Ground state solutions for a Kirchhoff type elliptic systems involving critical exponential growth nonlinearities

Shengbing Deng* and Tingting Huang

School of Mathematics and Statistics, Southwest University, Chongqing, 400715, P.R. China

Abstract

The aim of this paper is to study the ground state solution for a Kirchhoff type elliptic systems without the Ambrosetti-Rabinowitz condition.

Keywords: Kirchhoff type elliptic systems; ground state solution; critical exponential growth; without the Ambrosetti-Rabinowitz condition.

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1 Introduction

In this paper, we consider the existence of ground state solution for the following Kirchhoff type elliptic system

$$\begin{cases} -m(\|u\|^2)\Delta u = H_u(x,u,v) & \text{in} \quad \Omega, \\ -m(\|v\|^2)\Delta v = H_v(x,u,v) & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.1}$$

where Ω is a bounded domain in $\mathbb{R}^2$ with smooth boundary, m is a continuous Kirchhoff type function, H_u and H_v have the maximal growth which allows treating (1.1) variationally in the Sobolev space $H_0^1(\Omega,\mathbb{R}^2)$.

The system (1.1) is a Kirchhoff type problem with critical growth. We know that the Kirchhoff problem is nonlocal because of the term $m(\|u\|^2)$. In order to obtain the weak solution, we need the strong convergence. So, the presence of $m(\|u\|^2)$ cause some mathematical difficulties that makes the study of such class of problem interesting. And this class of problem also has physical motivation. In 1883, Kirchhoff studied the hyperbolic equation

$$\rho u_{tt} - \Big(\frac{\rho_0}{h} + \frac{E}{2L}\int_0^L |u_x|^2 dx\Big) u_{xx} = 0$$

that extends the classical D'Alembert wave equation by considering the effects of the changes in the strings during the vibrations. Where L is the length of the string, h is the area of cross-section, E is

*Corresponding author. E-mail address: shbdeng@swu.edu.cn (S. Deng), 117577734@qq.com (T. Huang)

the Young modulus of the material, ρ is the mass density and ρ_0 is the initial tension in [13]. Moreover, there is a lot of literature concerning the existence of solution for Kirchhoff type problem with critical growth. Figueiredo G.M. [10] get the existence of positive solution for a Kirchhoff problem with critical growth via truncation argument. G.M. Figueiredo and U.B. Severo [11] used minimax techniques combined with the Trudinger-Moser inequality to get the ground state solution for a Kirchhoff problem with exponential critical growth.

When the Kirchhoff functions is constant $m(t) = 1$, system (1.1) becomes the following system

$$\begin{cases} -\Delta u = H_u(x, u, v) & \text{in} \quad \Omega, \\ -\Delta v = H_v(x, u, v) & \text{in} \quad \Omega, \\ u = v = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.2}$$

It is a special case of [9]. By the Ekeland variational principle, the Mountain-Pass Theorem and a suitable Trudinger-Moser inequality, Manasses de Souza [9] has obtained the existence of solution to (1.2). (1.2) is a generalization of the well known Dirichlet boundary value problem for one single semilinear elliptic equation

$$\begin{cases} -\Delta u = f(x, u) & \text{in} \quad \Omega, \\ u = 0 & \text{on} \quad \partial\Omega, \end{cases} \tag{1.3}$$

We know critical point theory has become one of the main tools for us to find the solutions for elliptic equation where Ambrosetti-Rabinowitz(AR) condition plays an important role. The reason is (AR) condition ensure that the Palais-Smale sequence of the functional is bounded. The (AR) condition was originally introduced in [2]: there exist $\mu > 2$ and $r > 0$ such that

$$0 < \mu F(x, s) \leq s f(x, s), \ \forall \ |s| \geq r \ \text{ uniformly } \ a.e. \ \ x \in \Omega,$$

where $F(x, s) = \int_0^s f(x, t)dt$. In fact, (AR) condition implies that $F(x, s) \geq C|s|^\mu, \ \forall \ |s| \geq r$. Thus, (1.3) is called superlinear because of f is superlinear at infinity. However, there are many case that the nonlinear term f does not satisfy (AR) condition. So, many authors focused on how to weaken the (AR) condition. Jeanjean[12] replaced (AR) condition with the following conditions:

$$\lim_{s \to \infty} \frac{f(x, s)}{s} \to \infty \ \text{ uniformly for } \ x \in \mathbb{R}^N,$$

$$\exists p \in (2, 2^*), \lim_{s \to \infty} \frac{f(x, s)}{s^{p-1}} = 0 \ \text{ uniformly for } \ x \in \mathbb{R}^N,$$

$$DH(x, s) \geq H(x, t), 0 \leq t \leq s, \ \text{where } D \geq 1 \text{ and } H(x, s) = s f(x, s) - 2F(x, s).$$

They get that (PS) sequence was bounded by using a suitable Mountain Pass Theorem.

Li and Zhou [16] replaced (AR) condition with the following conditions:

$$\lim_{s \to \infty} \frac{f(x, s)}{s} \to \infty \ \text{ uniformly for } \ x \in \mathbb{R}^N,$$

$$\frac{f(x,s)}{s} \text{ is nondecreasing for } s>0, x\in\Omega.$$

They proved the existence of solution to (1.3).

Schechter and Zou[19] replaced (AR) condition with the following conditions:

$$\lim_{s\to\infty}\frac{f(x,s)}{s}\to\infty \text{ uniformly for } x\in\mathbb{R}^N,$$

$$\mu F(x,s)-sf(x,s)\le C(1+s^2),\ |s|\ge r, \text{ for some } \mu>2, r\ge 0.$$

Although it allows more freedom for the function f, we still eliminates many superlinear problems. Hence, recently, N.Lam and G.Lu [14] proved that Cerimi sequence was bounded and established the existence of nontrivial nonnegative solutions by using following conditions:

$$\lim_{s+\to\infty}\frac{f(x,s)}{s^2}\to+\infty \text{ uniformly for } x\in\Omega,$$

$$\theta H(x,s)\ge H(x,t), 0\le t\le s, \text{ where } \theta\ge 1 \text{ and } H(x,s)=sf(x,s)-2F(x,s).$$

Motivated by [9] and the result has been studied about Kirchhoff type problem with critical growth, we are interested in the ground state solution for a Kirchhoff type elliptic systems without the Ambrosetti-Rabinowitz Condition. First, let us introduce some notations:

$H_0^1(\Omega,\mathbb{R}^2)$ denotes the Sobolev modeled in $L^2(\Omega,\mathbb{R}^2)$ with the scalar product

$$\langle W,\Phi\rangle=\int_\Omega \nabla u\nabla\varphi dx+\int_\Omega\nabla v\nabla\psi dx,$$

where $W=(u,v)$, $\Phi=(\varphi,\psi)$, and $|W|=\sqrt{u^2+v^2}$, the corresponds norm $\|W\|=\langle W,W\rangle^{\frac{1}{2}}$.

We denote by $H_W(x,u,v)=(H_u(x,u,v),H_v(x,u,v))$, where $H_W(x,u,v)$ stands for the gradient of H in the variables $W=(u,v)\in\mathbb{R}^2$.

From the work of Adimurthi [1] and de Figueiredo et al. [8], we say that $h:\Omega\times\mathbb{R}^2$ has critical growth at $+\infty$ if there exists $\alpha_0>0$ such that

$$\lim_{|W|\to+\infty}\frac{|h(x,W)|}{e^{\alpha|W|^2}}=\begin{cases}0, & \text{uniformly on } x\in\Omega,\ \forall\alpha>\alpha_0,\\ +\infty, & \text{uniformly on } x\in\Omega,\ \forall\alpha<\alpha_0.\end{cases}$$

Then, in order to solve our problem we give following necessary condition:

(M_1) there exists $m_0>0$ such that $m(t)\ge m_0$ for all $t\ge 0$ and

$$M(t+s)\ge M(t)+M(s),\quad \forall\ s,t\ge 0.$$

where $M(t)=\int_0^t m(s)ds$.

(M_2) there exists $\theta>1$ such that $\frac{m(t)}{t^{\theta-1}}$ is nonincreasing for $t>0$.

Noting that the condition (M_1) shows that M is nondecreasing and $m(t)=m_0+ct, c>0$ is a valid example of a function m satisfying the conditions (M_1) and (M_2).

Remark 1.1. *We observe that* (M_2) *show that for* $0 < t_1 < t_2$

$$\begin{aligned}\theta M(t_1) - m(t_1)t_1 =& \theta M(t_2) - \theta \int_{t_1}^{t_2} m(t)dt - \frac{m(t_1)}{t_1^{\theta-1}} t_1^{\theta} \\ \leq& \theta M(t_2) - \frac{m(t_2)}{t_2^{\theta-1}}(t_2^{\theta} - t_1^{\theta}) - \frac{m(t_2)}{t_2^{\theta-1}} t_1^{\theta} \\ =& \theta M(t_2) - m(t_2)t_2.\end{aligned}$$

Thus, for $t > 0$

$$\theta M(t) - m(t)t \quad \textit{is nondecreasing.} \tag{1.4}$$

In particular, we can deduce that

$$M(t) \leq M(1)t^{\theta}, \quad \forall t > 1. \tag{1.5}$$

Here, we also require that $H : \Omega \times \mathbb{R}^2 \to \mathbb{R}$ is continuous and admits partial derivatives H_u and H_v of class $C(\Omega \times \mathbb{R}^2)$, $H(x, u, v) = H_u(x, u, v) = H_v(x, u, v) = 0$ if $u \leq 0$ or $v \leq 0$. Moreover, we assume H satisfy following condition:

(H_1) $\lim\limits_{|W|\to\infty} \frac{H(x,W)}{|W|^{2\theta}} = +\infty$ uniformly in Ω.

(H_2) $G(x, tW) \leq G(x, W)$ for all $0 < t < 1, x \in \Omega$. where

$$G(x, W) = H_W(x, W)W - 2\theta H(x, W).$$

(H_3) $\limsup\limits_{|W|\to 0} \frac{2H(x,W)}{|W|^2} < \lambda_1 m_0$, uniformly in $x \in \Omega$, where

$$\lambda_1 = \inf \Big\{ \frac{\|W\|^2}{\int_{\Omega} |W|^2 dx} : W \in H_0^1(\Omega, \mathbb{R}^2)\backslash\{0\} \Big\}.$$

Remark 1.2. *According to the condition* (H_2)*, we know that* $G(x, tu, tv)$ *is increasing where* $u, v \in H_0^1(\Omega)$ *with* $s \in \{u, v\}$*, it is easily to get that* $H_s'(x, tu, tv) \geq (2\theta - 1)\frac{H_s(x,tu,tv)}{ts}$*. This implies that* $\Big[\frac{H_s(x,tu,tv)}{(ts)^{2\theta-1}}\Big]' \geq 0$*. Thus, we get following important result,*

$$\frac{H_s(x, tu, tv)}{(ts)^{2\theta-1}} \quad \textit{is nondecreasing.} \tag{1.6}$$

Finally, we state our main result as follow:

Theorem 1.1. *Assume that* $(M_1) - (M_2)$*,* $(H_1) - (H_3)$ *hold and* H_u*,* H_v *satisfy critical exponential growth. Furthermore, assume that*

(H_4) *there exists* $\beta_0 > \frac{m(\frac{\pi}{\alpha_0})}{\alpha_0 d^2 e}$ *such that for some* $s \in \{u, v\}$*,*

$$\liminf_{|W|\to+\infty} \frac{sH_s(x, u, v)}{\exp(4\alpha_0 |W|^2)} \geq \beta_0, \quad \textit{uniformly in} \;\; x \in \Omega,$$

where d *is the inner radius of the largest open ball contain in* Ω*. Then, problem (1.1) has a positive ground state solution.*

This paper have some difficulties because of the presence of the Kirchhoff function and the nonlinearities do not satisfy Ambrosetti-Rabinowitz condition. Therefore, this paper is organized as follow: In Section 2, we give some necessary definition and lemmas in order to overcome the dificulty of the nonlinearities without Ambrosetti-Rabinowitz condition. In Section 3, we prove the energy functional satisfies geometric construction. In Section 4, we first prove the boundedness of $\{(u_n, v_n)\}$, then we will complete the proof of our main results.

2 Preliminary Result

The Euler-Lagrange functional associated with problem (1.1) is

$$I(W) = \frac{1}{2}M(\|u\|^2) + \frac{1}{2}M(\|v\|^2) - \int_\Omega H(x,u,v)dx,$$

where $W = (u, v)$. Under our assumptions about m, H_u, H_v are continuous and have critical growth at $+\infty$, by (H_3) and the Sobolev embedding it follows that $H(x,u,v) \in L^1(\Omega, \mathbb{R}^2)$, which implies that I is well defined. Moreover, we can see that $I \in C^1\big(H_0^1(\Omega,\mathbb{R}^2), \mathbb{R}\big)$ with

$$\langle I'(W), \Phi\rangle = m(||u||^2)\langle u, \varphi\rangle + m(||v||^2)\langle v, \psi\rangle - \int_\Omega \Phi \cdot H_W(x,u,v)dx, \quad \forall \Phi = (\varphi, \psi) \in H_0^1(\Omega, \mathbb{R}^2).$$

where $\langle u, \varphi\rangle = \int_\Omega \nabla u \nabla \varphi dx, \langle v, \psi\rangle = \int_\Omega \nabla v \nabla \psi dx$. Consequently, the critical points of the functional I are precisely the weak solutions of problem (1.1).

In this paper, we need deal with systems without the Ambrosetti-Rabinowitz condition. So, we will use some classical facts introduced by Cerami ([2, 5, 3, 4]) to prove Theorem 1.1.

Definition 2.1. *For $c \in \mathbb{R}$, we say that I satisfies the $(C)_c$ condition if for any sequence $\{u_n\} \subset H_0^1(\Omega)$ with*

$$(1 + \|u_n\|)\|I'(u_n)\| \to 0, \quad and \quad I(u_n) \to c,$$

there is a subsequence $\{u_n\}$ such that $\{u_n\}$ converges strongly in $H_0^1(\Omega)$.

Moreover, we give the following versions of Mountain Pass Theorem to prove our result.

Lemma 2.1. *[6] Let E be a real Banach space, $I \in C^1(E, \mathbb{R})$ satisfies $I(0) = 0$ and*
(i) There are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$.
(ii) There is an $e \in E \backslash B_\rho$ such that $I(e) \leq 0$.
Let c be characterized by $c = \inf\limits_{\gamma\in\Gamma} \max\limits_{0\leq t\leq 1} I(\gamma(t))$, where

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}.$$

Then I possesses a $(C)_c$ sequence.

Lemma 2.2. *[6] Let E be a real Banach space, $I \in C^1(E, \mathbb{R})$ satisfies the $(C)_c$ condition for any $c \in \mathbb{R}$, and*
(i) There are constants $\rho, \alpha > 0$ such that $I|_{\partial B_\rho} \geq \alpha$.
(ii) There is an $e \in E \backslash B_\rho$ such that $I(e) \leq 0$.
Then,

$$c = \inf_{\gamma\in\Gamma} \max_{0\leq t\leq 1} I(\gamma(t)) \geq \alpha$$

is a critical value of I, where

$$\Gamma = \{\gamma \in C([0,1], E) : \gamma(0) = 0, \gamma(1) = e\}.$$

Next, we introduce some famous inequalities for the equation.

Lemma 2.3. *[18] Let Ω be a bounded domain in $\mathbb{R}^2$ and $u \in H_0^1(\Omega)$. Then for every $\beta > 0$,*

$$\int_\Omega e^{\beta|u|^2} < +\infty.$$

Moreover, there exists a constant $C > 0$ such that

$$\sup_{u \in H_0^1(\Omega): ||u|| \le 1} \int_\Omega e^{\beta|u|^2} \le C|\Omega|, \quad \forall \beta \le 4\pi,$$

where 4π is the best constant, that is, the supreme in the left is $+\infty$ if $\beta > 4\pi$.

More specifically, P.-L.Lions proved the following:

Lemma 2.4. *[17] Let $\{u_n\}$ be a sequence of functions in $H_0^1(\Omega)$ with $||u_n|| = 1$ such that $u_n \rightharpoonup u \neq 0$ weakly in $H_0^1(\Omega)$. Then*

$$\sup_n \int_\Omega e^{p|u_n|^2} < +\infty, \ \forall \ 0 < p < 4\pi/(1 - ||u||^2).$$

Delighted by the above inequalities, we have the following result.

Lemma 2.5. *Let $\{W_n\}$ be a sequence of functions in $H_0^1(\Omega, \mathbb{R}^2)$ with $||W_n|| = 1$ such that $W_n \rightharpoonup W \neq 0$ weakly in $H_0^1(\Omega, \mathbb{R}^2)$. Then*

$$\sup_n \int_\Omega e^{\beta|W_n|^2} < +\infty, \ \forall \ 0 < \beta < 2\pi/(1 - ||W||^2).$$

Proof. Since $W_n \rightharpoonup W \neq 0$ and $||W_n|| = 1$, we have

$$||W_n - W_0||^2 = 1 - 2\langle W_n, W_0 \rangle + ||W_0||^2 \to 1 - ||W_0||^2 < \frac{2\pi}{\beta},$$

then

$$\beta ||W_n - W_0||^2 < 2\pi.$$

By Hölder inequality and Lemma 2.3, we obtain that

$$\int_\Omega e^{\beta r_1(1+\varepsilon^2)|W_n - W_0|^2} \le C_1 \Big[\int_\Omega e^{2\beta r_1(1+\varepsilon^2)||W_n - W_0||^2 (\frac{u_n - u_0}{||W_n - W_0||})^2} \Big]^{\frac{1}{2}} \Big[\int_\Omega e^{2\beta r_1(1+\varepsilon^2)||W_n - W_0||^2 (\frac{v_n - v_0}{||W_n - W_0||})^2} \Big]^{\frac{1}{2}} \le C_2.$$

where $r_1 > 1$ close to 1 and $\varepsilon > 0$ satisfying

$$\beta r_1(1 + \varepsilon^2)||W_n - W_0||^2 \le 2\pi.$$

Moreover, from Lemma 2.3, Hölder inequality and $|W| = \sqrt{u^2 + v^2}$

$$\int_\Omega e^{\beta|W|^2} = \int_\Omega e^{\beta(u^2+v^2)} = \int_\Omega e^{\beta|u|^2} e^{\beta|v|^2} \le \left(\int_\Omega e^{2\beta|u|^2} \right)^{\frac{1}{2}} \left(\int_\Omega e^{2\beta|v|^2} \right)^{\frac{1}{2}} < +\infty.$$

Using Young inequality, we have

$$\beta|W_n|^2 \le \beta(1+\varepsilon^2)|W_n - W_0|^2 + \beta(1+1/\varepsilon^2)|W_0|^2.$$

Thus, it togethers with Hölder inequality can deduce that

$$\int_\Omega e^{\beta|W_n|^2} \le \left(\int_\Omega e^{\beta r_1(1+\varepsilon^2)|W_n-W_0|^2}\right)^{\frac{1}{r_1}} \left(\int_\Omega e^{\beta r_2(1+1/\varepsilon^2)|W_0|^2}\right)^{\frac{1}{r_2}} \le C_1 \left(\int_\Omega e^{\beta r_2(1+1/\varepsilon^2)|W_0|^2}\right)$$

for large n, where $r_2 = \frac{r_1}{r_1-1}$. The second term in the last inequality is bounded and the result is proved. □

3 Mountain Pass Structure

Lemma 3.1. *Suppose that (M_1) and (H_3) hold, H_u and H_v satisfy critical exponential growth. Then, there exists $\eta, \rho > 0$ such that $I(W) \ge \eta$ if $\|W\| = \rho$.*

Proof. We know that H_u, H_v satisfy critical exponential growth, there exist suitable constant $C > 0, \tau > 0$ and $q > 2$, when $|W| \ge \tau$ and $\alpha > \alpha_0$, we have that

$$|H_u(x,u,v)| \le C|W|^{q-1}e^{\alpha|W|^2} \quad \text{and} \quad |H_v(x,u,v)| \le C|W|^{q-1}e^{\alpha|W|^2}$$

Given $\varepsilon > 0$, the above inequalities together with (H_3) yield that

$$|H(x,W)| \le \frac{1}{2}(\lambda_1 m_0 - \varepsilon)|W|^2 + C_1|W|^q e^{\alpha|w|^2}, \quad \forall (x,W) \in \Omega \times \mathbb{R}^2.$$

By Hölder inequality, Young inequality, Lemma 2.3 and Sobolev embedding, we can obtain that

$$\begin{aligned}
\int_\Omega |W|^q e^{\alpha|W|^2} \le & C_2\|W\|^q_{qr_3}\Big[\int_\Omega e^{r_4\alpha|W|^2}dx\Big]^{\frac{1}{r_4}} \\
\le & C_3\|W\|^q_{qr_3}\Big[\int_\Omega e^{2r_4\alpha\|W\|^2\cdot\left|\frac{u}{\|W\|}\right|^2}dx + \int_\Omega e^{2r_4\alpha\|W\|^2\cdot\left|\frac{v}{\|W\|}\right|^2}dx\Big]^{\frac{1}{r_4}} \\
\le & C_4\|W\|^q,
\end{aligned}$$

if $r_4 > 1$ close to 1 with $r_4\alpha\|W\|^2 = r_4\alpha\rho_1^2 < 2\pi$ and $\frac{1}{r_3} + \frac{1}{r_4} = 1$. Using the definition of λ_1, Sobolev embedding and (M_1), it follow that

$$\begin{aligned}
I(W) = & \frac{1}{2}M(\|u\|^2) + \frac{1}{2}M(\|v\|^2) - \int_\Omega H(x,u,v)dx \\
\ge & \frac{1}{2}m_0\|u\|^2 + \frac{1}{2}m_0\|v\|^2 - \frac{1}{2}(\lambda_1 m_0 - \varepsilon)\|W\|_2^2 - C_5\|W\|^q \\
\ge & \frac{1}{2}m_0\|W\|^2 - \frac{1}{2}(m_0 - \frac{\varepsilon}{\lambda_1})\|W\|^2 - C_5\|W\|^q \\
= & (\frac{\varepsilon}{2\lambda_1} - C_5\|W\|^{q-2})\|W\|^2.
\end{aligned}$$

Since $q > 2$, we can choose a suitable $\rho_2 > 0$ such that $\frac{\varepsilon}{2\lambda_1} - C\rho_2^{q-2} > 0$. Consequently, $I(W) \ge \eta$ when $\|W\| = \rho = \min\{\rho_1, \rho_2\}$, where $\eta = (\frac{\varepsilon}{2\lambda_1} - C\rho^{q-2})\rho^2$. □

Lemma 3.2. *Suppose that* (M_2) *and* (H_1) *hold, then* $I(tW) \to -\infty$ *as* $t \to +\infty$ *for all nonnegative* $W \in H_0^1(\Omega, \mathbb{R}^2) \setminus \{0\}$.

Proof. Fix $W \in H_0^1(\Omega, \mathbb{R}^2) \setminus \{0\}$ and $W \geq 0$, by (H_1), there exists $L > \frac{M(1)\|W\|^{2\theta}}{2\|W\|_{2\theta}^{2\theta}} > 0$ and constant $C > 0$ such that $H(x, W) \geq L|W|^{2\theta} - C$ when $|W| > L$. Then, using (1.5) follow that

$$
\begin{aligned}
I(tW) =& \frac{1}{2}M(\|tu\|^2) + \frac{1}{2}M(\|tv\|^2) - \int_\Omega H(x, tW)dx \\
\leq& \frac{1}{2}M(1)t^{2\theta}\|u\|^{2\theta} + \frac{1}{2}M(1)t^{2\theta}\|v\|^{2\theta} - Lt^{2\theta}\|W\|_{2\theta}^{2\theta} + C \\
\leq& t^{2\theta}[\frac{1}{2}M(1)\|W\|^{2\theta} - L\|W\|_{2\theta}^{2\theta}] + C.
\end{aligned}
$$

Consequently, we can conclude that $I(tW) \to -\infty$ as $t \to +\infty$. □

4 Proof of The Main Result

In order to get a more precise information about the minimax level c, let us consider the following sequence which was introduced in [7]:

$$
D_n(t) = \begin{cases} \frac{t}{n^{1/2}}(1-\delta_n)^{1/2}, & \text{if } \ 0 \leq t \leq n, \\ \frac{1}{\big[n(1-\delta_n)\big]^{1/2}} \log \frac{Z_n+1}{Z_n+e^{-(t-n)}} + \big[n(1-\delta_n)\big]^{1/2}, & \text{if } \quad t \geq n, \end{cases}
$$

where Z_n is defined as $Z_n = \frac{1}{en^2} + O(\frac{1}{n^4})$ and $\delta_n = \frac{2\log n}{n}$. We have

$$
\begin{cases} \{D_n\} \subset C\big([0, +\infty)\big), \text{ piecewise differentiable, with } D_n(0) = 0 \text{ and } D_n'(t) \geq 0; \\ \int_0^{+\infty} |D_n'(t)|^2 = 1; \\ \lim\limits_{n \to +\infty} \int_0^{+\infty} e^{D_n^2(t) - t} dt = 1 + e. \end{cases}
$$

Now, let $D_n(t) = 2\sqrt{\pi}\tilde{G}_n(e^{-\frac{t}{2}})$ with $|x| = e^{-\frac{t}{2}}$, define a function $\tilde{G}_n(x) = \tilde{G}_n(|x|)$ on $\overline{B_1(0)}$, which is nonnegative and radially symmetric. Moreover, we can conclude that

$$
\int_{B_1(0)} |\nabla \tilde{G}_n(x)|^2 dx = \int_0^{+\infty} |D_n'(t)|^2 = 1.
$$

Thus, we can get that $||\tilde{G}_n|| = 1$. Let $x_0 \in \Omega$ be such that the open ball $B_d(x_0)$ is contained in Ω, where d was given in (H_4). Considering

$$
E_{n,d}(x) := \big(G_{n,d}(x), 0\big), \quad \text{where} \quad G_{n,d}(x) := \tilde{G}_n\Big(\frac{x - x_0}{d}\Big),
$$

then $E_{n,d}(x)$ belongs to $H_0^1(\Omega, \mathbb{R}^2)$ with $||E_{n,d}|| = 1$, and the support of $E_{n,d}$ is contained in $B_d(x_0)$.

Lemma 4.1. *If* $(M_1), (M_2)$ *and* (H_4) *hold, then* $c < \frac{1}{2}M(\frac{\pi}{\alpha_0})$, *where* $c = \inf\limits_{\gamma \in \Gamma} \max\limits_{0 \leq t \leq 1} I(\gamma(t))$.

Proof. As in the proof of Lemma 3.2, we have that $I(tE_{n,d}) \to -\infty$ as $t \to +\infty$ because of $||E_{n,d}|| = 1$. Consequently, $c \leq \max\limits_{t>0} I(tE_{n,d})$ for all $n \in \mathbb{N}$. Thus, it suffices to show that $\max\limits_{t>0} I(tE_{n,d}) < \frac{1}{2}M(\frac{\pi}{\alpha_0})$ for some $n \in \mathbb{N}$. Suppose by contradiction that

$$\max_{t>0} I(tE_{n,d}) \geq \frac{1}{2}M\Big(\frac{\pi}{\alpha_0}\Big), \quad \forall\ n \in \mathbb{N}. \tag{4.1}$$

As I possess the Mountain-pass geometry, there exists $t_n > 0$ such that

$$I(t_nE_{n,d}) = \max_{t>0} I(tE_{n,d}).$$

Which means that $M(t_n^2) \geq M\big(\frac{\pi}{\alpha_0}\big)$ due to the fact that $\int_\Omega H(x, t_nG_{n,d}, 0) = 0$. From (M_1), we have

$$t_n^2 \geq \frac{\pi}{\alpha_0}. \tag{4.2}$$

On the other hand, since $\frac{d}{dt}I(tE_{n,d})\,|_{t=t_n} = 0$, it follows that

$$m(t_n^2)t_n^2 = \int_\Omega t_nE_{n,d} \cdot H_W(x, t_nE_{n,d})dx \geq \int_{B_{\frac{d}{n}}(x_0)} t_nE_{n,d} \cdot H_W(x, t_nE_{n,d})dx \tag{4.3}$$

By (H_4), given $\delta > 0$ there exists $s_\delta > 0$ such that

$$uH_u(x, W) \geq (\beta_0 - \delta)e^{4\alpha_0|W|^2}, \ \ \forall\ x \in \Omega, \ \ |W| \geq s_\delta. \tag{4.4}$$

According to above equations and using polar transformation, we can deduce that

$$\begin{aligned}
m(t_n^2)t_n^2 \geq &(\beta_0 - \delta) \int_{B_{\frac{d}{n}}(x_0)} e^{4\alpha_0|t_nG_{n,d}|^2}dx \\
=&(\beta_0 - \delta)\Big(\frac{d}{n}\Big)^2 \int_{B_1(0)} e^{4\alpha_0|t_n\tilde{G}_n|^2}dx \\
=&2\pi(\beta_0 - \delta)\Big(\frac{d}{n}\Big)^2 \int_0^1 e^{4\alpha_0|t_n\tilde{G}_n(\rho)|^2}\rho d\rho.
\end{aligned}$$

Setting $\rho = e^{-\frac{t}{2}}$, then

$$\begin{aligned}
m(t_n^2)t_n^2 \geq &\pi(\beta_0 - \delta)\Big(\frac{d}{n}\Big)^2 \int_0^{+\infty} e^{\frac{\alpha_0|t_nD_n(t)|^2}{\pi}}e^{-t}dt \\
\geq &\pi(\beta_0 - \delta)\Big(\frac{d}{n}\Big)^2 \int_n^{+\infty} e^{\frac{\alpha_0t_n^2(n-2\log n)}{\pi}}e^{-t}dt \\
=&\pi d^2(\beta_0 - \delta)e^{[\frac{\alpha_0t_n^2(n-2\log n)}{\pi} - 2\log n - n]} \\
=&\pi d^2(\beta_0 - \delta)e^{[(\frac{\alpha_0t_n^2}{\pi} - 1)n - (\frac{\alpha_0t_n^2}{\pi} + 1)2\log n]}.
\end{aligned} \tag{4.5}$$

According to (1.5), if $t_n \to +\infty$, we have that

$$\frac{m(t_n^2)t_n^2}{e^{t_n^2n\Big[\frac{\alpha_0(1-\frac{2\log n}{t_n^2n})}{\pi} - \frac{2\log n + n}{t_n^2n} - \frac{\log(m(t_n^2)t_n^2)}{t_n^2n}\Big]}} \to +\infty.$$

Which is a contradiction with (4.5). Thus $\{t_n\}$ is bounded. Moreover, using (4.5) again, we have $\frac{\alpha_0 t_0^2}{\pi} - 1 \leq 0$, this together with (4.2) deduce that

$$t_n^2 \to \frac{\pi}{\alpha_0}. \tag{4.6}$$

Next, in view of (4.4), for $0 < \delta < \beta_0$ and $n \in \mathbb{N}$ we set

$$U_{n,\delta} := \{x \in B_d(x_0) : t_n G_{n,d} \geq s_\delta\} \;\text{ and }\; V_{n,\delta} := B_d(x_0) \backslash U_{n,\delta}.$$

Thus, by splitting the integral (4.3) on $U_{n,\delta}$ and $V_{n,\delta}$, and using (4.4), it follows that

$$m(t_n^2)t_n^2 \geq (\beta_0 - \delta) \int_{B_d(x_0)} e^{4\alpha_0 t_n^2 G_{n,d}^2} dx - (\beta_0 - \delta) \int_{E_{n,\delta}} e^{4\alpha_0 t_n^2 G_{n,d}^2} dx + \int_{E_{n,\delta}} t_n G_n H_u(x, t_n E_{n,d}) dx. \tag{4.7}$$

Since $G_{n,d}(x) \to 0$ for almost everywhere $x \in B_d(x_0)$, we have that the characteristic functions $\chi_{E_{n,\delta}}$ satisfy

$$\chi_{V_{n,\delta}} \to 1 \;\text{ a.e. in }\; B_d(x_0) \;\text{ as }\; n \to +\infty.$$

Moreover, $t_n G_{n,d} < s_\delta$ in $V_{n,\delta}$. By the Lebesgue dominated convergence theorem, we have

$$\int_{V_{n,\delta}} e^{4\alpha_0 t_n^2 G_{n,d}^2} dx \to \pi d^2 \;\text{ and }\; \int_{E_{n,\delta}} t_n G_{n,d} H_u(x, t_n E_{n,d}) dx \to 0, \;\text{ as }\; n \to +\infty.$$

Note that the definition of $\widetilde{G}_n$, then using polar transformation can obtain that

$$\int_{B_d(x_0)} e^{4\alpha_0 t_n^2 G_{n,d}^2} dx = d^2 \int_{B_1(0)} e^{4\alpha_0 t_n^2 \tilde{G}_n^2} dx = 2\pi d^2 \int_0^1 e^{4\alpha_0 t_n^2 \tilde{G}_n^2(\rho)} \rho d\rho.$$

Setting $\rho = e^{-\frac{t}{2}}$, as $n \to +\infty$, we obtain

$$\begin{aligned} \int_{B_d(x_0)} e^{4\alpha_0 t_n^2 G_{n,d}^2} dx &= \pi d^2 \int_0^{+\infty} e^{\frac{\alpha_0 |t_n D_n(t)|^2}{\pi}} e^{-t} dt \\ &\geq \pi d^2 \int_0^{+\infty} e^{D_n^2(t) - t} dt \\ &\to \pi d^2 (1 + e), \;\text{ as }\; n \to +\infty. \end{aligned}$$

Passing to limit in (4.7) can obtain that

$$m\Big(\frac{\pi}{\alpha_0}\Big)\frac{\pi}{\alpha_0} \geq (\beta_0 - \delta)\Big[\pi d^2(1 + e) - \pi d^2\Big] = (\beta_0 - \delta)\pi d^2 e, \tag{4.8}$$

and let $\delta \to 0^+$, we get $\beta_0 \leq \frac{m(\frac{\pi}{\alpha_0})}{\alpha_0 d^2 e}$, which contradicts (H_4). Thus, this lemma is proved. $\square$

At this stage, we define the Nehari manifold associated to the functional I as

$$\mathcal{N} := \{W \in H_0^1(\Omega, \mathbb{R}^2) : \langle I'(W), W \rangle = 0, \; W \neq 0\}$$

and the number $b := \inf_{W \in \mathcal{N}} I(W)$. The ground state refers to minimizers of the corresponding energy within the set of nontrivial solutions. It is crucial to compare the minimax level c with b for our result. Let us describe this comparison in following Lemma:

Lemma 4.2. *If* (M_2) *and* (H_2) *hold, then* $c \leq b$, *where* $c = \inf_{\gamma\in\Gamma}\max_{0\leq t\leq 1} I(\gamma(t))$

Proof. Let $W \in \mathcal{N}$ and define $g : (0, +\infty) \to \mathbb{R}$ as $g(t) = I(tW)$. We have that g is differentiable and by $\langle I'(W), W\rangle = 0$ can deduce that

$$\begin{aligned}
g'(t) =&\langle I'(tW), W\rangle = m(t^2||u||^2)t||u||^2 + m(t^2||v||^2)t||v||^2 - \int_\Omega WH_W(x, tW)dx \\
=&[\frac{m(t^2||u||^2)}{(t^2||u||^2)^{\theta-1}} - \frac{m(||u||^2)}{(||u||^2)^{\theta-1}}]t^{2\theta-1}||u||^{2\theta} + [\frac{m(t^2||v||^2)}{(t^2||v||^2)^{\theta-1}} - \frac{m(||v||^2)}{(||v||^2)^{\theta-1}}]t^{2\theta-1}||v||^{2\theta} \\
&+ \int_\Omega [t^{2\theta-1}uH_u(x, u, v) - uH_u(x, tu, tv)]dx + \int_\Omega [t^{2\theta-1}vH_v(x, u, v) - vH_v(x, tu, tv)]dx.
\end{aligned}$$

We know that $H_u(x, u, v) = H_v(x, u, v) = 0$ if $u = 0$ or $v = 0$ and (H_2) can deduce that (1.4), then

$$\begin{aligned}
g'(t) =&[\frac{m(t^2||u||^2)}{(t^2||u||^2)^{\theta-1}} - \frac{m(||u||^2)}{(||u||^2)^{\theta-1}}]t^{2\theta-1}||u||^{2\theta} + [\frac{m(t^2||v||^2)}{(t^2||v||^2)^{\theta-1}} - \frac{m(||v||^2)}{(||v||^2)^{\theta-1}}]t^{2\theta-1}||v||^{2\theta} \\
&+t^{2\theta-1}\int_\Omega \Big[u^{2\theta}\Big[\Big(\frac{H_u(x, u, v)}{u^{2\theta-1}} - \frac{H_u(x, tu, tv)}{(tu)^{2\theta-1}}\Big) + v^{2\theta}\Big(\frac{H_v(x, u, v)}{v^{2\theta-1}} - \frac{H_v(x, tu, tv)}{(tv)^{2\theta-1}}\Big)dx\Big]dx.
\end{aligned}$$

Thus, from $(M_2), (1.4)$ can deduce that $g'(1) = 0, g'(t) \geq 0$ for $0 < t < 1$ and $g'(t) \leq 0$ for $t > 1$. Consequently,

$$I(W) = \max_{t\geq 0} I(tW).$$

Defining $h : [0, 1] \to H_0^1(\Omega, \mathbb{R}^2)$ by $(t) = tt_0W$, where t_0 is such that $I(t_0W) < 0$. Then,

$$h \in \Gamma \text{ and } c \leq \max_{t\in[0,1]} I(h(t)) \leq \max_{t\geq 0} I(tW) = I(W).$$

Therefore, we can deduce that $c \leq b$ because of $W \in \mathcal{N}$ is arbitrary. □

Lemma 4.3. *Any Cerami sequence associated with the functional* I *is bounded in* $H_0^1(\Omega, \mathbb{R}^2)$.

Proof. Let $\{(u_n, v_n)\}$ be a Cerami sequence, the definition of Cerami sequence shows that

$$(1 + \|(u_n, v_n)\|)\|I'(u_n, v_n)\| \to 0, \tag{4.9}$$

$$I(u_n, v_n) \to c. \tag{4.10}$$

Thus, there exists $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ as $n \to \infty$ such that

$$|\langle I'(u_n, v_n), (\varphi, \psi)\rangle| \leq \frac{\varepsilon_n\|(\varphi, \psi)\|}{1 + \|(u_n, v_n)\|}. \tag{4.11}$$

Let $(\varphi, \psi) = (u_n, v_n)$ in (4.11), we have

$$\begin{aligned}
|\langle I'(u_n, v_n), (u_n, v_n)\rangle| =&\Big|m(\|u_n\|^2)\|u_n\|^2 + m(\|v_n\|^2)\|v_n\|^2 - \int_\Omega u_nH_u(x, u_n, v_n) - \int_\Omega v_nH_v(x, u_n, v_n) \\
\leq&\frac{\varepsilon_n\|(u_n, v_n)\|}{1 + \|(u_n, v_n)\|}
\end{aligned}$$

$\leq C.$

Now, we will prove that $\{(u_n, v_n)\}$ is bounded. Suppose that $\{(u_n, v_n)\}$ is unbounded, then $\|(u_n, v_n)\| \to \infty$ as $n \to \infty$. We let $(\widehat{u_n}, \widehat{v_n}) = \frac{(u_n,v_n)}{\|(u_n,v_n)\|}$, then $\|\widehat{u_n}, \widehat{v_n}\| = 1$. Thus, there exists $(\widehat{u}, \widehat{v}) \in H_0^1(\Omega, \mathbb{R}^2)$ such that

$$\begin{cases} (\widehat{u_n}, \widehat{v_n}) \rightharpoonup (\widehat{u}, \widehat{v}) & \text{in} \quad H_0^1(\Omega, \mathbb{R}^2), \\ (\widehat{u_n}, \widehat{v_n}) \to (\widehat{u}, \widehat{v}) & \text{in} \quad L^p(\Omega, \mathbb{R}^2), \\ (\widehat{u_n}, \widehat{v_n}) \to (\widehat{u}, \widehat{v}) & \text{a.e} \quad \text{in} \quad (\Omega, \mathbb{R}^2). \end{cases}$$

Let $\widehat{u_n}^- = \min\{0, \widehat{u_n}\}, \widehat{v_n}^- = \min\{0, \widehat{v_n}\}$. Obviously, $\{(\widehat{u_n}^-, \widehat{v_n}^-)\}$ is also bounded in $H_0^1(\Omega, \mathbb{R}^2)$. Choosing $(\varphi, \psi) = (\widehat{u_n}^-, \widehat{v_n}^-)$ in (4.11) and $\|(u_n, v_n)\| \to \infty$, we get

$$\begin{aligned} o(1) =& \frac{\langle I'(u_n, v_n), (\widehat{u_n}^-, \widehat{v_n}^-)\rangle}{\|(u_n, v_n)\|} \\ =& \frac{m(\|u_n\|^2)\langle u_n, \widehat{u_n}^-\rangle + m(\|v_n\|^2)\langle v_n, \widehat{v_n}^-\rangle}{\|(u_n, v_n)\|} - \int_\Omega \frac{H_u(x, u_n, v_n)\widehat{u_n}^- + H_v(x, u_n, v_n)\widehat{v_n}^-}{\|(u_n, v_n)\|} dx \\ =& \frac{m(\|u_n\|^2)\langle u_n, u_n^-\rangle + m(\|v_n\|^2)\langle v_n, v_n^-\rangle}{\|(u_n, v_n)\|^2} - \int_\Omega \frac{H_u(x, u_n, v_n)u_n^- + H_v(x, u_n, v_n)v_n^-}{\|(u_n, v_n)\|^2} dx \\ =& \frac{1}{\|(u_n, v_n)\|^2}\left[m(\|u_n\|^2)\langle u_n, u_n^-\rangle + m(\|v_n\|^2)\langle v_n, v_n^-\rangle\right] \\ \geq& \frac{1}{\|(u_n, v_n)\|^2}\left[m(\|u_n\|^2)\|u_n^-\|^2 + m(\|v_n\|^2)\|v_n^-\|^2\right] \\ =& m(\|u_n\|^2)\|\widehat{u_n}^-\|^2 + m(\|v_n\|^2)\|\widehat{v_n}^-\|^2 \\ \geq& m_0(\|\widehat{u_n}^-\|^2 + \|\widehat{v_n}^-\|^2). \end{aligned}$$

We can find that $\|\widehat{u_n}^-\| \to 0$ and $\|\widehat{v_n}^-\| \to 0$ as $n \to \infty$. Thus, $(\widehat{u_n}^-, \widehat{v_n}^-) \to (0, 0)$ *a.e.* in Ω, that is, $\widehat{u} \geq 0$ and $\widehat{v} \geq 0$ *a.e.* in Ω. Since $\|(u_n, v_n)\| \to \infty$, we have $|(u_n, v_n)| = \|(u_n, v_n)\| \cdot |(\widehat{u_n}, \widehat{v_n})| \to \infty$ *a.e.* in Ω^+, where $\Omega^+ = \{x \in \Omega : \widehat{u}(x) > 0 \text{ or } \widehat{v}(x) > 0\}$. This together with (H_1) can get

$$\lim_{n\to\infty} \frac{H(x, u_n, v_n)}{\|(u_n, v_n)\|^{2\theta}} = \lim_{n\to\infty} \frac{H(x, u_n, v_n)|(\widehat{u_n}, \widehat{v_n})|^{2\theta}}{|(u_n, v_n)|^{2\theta}} dx = \infty \quad \text{a.e. in} \quad \Omega^+$$

Applying Fatou's lemma,

$$\lim_{n\to\infty} \int_\Omega \frac{H(x, u_n, v_n)}{\|(u_n, v_n)\|^{2\theta}} dx = \lim_{n\to\infty} \int_\Omega \frac{H(x, u_n, v_n)|(\widehat{u_n}, \widehat{v_n})|^{2\theta}}{|(u_n, v_n)|^{2\theta}} dx = \infty. \tag{4.12}$$

From (4.9) it follow that there exists a suitable constant $C > 0$ such that $|I(u_n, v_n)| \leq C$, then using (1.3), we have

$$\begin{aligned} \int_\Omega H(x, u_n, v_n) dx \leq& \frac{1}{2}M(\|u_n\|^2) + \frac{1}{2}M(\|v_n\|^2) + C \\ \leq& \frac{1}{2}M(1)\|u_n\|^{2\theta} + \frac{1}{2}M(1)\|v_n\|^{2\theta} + C \end{aligned}$$

$$\leq \frac{1}{2}M(1)\|(u_n,v_n)\|^{2\theta}+C.$$

It is easy to obtain that $\limsup_{n\to\infty}\int_\Omega \frac{H(x,u_n,v_n)}{\|(u_n,v_n)\|^{2\theta}}dx \leq \frac{1}{2}M(1)+\frac{C}{\|(u_n,v_n)\|^{2\theta}}$, which contradicts (4.12). Hence, Ω^+ has zero measure, that is, $(\widehat{u},\widehat{v})=(0,0)$ *a.e.* in Ω. Let $t_n\in[0,1]$ such that $I(t_nu_n,t_nv_n)=\max_{0\leq t\leq 1} I(tu_n,tv_n)$, let $\wp\in(0,\sqrt{\frac{2\pi}{\alpha_0}})$, we see that

$$\begin{aligned}I(t_nu_n,t_nv_n)\geq& I(\frac{\wp}{\|(u_n,v_n)\|}\cdot u_n,\frac{\wp}{\|(u_n,v_n)\|}\cdot v_n)=I(\wp\widehat{u_n},\wp\widehat{v_n})\\ =&\frac{1}{2}M(\|\wp\widehat{u_n}\|^2)+\frac{1}{2}M(\|\wp\widehat{v_n}\|^2)-\int_\Omega H(x,\wp\widehat{u_n},\wp\widehat{v_n})dx\end{aligned}$$

Since H_u,H_v satisfy critical exponential growth, and according to the assumption (H_3), there exists constant $C>0,\varepsilon>0$ such that

$$|H(x,W)|\leq C|W|^2+\varepsilon|W|^qe^{\alpha|W|^2} \tag{4.13}$$

By Hölder inequality, Young inequality, Lemma 2.2, we let $W=(\wp\widehat{u_n},\wp\widehat{v_n})$

$$\begin{aligned}\int_\Omega |W|^qe^{\alpha|W|^2}=&\wp^q\int_\Omega|(\widehat{u_n},\widehat{v_n})|^qe^{\alpha\wp^2|(\widehat{u_n},\widehat{v_n})|^2}\\ \leq& C\wp^q\|(\widehat{u_n},\widehat{v_n})\|^q_{qr_5}\Big[\int_\Omega e^{2r_6\alpha\wp^2\|(\widehat{u_n},\widehat{v_n})\|^2\cdot|\frac{\widehat{u_n}}{\|(\widehat{u_n},\widehat{v_n})\|}|^2}+\int_\Omega e^{2r_6\alpha\wp^2\|(\widehat{u_n},\widehat{v_n})\|^2\cdot|\frac{\widehat{u_n}}{\|(\widehat{u_n},\widehat{v_n})\|}|^2}\Big]^{\frac{1}{r_6}}\\ \leq& C\wp^q\|(\widehat{u_n},\widehat{v_n})\|^q_{qr_5},\end{aligned} \tag{4.14}$$

where $\frac{1}{r_5}+\frac{1}{r_6}$ and $r_6>1$ close to 1, $\alpha>\alpha_0$ close to α_0 and $r_6\alpha\wp^2<2\pi$. And since $\|(\widehat{u_n},\widehat{v_n})\|=1$, we have

$$\begin{cases}\|\widehat{v_n}\|^2\geq\frac{\|\widehat{u_n}\|^2+\|\widehat{v_n}\|^2}{2}, & \text{if } \|\widehat{u_n}\|\leq\|\widehat{v_n}\|.\\ \|\widehat{u_n}\|^2\geq\frac{\|\widehat{u_n}\|^2+\|\widehat{v_n}\|^2}{2}, & \text{if } \|\widehat{u_n}\|>\|\widehat{v_n}\|.\end{cases}$$

This together with (4.13), (4.14), we let $n\to\infty$ and $\wp\to\sqrt{\frac{2\pi}{\alpha_0}}$,

$$\begin{aligned}I(t_nu_n,t_nv_n)\geq&\frac{1}{2}M(\wp^2\|\widehat{u_n}\|^2)+\frac{1}{2}M(\wp^2\|\widehat{v_n}\|^2)-C\wp^2\int_\Omega|(\widehat{u_n}.\widehat{v_n})|^2dx-\varepsilon\wp^q\|(\widehat{u_n},\widehat{v_n})\|^q_{qr_5}\\ \geq&\max\{\frac{1}{2}M(\wp^2\|\widehat{u_n}\|^2),\frac{1}{2}M(\wp^2\|\widehat{v_n}\|^2)\}-C\wp^2\int_\Omega|(\widehat{u_n}.\widehat{v_n})|^2dx-\varepsilon\wp^q\|(\widehat{u_n},\widehat{v_n})\|^q_{qr_5}\\ \geq&\frac{1}{2}M(\wp^2\cdot\frac{\|\widehat{u_n}\|^2+\|\widehat{v_n}\|^2}{2})-C\wp^2\int_\Omega|(\widehat{u_n}.\widehat{v_n})|^2dx-\varepsilon\wp^q\|(\widehat{u_n},\widehat{v_n})\|^q_{qr_5}\\ =&\frac{1}{2}M(\frac{1}{2}\wp^2)+o(1)\\ >&c.\end{aligned} \tag{4.15}$$

We know the fact that $I(0,0)=0$ and $I(u_n,v_n)\to c$, by (4.15) we can assume that $t_n\in(0,1)$, we see that $\langle I'(t_nW_n),t_nW_n\rangle=0$ implies that

$$I(t_nu_n,t_nv_n)=I(t_nu_n,t_nv_n)-\frac{1}{2\theta}\langle I'(t_nu_n,t_nv_n),(t_nu_n,t_nv_n)\rangle$$

$$
\begin{aligned}
=&\frac{1}{2}M(\|t_nu_n\|^2)-\frac{1}{2\theta}m(\|t_nu_n\|^2)\|t_nu_n\|^2+\frac{1}{2}M(\|t_nv_n\|^2)-\frac{1}{2\theta}m(\|t_nv_n\|^2)\|t_nv_n\|^2\\
+&\frac{1}{2\theta}\int_\Omega[H_u(x,t_nu_n,t_nv_n)t_nu_n+H_v(x,t_nu_n,t_nv_n)t_nv_n]dx-\int_\Omega H(x,t_nu_n,t_nv_n)dx\\
\le&\frac{1}{2\theta}[\theta M(\|u_n\|^2)-m(\|u_n\|^2)\|u_n\|^2]+\frac{1}{2\theta}[\theta M(\|v_n\|^2)-m(\|v_n\|^2)\|v_n\|^2]\\
+&\frac{1}{2\theta}\int_\Omega[H_u(x,u_n,v_n)u_n+H_v(x,u_n,v_n)v_n]dx-\int_\Omega H(x,u_n,v_n)dx\\
=&I(u_n,v_n)-\frac{1}{2\theta}\langle I'(u_n,v_n),(u_n,v_n)\rangle\\
\le&c,
\end{aligned}
$$

which contradicts (4.15). Consequently, $\{(u_n,v_n)\}$ is bounded in $H_0^1(\Omega,\mathbb{R}^2)$. $\square$

5 Proof of Theorem 1.1

Lemma 5.1. *[8] Suppose Ω is a bounded domain in $\mathbb{R}^2$. Let $\{u_n\}$ be in $L^1(\Omega)$ such that $u_n\to u$ in $L^1(\Omega)$ and $f(x,s)$ be a continuous function. Then $f(x,u_n)\to f(x,u)$ in $L^1(\Omega)$ provided that $f(x,u_n)\in L^1(\Omega)$ for all n and $\int_\Omega|f(x,u_n)u_n|dx\le C$.*

Proof of Theorem 1.1: Without loss of generality, for some $(u_0,v_0)\in H_0^1(\Omega,\mathbb{R}^2)$, we have

$$
\begin{cases}
(u_n,v_n)\rightharpoonup(u_0,v_0) & \text{in}\quad H_0^1(\Omega,\mathbb{R}^2),\\
(u_n,v_n)\to(u_0,v_0) & \text{in}\quad L^p(\Omega,\mathbb{R}^2),\\
(u_n,v_n)\to(u_0,v_0) & \text{a.e}\quad\text{in}\quad(\Omega,\mathbb{R}^2).
\end{cases}
\tag{5.1}
$$

From (4.9) and (4.10), it follows that

$$
\int_\Omega W_nH_W(x,W_n)dx\le C\quad\text{and}\quad\int_\Omega H(x,W_n)dx\le C.
\tag{5.2}
$$

Moreover, $W_n\to W_0$ for almost every $x\in\Omega$ and $\|W_n\|\le C$, we have $\int_\Omega|H(x,W_n)W_n|dx\le C$. By Lemma5.1 and generalized Lebesgue dominated convergence theorem, we can get $\int_\Omega H(x,u_n,v_n)\to\int_\Omega H(x,u,v)$. We can suppose that $\|u_n\|\to\rho_1>0$ and $\|v_n\|\to\rho_2>0$ because of the boundness of $\{(u_n,v_n)\}$. Thus, $I'(u_n,v_n)\to0$ implies that

$$
m(\rho_1^2)\int_\Omega\nabla u_0\nabla\varphi dx+m(\rho_2^2)\int_\Omega\nabla v_0\nabla\psi dx=\int_\Omega(\varphi,\psi)H_W(x,W)dx,\quad\forall(\varphi,\psi)\in C_0^\infty(\Omega,\mathbb{R}^2).
\tag{5.3}
$$

We want to prove (u_0,v_0) is a solution of this problem, it suffices to show that $\rho_1=\|u_0\|$ and $\rho_2=\|v_0\|$. This together with semicontinuity of norm show that $I(W_0)\le c$. Next, we prove the case of $I(W_0)<c$ cannot occur. If $I(W_0)<c$, we have $\|u_0\|^2<\rho_1^2$, $\|v_0\|^2<\rho_2^2$ and

$$
\frac{1}{2}M(\rho_1^2)+\frac{1}{2}M(\rho_2^2)=\lim_{n\to\infty}[\frac{1}{2}M(\|u_n\|^2)+\frac{1}{2}M(\|v_n\|^2)]=c+\int_\Omega H(x,W_0)dx.
\tag{5.4}
$$

Then

$$2c - 2I(W_0) = M(\rho_1^2) - M(\|u_0\|^2) + M(\rho_2^2) - M(\|v_0\|^2) \tag{5.5}$$

Otherwise, we claim that $m(\|u_0\|^2)\|u_0\|^2 + m(\|v_0\|^2)\|v_0\|^2 \geq \int_\Omega W_0 H_W(x, W_0)dx$. Suppose by contradiction that $m(\|u_0\|^2)\|u_0\|^2 + m(\|v_0\|^2)\|v_0\|^2 < \int_\Omega W_0 H_W(x, W_0)dx$, that is, $\langle I'(W_0), W_0\rangle < 0$. According to (1.4) and the assumption of (M_1), we obtain that

$$\langle I'(tW_0), W_0\rangle \geq m_0\|u_0\|^2 t^2 + m_0\|v_0\|^2 t^2 - t^\mu \|W_0\|_{\mu+1}^{\mu+1} > 0,$$

for $\mu > 2$ and t sufficiently small. The information above tell us that there exists $\sigma \in (0, 1)$ such that $\langle I'(\sigma W_0), W_0\rangle = 0$. This implies that $\sigma W_0 \in \mathcal{N}$. According to (1.4), (H_2), semicontinuity of norm and using Fatou Lemma, we can get

$$\begin{aligned}
c \leq& b \leq I(\sigma W_0) = I(\sigma W_0) - \frac{1}{2\theta}\langle I'(\sigma W_0), \sigma W_0\rangle \\
=&\frac{1}{2\theta}[\theta M(\|\sigma u_0\|^2) - m(\|\sigma u_0\|^2)\|\sigma u_0\|^2] + \frac{1}{2\theta}[\theta M(\|\sigma v_0\|^2) - m(\|\sigma v_0\|^2)\|\sigma v_0\|^2] \\
&+\frac{1}{2\theta}\int_\Omega [H_W(x, \sigma W_0)\sigma W_0 - 2\theta H(x, \sigma W_0)]dx \\
<&\frac{1}{2\theta}[\theta M(\|u_0\|^2) - m(\|u_0\|^2)\|u_0\|^2] + \frac{1}{2\theta}[\theta M(\|v_0\|^2) - m(\|v_0\|^2)\|v_0\|^2] \\
&+\frac{1}{2\theta}\int_\Omega [H_W(x, W_0)W_0 - 2\theta H(x, W_0)]dx \\
\leq& \liminf_{n\to\infty}[\frac{1}{2}M(\|u_n\|^2) - \frac{1}{2\theta}m(\|u_n\|^2)\|u_n\|^2] + \liminf_{n\to\infty}[\frac{1}{2}M(\|v_n\|^2) - \frac{1}{2\theta}m(\|v_n\|^2)\|v_n\|^2] \\
&+\liminf_{n\to\infty}\int_\Omega [\frac{1}{2\theta}H_W(x, W_n)W_n - H(x, W_n)]dx \\
\leq& \lim_{n\to\infty}[I(W_n) - \frac{1}{2\theta}\langle I'(W_n), W_n\rangle] \\
=&c,
\end{aligned}$$

this case is impossible. This together with (1.4) and (H_2), we can get following estimate

$$\begin{aligned}
I(W_0) \geq& \frac{1}{2\theta}[\theta M(\|u_0\|^2) - m(\|u_0\|^2)\|u_0\|^2] + \frac{1}{2\theta}[\theta M(\|v_0\|^2) - m(\|v_0\|^2)\|v_0\|^2] \\
&+\frac{1}{2\theta}\int_\Omega [H_W(x, W_0)W_0 - 2\theta H(x, W_0)]dx \\
\geq& 0.
\end{aligned} \tag{5.6}$$

Therefore, above inequality and the fact of $c < \frac{1}{2}M(\frac{\pi}{\alpha_0})$ and (5.5) yield that

$$M(\rho_1^2) + M(\rho_2^2) < M(\frac{\pi}{\alpha_0}) + M(\|u_0\|^2) + M(\|v_0\|^2).$$

By semicontinuity of norm, we have $\|u_0\|^2 \leq \rho_1^2, \|v_0\|^2 \leq \rho_2^2$. And applying the condition (M_2), we can get

$$M(\rho_1^2) < M(\frac{\pi}{\alpha_0}) + M(\|u_0\|^2) \;\text{ and }\; M(\rho_2^2) < M(\frac{\pi}{\alpha_0}) + M(\|v_0\|^2),$$

this implies that $\rho_1^2 < M^{-1}[M(\frac{\pi}{\alpha_0})+M(\|u_0\|^2)]$ and $\rho_2^2 < M^{-1}[M(\frac{\pi}{\alpha_0})+M(\|v_0\|^2)]$. We can conclude that

$$\rho_1^2+\rho_2^2 < \frac{2\pi}{\alpha_0}+\|u_0\|^2+\|v_0\|^2.$$

Defining $A_n = \frac{W_n}{\|W_n\|}$ and $A_0 = \frac{W_0}{\sqrt{\rho^2+\sigma^2}}$, it easily follows $A_n \rightharpoonup A_0$ in $H_0^1(\Omega,\mathbb{R}^2)$ and $\|A_0\| \le 1$, by Lemma 2.4 we have

$$\sup_{n\in N}\int_\Omega e^{\beta|A_n|^2}dx < +\infty, \quad \forall \beta < \frac{2\pi}{1-\|A_n\|^2}. \tag{5.7}$$

Above inequality together with $\rho_1^2+\rho_2^2 = \frac{\rho_1^2+\rho_2^2-\|W_0\|^2}{1-\|A_0\|^2}$ shows that $\rho_1^2+\rho_2^2 < \frac{\frac{2\pi}{\alpha_0}}{1-\|A_0\|^2}$. Therefore, there exists $\beta > 0$ such that $\alpha_0\|W_n\|^2 < \beta < \frac{2\pi}{1-\|A_0\|^2}$ for n sufficiently large. For $q > 1$ close to 1, $\alpha > \alpha_0$ close to α_0, we also have $q\alpha\|W_n\|^2 \le \beta < \frac{2\pi}{1-\|A_0\|^2}$, thus consider (5.7), we can get

$$\int_\Omega e^{q\alpha|W_n|^2}dx \le \int_\Omega e^{\beta|A_n|^2}dx \le C.$$

Hence, using Hölder inequality, Trudinger-Moser inequality, (H_3), (5.1) and Sobolev embedding,

$$\begin{aligned}\Big|\int_\Omega H_W(x,W_n)(W_n-W_0)dx\Big| \le & C\int_\Omega |W_n|\cdot|W_n-W_0|dx + C\int_\Omega |W_n-W_0|\cdot e^{\alpha|W_n|^2}dx\\ \le & C\|W_n\|_{r_7}\|W_n-W_0\|_{r_8} + C\|W_n-W_0\|_{r_9}\Big[\int_\Omega e^{r_{10}\alpha\|W_n\|^2\cdot|\frac{W_n}{\|W_n\|}|}dx\Big]^{\frac{1}{r_{10}}}\\ \to & 0,\end{aligned}$$

where $\frac{1}{r_7}+\frac{1}{r_8}=1, \frac{1}{r_9}+\frac{1}{r_{10}}$. According to the fact that $\langle I'(W_n), W_n-W_0\rangle \to 0$, we get

$$m(\|u_n\|^2)\int_\Omega \nabla u_n\cdot\nabla(u_n-u_0)dx + m(\|v_n\|^2)\int_\Omega \nabla v_n\cdot\nabla(v_n-v_0)dx \to 0. \tag{5.8}$$

However, the boundness of $\{(u_n,v_n)\}$ shows that

$$\begin{aligned}m(\|u_n\|^2)\int_\Omega \nabla u_n\cdot\nabla(u_n-u_0)dx =& m(\|u_n\|^2)\|u_n\|^2 - m(\|u_n\|^2)\int_\Omega \nabla u_n\nabla u_0 dx\\ \to& m(\rho_1^2)\rho_1^2 - m(\rho_1^2)\|u_0\|^2,\end{aligned} \tag{5.9}$$

$$\begin{aligned}m(\|v_n\|^2)\int_\Omega \nabla v_n\cdot\nabla(v_n-v_0)dx =& m(\|v_n\|^2)\|v_n\|^2 - m(\|v_n\|^2)\int_\Omega \nabla v_n\nabla v_0 dx\\ \to& m(\rho_2^2)\rho_2^2 - m(\rho_2^2)\|v_0\|^2.\end{aligned} \tag{5.10}$$

Thus, from (5.8), (5.9), (5.10), we can get $\rho_1 = \|u_0\|, \rho_2 = \|v_0\|$. This shows that we have $I(W_0) = c_M$, which contradicts with our assumption. Therefore, we have

$$M(\rho_1^2)+M(\rho_2^2) = M(\|u_0\|^2)+M(\|v_0\|^2),$$

that is,

$$\rho_1 = \|u_0\| \text{ and } \rho_2 = \|v_0\|.$$

Thus, we have $u_n \to u_0$ and $v_n \to v_0$ in $H_0^1(\Omega)$. From (5.3) and above information we can conclude that

$$m(\|u_0\|^2)\int_\Omega \nabla u_0 \nabla \varphi dx + m(\|v_0\|^2)\int_\Omega \nabla v_0 \nabla \psi dx = \int_\Omega (\varphi,\psi) H_w(x,u_0,v_0)dx,\ \forall (\varphi,\psi)\in C_0^\infty(\Omega,\mathbb{R}^2). \tag{5.11}$$

Finally, we need to prove $u_0 \neq 0$ and $v_0 \neq 0$. If $u_0 = 0$ and $v_0 \neq 0$, then $\int_\Omega H(x,u_0,v_0) = 0$. By (4.19) we can get that

$$\frac{1}{2}M(\|u_n\|^2) + \frac{1}{2}M(\|v_n\|^2) \to c < \frac{1}{2}M(\frac{2\pi}{\alpha_0})$$

This means that $\|W_n\|^2 = \|u_n\|^2 + \|v_n\|^2 < \frac{2\pi}{\alpha_0}$ because of M is increasing. So, there exists $n_0 \in \mathbb{N}$ and $\beta > 0$ such that $\alpha_0 ||W_n||^2 < \beta < 2\pi$ for all $n > n_0$. We can choose $r' > 1$ close to 1 and $\alpha > \alpha_0$ close to α_0 such that we still have $r'\alpha||W_n||^2 \leq \beta < 2\pi$. Applying Lemma2.2, Sobolev embedding and Hölder inequality, we have

$$\begin{aligned}
&|\int_\Omega W_n \cdot H_W(x,u_n,v_n)dx| \\
&\leq (\lambda m_0 - \varepsilon)\int_\Omega |W_n|^2 dx + C_1 \int_\Omega |W_n| e^{\alpha |W_n|^2} dx \\
&\leq (\lambda m_0 - \varepsilon)||W_n||_2^2 + C_2 ||W_n||_{\frac{r'}{r'-1}} \Big[\int_\Omega e^{2r'\alpha ||W_n||^2 \left(\frac{u_n}{||W_n||}\right)^2} + \int_\Omega e^{2r'\alpha ||W_n||^2 \left(\frac{v_n}{||W_n||}\right)^2} \Big]^{\frac{1}{r'}} \\
&\leq (\lambda m_0 - \varepsilon)||W_n||_2^2 + C_3 ||W_n||_{\frac{r'}{r'-1}} \\
&\to 0 \quad \text{as} \quad n \to +\infty.
\end{aligned}$$

We know that $\langle I'(W_n), W_n\rangle = M(||u_n||^2)||u_n||^2 + M(||v_n||^2)||v_n||^2 - \int_\Omega W_n \cdot H(x,W_n)dx \to 0$. This information together with (M_1) shows that $||u_n||^2 \to 0$ and $||v_n||^2 \to 0$. Obviously, this contradict with our assumption. And we can prove the case $u_0 \neq 0, v_0 = 0$ and the case $u_0 = 0, v_0 = 0$ can not occur by same way. Thus, we say that $W_0 = (u_0, v_0)$ is a nontrival nonnegative ground state solution of problem (1.1).

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