^n+4u_j^n+u_{j-1}^n).
\end{aligned}
$$

The compact operators $\mathcal{A},\mathcal{B}$ satisfy

$$
\mathcal{A}\partial_x^2 u(x_j,t_n)=\delta_x^2 u(x_j,t_n)+\mathcal{O}(h^4),\quad \mathcal{B}\partial_x u(x_j,t_n)=\delta_{\hat{x}}u(x_j,t_n)+\mathcal{O}(h^4),
$$

which are diagonally dominant and invertible, i.e.

$$
\begin{aligned}
&\partial_x^2 u(x_j,t_n)=\mathcal{A}^{-1}\delta_x^2 u(x_j,t_n)+\mathcal{O}(h^4),\quad \partial_x u(x_j,t_n)=\mathcal{B}^{-1}\delta_{\hat{x}}u(x_j,t_n)+\mathcal{O}(h^4),\\
&\partial_x^4 u(x_j,t_n)=\mathcal{A}^{-2}\delta_x^4 u(x_j,t_n)+\mathcal{O}(h^4).
\end{aligned}
$$

Denote the discrete space

$$
\mathcal{U}_{per}^h=\{u\mid u=(u_j),\ u_{j+J}=u_j,\ j\in\mathbb{Z}\}.
$$

For any grid function $u,v\in\mathcal{U}_{per}^h$, we define the discrete inner products and the corresponding norms as follows

$$
\begin{aligned}
&(u,v)=h\sum_{j=1}^{J}u_jv_j,\quad \|u\|=\sqrt{(u,u)},\quad \|u\|_\infty=\max_{1\leq j\leq J}|u_j|,\\
&\|\delta_x u\|=\sqrt{(\delta_x u,\delta_x u)}=\sqrt{(\delta_x^2 u,u)},\quad \|\delta_x^2 u\|=\sqrt{(\delta_x^2 u,\delta_x^2 u)}=\sqrt{(\delta_x^4 u,u)}.
\end{aligned}
$$

Throughout this paper, C is a positive real constant independent of mesh size h and time step τ.

We introduce the following matrix notations as

$$\mathbf{S}_1 = \begin{bmatrix} 0 & 1 & 0 & \cdots & 1 \\ 1 & 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & 1 \\ 1 & \cdots & 0 & 1 & 0 \end{bmatrix}_{J\times J}, \quad \mathbf{S}_2 = \begin{bmatrix} 0 & -1 & 0 & \cdots & 1 \\ 1 & 0 & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & \cdots & 1 & 0 & -1 \\ -1 & \cdots & 0 & 1 & 0 \end{bmatrix}_{J\times J},$$

and $\mathbf{A} = \frac{1}{12}(\mathbf{S}_1 + 10\mathbf{I})$, $\mathbf{B} = \frac{1}{6}(\mathbf{S}_1 + 4\mathbf{I})$, $\mathbf{D}_1 = \frac{1}{2h}\mathbf{S}_2$, $\mathbf{D}_2 = \frac{1}{h^2}(\mathbf{S}_1 - 2\mathbf{I})$, where $\mathbf{I}$ is an identity matrix with size J. It can be easily see that the matrices $\mathbf{A}$, $\mathbf{B}$, $\mathbf{D}_2$ and $\mathbf{H}_1 = \mathbf{A}^{-1}$, $\mathbf{H}_2 = \mathbf{B}^{-1}$ are $J \times J$ real circulate symmetric positive define. We set $\mathbf{H}_1$, $\mathbf{H}_2$, $\mathbf{D_1}$, $\mathbf{D}_2$ be the circulant matrices corresponding to the inverse operators of $\mathcal{H}_1 = \mathcal{A}^{-1}$, $\mathcal{H}_2 = \mathcal{B}^{-1}$ and the operators $\delta_{\hat{x}}$, δ_x^2.

We now present some lemmas here, which are useful in the subsequent analysis.

Lemma 2.1. *(See [34]) For any grid function $u \in \mathcal{U}_{per}^h$, we have*

$$\|u\|_\infty \le \frac{\sqrt{L}}{2}\|\delta_x u\|, \quad \|u\| \le \frac{L}{\sqrt{6}}\|\delta_x u\|, \quad \|\delta_{\hat{x}} u\| \le \|\delta_x u\|.$$

In addition, for arbitrary $\varepsilon > 0$, we have

$$\|u\|_\infty \le \varepsilon\|\delta_x u\| + \frac{1}{2\varepsilon}\|u\|.$$

Lemma 2.2. *(See [35]) For any grid function $u \in \mathcal{U}_{per}^h$ and $\mathcal{H}_1 = \mathcal{Q}^\top\mathcal{Q}$, $\mathcal{H}_2 = \mathcal{R}^\top\mathcal{R}$, we have*

$$(\mathcal{H}_1\delta_x^2 u, u) = -\|\mathcal{Q}\delta_x u\|^2, \quad (\mathcal{H}_1^2\delta_x^4 u, u) = \|\mathcal{H}_1\delta_x^2 u\|^2, \quad (\mathcal{H}_2\delta_{\hat{x}} u, u) = 0,$$

and

$$\begin{aligned} &\|u\|^2 \le (\mathcal{H}_1 u, u) = \|\mathcal{Q}u\|^2 \le \frac{3}{2}\|u\|^2, \quad \|u\|^2 \le (\mathcal{H}_2 u, u) = \|\mathcal{R}u\|^2 \le 3\|u\|^2, \\ &\|u\|^2 \le (\mathcal{H}_1^2 u, u) = \|\mathcal{H}_1 u\|^2 \le \frac{9}{4}\|u\|^2, \end{aligned}$$

where $\mathcal{Q} = Chol(\mathcal{H}_1)$, $\mathcal{R} = Chol(\mathcal{H}_2)$ are the Cholesky factorizations of $\mathcal{H}_1$ and $\mathcal{H}_2$, respectively.

Lemma 2.3. *(See [36]) For any grid function $u \in \mathcal{U}_{per}^h$, we have*

$$\sum_{j=1}^{J}\mathcal{B}^{-1}u_j^n = \sum_{j=1}^{J}u_j^n = \sum_{j=1}^{J}\mathcal{B}u_j^n.$$

3. Compact difference scheme based on SAV method

In this section, we reformulate the generalized Kawahara equation in (1.1) into an equivalent form based on quadratic energy function, which provides and platform for constructing our scheme. Firstly, it can be see that Eq. (1.1) can be rewritten equivalently into the Hamiltonian system as follows

$$\frac{\partial u}{\partial t} = -\mathcal{D}\frac{\delta\mathcal{H}}{\delta u}, \quad \mathcal{D} = \partial_x,$$

where $\mathcal{H} = \int_\Omega H(u, u_x, u_{xx})dx$ and $\delta\mathcal{H}/\delta u$ denotes the variational derivative of $\mathcal{H}$ with respect to u (see [37] and many works afterwards):

$$\frac{\delta \mathcal{H}}{\delta u} = \frac{\partial H}{\partial u} - \frac{\partial}{\partial_x}\frac{\partial H}{\partial u_x}.$$

We introduce the scalar auxiliary variable

$$r(t) = \sqrt{\mathcal{S}(u)}, \quad \mathcal{S}(u) = \int_\Omega \left(\frac{\gamma}{2}u^2 + \frac{\mu}{k+1}u^{k+1}\right) dx + C_0,$$

where C_0 is a positive constant, then the system (1.1) can be rewritten as

$$u_t = -v_x, \tag{3.1a}$$

$$v = -\alpha u_{xxxx} + \beta u_{xx} + \frac{rG(u)}{\sqrt{\mathcal{S}(u)}}, \tag{3.1b}$$

$$r_t = \frac{(G(u), u_t)}{2\sqrt{\mathcal{S}(u)}}, \tag{3.1c}$$

where $G(u) = \gamma u + \mu u^k$ and $(\cdot,\cdot)$ means the L^2-inner product over Ω defined by $(f, g) = \int_\Omega fg\ dx$. Taking the inner product of (3.1a) with v, of (3.1b) with $-u_t$, multiplying (3.1c) with $2r$, and summing the resulting formulas, we have

$$\frac{d}{dt}\left(\frac{\alpha}{2}\|u_{xx}\|^2 + \frac{\beta}{2}\|u_x\|^2 - r^2\right) = 0,$$

which corresponds to the Hamiltonian energy (1.4).

Define the grid function

$$u_j^n \approx U_j^n = u(x_j, t_n), \quad v_j^n \approx V_j^n = v(x_j, t_n), \quad r^n \approx R^n = r(t_n),$$

for $0 \le j \le J$, $0 \le n \le N$. Considering the system (3.1) at the point (x_j, t_n), and applying the Crank-Nicolson method for temporal discretization and the compact difference method for spatial discretization, we obtain

$$\delta_t U_j^n = -\mathcal{H}_2\delta_{\hat{x}}V_j^{n+\frac{1}{2}} + T_1^n, \quad 1 \le j \le J, \quad 0 \le n \le N-1, \tag{3.2a}$$

$$V_j^{n+\frac{1}{2}} = -\alpha\mathcal{H}_1^2\delta_x^4 U_j^{n+\frac{1}{2}} + \beta\mathcal{H}_1\delta_x^2 U_j^{n+\frac{1}{2}} + \frac{r^{n+\frac{1}{2}}G(\widetilde{U}_j^n)}{\sqrt{S(\widetilde{U}^n)}} + T_2^n, \quad 1 \le j \le J, \quad 0 \le n \le N-1, \tag{3.2b}$$

$$\delta_t R^n = \frac{\left(G(\widetilde{U}^n), \delta_t U^n\right)}{2\sqrt{S(\widetilde{U}^n)}} + T_3^n, \quad 0 \le n \le N-1, \tag{3.2c}$$

$$U_j^0 = u_0(x_j), \quad R^0 = \sqrt{S(U^0)}, \quad 0 \le j \le J, \tag{3.2d}$$

$$U_{j+J}^n = U_j^n, \quad 1 \le j \le J, \quad 1 \le n \le N, \tag{3.2e}$$

where $\widetilde{U}^n = (3U^n - U^{n-1})/2$ when $n \ge 1$, $\widetilde{U}^n = U^n$ when $n = 0$, and

$$\max\{|T_1^n|,\ |T_2^n|,\ |T_3^n|\} \le C(\tau^2 + h^4), \quad \max\{|\delta_t T_1^n|,\ |\delta_t T_2^n|\} \le Ch^4. \tag{3.3}$$

$S(\widetilde{U}^n)$ is the composite Simpson formula of $\mathcal{S}(\widetilde{U}^n)$, that is,

$$S(\widetilde{U}^n) = \frac{h}{3}\sum_{j=1}^{J}\left[\frac{\gamma}{2}(\widetilde{U}_j^n)^2 + \frac{\mu}{k+1}(\widetilde{U}_j^n)^{k+1} + \frac{\gamma}{4}(\widetilde{U}_{j+1}^n + \widetilde{U}_j^n)^2 + \frac{\mu}{2^k(k+1)}(\widetilde{U}_{j+1}^n + \widetilde{U}_j^n)^{k+1}\right].$$

Replacing the exact solutions U_j^n, V_j^n, R^n by u_j^n, v_j^n, r^n in (3.2), and omitting the truncation errors T_1^n, T_2^n, T_3^n, we construct the following compact difference scheme for the system (3.1):

$$\delta_t u_j^n = -\mathcal{H}_2\delta_{\hat{x}} v_j^{n+\frac{1}{2}}, \quad 1 \le j \le J, \quad 0 \le n \le N-1, \tag{3.4a}$$

$$v_j^{n+\frac{1}{2}} = -\alpha\mathcal{H}_1^2\delta_x^4 u_j^{n+\frac{1}{2}} + \beta\mathcal{H}_1\delta_x^2 u_j^{n+\frac{1}{2}} + \frac{r^{n+\frac{1}{2}}G(\widetilde{u}_j^n)}{\sqrt{S(\widetilde{u}^n)}}, \quad 1 \le j \le J, \quad 0 \le n \le N-1, \tag{3.4b}$$

$$\delta_t r^n = \frac{(G(\widetilde{u}^n), \delta_t u^n)}{2\sqrt{S(\widetilde{u}^n)}}, \quad 0 \le n \le N-1, \tag{3.4c}$$

$$u_j^0 = u_0(x_j), \quad r^0 = \sqrt{S(u^0)}, \quad 0 \le j \le J, \tag{3.4d}$$

$$u_{j+J}^n = u_j^n, \quad 0 \le j \le J, \quad 1 \le n \le N. \tag{3.4e}$$

4. Discrete conservative laws and boundedness

In this section, we give the analysis of the discrete conservative laws and boundedness for the presented compact difference scheme (3.4) in detail. In the following analysis, we assume the exact solutions (u, v, r) of the system (3.1) satisfy the following regularity condition

$$u(x,t) \in L^\infty(0,T; H^6(\Omega)), \quad u_t(x,t) \in L^2(0,T; H^4(\Omega)) \cap L^\infty(0,T; L^\infty(\Omega)). \tag{4.1}$$

Supposed that there exists a positive constant C, s.t.

$$\max_{0\le n\le N}\{\|U^n\|_{L^\infty}, \|U_x^n\|_{L^\infty}, \|U_{xx}^n\|_{L^\infty}, \|U_t^n\|_{L^\infty}\} \le C.$$

Theorem 4.1. *The compact difference scheme (3.4) unconditionally preserves the discrete mass and Hamiltonian energy in the sense of*

$$M^{n+1} = M^0 \quad \textit{with} \quad M^n = \frac{h}{2}\sum_{j=1}^{J} u_j^n, \tag{4.2}$$

$$H^{n+1} = H^0 \quad \textit{with} \quad H^n = \frac{\alpha}{2}\|\mathcal{H}_1\delta_x^2 u^n\|^2 + \frac{\beta}{2}\|\mathcal{Q}\delta_x u^n\|^2 - (r^n)^2, \tag{4.3}$$

for $1 \le n \le N$.

proof. Multiplying both-hand sides of (3.4a) with h, summing up j from 1 to J, and using Lemma 2.3, we obatin

$$\frac{h}{\tau}\sum_{j=1}^{J}(u_j^{n+1} - u_j^n) = 0,$$

which implies (4.2) by the definition of M^n.

Taking the inner product of (3.4a) with $v^{n+\frac{1}{2}}$, of (3.4b) with $-\delta_t u^n$, multiplying (3.4c) with $2r^{n+\frac{1}{2}}$, using Lemma 2.2 and summing the resulting formulas, we obatin

$$\frac{\alpha}{2\tau}(\|\mathcal{H}_1\delta_x^2 u^{n+1}\|^2 - \|\mathcal{H}_1\delta_x^2 u^n\|^2) + \frac{\beta}{2\tau}(\|\mathcal{Q}\delta_x u^{n+1}\|^2 - \|\mathcal{Q}\delta_x u^n\|^2) - \frac{1}{\tau}((r^{n+1})^2 - (r^n)^2) = 0.$$

which implies (4.3) by the definition of H^n. This completes the proof. □

Theorem 4.2. *Suppose that $S(U^n) \geq 0$, then we have the following estimates*

$$\|u^n\| + \|\delta_x u^n\| + \|\delta_x^2 u^n\| + r^n \leq C, \quad \|u^n\|_\infty + \|\delta_x u^n\|_\infty \leq C, \tag{4.4}$$

for $1 \leq n \leq N$.

proof. First, taking the inner product of (3.4a) with $v^{\frac{1}{2}}$, of (3.4b) with $-\delta_t u^0$, multiplying (3.4c) with $-2r^{\frac{1}{2}}$ and applying Lemmas 2.1 and 2.2, we obtain

$$\begin{aligned}
&\frac{\alpha}{2\tau}(\|\mathcal{H}_1\delta_x^2 u^1\|^2 - \|\mathcal{H}_1\delta_x^2 u^0\|^2) + \frac{\beta}{2\tau}(\|\mathcal{Q}\delta_x u^1\|^2 - \|\mathcal{Q}\delta_x u^0\|^2) + \frac{1}{\tau}((r^1)^2 - (r^0)^2)\\
= \;& r^{\frac{1}{2}}\left(\frac{G(u^0)}{\sqrt{S(u^0)}}, \delta_t u^0\right)\\
\leq \;& \frac{1}{4\tau}\left(\frac{6\beta}{L^2}\|u^1\|^2 + (r^1)^2\right) + C\\
\leq \;& \frac{1}{4\tau}\left(\beta\|\delta_x u^1\|^2 + (r^1)^2\right) + C.
\end{aligned}$$

Using Lemma 2.3, we have

$$\alpha\|\delta_x^2 u^1\|^2 + \beta\|\delta_x u^1\|^2 + 2(r^1)^2 \leq C,$$

which holds (4.4) by applying Lemma 2.1 and the discrete Sobolev inequality.

Next, we use the mathematical induction. Assume that

$$\max_{1\leq l\leq n}\{\|u^l\|, \ \|\delta_x u^l\|, \ \|\delta_x^2 u^l\|, \ r^l\} \leq C,$$

then using Lemma 2.1 and the discrete Sobolev inequality, we have

$$\max\{\|u^l\|_\infty, \|\delta_x u^l\|_\infty\} \leq C.$$

Taking the inner product of (3.4a) with $v^{n+\frac{1}{2}}$, of (3.4b) with $-\delta_t u^n$, multiplying (3.4c) with $-2r^{n+\frac{1}{2}}$, then replacing n by l, summing up for l from 1 to n and applying Lemma 2.2, we obtain

$$\begin{aligned}
&\frac{\alpha}{2}(\|\mathcal{H}_1\delta_x^2 u^{n+1}\|^2 - \|\mathcal{H}_1\delta_x^2 u^1\|^2) + \frac{\beta}{2}(\|\mathcal{Q}\delta_x u^{n+1}\|^2 - \|\mathcal{Q}\delta_x u^1\|^2) + ((r^{n+1})^2 - (r^1)^2)\\
= \;& r^{n+\frac{1}{2}}\left(\frac{G(\widetilde{u}^n)}{\sqrt{S(\widetilde{u}^n)}}, u^{n+1} - u^n\right) + \tau\sum_{l=1}^{n-1} r^{l+\frac{1}{2}}\left(\frac{G(\widetilde{u}^l)}{\sqrt{S(\widetilde{u}^l)}}, \delta_t u^l\right)\\
\leq \;& \frac{1}{4\tau}\left(\frac{6\beta}{L^2}\|u^{n+1}\|^2 + (r^{n+1})^2\right) + C\\
\leq \;& \frac{1}{4\tau}\left(\beta\|\delta_x u^{n+1}\|^2 + (r^{n+1})^2\right) + C,
\end{aligned}$$

which implies

$$\alpha\|\delta_x^2 u^{n+1}\|^2 + \beta\|\delta_x u^{n+1}\|^2 + 2(r^{n+1})^2 \leq C.$$

Thus, we get

$$\|u^{n+1}\| \leq C, \quad \|u^{n+1}\|_\infty + \|\delta_x u^{n+1}\|_\infty \leq C.$$

It can be seen that the conclusions are also valid for $l = n+1$. This completes the proof. □

5. Convergence analysis

This section will given the strictly proof of the error estimates of the compact difference scheme (3.4). We denote the error function as

$$e^n = U^n - u^n, \quad f^n = V^n - v^n, \quad g^n = R^n - r^n.$$

Substituting (3.2) from the compact difference scheme (3.4), we derive the error equations

$$\delta_t e_j^n = -\mathcal{H}_2 \delta_{\hat{x}} f_j^{n+\frac{1}{2}} + T_1^n, \quad 1 \le j \le J, \quad 0 \le n \le N-1, \tag{5.1a}$$

$$f_j^{n+\frac{1}{2}} = -\alpha \mathcal{H}_1^2 \delta_x^4 e_j^{n+\frac{1}{2}} + \beta \mathcal{H}_1 \delta_x^2 e_j^{n+\frac{1}{2}} + P_1^{n+\frac{1}{2}} + T_2^n, \quad 1 \le j \le J, \quad 0 \le n \le N-1, \tag{5.1b}$$

$$\delta_t g^n = P_2^{n+\frac{1}{2}} + T_3^n, \quad 0 \le n \le N-1, \tag{5.1c}$$

$$e_j^0 = 0, \quad g^0 = 0, \quad 0 \le j \le J, \tag{5.1d}$$

$$e_{j+J}^n = e_j^n, \quad 1 \le j \le J, \quad 1 \le n \le N, \tag{5.1e}$$

where

$$P_1^{n+\frac{1}{2}} = \frac{R^{n+\frac{1}{2}} G(\widetilde{U}^n)}{\sqrt{S(\widetilde{U}^n)}} - \frac{r^{n+\frac{1}{2}} G(\widetilde{u}^n)}{\sqrt{S(\widetilde{u}^n)}}, \quad P_2^{n+\frac{1}{2}} = \left(\frac{G(\widetilde{U}^n)}{\sqrt{S(\widetilde{U}^n)}}, \delta_t U^n \right) - \left(\frac{G(\widetilde{u}^n)}{\sqrt{S(\widetilde{u}^n)}}, \delta_t u^n \right).$$

Theorem 5.1. *Suppose that the exact solutions of (3.1) satisfy the condition (4.1) and $h^4\tau^{-1} = o(1)$, then we have the following error estimates*

$$\|e^n\| + \|\delta_x e^n\| + \|\delta_x^2 e^n\| \le C(\tau^2 + h^4), \quad \|e^n\|_\infty + \|\delta_x e^n\|_\infty \le C(\tau^2 + h^4), \tag{5.2}$$

for $1 \le n \le N$.

proof. At first, we consider the convergence results when $n = 0$. Taking the inner product of (5.1a) with $\frac{1}{2} f^1$, of (5.1b) with $-\frac{1}{\tau} e^1$ and multiplying (5.1c) with $-g^1$, we have

$$\begin{aligned} & \frac{\alpha}{2\tau} \|\mathcal{H}_1 \delta_x^2 e^1\|^2 + \frac{\beta}{2\tau} \|\mathcal{Q} \delta_x e^1\|^2 + \frac{1}{\tau} \|g^1\|^2 \\ = \quad & -(f^1, T_1^0) + \frac{1}{\tau}(T_2^0, e^1) + g^1 T_3^0 + \frac{1}{\tau} \left(\frac{g^1 G(\widetilde{U}^0)}{\sqrt{S(\widetilde{U}^0)}}, e^1 \right) \\ \le \quad & \|f^1\| \|T_1^0\| + \frac{2}{\tau} \|T_2^0\| \|e^1\| + \|g^1\| \|T_3^0\| + \frac{1}{\tau} \|g^1\| \|e^1\| \left\| \frac{G(\widetilde{U}^0)}{\sqrt{S(\widetilde{U}^0)}} \right\|. \end{aligned} \tag{5.3}$$

It follows from (5.1a) and Lemmas 2.1, 2.3 that

$$\frac{\sqrt{6}}{L} \|f^1\| \le \|\mathcal{H}_2 \delta_{\hat{x}} f^1\| \le \frac{2}{\tau} \|e^1\| + 2\|T_1^0\|.$$

Applying Lemma 2.3 to (5.3), we obatin

$$\begin{aligned} \alpha \|\delta_x e^1\|^2 + \beta \|\delta_x^2 e^1\|^2 + 2\|g^1\|^2 \quad & \le \quad C(\|e^1\|^2 + \|g^1\|^2) + C(1+\tau)\|T_1^0\|^2 + \|T_2^0\|^2 + \tau \|T_3^0\|^2 \\ & \le \quad C(\beta \|\delta_x e^1\|^2 + \|g^1\|^2) + C(\tau^2 + h^4)^2, \end{aligned}$$

which holds (5.2) by using Lemma 2.1 and the discrete Sobolev inequality.

Next, we adopt mathematical induction as in [38] to further analyze the error estimate. Assume that there exists a constant $h_0 > 0$, $\tau_0 > 0$ such that, for $0 < h < h_0$, $0 < \tau < \tau_0$,

$$\|\delta_t e^{l-1}\| \leq 1, \quad \max\{\|e^l\|,\ \|\delta_x e^l\|,\ \|\delta_x^2 e^l\|,\ \|g^l\|\} \leq C(\tau^2 + h^4),$$

for $1 \leq l \leq n$, which implies

$$\max\{\|e^l\|_\infty,\ \|\delta_x e^l\|_\infty\} \leq C(\tau^2 + h^4).$$

Taking the inner product of (5.1a) with $f^{n+\frac{1}{2}}$, of (5.1b) with $-\delta_t e^n$ and multiplying (5.1c) with $-g^{n+\frac{1}{2}}$, summing up them, we have

$$\begin{aligned} & \frac{\alpha}{2\tau}(\|\mathcal{H}_1\delta_x^2 e^{n+1}\|^2 - \|\mathcal{H}_1\delta_x^2 e^n\|^2) + \frac{\beta}{2\tau}(\|\mathcal{Q}\delta_x e^{n+1}\|^2 - \|\mathcal{Q}\delta_x e^n\|^2) + \frac{1}{\tau}\left(\|g^{n+1}\|^2 - \|g^n\|^2\right) \\ = \ & -(T_1^n, f^{n+\frac{1}{2}}) + (P_1^{n+\frac{1}{2}}, \delta_t e^n) + (T_2^n, \delta_t e^n) + g^{n+\frac{1}{2}}(P_2^{n+\frac{1}{2}} + T_3^n). \end{aligned} \tag{5.4}$$

It follows from (5.1a) that

$$\frac{\sqrt{6}}{L}\|f^{n+\frac{1}{2}}\| \leq \|\mathcal{H}_2\delta_{\hat{x}} f^{n+\frac{1}{2}}\| \leq \|\delta_t e^n\| + \|T_1^n\|,$$

we have

$$|(T_1^n, f^{n+\frac{1}{2}})| \leq C\|T_1^n\|\|\delta_t e^n\| + C\|T_1^n\|^2 \leq C(\|e^{n+1}\|^2 + \|e^n\|^2 + \|\delta_t T_1^n\|^2 + \|T_1^n\|^2). \tag{5.5}$$

Since

$$\begin{aligned} P_1^{n+\frac{1}{2}} &= \frac{g^{n+\frac{1}{2}} G(\widetilde{U}^n)}{\sqrt{S(\widetilde{U}^n)}} + r^{n+\frac{1}{2}}\left(\frac{G(\widetilde{U}^n)}{\sqrt{S(\widetilde{U}^n)}} - \frac{G(\widetilde{u}^n)}{\sqrt{S(\widetilde{u}^n)}}\right) \\ &= \frac{g^{n+\frac{1}{2}} G(\widetilde{U}^n)}{\sqrt{S(\widetilde{U}^n)}} + \frac{r^{n+\frac{1}{2}}\left(G(\widetilde{U}^n) - G(\widetilde{u}^n)\right)}{\sqrt{S(\widetilde{U}^n)}} + \frac{r^{n+\frac{1}{2}} G(\widetilde{u}^n)\left(S(\widetilde{U}^n) - S(\widetilde{u}^n)\right)}{\sqrt{S(\widetilde{U}^n) S(\widetilde{u}^n)\left(S(\widetilde{U}^n) + S(\widetilde{u}^n)\right)}} \\ &:= g^{n+\frac{1}{2}} M_1 + r^{n+\frac{1}{2}}(M_2 + M_3), \end{aligned} \tag{5.6}$$

$$\begin{aligned} P_2^{n+\frac{1}{2}} &= \left(\frac{G(\widetilde{U}^n)}{\sqrt{S(\widetilde{U}^n)}}, \delta_t e^n\right) + \left(\frac{G(\widetilde{U}^n)}{\sqrt{S(\widetilde{U}^n)}} - \frac{G(\widetilde{u}^n)}{\sqrt{S(\widetilde{u}^n)}}, \delta_t u^n\right) \\ &= (M_1, \delta_t e^n) + (M_2 + M_3, \delta_t u^n), \end{aligned} \tag{5.7}$$

we have

$$\|M_1\| \leq C, \tag{5.8}$$

$$\|M_2\| \leq C\left\|G'(\widetilde{\xi}^n)(\widetilde{U}^n - \widetilde{u}^n)\right\| \leq C\|\widetilde{e}^n\|, \tag{5.9}$$

$$\|M_3\| \leq C\left\|S'(\widetilde{\eta}^n)(\widetilde{U}^n - \widetilde{u}^n)\right\| \leq C\|\widetilde{e}^n\|, \tag{5.10}$$

where $\widetilde{\xi}^n$, $\widetilde{\eta}^n$ are on the segment that connects $\widetilde{U}^n$ and $\widetilde{u}^n$, then it implies from (5.6)-(5.10) that

$$\begin{aligned}(P_1^{n+\frac{1}{2}},\delta_t e^n)+g^{n+\frac{1}{2}}P_2^{n+\frac{1}{2}} &= \left(2g^{n+\frac{1}{2}}M_1+r^{n+\frac{1}{2}}(M_2+M_3),\delta_t e^n\right)+g^{n+\frac{1}{2}}(M_2+M_3,\delta_t u^n)\\ &\leq C(2\|g^{n+\frac{1}{2}}\|\|\delta_t e^n\|+\|\delta_t e^n\|\|\widetilde{e}^n\|+\|g^{n+\frac{1}{2}}\|\|\delta_t e^{n-1}\|). \qquad (5.11)\end{aligned}$$

Applying the Cauchy-Schwarz inequality, we have

$$(T_2^n,\delta_t e^n)+g^{n+\frac{1}{2}}T_3^n\leq C(\|e^{n+1}\|^2+\|e^n\|^2+\|g^{n+1}\|^2+\|g^n\|^2)+\frac{1}{2}(\|\delta_t T_2^n\|^2+\|T_3^n\|^2). \quad (5.12)$$

Substituting (5.5), (5.11) and (5.12) into (5.4), replacing n by l and summing up for l from 0 to n, we have

$$\begin{aligned}&\alpha\|\mathcal{H}_1\delta_x^2 e^{n+1}\|^2+\beta\|\mathcal{Q}\delta_x e^{n+1}\|^2+2\|g^{n+1}\|^2\\ \leq\ & C\tau\sum_{l=0}^{n+1}(\|e^l\|^2+\|g^l\|^2)+C\tau\sum_{l=0}^{n}(2\|g^{l+\frac{1}{2}}\|\|\delta_t e^l\|+\|\delta_t e^l\|\|\widetilde{e}^l\|+\|g^{l+\frac{1}{2}}\|\|\delta_t e^{l-1}\|)\\ &+C\tau\sum_{l=0}^{n}(\|\delta_t T_1^l\|^2+\|T_1^l\|^2+\|\delta_t T_2^l\|^2+\|T_3^l\|^2)\\ \leq\ & C\tau(\|e^{n+1}\|^2+\|g^{n+1}\|^2)+C\tau\left[2\|g^{n+\frac{1}{2}}\|\|\delta_t e^n\|+\|\delta_t e^n\|\|\widetilde{e}^n\|+\|g^{n+\frac{1}{2}}\|\|\delta_t e^{n-1}\|\right]\\ &+C\tau\sum_{l=0}^{n}(\|e^l\|^2+\|g^l\|^2)+C\tau\sum_{l=0}^{n-1}(2\|g^{l+\frac{1}{2}}\|\|\delta_t e^l\|+\|\delta_t e^l\|\|\widetilde{e}^l\|+\|g^{l+\frac{1}{2}}\|\|\delta_t e^{l-1}\|)+C(\tau^2+h^4)^2\\ \leq\ & \left(\frac{1}{2}+C\tau\right)(\beta\|e^{n+1}\|^2+\|g^{n+1}\|^2)+C\tau\sum_{l=0}^{n}(\|e^l\|^2+\|g^l\|^2)+C(\tau^2+h^4)^2, \qquad (5.13)\end{aligned}$$

where

$$\begin{aligned}&C\tau\sum_{l=0}^{n}(\|\delta_t T_1^l\|^2+\|T_1^l\|^2+\|\delta_t T_2^l\|^2+\|T_3^l\|^2)\\ \leq\ & Cn\tau\cdot\max_{0\leq l\leq n}\{\|\delta_t T_1^l\|^2,\|T_1^l\|^2,\|\delta_t T_2^l\|^2,\|T_3^l\|^2\}\leq C(\tau^2+h^4)^2.\end{aligned}$$

It follows from Lemmas 2.1, 2.3 that

$$\|\delta_x^2 e^{n+1}\|^2+\|\delta_x e^{n+1}\|^2+\|g^{n+1}\|^2\leq C\tau\sum_{l=0}^{n}(\|\delta_x^2 e^l\|^2+\|\delta_x e^l\|^2+\|g^l\|^2)+C(\tau^2+h^4)^2. \quad (5.14)$$

for τ sufficiently small, s.t. $\tau<1/(2C)$. By using the discrete Gronwall inequality, we have

$$\|\delta_x^2 e^{n+1}\|^2+\|\delta_x e^{n+1}\|^2+\|g^{n+1}\|^2\leq C(\tau^2+h^4)^2.$$

Applying the discrete Sobolev inequality, we obtain

$$\|e^{n+1}\|\leq C(\tau^2+h^4),\quad \|e^{n+1}\|_\infty+\|\delta_x e^{n+1}\|_\infty\leq C(\tau^2+h^4)$$

and

$$\|\delta_t e^n\|\leq\frac{1}{\tau}\|e^{n+1}-e^n\|\leq C(\tau+h^4\tau^{-1}).$$

We can see that, from the inequalities above, our assumptions holds for $l=n+1$ by setting $\tau+h^4\tau^{-1}\to 0$ as $h\to 0$, $\tau\to 0$ and taking h sufficiently small, which implies that our assumption is valid for $l=n+1$ and holds (5.2). This completes the mathematical induction and proof. $\square$

6. Algorithm

In this section, we propose two algorithms of the presented compact scheme (3.4) for solving the generalized Kawahara equation.

The compact scheme (3.4) can be rewritten as the following matrix-vector equations:

$$\mathbf{B}\delta_t\mathbf{u}^n = -\mathbf{D}_1\mathbf{v}^{n+\frac{1}{2}}, \tag{6.1a}$$

$$\mathbf{A}^2\mathbf{v}^{n+\frac{1}{2}} = -\alpha\mathbf{D}_2^2\mathbf{u}^{n+\frac{1}{2}} + \beta\mathbf{A}\mathbf{D}_2\mathbf{u}^{n+\frac{1}{2}} + \frac{r^{n+\frac{1}{2}}}{\sqrt{S(\widetilde{u}^n)}}\mathbf{A}^2G(\widetilde{\mathbf{u}}^n), \tag{6.1b}$$

$$\delta_t r^n = \frac{1}{2\sqrt{S(\widetilde{u}^n)}}\left(G(\widetilde{\mathbf{u}}^n), \delta_t\mathbf{u}^n\right), \tag{6.1c}$$

for $0 \le n \le N-1$, then the we conclude (6.1) in the following form

$$\begin{bmatrix} \mathbf{A}_1 & \mathbf{A}_2 & \mathbf{0} \\ \mathbf{A}_3 & \mathbf{A}_4 & \mathbf{a}_1 \\ \mathbf{a}_2^\top & \mathbf{0}^\top & 1 \end{bmatrix}\begin{bmatrix} \mathbf{u}^{n+1} \\ \mathbf{v}^{n+1} \\ r^{n+1} \end{bmatrix} = \begin{bmatrix} \mathbf{A}_1 & -\mathbf{A}_2 & \mathbf{0} \\ -\mathbf{A}_3 & -\mathbf{A}_4 & -\mathbf{a}_1 \\ \mathbf{a}_2^\top & \mathbf{0}^\top & 1 \end{bmatrix}\begin{bmatrix} \mathbf{u}^{n} \\ \mathbf{v}^{n} \\ r^{n} \end{bmatrix}, \tag{6.2}$$

where

$$\mathbf{A}_1 = \frac{1}{\tau}\mathbf{B}, \quad \mathbf{A}_2 = \frac{1}{2}\mathbf{D}_1, \quad \mathbf{A}_3 = \frac{\alpha}{2}\mathbf{D}_2^2 - \frac{\beta}{2}\mathbf{A}\mathbf{D}_2, \quad \mathbf{A}_4 = \frac{1}{2}\mathbf{A}^2,$$
$$G(\widetilde{\mathbf{u}}^n) = \left[G(\widetilde{u}_1^n),\ G(\widetilde{u}_2^n),\ \ldots,\ G(\widetilde{u}_J^n)\right]^\top, \quad \mathbf{a}_1 = -\frac{1}{2\sqrt{S(\widetilde{u}^n)}}\mathbf{A}^2G(\widetilde{\mathbf{u}}^n), \quad \mathbf{a}_1 = -\frac{h}{2\sqrt{S(\widetilde{u}^n)}}G(\widetilde{\mathbf{u}}^n).$$

It can be easy to see that this method needs the condition v^0 and solve a $(2J+1)\times(2J+1)$ matrix, which may leads to slow calculation efficiency when h takes a sufficiently small value.

Note that $\mathbf{v}^{n+1}$ is just an intermediate variable which can be eliminated and $\mathbf{u}^{n+1}$ is coupled by r^{n+1} in system (6.1). Therefore, the system (6.1) can be decoupled by using a block-Gaussian elimination process. From (6.1c), we have

$$r^{n+1} = r^n + \left(\frac{G(\widetilde{\mathbf{u}}^n)}{2\sqrt{S(\widetilde{u}^n)}}, \mathbf{u}^{n+1} - \mathbf{u}^n\right). \tag{6.3}$$

Setting

$$\eta^{n+1} = \left(\frac{G(\widetilde{\mathbf{u}}^n)}{2\sqrt{S(\widetilde{u}^n)}}, \mathbf{u}^{n+1}\right),$$

and substituting (6.1b) and (6.3) into (6.1a), we obtain

$$\begin{aligned} &\left(\frac{1}{\tau}\mathbf{A}^2\mathbf{B} - \frac{\alpha}{2}\mathbf{D}_1\mathbf{D}_2^2 + \frac{\beta}{2}\mathbf{A}\mathbf{D}_1\mathbf{D}_2\right)\mathbf{u}^{n+1} + \frac{\eta^{n+1}}{2\sqrt{S(\widetilde{u}^n)}}\mathbf{A}^2\mathbf{D}_1G(\widetilde{\mathbf{u}}^n) \\ =\ &\left(\frac{1}{\tau}\mathbf{A}^2\mathbf{B} + \frac{\alpha}{2}\mathbf{D}_1\mathbf{D}_2^2 - \frac{\beta}{2}\mathbf{A}\mathbf{D}_1\mathbf{D}_2\right)\mathbf{u}^{n} - \frac{2r^n - \eta^n}{2\sqrt{S(\widetilde{u}^n)}}\mathbf{A}^2\mathbf{D}_1G(\widetilde{\mathbf{u}}^n). \end{aligned} \tag{6.4}$$

Let

$$\mathbf{u}^{n+1} = \mathbf{u}_1^{n+1} + \eta^{n+1}\mathbf{u}_2^{n+1}, \tag{6.5}$$

in (6.4), we obtain the following two decoupled equations:

$$\left(\frac{1}{\tau}\mathbf{A}^2\mathbf{B}-\mathbf{C}\right)\mathbf{u}_1^{n+1}=\left(\frac{1}{\tau}\mathbf{A}^2\mathbf{B}+\mathbf{C}\right)\mathbf{u}^n-\frac{2r^n-\eta^n}{2\sqrt{S(\widetilde{u}^n)}}\mathbf{A}^2\mathbf{D}_1G(\widetilde{\mathbf{u}}^n), \tag{6.6a}$$

$$\left(\frac{1}{\tau}\mathbf{A}^2\mathbf{B}-\mathbf{C}\right)\mathbf{u}_2^{n+1}=-\frac{1}{2\sqrt{S(\widetilde{u}^n)}}\mathbf{A}^2\mathbf{D}_1G(\widetilde{\mathbf{u}}^n), \tag{6.6b}$$

where $\mathbf{C}=\frac{\alpha}{2}\mathbf{D}_1\mathbf{D}_2^2-\frac{\beta}{2}\mathbf{A}\mathbf{D}_1\mathbf{D}_2$. Taking the inner product of (6.5) with $G(\widetilde{\mathbf{u}}^n)/\left(2\sqrt{S(\widetilde{u}^n)}\right)$, we have

$$\left[1-\left(\frac{G(\widetilde{\mathbf{u}}^n)}{2\sqrt{S(\widetilde{u}^n)}},\mathbf{u}_2^{n+1}\right)\right]\eta^{n+1}=\left(\frac{G(\widetilde{\mathbf{u}}^n)}{2\sqrt{S(\widetilde{u}^n)}},\mathbf{u}_1^{n+1}\right), \tag{6.7}$$

from which η^{n+1} can be determined.

To summarize, we can obtain $\mathbf{u}^{n+1}$ and r^{n+1} from (6.1) as follows:

Step 1. Solve $(\mathbf{u}_1^{n+1},\mathbf{u}_2^{n+1})$ from (6.6);

Step 2. Determine η^{n+1} from (6.7);

Step 3. Compute $\mathbf{u}^{n+1}$ from (6.5), then determined r^{n+1} from (6.3).

7. Numerical experiments

In this section, numerical results are presented to test our numerical schemes. The accuracy of the present scheme is measured by the discrete L^2-norm and L^∞-norm. We take $C_0=0$. The momentum $\mathcal{E}$ in (1.3) is approximated by

$$E^n=\frac{h}{2}\sum_{j=1}^{J}(u_j^n)^2.$$

Example 1. *We consider the parameters to be $\alpha=\beta=1$, $\gamma=0,1$, $\mu=1/2$ and $k=2$. For this case, the system (1.1) is called to the classical Kawahara equation [15]. The exact traveling wave solution is*

$$u(x,t)=\frac{105}{169}\operatorname{sech}^4\left(\frac{1}{2\sqrt{13}}\left(x-\frac{36t}{169}-x_0\right)\right),$$

when $\gamma=0$ and

$$u(x,t)=\frac{105}{169}\operatorname{sech}^4\left(\frac{1}{2\sqrt{13}}\left(x-\frac{205t}{169}-x_0\right)\right),$$

when $\gamma=1$.

First, we chose $x_0=0$, $\Omega=[-60,100]$ and $T=120$. The results of errors and convergence rates with $\gamma=0$, $h=0.5$, $\tau=h^2$ are reported in Table 1. Table 2 lists the error results at $T=5,15,25$ with $\gamma=0$, $x_0=2$ and $h=0.2$, which are compared with MQ, GA in [15], septic B-spline collocation method (SBC) in [16], MCBC-DQM in [17] and Scheme 3 in [24]. We can be seen that the present scheme is fourth-order accuracy in space and has much small error than the schemes in [15, 16, 17, 24] even if a larger temporal step was used. Numerical traveling waves and absolute errors at difference times with $\gamma=1$, $h=0.0625$, $\tau=h^2$ are shown in Figure 1. It can be seen that the maximum error occurs near each highest peak.

Table 1: The errors and convergence rates at $T = 120$ with $\gamma = 0$, $h = 0.5$, $\tau = h^2$ for Example 1.

	$\|e^n\|_\infty$	Rate	$\|e^n\|$	Rate
$h = 0.5$	8.7823E-4	–	2.5964E-3	–
$h = 0.25$	5.3151E-5	4.0464	1.5722E-4	4.0457
$h = 0.125$	3.2843E-6	4.0165	9.7499E-6	4.0113

Table 2: The comparisons of errors with $\gamma = 0$ and $h = 0.2$ for Example 1.

T	Scheme	$\|e^n\|$	$\|e^n\|_\infty$	Scheme	$\|e^n\|$	$\|e^n\|_\infty$
5	Present scheme	4.867E-6	1.2888E-5	Scheme 3 [24]	2.85426E-4	1.02518E-4
15	($\tau = 0.05$)	6.933E-6	1.7388E-5	($\tau = 0.01$)	4.52481E-4	1.76969E-4
25		8.221E-6	2.4561E-5		6.16321E-4	2.43811E-4
5	MQ [15]	9.468E-5	4.669E-5	GA [15]	1.0075E-4	3.4297E-5
15	($\tau = 0.001$)	1.5362E-4	5.939E-5	($\tau = 0.001$)	1.0113E-4	3.830E-5
25		1.6818E-4	4.660E-5		1.3160E-4	3.990E-5
5	SBC [16]	3.249E-4	1.116E-4	MCBC-DQM [17]	6.3E-5	2.8E-5
15	($\tau = 0.01$)	1.807E-4	7.44E-5	($\tau = 0.01$)	5.6E-5	1.9E-5
25		1.395E-4	5.11E-5		7.2E-5	2.9E-5

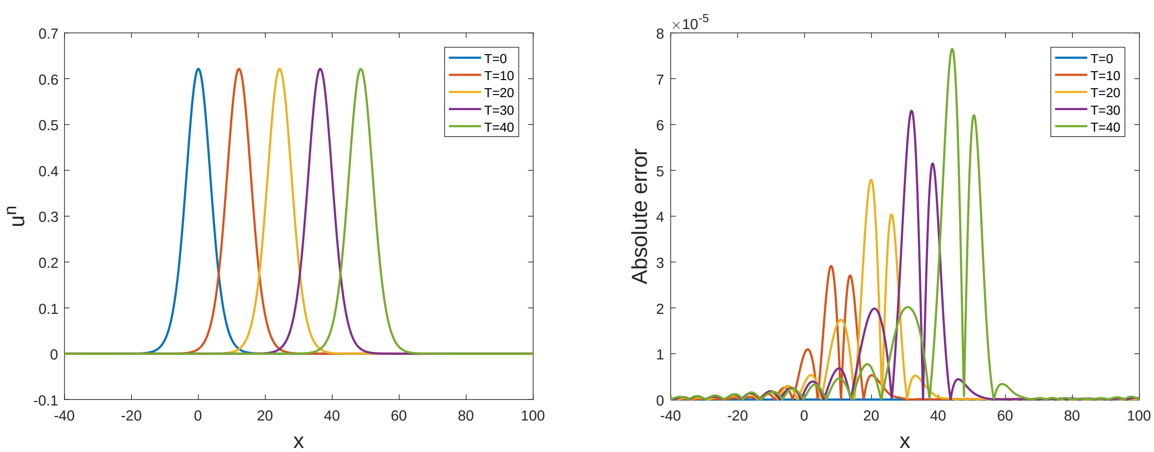

Figure 1: Numerical traveling waves (left) and absolute errors (right) at different times with $\gamma = 1$, $h = 0.0625$, $\tau = h^2$ for Example 1.

Table 3: The errors and convergence rates at $T = 40$ with $\gamma = 0$, $h = 0.5$, $\tau = h^2$ for Example 2.

	$\|e^n\|_\infty$	Rate	$\|e^n\|$	Rate
$h = 0.5$	1.2079E-3	–	3.3931E-3	–
$h = 0.25$	7.4071E-5	4.0274	2.0814E-4	4.0270
$h = 0.125$	4.6994E-6	3.9784	1.2953E-5	4.0062

Table 4: The comparisons of errors with $\gamma = 0$ and $h = 0.4$ for Example 2.

Scheme		$T = 0.1$	$T = 0.2$	$T = 0.4$	$T = 0.5$
DPGM [12]	$\tau = 0.0002$	7.071E-5	1.173E-4	2.076E-4	2.531E-4
	$\tau = 0.0001$	1.77E-5	2.93E-5	5.19E-5	6.33E-5
MSFPM [19]	$\tau = 0.0002$	2.1300E-5	3.9106E-5	7.3106E-5	9.0011E-5
	$\tau = 0.0001$	5.3255E-6	9.7775E-5	1.8279E-5	2.2506E-5
Present scheme	$\tau = 0.005$	4.4644E-6	5.0265E-6	5.9881E-6	6.3340E-6

Example 2. *We consider the parameters to be $\gamma = 0, 1$ and $k = 3$. For this case, the system (1.1) is called to the modified Kawahara equation [18]. The exact traveling wave solution is*

$$u(x,t) = -\frac{3\beta}{\sqrt{30\alpha\mu}}\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{\beta}{5\alpha}}\left(x - \frac{4\beta^2}{25\alpha}t\right)\right),$$

when $\gamma = 0$ and

$$u(x,t) = \pm\frac{3\beta}{\sqrt{30\alpha\mu}}\,\mathrm{sech}^2\left(\frac{1}{2}\sqrt{\frac{\beta}{5\alpha}}\left(x + \frac{25\alpha + 4\beta^2}{25\alpha}t\right)\right),$$

when $\gamma = 1$.

We chose the parameters $\alpha = \beta = 1$, $\mu = 1/3$ and $\Omega = [-40, 100]$, $T = 40$. Table 3 shows the results of errors and convergence rates with $\gamma = 0$, $h = 0.5$, $\tau = h^2$. It means that the present scheme is fourth-order accuracy. Tables 4 and 5 list the error results at $T = 0.1, 0.2, 0.4, 0.5, 1$ with $\gamma = 0$ and $h = 0.4$, which are compared with dual-petrov-Galerkin method (DPGM) in [12], multi-symplectic Fourier pseudo-spectral method (MSFPM) in [19] and three-level linear difference scheme (TLDS) in [23]. We can be seen that the present scheme has much small error than the schemes in [12, 19, 23]. The errors of the long time discrete mass M^n, momentum E^n and Hamiltonian energy H^n with $\gamma = 1$, $h = 0.0625$ and $\tau = h^2$ are shown in Figure 2, which illustrates that Theorem 4.1 holds for the present scheme with small variation error. Numerical traveling waves at different times with $h = 0.125$, $\tau = h^2$ are displayed in Figure 3. We see that numerical solutions agree with the exact solutions in Figure 3.

Example 3. *We now consider oscillatory solitary waves which consist of a packet of solitary waves with arbitrary small perturbations (see [6, 12] and the references therein). It can be described as the following perturbed KdV equation [1]:*

$$u_t + 6uu_x + u_{xxx} + \epsilon^2 u_{xxxxx} = 0. \tag{7.1}$$

The asymptotic solution can be fined by assuming that the solution of (7.1) takes form of a small-

Table 5: The comparisons of errors at $T = 1$ with $\gamma = 0$ and different h and τ for Example 2.

		Present scheme		TLDS [23]	
		$\lVert e^n \rVert$	$\lVert e^n \rVert_\infty$	$\lVert e^n \rVert$	$\lVert e^n \rVert_\infty$
$\tau = 0.001$	$h = 0.4$	4.6815E-6	9.9437E-6	1.5096E-3	6.8396E-4
	$h = 0.2$	5.7421E-7	9.7182E-7	3.8035E-4	1.7360E-4
	$h = 0.1$	3.8538E-7	9.5274E-7	9.5267E-5	4.3286E-5
	$h = 0.05$	3.9368E-7	9.6407E-7	2.3829E-5	1.0833E-5
	$h = 0.025$	3.9311E-7	9.7477E-7	5.9592E-6	2.7087E-6
$\tau/h = 0.1$	$\tau = 0.04$	1.5589E-5	9.2197E-6	1.5056E-3	6.8371E-4
	$\tau = 0.02$	3.7049E-6	1.6793E-6	3.8091E-4	1.7364E-4
	$\tau = 0.01$	9.5274E-7	3.8538E-7	9.5405E-5	4.3389E-5
	$\tau = 0.005$	2.4068E-7	9.6870E-8	2.3862E-5	1.0855E-5
	$\tau = 0.0025$	7.7179E-8	3.7225E-8	5.9663E-6	2.7121E-6

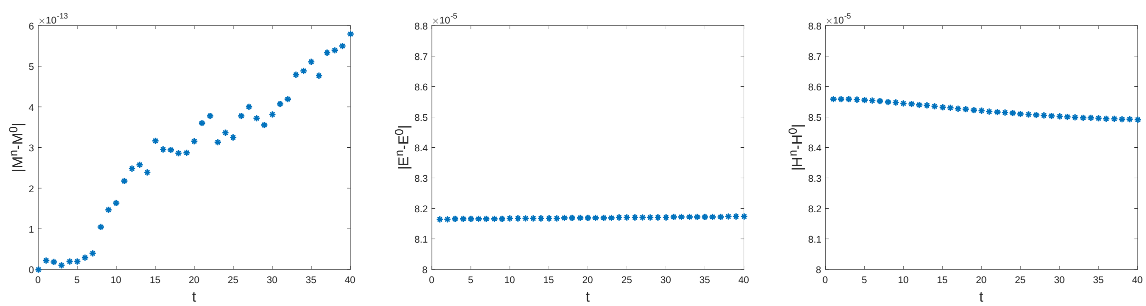

Figure 2: The errors of the long time discrete conservative properties with $\gamma = 1$, $h = 0.0625$ and $\tau = h^2$ for Example 2.

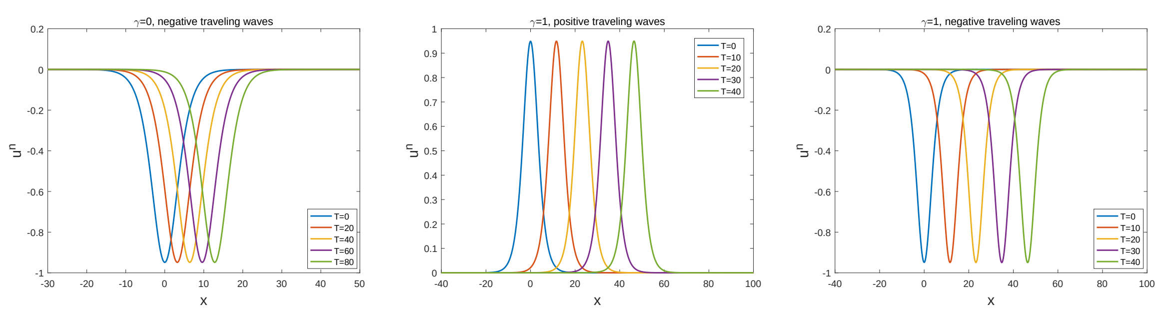

Figure 3: Numerical traveling waves at different times with $h = 0.125$, $\tau = h^2$ for Example 2.

Table 6: The errors between the numerical solutions and the asymptotic solution at different times for Example 3.

	$T = 0.05$	$T = 0.1$	$T = 0.15$	$T = 0.2$
$\|e^n\|$	2.1864E-5	2.9656E-5	2.9719E-5	3.5008E-5
$\|e^n\|_\infty$	7.6791E-5	1.2620E-5	1.3258E-5	1.7396E-5

amplitude modulated wave packet and using two-scale expansion [12] correct to $\mathcal{O}(\epsilon^2)$*,*

$$\begin{aligned} u_{ex}(x,t) &= \sqrt{\frac{2}{19}}\epsilon\cos(\theta\xi+\phi_0)\operatorname{sech} X + \epsilon^2\left\{\frac{187}{57\sqrt{19}}\sin(\theta\xi+\phi_0)\operatorname{sech} X\tanh X\right. \\ &\quad \left. -\frac{4}{19}\left(3+\frac{1}{3}\cos(2\theta\xi+2\phi_0)\right)\operatorname{sech}^2 X\right\} + \mathcal{O}(\epsilon^3) \\ &:= \bar{u}(x,t) + \mathcal{O}(\epsilon^3), \end{aligned} \tag{7.2}$$

where $\xi = x - ct$ *and* $X = \epsilon\xi$, $0 < \epsilon \ll 1$.

We rescale (7.1) with $(\tilde{x},\tilde{t}) = (-L^{-1}x, L^{-1}t)$*, still use* (x,t) *to denote* $(\tilde{x},\tilde{t})$*, we are leads to consider the following problem [12]:*

$$u_t - 6uu_x - \frac{1}{L^2}u_{xxx} - \frac{1}{L^4}u_{xxxxx} = 0, \tag{7.3a}$$

$$u(\pm1,t) = u_x(\pm1,t) = u_{xx}(1,t) = 0, \tag{7.3b}$$

$$u(x,0) = \bar{u}(Lx,0). \tag{7.3c}$$

We take $\epsilon = 0.01$, $\theta = \sqrt{0.5}$, $\phi_0 = 0$, $c = 0.25$ *and* $L = 2000$*. In this example, we use* $\tau = 10^{-5}$, $h = 10^{-3}$*. Table 6 lists the errors between the numerical solutions of (7.3) and the asymptotic solution at different times* $T = 0.05, 0.1, 0.15, 0.2$ *which correspond to original times* $T = 100, 200, 300, 400$*. It can be seen that the numerical accuracy is affected by the accuracy of the asymptotic solution which is accurate to the order of* ϵ^3*. We display the numerical solutions and the asymptotic solutions at three different times and shown the errors of the discrete conservative properties in Figures 4-7. These solutions exhibit a highly oscillating behavior in Figures 4-6, but which are well computed by our present scheme.*

8. Conclusion

In this work, we develop a new conservative compact difference scheme based on SAV approach for solving the generalized Kawahara equation. The present compact scheme unconditionally preserves the discrete mass and Hamiltonian energy. The boundedness and convergence estimates are strictly analyzed in detail. Further, We present a fast algorithm of the present scheme and verify the theoretical analysis in the numerical experiments.

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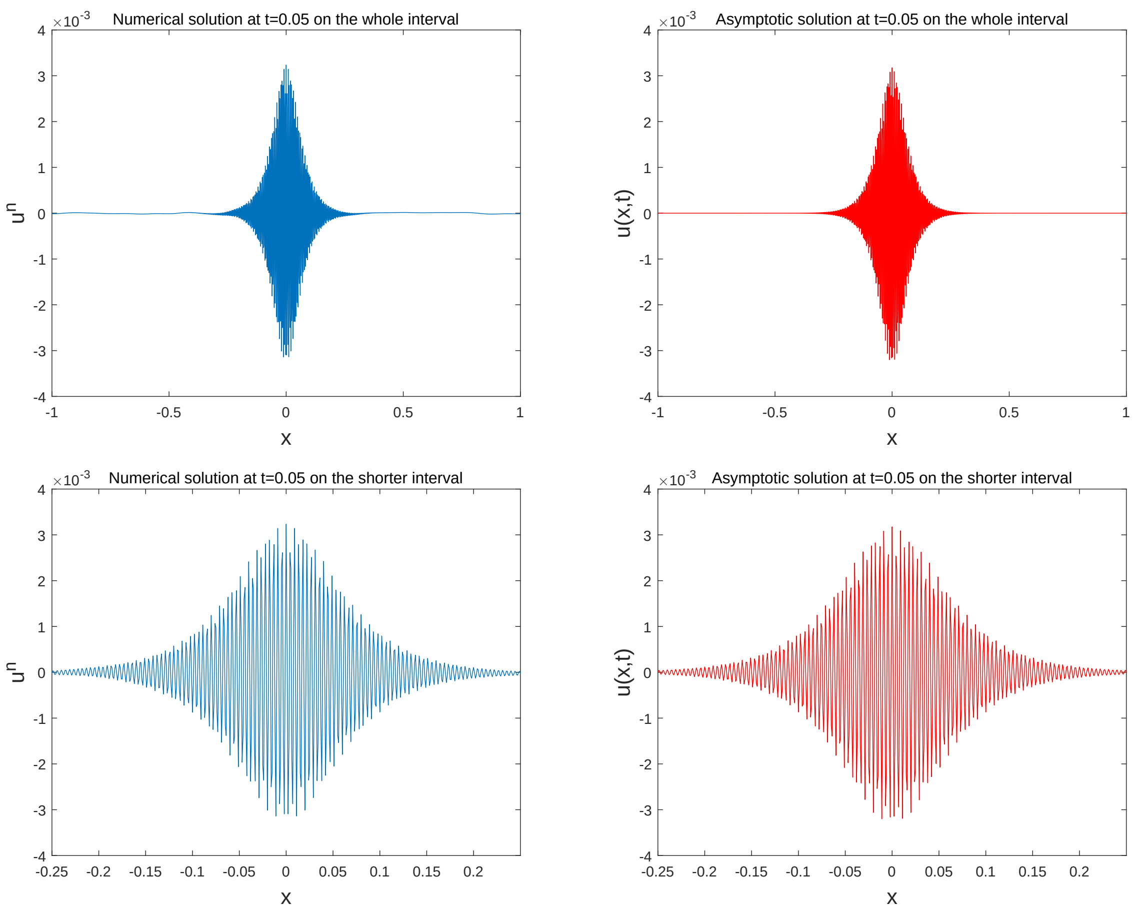

Figure 4: Numerical solution (left) and asymptotic solution (right) at $t = 0.05$ for Example 3.

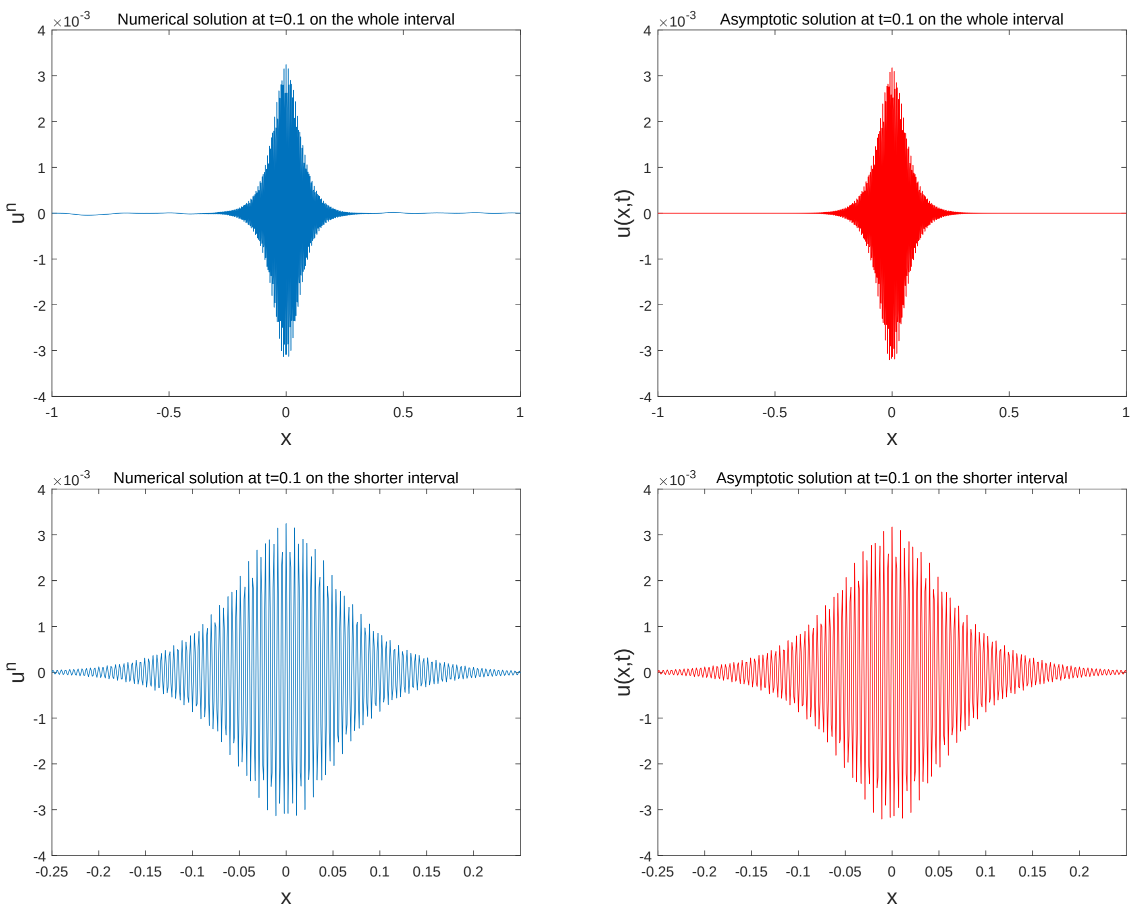

Figure 5: Numerical solution (left) and asymptotic solution (right) at $t = 0.1$ for Example 3.

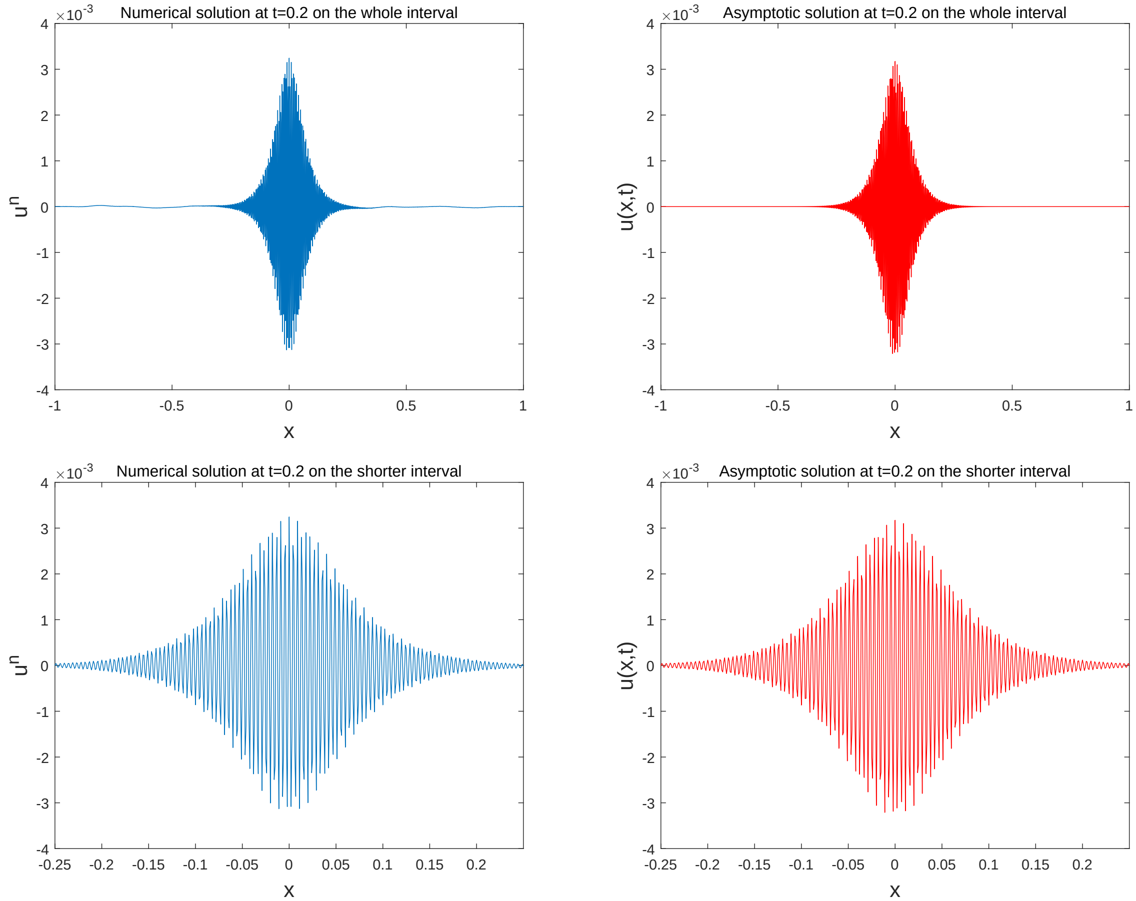

Figure 6: Numerical solution (left) and asymptotic solution (right) at $t = 0.2$ for Example 3.

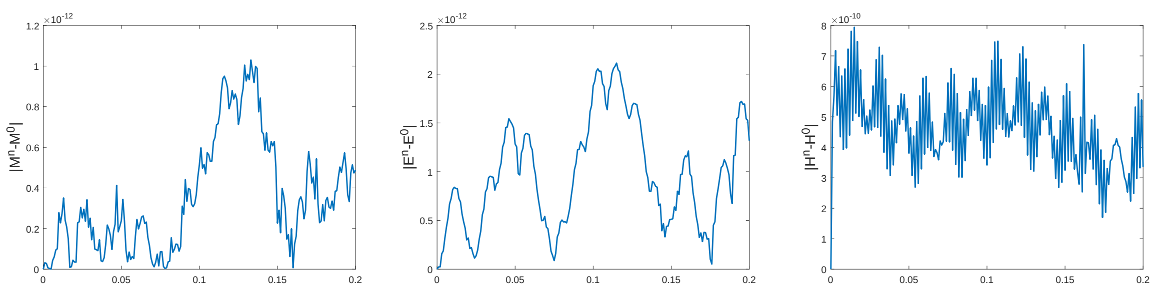

Figure 7: The errors of the discrete conservative properties at different times for Example 3.

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