ON HYPONORMAL AND DISSIPATIVE CORRECT EXTENSIONS AND RESTRICTIONS

N. KAKHARMAN, K. TULENOV, AND L. ZHUMANOVA

ABSTRACT. The main aim of this paper is to study hyponormal and dissipative correct restrictions and extensions as well as their applications to differential operators.

1. INTRODUCTION

The theory of extensions and restrictions started from a pioneer work of J. Von Neumann [23] and it is further developed to several directions by M.A. Krasnosel'skii [18], M.A. Naimark [22], M.I. Vishik [27], A.V. Bicadze and A. A. Samarskiĭ [1], and A.A. Dezin [5, 6]. Later, Bitsadze-Samarsky type problems were investigated by M. Otelbaev [24] and his disciples [16, 17]. A description of regular extensions of hyperbolic and mixed type operators in terms of boundary conditions was studied in [10, 11, 12]. Various applications of this theory and its further development can be found from the papers [2, 3, 4, 6, 7, 9, 13, 14, 15, 19, 20, 21, 25, 26] and references therein.

In this paper, we investigate correct restrictions of hyponormal and dissipative operators, and present some applications to differential operators. Many spectral properties of linear operators are closely related to these notions. Hyponormal operators give a lot of information about the spectral properties of linear operators. For example, a compact hyponormal operator has a nonzero eigenvalue and if a compact hyponormal operator is invertible, then its system of root vectors is complete in a Hilbert space H. Furthermore, if the spectrum of the hyponormal operator lies in the right half-plane, then for all $f \in D(A)$ we have $(\Re e(A)f, f) \geq 0$. All these interesting properties hyponormal operators motivate us to study their correct restrictions and extensions.

The structure of this paper is as follows. Section 2 recalls a necessary theory on correct restrictions and extensions of linear operators. In Section 3, we describe all possible correct hyponormal restrictions of the maximal and correct hyponormal extensions of the minimal operators. Section 4 is devoted to study correct hyponormal restrictions and extensions whenever minimal operator is symmetric. Finally, we investigate dissipative correct restrictions of arbitrary maximal operators, which we address section 5. The results on dissipative restrictions obtained in this section are maximally dissipative restrictions due to their correctness. Such operators generate a continuous semigroup. In addition, they are necessary for proving many theorems on the completeness of the system of root vectors such as theorems of M.S. Livshits [28] and V.B. Lidskii [29] (see also [8]). The obtained results allow to single out whole classes of disspative correct operators, which, in the case of Nuclearity, are automatically operators with a complete system of root vectors. For example, the dissipative operators obtained from Theorem 5.2 for the differential operator for $n \geq 2$ have a complete system of root vectors.

2. PRELIMINARIES

This paper is a continuation of the research in [26]. Therefore, for convenience, we keep almost all notations as in [26]. So, for undefined notations and notions below, we refer the reader to [26]. Recall that an operator A with domain $D(A)$ is said to be a restriction of an operator B or B is called an extension of A, if

(i) $D(A) \subset D(B)$;
(ii) $Ax = Bx$ for all $x \in D(A)$.

Definition 2.1. (i) A linear closed operator A_0 in H will be called minimal, if $\overline{R(A_0)} \neq H$;
(ii) A linear closed operator $\widehat{A}$ in H is called maximal, if $R(\widehat{A}) = H$ and $Ker\widehat{A} \neq \{0\}$.

Definition 2.2. A correctly and everywhere solvable extension A of the minimal operator A_0 will be called a correct extension, and a correctly and everywhere solvable restriction A of the maximal operator $\widehat{A}$ will be called a correct restriction.

2010 *Mathematics Subject Classification.* 47A05, 34B05, 47B25, 34B09, 34L10, 35P05, 35P10, 34L05, 34B24.
Key words and phrases. Correct restrictions, hyponormal operator, minimal operator, dissipative operator, symmetric operator.

Definition 2.3. A unbounded linear operator is called *normal* if $D(A) = D(A^*)$ and $\|A^*x\| = \|Ax\|$ holds for all $x \in D(A)$. A closed densely defined operator is called *hyponormal* if $D(A) = D(A^*)$ and $\|A^*x\| \leq \|Ax\|$ holds for all $x \in D(A)$. An unbounded operator A is called a *formally hyponormal* operator if for all f in $D(A) \subset D(A^*)$, such that $\|A^*f\| \leq \|Af\|$, and is called a hyponormal operator in case $D(A) = D(A^*)$.

In this regard, we are interested in hyponormal correct restrictions of the maximal operator $\widehat{L}$, and hyponormal correct extensions of minimal operator L_0. We shall notice, that for hyponormality, the density of the domain is necessary. And hyponormality of restriction L_K means for all $f \in D(L_K^*) = D(L_K)$ such that $\|L_K^* f\| \leq \|L_K f\|$. If there is some known correct restriction of the maximal operator and we take a correct self-adjoint restriction L_s of the maximal operator $\widehat{L}$, then the inverse operator to the arbitrary correct restriction of L_K has the form

$$L_K^{-1} f = L_s^{-1} f + Kf, \tag{2.1}$$

where K is a linear bounded operator in H such that $R(K) \subset Ker\widehat{L}$.

Definition 2.4. If minimal operator L_0 and maximal operator $\widehat{L}$ are connected each other as $L_0 \subset \widehat{L}$, then a correct extension L of the minimal operator L_0 will be called a boundary correct extension of the minimal operator L_0 with respect to $\widehat{L}$, if it is also a correct restriction of the maximal operator $\widehat{L}$, that is $L_0 \subset L \subset \widehat{L}$.

It was shown in [2] (see also [3]) that all kinds of correct extensions of the minimal operator L_0 are exhausted by adjoint operators to all kinds of correct restrictions with dense domains. Due to this connection, it is enough to study many spectral properties for correct restrictions. It follows that, the correct restrictions of L_K from (2.1) turn out to be boundary correct extension if and only if $R(L_0) \subset KerK$ [6]. Let L be any known boundary correct extension of L_0, i.e., $L_0 \subset L \subset \widehat{L}$. The existence of at least one boundary correct extension L was proved by Vishik in [27]. It is known that any self-adjoint correct extension L_s of the minimal operator L_0 is also a restriction of the maximal operator $\widehat{L}$, that is, L_s is a boundary correct extension. The domain of definition of the operator L_s is usually defined as a kernel of some linear boundary operator Γ_s, i.e.

$$D(L_s) = \{u \in D(\widehat{L}) \ : \ \Gamma_s u = 0\}.$$

In the case of differential operators, Γ_s is a boundary operator, and $\Gamma_s u = 0$ is a boundary condition.

Let us consider some minimal operator L_0 in a Hilbert space H. Then, $\widehat{L} = L_0^*$ is a maximal operator. We will denote by L_s some self-adjoint boundary extension. It is well known that $L_0 \subset L_s \subset \widehat{L}$, and conversely all correct restrictions L_K of the maximal operator $\widehat{L}$ have the following form

$$u = L_K^{-1} f = L_s^{-1} f + Kf, \tag{2.2}$$

where K is an arbitrary bounded operator acting from H into $Ker\widehat{L}$. Moreover, it is only correct restriction of the maximal operator $\widehat{L}$. We shall notice, that the normal correct restriction of the maximal operator $\widehat{L}$ coincides with self-adjoint if only if L_0 is symmetric and $\widehat{L} = L_0^*$. We also need the following definition.

Definition 2.5. A linear operator $A : H \to H$ with the dense domain $D(A)$ is called dissipative if $\Im m A \geqslant 0$.

It is known that if A is a dissipative operator, then operators A^{-1} and A^* are also dissipative.

As usual, we denote by $W_2^k[0,1]$, $k \in \mathbb{N}$, the Sobolev space with respect to the norm

$$\|f\|_{W_2^k[0,1]} = \left(\int_0^1 (|f(t)|^2 + |f^{(k)}(t)|^2) dt \right)^{1/2}.$$

Moreover, this space is a Hilbert space with the inner product

$$\langle f, g \rangle_{W_2^k[0,1]} = \int_0^1 \left(f(t)\overline{g(t)} + f^{(k)}(t)\overline{g^{(k)}(t)} \right) dt.$$

3. On coincidence of hyponormal and self-adjoint correct extensions of symmetric minimal operator

The following is the main results of this section.

Theorem 3.1. *Hyponormal correct extensions L_K^* of the symmetric operator L_0 and hyponormal correct restrictions L_K of the maximal operator $\widehat{L} = L_0^*$ are exhausted by self-adjoint boundary extensions of the minimal operator L_0.*

Proof. For the hyponormality of L_K^*, it is necessary that $D(L_K^*) = D(L_K)$. Then

$$D(L_0) \subset D(L_K^*) = D(L_K) \subset D(\widehat{L}).$$

Moreover, we have representation (2.2) for the operator L_K^{-1}. Let us take arbitrary element $u_0 \in D(L_0)$. Then, by (2.2) we have

$$u_0 = L_s^{-1} f_0 + K f_0. \tag{3.1}$$

Hence,

$$\widehat{L} u_0 = f_0 \in R(L_0).$$

It is well known that $L_s^{-1} f_0 = L_0^{-1} f_0 \in D(L_0)$. By (3.1) we have that $K f_0 \in D(L_0)$. Since $R(K) \subset Ker \widehat{L}$, it follows from the uniquely solvability of L_0 that $K f_0 = 0$. In other words, $R(L_0) \subset Ker K$. Thus, we obtain

$$L_0 \subset L_K \subset \widehat{L}.$$

Further, for all $g \in H$, there exists inverse image $v = (L_K^*)^{-1} g$, and for any v by $D(L_K^*) = D(L_K)$ there exists $f \in H$ such that $v = L_K^{-1} f$. Therefore, for every f and g in H we have

$$L_K^{-1} f = (L_K^*)^{-1} g.$$

Since $(L_K^*)^{-1} = (L_K^{-1})^*$, we obtain

$$L_s^{-1} f + K f = L_s^{-1} g + K^* g. \tag{3.2}$$

It follows from $L_0 \subset L_K \subset \widehat{L}$ that $K^* g \in Ker \widehat{L}$. Applying the maximal operator $\widehat{L}$ to both side of (3.2) we obtain that $f = g$. Since g is arbitrary, it follows that L_K^* is self-adjoint. Thus, correct hyponormal extensions of the minimal operator L_0 coincide with self-adjoint correct extensions. Here, we used only $D(L_K) = D(L_K^*)$ and this condition is also figured in condition of hyponormality of L_K. This concludes the proof. □

Example 3.1. *Hyponormal correct extensions or restrictions of the equation*

$$Ly = i \frac{dy}{dx} = f$$

in $L_2(0,1)$ are operators L with domains

$$D(L) = \left\{ y \in W_2^1[0,1] : y(0) = \alpha y(1), |\alpha| = 1 \right\}.$$

They are all possible self-adjoint boundary extensions.

4. On hyponormal correct extensions in case the minimal operator is not symmetric

In a Hilbert space H we will consider a correct operator L_0 with the dense domain and not dense range. Let there exist other correct solvable operator M_0 satisfying

$$(L_0 u, v) = (u, M_0 v), \ \forall u, v \in D(L_0) = D(M_0).$$

It is well known that, if there is a normal correct restriction L_H of the maximal operator $\widehat{L} = M_0^*$ or a normal correct extension L_H of the minimal operator L_0, then it is a boundary extension. In other words, $L_0 \subset L_H \subset \widehat{L}$. Therefore, L_H^* is also a normal operator and $M_0 \subset L_H^* \subset \widehat{M} = L_0^*$. The following theorem shows that hyponormal correct extensions or restrictions have the same properties.

Theorem 4.1. *If there is only one correct boundary extension L_H of the minimal operator L_0, then all hyponormal correct extensions L_K of the minimal operator L_0 and hyponormal correct restrictions L_K of the maximal operator $\widehat{L}$ are boundary, i.e.*

$$L_0 \subset L_K \subset \widehat{L}.$$

Proof. For the hyponormality of correct restrictions and correct extensions, the condition

$$D(L_k) = D(L_k^*)$$

is required. Let us consider the case when L_k is a correct restriction. If L_k is a correct extension, then working with the adjoint operator the problem is proved similarly to the case of correct restriction. Therefore, it is sufficient to prove the theorem in the case of correct restriction.

Indeed, using the representation

$$u = L_k^{-1} f = L_H^{-1} f + K f$$

of the correct restriction, where L_H is a normal restriction of the maximal operator $\widehat{L} = M_0^*$ and K is a linear bounded operator from H into $Ker\widehat{L}$. Hence, by the equality $D(L_k) = D(L_k^*)$, for each f there exists $g \in H$ such that

$$u = \left(L_k^*\right)^{-1} g = \left(L_H^*\right)^{-1} g + K^* g.$$

Thus,

$$L_H^{-1} f + K f = (L_H^*)^{-1} g + K^* g. \tag{4.1}$$

It follows from (4.1) that $K^* g \in D(\widehat{L})$ and $K f \in D(\widehat{M})$. Acting on both sides of (4.1) with the operators $\widehat{L}$ and $\widehat{M}$ we obtain the following equalities

$$f = L_H (L_H^*)^{-1} g + \widehat{L} K^* g, \tag{4.2}$$

$$g = L_H^* L_H^{-1} f + \widehat{M} K f,$$

respectively. Substituting (4.2) into (4.1) we get

$$K L_H (L_H^*)^{-1} g + K \widehat{L} K^* g + L_H^{-1} \widehat{L} K^* g = K^* g \tag{4.3}$$

for all g in H. If in (4.3) elements g are taken from $R(M_0)$, then, since $R(M_0) \subset Ker K^*$ and $R(L_0) = L_H (L_H^*)^{-1} R(M_0)$, we obtain

$$R(L_0) \subset Ker K.$$

In other words, we have that $L_0 \subset L_k \subset \widehat{L}$, thereby completing the proof. □

Example 4.1. *Let us consider the equation*

$$L y \equiv \frac{dy}{dx} = f$$

in the Hilbert space $L_2(0,1)$. Then, all possible hyponormal correct extensions and restrictions of this equation are operators L with domains

$$D(L) = \left\{ y \in W_2^1[0,1] : \left(\frac{1}{2} - ic\right) y(0) + \left(\frac{1}{2} + ic\right) y(1) = 0, c \in \mathbb{R} \right\}.$$

It is clear that, they are boundary extensions, i.e.

$$L_0 \subset L \subset \widehat{L}.$$

Remark 4.1. *This example shows that hyponormal correct operators L are exhausted by normal boundary extensions. Also, note that hyponormal correct extensions of a symmetric minimal operator are exhausted by self-adjoint extensions.*

There is an essential question that

Question 4.1. *Do not hyponormal correct operators have the same property as in Remark 4.1 under the condition of Theorem 4.1?*

In order to answer this question, first it is necessary to describe all normal correct restrictions. Although normal operators have been studied by many authors for many years, there is still no effective approach to describe normal boundary extensions. The next theorem is devoted to the above question.

Theorem 4.2. *Suppose that one normal boundary extension L_H of the minimal operator L_0 is known. Then the correct restriction L_k is normal if and only if $D(L_k) = D(L_k^*)$ and $(\widehat{M} K)^* = \widehat{L} K^*$, where K is a linear bounded operator from H into $Ker\widehat{L}$ which satisfies*

$$u \equiv L_k^{-1} f = L_H^{-1} f + K f. \tag{4.4}$$

Proof. Let L_k be a normal boundary extension. Then $L_k^{-1}(L_k^*)^{-1} = (L_k^*)^{-1}L_k^{-1}$. By virtue of representation (4.4) we obtain

$$(L_H^{-1} + K)((L_H^*)^{-1} + K^*)f = ((L_H^*)^{-1} + K^*)(L_H^{-1} + K)f, \ \ \forall f \in H.$$

Hence,

$$L_H^{-1}K^*f + K(L_H^*)^{-1}f + KK^*f = (L_H^*)^{-1}Kf + K^*L_H^{-1}f + K^*Kf. \tag{4.5}$$

It follows from the necessary condition of normality $D(L_k) = D(L_k^*)$ that

$$R(K) \subset D(\widehat{M}), \ \ R(K^*) \subset D(\widehat{L}), \ \ R(K^*) \subset Ker\widehat{M}.$$

Therefore, it follows from the closedness of the operators $\widehat{L}$ and $\widehat{M}$ we obtain the boundedness of the operators $\widehat{M}K$ and $\widehat{L}K^*$. Acting on both sides of (4.5) from the left with the maximal operator $\widehat{L}$, we obtain

$$K^*f = L_H(L_H^*)^{-1}Kf + \widehat{L}K^*L_H^{-1}f + \widehat{L}K^*Kf.$$

It follows that

$$Kf = K^*L_H^*L_H^{-1}f + (L_H^*)^{-1}(\widehat{L}K^*)^*f + K^*(\widehat{L}K^*)^*f.$$

Again, acting on both sides of above equation with the operator $\widehat{M}$ we have

$$\widehat{M}Kf = (\widehat{L}K^*)^*f, \ \ \forall f \in H.$$

This is equivalent to $\widehat{L}K^* = (\widehat{M}K)^*$.

Conversely, let $D(L_k) \in D(L_k^*)$ and $\widehat{L}K^* = (\widehat{M}K)^*$. Then, for any $f \in H$, there is a function $g \in H$ such that $u = L_k^{-1}f = L_k^{*-1}g$, where g ranges over the entire H. This equality will be rewritten as

$$L_H^{-1}f + Kf = (L_H^*)^{-1}g + K^*g. \tag{4.6}$$

Acting on both sides of the equality with the operator $\widehat{M}$, we obtain

$$g = L_H^*L_H^{-1}f + \widehat{M}Kf.$$

Substituting g into (4.6), we have

$$Kf = (L_H^*)^{-1}\widehat{M}Kf + K^*L_H^*L_H^{-1}f + K^*\widehat{M}Kf, \ \ \forall f \in H.$$

Then

$$K^*f = (\widehat{M}K)^*L_H^{-1}f + L_H(L_H^*)^{-1}Kf + (\widehat{M}K)^*Kf.$$

Since $(\widehat{M}K)^* = \widehat{L}K^*$, we have

$$K^*f = \widehat{L}K^*L_H^{-1}f + L_H(L_H^*)^{-1}Kf + \widehat{L}K^*Kf. \tag{4.7}$$

Adding $(L_H^*)^{-1}f$ to both sides of equality (4.7) we obtain

$$K^*f + (L_H^*)^{-1}f = (L_H^*)^{-1}f + \widehat{L}K^*L_H^{-1}f + L_H(L_H^*)^{-1}Kf + \widehat{L}K^*Kf.$$

This is equivalent to the following equality

$$K^*f + (L_H^*)^{-1}f = L_H(L_H^*)^{-1}f + \widehat{L}K^*L_H^{-1}f + L_H(L_H^*)^{-1}Kf + \widehat{L}K^*Kf.$$

It follows that

$$(L_H^*)^{-1}f = \widehat{L}((L_H^*)^{-1} + K^*)L_H^{-1}f + \widehat{L}((L_H^*)^{-1} + K^*)Kf.$$

This is equivalent to the following equality

$$(L_k^*)^{-1}f = L_k(L_k^*)^{-1}L_k^{-1}f.$$

Thus, we have

$$L_k^{-1}(L_k^*)^{-1}f = (L_k^*)^{-1}L_k^{-1}f, \ \ \forall f \in H.$$

This completes the proof. ∎

Example 4.2. *Let us consider the differential equation*

$$Ly \equiv y'' + y' = f \tag{4.8}$$

in $L_2(0,1)$. *Then, it is easy to see that formally adjoint equation has the form*

$$Mv = v'' - v' = g.$$

Hence,

$$D(L_0) = D(M_0) = \left\{ g \in W_2^2[0,1] : y(0) = y(1) = y'(0) = y'(1) = 0 \right\},$$
$$D(\widehat{L}) = \{ y \in W_2^2[0,1] \},$$
$$D(\widehat{M}) = \left\{ v \in W_2^2[0,1] \right\}.$$

Moreover, it is easy to check that the operator L_H, *corresponding to the equation* (4.8) *with conditions*

$$y(0) + y(1) = 0, \;\; y'(0) + y'(1) = 0,$$

is normal. Thus, using Theorem 4.2, we find all possible correct normal operators. To do this, first we need to check the condition $\widehat{L}K^* = (\widehat{M}K)^*$. *Any correct restriction* L_k *of the maximal operator* $\widehat{L}$ *has the following inverse*

$$y \equiv L_k^{-1} f = L_H^{-1} f + Kf.$$

Taking into account the fact that normal correct operators lie between L_0 *and* $\widehat{L}$ *we have a general form for* K

$$(Kf)(x) = \int_0^1 f(t)(\overline{a}_{11} + \overline{a}_{12} e^t) dt + e^{-x} \int_0^1 f(t)(\overline{a}_{21} + \overline{a}_{22} e^t) dt.$$

Then,

$$(K^* f)(x) = (\overline{a}_{11} + \overline{a}_{12} e^x) \int_0^1 f(t) dt + (\overline{a}_{21} + \overline{a}_{22} e^x) \int_0^1 e^{-t} f(t) dt.$$

Now, acting on K *and* K^* *with the operators* $\widehat{M}$ *and* $\widehat{L}$, *respectively, we have*

$$(\widehat{M}Kf)(x) = 2e^{-x} \int_0^1 (\overline{a}_{21} + \overline{a}_{22} e^t) f(t) dt,$$

$$(\widehat{L}K^* f)(x) = 2a_{12} e^x \int_0^1 f(t) dt + 2a_{22} e^x \int_0^1 e^{-t} f(t) dt.$$

Finding $(\widehat{M}K)^*$,

$$(\widehat{M}K^* f)(x) = 2(a_{21} + a_{22} e^x) \int_0^1 e^{-t} f(t) dt$$

from the condition $(\widehat{M}K)^* = \widehat{L}K^*$, *we obtain* $a_{21} = 0$ *and* $a_{12} = 0$. *Therefore, the operator* K *has the form*

$$(Kf)(x) = \overline{a}_{11} \int_0^1 f(t) dt + \overline{a}_{22} e^{-x} \int_0^1 e^t f(t) dt.$$

Thus,

$$(K^* f)(x) = a_{11} \int_0^1 f(t) dt + a_{22} e^x \int_0^1 e^{-t} f(t) dt.$$

Now, from the condition $D(L_k) = D(L_k^*)$ *it is easy to obtain conditions for* a_{11} *and* a_{22}. *It follows that all possible normal correct operators* L_k *correspond to the following boundary conditions for equation* (4.8)

$$y(0) = \alpha y(1), \;\; y'(0) = \alpha y'(1), \;\; |\alpha| = 1, \;\; \alpha \neq 1.$$

Example 4.3. *Consider the linear differential equation*

$$Ly \equiv \frac{d^{2n+1} y}{dx^{2n+1}} = f(x) \tag{4.9}$$

in $L_2(0,1)$. *Then the formally adjoint differential equation has the form*

$$Mv \equiv -\frac{d^{2n+1} v}{dx^{2n+1}} = g(x).$$

Since $D(\widehat{L}) = D(\widehat{M})$, then the condition $\widehat{L}K^* = (\widehat{M}K)^*$ is satisfied. Therefore, any boundary extension of the minimal operator L_0, generated by the differential equation (4.9), is normal for all self-adjoint boundary conditions, that is, $D(L_k) = D(L_k^*)$. In this case, all possible hyponormal boundary extensions are exhausted by normal ones. If we do not assume the existence of a normal regular extension, then difficulties arise in describing and even proving the existence of hyponormal operators. The next example shows that for differential operators for which normal ones do not exist, one can find a formally hyponormal operator.

Example 4.4. *Let us given the following differential expression*

$$Lu = -\frac{1}{x}\frac{du}{dx} + \frac{u}{2x^2} + x^2 u$$

in $L_2(a,b)$, $0 < a < b < +\infty$. Then, the formally adjoint differential expression has the form

$$L^+ v = \frac{1}{x}\frac{dv}{dx} - \frac{1}{2x^2}v + x^2 v.$$

If L_0 is a differential operator generated by the differential expression L with the domain

$$D(L) = \overset{o}{W}{}_2^1[a,b],$$

then L^* is defined by the differential expression L^+ with the domain

$$D(L^*) = W_2^1[a,b].$$

Hence, it is easy to show that on $D(L_0) \subset D(L^*)$ the inequality

$$\|L^* u\| \leqslant \|L_0 u\|, \;\; u \in D(L_0)$$

holds.

5. On dissipative correct restrictions of one class of maximal operators

Let us consider a minimal symmetric operator L_0 in a Hilbert space H. Then the maximal operator is $L = L_0^*$ and there exists at least one self-adjoint boundary extension $\tilde{L}$ of the minimal operator L_0, i.e. $L_0 \subset \tilde{L} \subset L$. Hence, for all correct restriction of the maximal operator L_K the representation $L_K^{-1} f = \tilde{L}^{-1} f + Kf$ holds, where f is an arbitrary function in H and K is a linear operator acting from H into $KerL$.

The following abstract theorem is the main result of this section.

Theorem 5.1. *The imaginary part of the operator L_k is sign-defined if and only if $R(L_0) \subset KerK$ and $\mathfrak{Im}(K)$ is sign-defined on $KerL$.*

Proof. It is known that $H = R(L_0) \oplus KerL$. In other words, each element of $f \in H$ is uniquely decomposed as $f = f_0 + u_0$, where $f_0 \in R(L_0)$ and $u_0 \in KerL$. The sign-definiteness of $\mathfrak{Im}(L_k^{-1})f$ means the sign-definiteness of $(\mathfrak{Im}(L_k^{-1})f, f)$ for all f in H. Therefore,

$$\begin{aligned}
\left(\mathfrak{Im}(L_k^{-1})f, f\right) &= \left(\frac{L_k^{-1} - \left(L_k^{-1}\right)^*}{2i} f, f\right) = \left(\frac{K - K^*}{2i} f, f\right) \\
&= \frac{1}{2i}(Kf, f) - \frac{1}{2i}(K^* f, f) = \frac{1}{2i}(Kf, u_0) - \frac{1}{2i}\left(K^*(f_0 + u_0), f\right) \\
&= \frac{1}{2i}(Kf, u_0) - \frac{1}{2i}(f_0 + u_0, Kf) = \frac{1}{2i}(Kf, u_0) - \frac{1}{2i}(u_0, Kf) \\
&= \frac{1}{2i}(K(f_0 + u_0), u_0) - \frac{1}{2i}(K^* u_0, f_0 + u_0) \\
&= \frac{1}{2i}\left[(Kf_0, u_0) - (u_0, Kf_0)\right] + \frac{1}{2i}\left[(Ku_0, u_0) - (K^* u_0, u_0)\right].
\end{aligned} \tag{5.1}$$

There are two possible cases.

Case I. $Kf_0 = 0$, $\forall f_0 \in R(L_0)$. This means that $R(L_0) \subset Ker\widehat{L}$, and it can be seen from (5.1) that the sign-definiteness of $\mathfrak{Im}(L_k^{-1})$ is equivalent to the sign-definiteness of $\mathfrak{Im}(K)$ on $KerL$.

Case II. Let there exist $f_0 \in R(L_0)$ satisfying $Kf_0 \not\equiv 0$. We will show that, in this case the sign-definiteness of $\Im m(L_k^{-1})$ is impossible. It is well known that $Kf_0 \in KerL$. Set $u_2 = Kf_0$. As u_0 we take $u_0 = \frac{u_2}{\lambda}$, where λ is a non-zero complex parameter. Then (5.1) will take the form

$$\frac{\lambda - \bar{\lambda}}{2i}(u_0, u_0) + (\Im m(K)u_0, u_0), \quad u_0 \in KerL.$$

The sign-definiteness of this expression is equivalent to the sign-definiteness of the following expression

$$\frac{\lambda - \bar{\lambda}}{2i} + \frac{(\Im m(K)u_0, u_0)}{(u_0, u_0)}.$$

Since K is bounded and $\frac{\lambda-\bar{\lambda}}{2i}$ is arbitrary, it follows that $\Im m(L_k^{-1})$ cannot be sign-defined. This completes the proof. □

Consider the maximal operator L , generated by the differential expression $\ell(y) \equiv (i)^n \frac{d^n y}{dx^n}$ in the space $L_2(0,1)$. In this case the domain of the minimal operator L_0 coincides with $W_2^n[0,1]$ and domains of all boundary extensions of L_k of the minimal operator L_0 consists of functions in $W_2^{n-1}[0,1]$ satisfying the conditions

$$\begin{aligned} y(0) &= i^n \left\{ \sum_{k=1}^{n} (-1)^k \bar{a}_{1k} y^{(n-k)}(0) + \sum_{j=1}^{n}\sum_{k=j}^{n} \frac{(-1)^{j-1}\bar{a}_{1k}}{(k-j)!} y^{(n-j)}(1) \right\}, \\ y'(0) &= i^n \left\{ \sum_{k=1}^{n} (-1)^k \bar{a}_{2k} y^{(n-k)}(0) + \sum_{j=1}^{n}\sum_{k=j}^{n} \frac{(-1)^{j-1}\bar{a}_{2k}}{(k-j)!} y^{(n-j)}(1) \right\}, \\ &\vdots \\ y^{(n-1)}(0) &= i^n \left\{ \sum_{k=1}^{n} (-1)^k \bar{a}_{nk} y^{(n-k)}(0) + \sum_{j=1}^{n}\sum_{k=j}^{n} \frac{(-1)^{j-1}\bar{a}_{nk}}{(k-j)!} y^{(n-j)}(1) \right\}. \end{aligned} \tag{5.2}$$

It can be seen that these boundary conditions exhaust all correct boundary value problems for the differential expression $\ell(y)$.

Theorem 5.2. *A correct restriction L_k of the maximal operator L, generated by the differential expression $\ell(y)$, is dissipative if and only if L_k is a boundaty extension, i.e. $D(L_k)$ is determined by the boundary conditions (5.2) and the sums of minors*

$$S_p = \sum_{1\le i_1<\ldots<i_p<k_1<\ldots k_p\le 2n} A = \begin{pmatrix} i_1 & \cdots & i_p \\ k_1 & \cdots & k_p \end{pmatrix}, \quad p = 1, \ldots, 2n,$$

of the matrix

$$\begin{bmatrix} b_{11} & \cdots & b_{n1} & c_{11} & \cdots & c_{n1} \\ b_{12} & \cdots & b_{n2} & c_{12} & \cdots & c_{n2} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ b_{1n} & \cdots & b_{nn} & c_{1n} & \cdots & c_{nn} \\ 1 & \cdots & \frac{1}{n} & \bar{b}_{11} & \cdots & \bar{b}_{1n} \\ \frac{1}{2} & \cdots & \frac{1}{n+1} & \bar{b}_{21} & \cdots & \bar{b}_{2n} \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ \frac{1}{n} & \cdots & \frac{1}{2n-1} & \bar{b}_{n1} & \cdots & \bar{b}_{nn} \end{bmatrix},$$

where $b_{ki} = \sum_{j=1} \frac{\bar{a_{ij}}}{(j-1)!(j+k-1)}$, $c_{ik} = a_{i1} \sum_{\ell=1}^{n} \frac{\bar{a}_{kl}}{\ell!} + a_{i2} \sum_{\ell=1}^{n} \frac{\bar{a}_{kl}}{\ell!(\ell+1)} + \cdots + \frac{a_{in}}{(n-1)!} \sum_{\ell=1}^{n} \frac{\bar{a}_{kl}}{\ell!(\ell+n-1)}$ *are non-negative numbers.*

Proof. The dissipativity of L_k is equivalent to the dissipativity of L_k^{-1}. By Theorem 5.1, we only need to check the condition $\Im m(K) = \frac{K-K^*}{2i} \le 0$ on $KerL$, where

$$(Kf)(x) = \sum_{i=1}^{n}\sum_{j=1}^{n} \frac{x^{i-1}}{(i-1)!} \bar{a}_{ij} \int_0^1 \frac{t^{j-1}}{(j-1)!} f(t)dt,$$

$$(K^*f)(x) = \sum_{i=1}^{n}\sum_{j=1}^{n}\frac{x^{j-1}}{(j-1)!}a_{ij}\int_0^1 \frac{t^{i-1}}{(i-1)!}f(t)dt.$$

Since K is a finite-dimensional operator, it follows that $\Im m(K) \le 0$ if and only if its all eigenvalues λ are non-positive. Therefore, we need to find the criterion of non-positivity of the operator $\Im m(K)$. Its Fredholm determinant is

$$\Delta(\lambda) = |A - \lambda E| = (-\lambda)^{2n} + S_1(-\lambda)^{2n-1} + S_2(-\lambda)^{2n-2} + \cdots + S_{2n} = 0,$$

where

$$S_p = \sum_{1\le i_1<\ldots<i_p<k_1<\ldots k_p\le 2n} A = \left(\begin{array}{ccc} i_1 & \ldots & i_p \\ k_1 & \ldots & k_p \end{array}\right), \quad p = 1, \ldots, 2n,$$

is the sum of principal minors of order p of the matrix A. But, by the Hurwitz criterion for the stability of a polynomial with real coefficients, the eigenvalues λ are non-positive if and only if the principal minors of the Hurwitz matrix for $\Delta(\lambda)$ are non-negative. This concludes the proof. □

Example 5.1. *Consider the case $n = 1$. Then*

$$\ell(y) \equiv i\frac{dy}{dx}. \tag{5.3}$$

Hence, operators, which are inverse for all correct constrictions of the maximal operator generated by the differential expression (5.3), have the form

$$L_k^{-1}f = L_\phi^{-1}f + \int_0^1 f(t)\overline{\sigma(t)}dt,$$

where L_ϕ is a self-adjoint operator corresponding to the correct the problem

$$\begin{cases} iy'(x) = f(x), \\ y(0) = -y(1). \end{cases}$$

By Theorem 5.1, dissipative correct restrictions lie among boundary extensions. In other words, $\sigma(x) = \alpha + i\beta \equiv const$. Therefore, by Theorem 5.2, those correct constrictions for which $\Im m(\sigma) = \beta \le 0$ are dissipative. Thus, only the following boundary value problems are dissipative correct problems

$$\begin{cases} iy'(x) = f(x), \\ (1 + 2\beta + 2i\alpha)y(0) + (1 - 2\beta - 2i\alpha)y(1) = 0, \end{cases}$$

where $\beta \le 0$ and α is an arbitrary real number.

Let L be a maximal operator in a Hilbert space H, which has at least one self-adjoint correct restriction L_s. Generally, the domain of the operator $L_0 = L^*$ is not always dense in H. Therefore, L_0 is not a symmetric operator. In this case, despite the absence of such a minimal operator, all correct dissipative restrictions can be described. For this purpose, we obtain the following result.

Theorem 5.3. *Imaginary part of the operator*

$$L_k^{-1} = L_s^{-1} + K,$$

where K is a linear bounded operator acting from H into $KerL$, is sign-defined if and only if $(KerL)^\perp \subset KerK$ and $\Im m(K)$ is sign-defined on $KerL$.

Proof. If we replace the set $R(L_0)$ with $(KerL)^\perp$ in the proof of Theorem 5.1, then the same approach completes the proof. □

6. Acknowledgment

The work was partially supported by the grant (No. AP09258335 and No. AP08052004) of the Science Committee of the Ministry of Education and Science of the Republic of Kazakhstan.

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Nurbek Kakharman:
Al-Farabi Kazakh National University
71 Al-Farabi avenue
050040 Almaty
Kazakhstan
and
Institute of Mathematics and Mathematical Modeling
125 Pushkin str.
050010 Almaty
Kazakhstan
E-mail address: n.kakharman@math.kz

Kanat Tulenov:
Al-Farabi Kazakh National University
71 Al-Farabi avenue
050040 Almaty
Kazakhstan
and

Institute of Mathematics and Mathematical Modeling
125 Pushkin str.
050010 Almaty
Kazakhstan
E-mail address: tulenov@math.kz

Lyazzat Zhumanova:
Al-Farabi Kazakh National University
71 Al-Farabi avenue
050040 Almaty
Kazakhstan
E-mail address: lyazzatzhuman@gmail.com