Received ** April 20**; | Revised ** June 20**; | Accepted ** June 20**

DOI: xxx/xxxx

ARTICLE TYPE

Recovery of unknown coefficients in a two-dimensional hyperbolic equation with additional conditions

Mehraliyev Y T.[1] | Yang H.[2] | Azizbayov E I.[1,3]

[1]Department of Differential and Integral Equations, Baku State University, Baku, Azerbaijan
[2]Department of Mathematics, Augusta University, Augusta, United States of America
[3]Department of Control for Intelligent Systems, The Academy of Public Administration under the President of the Republic of Azerbaijan, Baku, Azerbaijan

Correspondence
*Azizbayov E.I., AZ1001, 74, Lermontov Str., Baku, Azerbaijan
Email: eazizbayov@bsu.edu.az

Present Address
The Academy of Public Administration under the President of the Republic of Azerbaijan, 74, Lermontov Str., Baku, AZ1001, Azerbaijan.

MSC Classification: Primary 35R30, 35L10; Secondary 35A01, 35A02, 35A09

Summary

In this paper, a nonlocal inverse boundary value problem for a two-dimensional hyperbolic equation with overdetermination conditions is studied. To investigate the solvability of the original problem, we first consider an auxiliary inverse boundary value problem and prove its equivalence (in a certain sense) to the original problem. Then using the Fourier method, solving an equivalent problem is reduced to solving a system of integral equations and by the contraction mappings principle the existence and uniqueness theorem for auxiliary problem is proved. Further, on the basis of the equivalency of these problems the uniquely existence theorem for the classical solution of the considered inverse problem is proved and some considerations on the numerical solution for this inverse problem are presented with the examples.

KEYWORDS:
Inverse coefficient problem, two-dimensional hyperbolic equation, Fourier method, classical solution, overdetermination condition

1 | INTRODUCTION AND FORMULATION OF INVERSE PROBLEM

Let $T > 0$ be a fixed time moment and let $D_T = \bar{Q}_{xy} \times \{0 \le t \le T\}$ denotes a closed bounded region in space, where Q_{xy} defined by the inequalities $0 < x < 1,\ 0 < y < 1$. We further suppose that $f(x, y, t), \varphi(x, y), \psi(x, y)$, and $h_i(t)$ $(i = 1, 2)$ are given functions of $x, y \in [0, 1]$ and $t \in [0, T]$. Consider the two-dimensional inverse boundary value problem of identifying an unknown triple of functions $\{u(x, y, t), a(t), b(t)\}$ for the equation

$$u_{tt}(x, y, t) = u_{xx}(x, y, t) + u_{yy}(x, y, t) + a(t)u(x, y, t) + b(t)u_t(x, y, t) + f(x, y, t) \ \ (x, y, t) \in D_T, \tag{1}$$

with the initial conditions

$$u(x, y, 0) = \varphi(x, y),\ \ u_t(x, y, 0) = \psi(x, y),\ \ 0 \le x, y \le 1, \tag{2}$$

the boundary conditions

$$u_x(0, y, t) = u(1, y, t) = 0,\ \ 0 \le y \le 1,\ 0 \le t \le T, \tag{3}$$

$$u(x, 0, t) = u_y(x, 1, t) = 0,\ \ 0 \le x \le 1,\ 0 \le t \le T, \tag{4}$$

and the overdetermination conditions

$$u(0, 1, t) = h_1(t),\ \ 0 \le t \le T, \tag{5}$$

$$\int_0^1\int_0^1 u(x,y,t)dxdy = h_2(t),\ \ 0 \le t \le T. \tag{6}$$

Definition 1. The triple $\{u(x,y,t), a(t), b(t)\}$ is said to be a classical solution of the problem (1)-(6), if the functions $u(x,y,t) \in C^2(D_T)$ and $a(t), b(t) \in C[0,T]$ satisfy the conditions (1)-(6) in the usual (classical) sense.

The primary purpose of the work is to study the problem of determining the coefficients on the right-hand side of the two-dimensional hyperbolic equation from the additional data. Such problems are a prominent branch of the theory of differential equations and are called inverse problems. The applied importance of inverse problems is so great (they arise in various fields of human activity, such as seismology, mineral exploration, biology, medicine, desalination of seawater, movement of liquid in a porous medium, acoustics, electromagnetics, fluid dynamics, remote sensing, nondestructive evaluation, and many other areas. etc.) which puts them a series of the most actual problems of modern mathematics. When solving so-called direct problems, the solution of a given differential equation or a system of equations is obtained using the initial and boundary conditions, but in inverse problems the equation itself is also unknown. Thus, the definition of both the equation itself and its solution requires the imposition of some additional conditions in comparison with the corresponding direct problem.

Recently, the inverse and ill-posed problems associated with the one-dimensional hyperbolic/wave equation has drawn the attention of many authors (see for example, [1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16], and the references therein). Nevertheless, the inverse problems for two-dimensional partial differential equations associated with the recovery of the coefficient have been studied scarce (see [17,18,19,20,21,22]) and need additional consideration.

A distinctive feature of presented article is the investigation of an inverse problem for two-dimensional hyperbolic equation with additional nonlocal integral conditions. It should be noted that one of the classes of qualitatively new problems is problems with nonlocal conditions. In the literature, the term “nonlocal boundary value problems” refers to problems that contain conditions relating the values of the solution and/or its derivatives either at different points of the boundary or at boundary points and some interior points. The term “nonlocal conditions” and their classification were introduced by A.A. Dezin[23].

The applied significance of problems with nonlocal conditions is associated with the study of physical processes, the mathematical models of which are such problems. These are processes occurring in the turbulent plasma, the processes of diffusion, thermal conductivity, moisture transfer in a capillary-porous medium, problems of mathematical biology, as well as some inverse problems of mathematical physics.

Systematic investigations of nonlocal direct problems for partial differential equations began with an article by A.V. Bitsadze and A.A. Samarskii[24]. In the article, considered the problem of finding a solution to an elliptic equation (and a system of equations) of the second order, whose values at points of some part of the boundary of the considered region are equal to its values in the images of these points for a given diffeomorphism. The simplest direct problems for one-dimensional hyperbolic equation with nonlocal conditions have been well studied by many authors using different methods (in particular, [25,26,27,28,29], et al.). Moreover, in[30,31] the authors present a regularity result for solutions of partial differential equations in the framework of mixed Morrey spaces.

The following theorem is valid.

Theorem 1. Assume that $\varphi(x,y), \psi(x,y) \in C(Q_{xy})$, $f(x,y,t) \in C(D_T)$, $h_i(t) \in C^2[0,T]$ $(i = 1,2)$, $h(t) \equiv h_1(t)h_2'(t) - h_2(t)h_1'(t) \neq 0$, $0 \le t \le T$ and the compatibility conditions

$$\varphi(0,1) = h_1(0),\ \ \psi(0,1) = h_1'(0), \tag{7}$$

$$\int_0^1\int_0^1 \varphi(x,y)dxdy = h_2(0),\ \ \int_0^1\int_0^1 \psi(x,y)dxdy = h_2'(0), \tag{8}$$

hold. Then the problem of finding a classical solution of (1)-(6) is equivalent to the problem of determining the functions $u(x,y,t) \in C^2(D_T)$, $a(t) \in C[0,T]$, and $b(t) \in C[0,T]$ satisfying (1)-(4), and the conditions

$$h_1''(t) = u_{xx}(0,1,t) + u_{yy}(0,1,t) + a(t)h_1(t) + b(t)h_1'(t) + f(0,1,t),\ \ 0 \le t \le T, \tag{9}$$

$$h_2''(t) = \int_0^1 u_x(1, y, t)dy - \int_0^1 u_y(x, 0, t)dx + a(t)h_2(t) + b(t)h_2'(t) + \int_0^1\int_0^1 f(x, y, t)dxdy, \ \ 0 \le t \le T. \tag{10}$$

Proof. Let the triple $\{u(x, y, t), a(t), b(t)\}$ be a classical solution of problem (1)-(6). Taking into account the condition $h_i(t) \in C^2[0, T]$ $(i = 1, 2)$, and differentiating twice both sides of (5) and (6) with respect to t gives

$$u_t(0, 1, t) = h_1'(t), \ \ u_{tt}(0, 1, t) = h_1''(t), \ \ 0 \le t \le T, \tag{11}$$

$$\int_0^1\int_0^1 u_t(x, y, t)dxdy = h_2'(t), \ \ \int_0^1\int_0^1 u_{tt}(x, y, t)dxdy = h_2''(t), \ \ 0 \le t \le T. \tag{12}$$

Setting $x = 0$ and $y = 1$ in Equation (1), the procedure yields

$$u_{tt}(0, 1, t) = u_{xx}(0, 1, t) + u_{yy}(0, 1, t) + a(t)u(0, 1, t) + b(t)u_t(0, 1, t) + f(0, 1, t), \ \ 0 \le t \le T. \tag{13}$$

From (13), taking into account (5) and (11), we conclude that condition (9) is satisfied.

Further, integrating Equation (1) with respect to x and y over the interval $[0, 1]$ gives

$$\frac{d^2}{dt^2}\int_0^1\int_0^1 u(x, y, t)dxdy = \int_0^1 (u_x(1, y, t) - u_x(0, y, t))dy + \int_0^1 (u_y(x, 1, t) - u_y(x, 0, t))dx$$

$$+a(t)\int_0^1\int_0^1 u(x, y, t)dxdy + b(t)\int_0^1\int_0^1 u_t(x, y, t)dxdy + \int_0^1\int_0^1 f(x, y, t)dxdy, \ \ 0 \le t \le T.$$

From the last relation, taking into account (3) and (4), we obtain

$$\frac{d^2}{dt^2}\int_0^1\int_0^1 u(x, y, t)dxdy = \int_0^1 u_x(1, y, t)dy - \int_0^1 u_y(x, 0, t)dx$$

$$+a(t)\int_0^1\int_0^1 u(x, y, t)dxdy + b(t)\int_0^1\int_0^1 u_t(x, y, t)dxdy + \int_0^1\int_0^1 f(x, y, t)dxdy, \ \ 0 \le t \le T. \tag{14}$$

Hence, from (14), taking into account (6) and (12), we arrive at (10).

Now suppose that the triple $\{u(x, y, t), a(t), b(t)\}$ is a solution to the problem (1)–(4), (9), (10). Then from (9) and (13), we get

$$\frac{d^2}{dt^2}(u(0, 1, t) - h_1(t)) = b(t)\frac{d}{dt}(u(0, 1, t) - h_1(t)) + a(t)(u(0, 1, t) - h_1(t)), \ \ 0 \le t \le T. \tag{15}$$

Using (2) and the compatibility condition (7), we obtain the following relation

$$u(0, 1, 0) - h_1(0) = \varphi(0, 1) - h_1(0) = 0, \quad u_t(0, 1, 0) - h_1'(0) = \psi(0, 1) - h_1'(0) = 0. \tag{16}$$

Since problem (15), (16) has only a trivial solution, so from $u(0, 1, t) - h_1(t) = 0, \ \ 0 \le t \le T$, we obtain that the condition (5) is satisfied.

Now, from (10) and (14) we find:

$$\frac{d^2}{dt^2}\left(\int_0^1\int_0^1 u(x, y, t)dxdy - h_2(t)\right)$$

$$= b(t)\frac{d}{dt}\left(\int_0^1\int_0^1 u(x, y, t)dxdy - h_2(t)\right) + a(t)\left(\int_0^1\int_0^1 u(x, y, t)dxdy - h_2(t)\right), \ \ 0 \le t \le T. \tag{17}$$

By using the initial conditions (2) and the compatibility conditions (8), we may write

$$\begin{gathered}\int_0^1\int_0^1 u(x,y,0)dxdy - h_2(0) = \int_0^1\int_0^1 \varphi(x,y)dxdy - h_2(0) = 0,\\ \int_0^1\int_0^1 u_t(x,y,0)dxdy - h_2'(0) = \int_0^1\int_0^1 \psi(x,y)dxdy - h_2'(0) = 0.\end{gathered} \tag{18}$$

Hence relations (14), (18) enable us to conclude that

$$\int_0^1\int_0^1 u(x,y,t)dxdy - h_2(t) = 0, \quad 0 \le t \le T,$$

i.e., the condition (4) is satisfied. The proof is complete. □

2 | CLASSICAL SOLVABILITY OF INVERSE BOUNDARY VALUE PROBLEM

We seek the first component of classical solution $\{u(x,y,t), a(t), b(t)\}$ of the problem (1)–(4), (9), (10) in the form

$$u(x,y,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} u_{k,n}(t)\cos\lambda_k x \sin\gamma_n y, \tag{19}$$

where

$$\lambda_k = \frac{\pi}{2}(2k-1), \quad k = 1,2,..., \quad \gamma_n = \frac{\pi}{2}(2n-1), \quad n = 1,2,...,$$

$$u_{k,n}(t) = 4\int_0^1\int_0^1 u(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy, \quad k,n = 1,2,....$$

Applying the method of separation of variables to determine the desired coefficients $u_{k,n}(t)$ $(k,n = 1,2,...)$ of the function $u(x,y,t)$ from (1), (2), we obtain:

$$u''_{k,n}(t) + \mu_{k,n}^2 u_{k,n}(t) = F_{k,n}(t;u,a,b), \quad k,n = 1,2,..., \quad 0 \le t \le T, \tag{20}$$

$$u_{k,n}(0) = \varphi_{k,n}, u'_{k,n}(0) = \psi_{k,n}, \quad k,n = 1,2,..., \tag{21}$$

where

$$\mu_{k,n}^2 = \lambda_k^2 + \gamma_n^2, \quad k,n = 1,2,...,$$

$$F_{k,n}(t;u,a,b) = f_{k,n}(t) + a(t)u_{k,n}(t) + b(t)u'_{k,n}(t), \quad k,n = 1,2,...,$$

$$f_{k,n}(t) = 4\int_0^1\int_0^1 f(x,y,t)\cos\lambda_k x\sin\gamma_n y dxdy, \quad k,n = 1,2,...,$$

$$\varphi_{k,n} = 4\int_0^1\int_0^1 \varphi(x,y)\cos\lambda_k x\sin\gamma_n y dxdy, \quad k,n = 1,2,...,$$

$$\psi_{k,n} = 4\int_0^1\int_0^1 \psi(x,y)\cos\lambda_k x\sin\gamma_n y dxdy, \quad k,n = 1,2,....$$

Solving the problem (20), (21) gives

$$u_{k,n}(t) = \varphi_{k,n}\cos\mu_{k,n}t + \frac{1}{\mu_{k,n}}\psi_{k,n} sin\mu_{k,n}t$$

$$+\frac{1}{\mu_{k,n}}\int_0^t F_{k,n}(\tau;u,a,b)\sin\mu_{k,n}(t-\tau)d\tau, \quad k,n = 1,2,..., \quad 0 \le t \le T. \tag{22}$$

Obviously,

$$u'_{k,n}(t) = -\mu_{k,n}\varphi_{k,n}\sin\mu_{k,n}t + \psi_{_{k,n}}\cos\mu_{k,n}t$$

$$+\int\limits_0^t F_{k,n}(\tau;u,a,b)\cos\mu_{k,n}(t-\tau)d\tau,\quad k,n=1,2,...,\quad 0\le t\le T. \tag{23}$$

Substituting the expression of $u_{k,n}(t),\quad k,n=1,2,...,$ into (19), we find

$$u(x,y,t) = \sum_{k=1}^{\infty}\left\{\varphi_{k,n}\cos\mu_{k,n}t + \frac{1}{\mu_{k,n}}\psi_{k,n}\sin\mu_{k,n}t + \frac{1}{\mu_{k,n}}\int\limits_0^t F_k(\tau;u,a,b)\sin\mu_{k,n}(t-\tau)d\tau\right\}\cos\lambda_k x. \tag{24}$$

Now from (9) and (10), taking into account (12), respectively, we get:

$$a(t)h_1(t) + b(t)h'_1(t) = h''_1(t) - f(0,1,t) + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{n+1}\mu_{k,n}^2 u_{k,n}(t),\quad 0\le t\le T, \tag{25}$$

$$a(t)h_2(t) + b(t)h'_2(t) = h''_2(t) - \int\limits_0^1\int\limits_0^1 f(x,y,t)dxdy + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}(-1)^{k+1}\frac{\mu_{k,n}^2}{\lambda_k\gamma_n}u_{k,n}(t),\quad 0\le t\le T. \tag{26}$$

Let us assume that

$$h(t) \equiv h_1(t)h'_2(t) - h_2(t)h'_1(t) \ne 0,\quad 0\le t\le T.$$

Then from (25) and (26), we find:

$$a(t) = [h(t)]^{-1}\left\{(h''_1(t) - f(0,1,t))h'_2(t) - \left(h''_2(t) - \int\limits_0^1\int\limits_0^1 f(x,y,t)dxdy\right)h'_1(t)\right.$$

$$\left.+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left((-1)^{n+1}h'_2(t) - (-1)^{k+1}\frac{h'_1(t)}{\lambda_k\gamma_n}\right)\mu_{k,n}^2 u_{k,n}(t)\right\},\ 0\le t\le T, \tag{27}$$

$$b(t) = [h(t)]^{-1}\left\{\left(h''_2(t) - \int\limits_0^1\int\limits_0^1 f(x,y,t)dxdy\right)h_1(t) - (h''_1(t) - f(0,1,t))h_2(t)\right.$$

$$\left.+\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left((-1)^{k+1}\frac{h_1(t)}{\lambda_k\gamma_n} - (-1)^{n+1}h_2(t)\right)\mu_{k,n}^2 u_{k,n}(t)\right\},\ 0\le t\le T. \tag{28}$$

The following expressions for the second and third components of the solution $\{u(x,y,t),a(t),b(t)\}$ to problem (1)–(4), (9), (10)

$$a(t) = [h(t)]^{-1}\left\{(h''_1(t) - f(0,1,t))h'_2(t) - \left(h''_2(t) - \int\limits_0^1\int\limits_0^1 f(x,y,t)dxdy\right)h'_1(t) + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left((-1)^{n+1}h'_2(t) - (-1)^{k+1}\frac{h'_1(t)}{\lambda_k\gamma_n}\right)\right.$$

$$\left.\times\mu_{k,n}^2\left[\varphi_{k,n}\cos\mu_{k,n}t + \frac{1}{\mu_{k,n}}\psi_{k,n}sin\mu_{k,n}t + \frac{1}{\mu_{k,n}}\int\limits_0^t F_{k,n}(\tau;u,a,b)\sin\mu_{k,n}(t-\tau)d\tau\right]\right\}, \tag{29}$$

$$b(t) = [h(t)]^{-1}\left\{\left(h''_2(t) - \int\limits_0^1\int\limits_0^1 f(x,y,t)dxdy\right)h_1(t) - (h''_1(t) - f(0,1,t))h_2(t) + \sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\left((-1)^{k+1}\frac{h_1(t)}{\lambda_k\gamma_n} - (-1)^{n+1}h_2(t)\right)\right.$$

$$\left.\times\mu_{k,n}^2\left[\varphi_{k,n}\cos\mu_{k,n}t + \frac{1}{\mu_{k,n}}\psi_{k,n}sin\mu_{k,n}t + \frac{1}{\mu_{k,n}}\int\limits_0^t F_{k,n}(\tau;u,a,b)\sin\mu_{k,n}(t-\tau)d\tau\right]\right\}, \tag{30}$$

respectively, were obtained by substituting (24) into (27) and (28).

Thus the solution of problem (1)–(4), (9), (10) was reduced to the solution of systems (24), (29), (30) with respect to unknown functions $u(x, y, t)$, $a(t)$, and $b(t)$.

The following lemma plays an important role in studying the uniqueness of the solution to problem (1)–(4), (9), (10).

Lemma 1. If $\{u(x, y, t), a(t), b(t)\}$ is any solution to problem (1)–(4), (9), (10), then the functions

$$u_{k,n}(t) = 4\int_0^1\int_0^1 u(x, y, t)\cos\lambda_k x\sin\gamma_n y dx dy, \quad k, n = 1, 2, ...$$

satisfy the system (22) on the interval $[0, T]$.

Proof. Let $\{u(x, y, t), a(t), b(t)\}$ be any solution of the problem (1)–(4), (9), (10). Multiplying both sides of the Equation (1) by the special functions $4\cos\lambda_k x\sin\gamma_n y, \quad k, n = 1, 2, ...$, integrating with respect to x and y over the interval $[0, T]$ and using the relations

$$4\int_0^1\int_0^1 u_{tt}(x, y, t)\cos\lambda_k x\sin\gamma_n y dx dy = \frac{d^2}{dt^2}\left(4\int_0^1\int_0^1 u(x, y, t)\cos\lambda_k x\sin\gamma_n y dx dy\right) = u''_{k,n}(t), \quad k, n = 1, 2, ...,$$

$$4\int_0^1\int_0^1 u_{xx}(x, y, t)\cos\lambda_k x\sin\gamma_n y dx dy = -\lambda_k^2\left(4\int_0^1\int_0^1 u(x, y, t)\cos\lambda_k x\sin\gamma_n y dx dy\right) = -\lambda_k^2 u_{k,n}(t), \quad k, n = 1, 2, ...,$$

$$4\int_0^1\int_0^1 u_{yy}(x, y, t)\cos\lambda_k x\sin\gamma_n y dx dy = -\gamma_n^2\left(4\int_0^1\int_0^1 u(x, y, t)\cos\lambda_k x\sin\gamma_n y dx dy\right) = -\gamma_k^2 u_{k,n}(t), \quad k, n = 1, 2, ...,$$

we obtain that the Equation (20) is satisfied.

In like manner, it follows from (2) that condition (21) is also satisfied.

Thus, the system of functions $u_{k,n}(t) \ \ (k, n = 1, 2, ...)$ is a solution of problem (20), (21). Hence it follows directly that the functions $u_{k,n}(t) \ \ (k, n = 1, 2, ...)$ are also satisfy the system (22) on $[0, T]$. The lemma is proved. □

Obviously, if $u_{k,n}(t) = 4\int_0^1\int_0^1 u(x, y, t)\cos\lambda_k x\sin\gamma_n y dx dy \ \ (k, n = 1, 2, ...)$ is a solution to system (22), then the triple $\{u(x, y, t), a(t), b(t)\}$ of functions $u(x, y, t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} u_{k,n}(t)\cos\lambda_k x\sin\gamma_n y$, $a(t)$, and $b(t)$ is also a solution to system (24), (29), (30).

It follows from the Lemma 1 that

Corollary 1. Assume that the system (24), (29), (30) has a unique solution. Then the problem (1)–(4), (9), (10) has at most one solution, i.e., if the problem (1)–(4), (9), (10) has a solution, then it is unique.

Let us consider the functional space $B_{2,T}^{3,2}$ that is introduced in the study of[32], where $B_{2,T}^{3,2}$ denotes a set of all functions of the form

$$u(x, y, t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} u_{k,n}(t)\cos\lambda_k x\sin\gamma_n y,$$

considered in D_T. Moreover, the functions $u_{k,n}(t) \ \ (k, n = 1, 2, ...)$ contained in last sum are continuously differentiable on $[0, T]$ and

$$J(u) \equiv \left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3 \left\|u_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} + \left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^2 \left\|u'_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}} < +\infty.$$

The norm on the set $J(u)$ is established as follows:

$$\|u(x, y, t)\|_{B_{2,T}^{3,2}} = J(u).$$

Let $E_T^{3,2}$ denote the space consisting of the topological product $B_{2,T}^{3,2} \times C[0, T] \times C[0, T]$, which is the norm of the element $z = \{u, a, b\}$ defined by the formula

$$\|z\|_{E_T^{3,2}} = \|u(x, y, t)\|_{B_{2,T}^{3,2}} + \|a(t)\|_{C[0,T]} + \|b(t)\|_{C[0,T]} .$$

It is known that the spaces $B^{3,2}_{2,T}$ and $E^{3,2}_T$ are Banach spaces.

Let us now consider the operator

$$\Phi(u,a,b)=\{\Phi_1(u,a,b),\Phi_2(u,a,b),\Phi_3(u,a,b)\},$$

in the space $E^{3,2}_T$, where

$$\Phi_1(u,a,b)=\tilde{u}(x,y,t)\equiv\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\tilde{u}_{k,n}(t)\cos\lambda_k x\sin\gamma_n y,\ \ \Phi_2(u,a,b)=\tilde{a}(t),\ \ \Phi_3(u,a,b)=\tilde{b}(t),$$

and the functions $\tilde{u}_{k,n}(t)\ \ (k,n=1,2,...)$, $\tilde{a}(t)$, and $\tilde{b}(t)$ are equal to the right-hand sides of (24), (29), and (30), respectively. Obviously, $\tilde{u}'_{k,n}(t)\ \ (k,n=1,2,...)$ are determined by right-hand side of (23).

It is easy to see that

$$\mu^3_{k,n}\le(\lambda_k^2+\gamma_k^2)(\lambda_k+\gamma_n)=\lambda_k^3+\lambda_k^2\gamma_n+\gamma_k^2\lambda_k+\gamma_k^3.$$

Taking into account this relation, we obtain

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu^3_{k,n}\left\|\tilde{u}_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}\le\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\gamma_n\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\gamma_n^2\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^3\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\psi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^2\left|\psi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{10T}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\left|f_{k,n}(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}+\sqrt{10T}\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^2\left|f_{k,n}(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}\right)$$

$$+\sqrt{10}T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu^3_{k,n}\left\|u_{k,n}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\sqrt{10}T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu^2_{k,n}\left\|u'_{k,n}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}},\tag{31}$$

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu^2_{k,n}\left\|\tilde{u}'_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}\le\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\gamma_n\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\gamma_n^2\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^3\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\psi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{10}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^2\left|\psi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\sqrt{10T}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\left|f_{k,n}(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}+\sqrt{10T}\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^2\left|f_{k,n}(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}\right)$$

$$+\sqrt{10}T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu^3_{k,n}\left\|u_{k,n}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+\sqrt{10}T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu^2_{k,n}\left\|u'_{k,n}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}},\tag{32}$$

$$\left\|\tilde{a}(t)\right\|_{C[0,T]}\le\left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|(h_1''(t)-f(0,1,t))h_2'(t)-\left(h_2''(t)-\int_0^1\int_0^1 f(x,y,t)dxdy\right)h_1'(t)\right\|_{C[0,T]}\right.$$

$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu^{-2}_{k,n}\right)^{\frac{1}{2}}\left\|\left|h_1'(t)\right|+\left|h_2'(t)\right|\right\|_{C[0,T]}\left[\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\gamma_n\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}\right.$$

$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\gamma_n^2\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^3\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\psi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^2\left|\psi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{T}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\left|f_{k,n}(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^2\left|f_{k,n}(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}\right)+T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\left\|u_{k,n}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}$$

$$+T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^2\left\|u'_{k,n}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\Bigg]\Bigg\}, \tag{33}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]}\leq\left\|[h(t)]^{-1}\right\|_{C[0,T]}\Bigg\{\left\|\left(h_2''(t)-\int_0^1\int_0^1 f(x,y,t)dxdy\right)h_1(t)-(h_1''(t)-f(0,1,t))h_2(t)\right\|_{C[0,T]}$$

$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_k^{-2}\right)^{\frac{1}{2}}\left\|\left|h_1(t)\right|+\left|h_2(t)\right|\right\|_{C[0,T]}\Bigg[\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^3\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\gamma_n\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}$$

$$+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k\gamma_n^2\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^3\left|\varphi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\left|\psi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}+\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^2\left|\psi_{k,n}\right|\right)^2\right)^{\frac{1}{2}}$$

$$+\sqrt{T}\left(\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\lambda_k^2\left|f_{k,n}(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}+\left(\int_0^T\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\gamma_n^2\left|f_{k,n}(\tau)\right|\right)^2d\tau\right)^{\frac{1}{2}}\right)$$

$$+T\left\|a(t)\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\left\|u_{k,n}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}+T\left\|b(t)\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^2\left\|u'_{k,n}(t)\right\|_{C[0,T]}\right)^2\right)^{\frac{1}{2}}\Bigg]\Bigg\}. \tag{34}$$

We impose the following conditions on the data of problem (1)–(4), (9), (10):

C_1. $\varphi(x,y),\varphi_x(x,y),\varphi_{xx}(x,y),\varphi_y(x,y),\varphi_{xy}(x,y),\varphi_{yy}(x,y)\in C(\bar{Q}_{xy})$,
$\varphi_{xxy}(x,y),\varphi_{xyy}(x,y),\varphi_{xxx}(x,y),\varphi_{yyy}(x,y)\in L_2(Q_{xy})$,
$\varphi_x(0,y)=\varphi(1,y)=\varphi_{xx}(1,y)=0,\ \ 0\leq y\leq 1$,
$\varphi(x,0)=\varphi_y(x,1)=\varphi_{yy}(x,0)=0,\ \ 0\leq x\leq 1$.

C_2. $\psi(x,y),\psi_x(x,y),\psi_y(x,y)\in C(\bar{Q}_{xy}),\ \ \psi_{xx}(x,y),\psi_{yy}(x,y)\in L_2(Q_{xy})$,
$\psi_x(0,y)=\psi(1,y)=0,\ \ 0\leq y\leq 1$,
$\psi(x,0)=\psi_y(x,1)=0,\ \ 0\leq x\leq 1$.

C_3. $f(x,y,t)\in C(D_T),\ \ f_x(x,y,t),f_y(x,y,t)\in L_2(D_T)$,
$f_x(0,y,t)=f(1,y,t)=0,\ \ 0\leq y\leq 1,\ \ 0\leq t\leq T$,
$f(x,0,t)=f_y(x,1,t)=0,\ \ 0\leq x\leq 1,\ \ 0\leq t\leq T$.

C_4. $h_i(t)\in C^2[0,T]\ \ (i=1,2),\ \ h(t)\equiv h_1(t)h_2'(t)-h_2(t)h_1'(t)\neq 0,\ \ 0\leq t\leq T$.

Then, from (31)–(34), respectively, we obtain

$$\left\|u(x,y,t)\right\|_{B_{2,T}^{3,2}}=\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^3\left\|u_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}+\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\left(\mu_{k,n}^2\left\|u'_{k,n}(t)\right\|_{C[0,T]}\right)^2\right\}^{\frac{1}{2}}$$

$$\leq A_1(T)+B_1(T)\left(\left\|a(t)\right\|_{C[0,T]}+\left\|b(t)\right\|_{C[0,T]}\right)\left\|u(x,y,t)\right\|_{B_{2,T}^{3,2}}, \tag{35}$$

$$\left\|\tilde{a}(t)\right\|_{C[0,T]}\leq A_2(T)+B_2(T)\left(\left\|a(t)\right\|_{C[0,T]}+\left\|b(t)\right\|_{C[0,T]}\right)\left\|u(x,y,t)\right\|_{B_{2,T}^{3,2}}, \tag{36}$$

$$\left\|\tilde{b}(t)\right\|_{C[0,T]}\leq A_3(T)+B_3(T)\left(\left\|a(t)\right\|_{C[0,T]}+\left\|b(t)\right\|_{C[0,T]}\right)\left\|u(x,y,t)\right\|_{B_{2,T}^{3,2}}, \tag{37}$$

where

$$A_1(T) = 2\sqrt{10}\,\|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})} + 2\sqrt{10}\,\left\|\varphi_{xyy}(x,y)\right\|_{L_2(Q_{xy})} + 2\sqrt{10}\,\left\|\varphi_{xxy}(x,y)\right\|_{L_2(Q_{xy})} + 2\sqrt{10}\,\left\|\varphi_{yyy}(x,y)\right\|_{L_2(Q_{xy})}$$

$$+2\sqrt{10}\,\left\|\psi_{yy}(x,y)\right\|_{L_2(Q_{xy})} + 2\sqrt{10}\,\left\|\varphi_{yy}(x,y)\right\|_{L_2(Q_{xy})} + 2\sqrt{10T}\left(\|f_{xx}(x,y,t)\|_{L_2(D_T)} + \left\|f_{yy}(x,y,t)\right\|_{L_2(D_T)}\right),$$

$$B_1(T) = 2\sqrt{10}T,$$

$$A_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|(h_1''(t) - f(0,1,t))h_2'(t) - \left(h_2''(t) - \int_0^1\int_0^1 f(x,y,t)dxdy\right)h_1'(t)\right\|_{C[0,T]}\right.$$

$$+\left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}}\left\|\left|h_1'(t)\right| + \left|h_2'(t)\right|\right\|_{C[0,T]}\left[\|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})} + \left\|\varphi_{xyy}(x,y)\right\|_{L_2(Q_{xy})} + \left\|\varphi_{xxy}(x,y)\right\|_{L_2(Q_{xy})}\right.$$

$$\left.\left.+\left\|\varphi_{yyy}(x,y)\right\|_{L_2(Q_{xy})} + \left\|\psi_{yy}(x,y)\right\|_{L_2(Q_{xy})} + \left\|\varphi_{yy}(x,y)\right\|_{L_2(Q_{xy})} + \sqrt{T}\left(\|f_{xx}(x,y,t)\|_{L_2(D_T)} + \left\|f_{yy}(x,y,t)\right\|_{L_2(D_T)}\right)\right]\right\},$$

$$B_2(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}}\left\|\left|h_1'(t)\right| + \left|h_2'(t)\right|\right\|_{C[0,T]}T,$$

$$A_3(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left\{\left\|\left(h_2''(t) - \int_0^1\int_0^1 f(x,y,t)dxdy\right)h_1(t) - (h_1''(t) - f(0,1,t))h_2(t)\right\|_{C[0,T]}\right.$$

$$+\left(\sum_{k=1}^{\infty}\sum_{n=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}}\left\|\left|h_1(t)\right| + \left|h_2(t)\right|\right\|_{C[0,T]}\left[\|\varphi_{xxx}(x,y)\|_{L_2(Q_{xy})} + \left\|\varphi_{xyy}(x,y)\right\|_{L_2(Q_{xy})} + \left\|\varphi_{xxy}(x,y)\right\|_{L_2(Q_{xy})}\right.$$

$$\left.\left.+\left\|\varphi_{yyy}(x,y)\right\|_{L_2(Q_{xy})} + \left\|\psi_{yy}(x,y)\right\|_{L_2(Q_{xy})} + \left\|\varphi_{yy}(x,y)\right\|_{L_2(Q_{xy})} + \sqrt{T}\left(\|f_{xx}(x,y,t)\|_{L_2(D_T)} + \left\|f_{yy}(x,y,t)\right\|_{L_2(D_T)}\right)\right]\right\},$$

$$B_3(T) = \left\|[h(t)]^{-1}\right\|_{C[0,T]}\left(\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}\mu_{k,n}^{-2}\right)^{\frac{1}{2}}\left\|\left|h_1(t)\right| + \left|h_2(t)\right|\right\|_{C[0,T]}T.$$

From inequalities (35)–(37), we conclude

$$\|\tilde{u}(x,y,t)\|_{B_{2,T}^{3,2}} + \|\tilde{a}(t)\|_{C[0,T]} + \left\|\tilde{b}(t)\right\|_{C[0,T]} \le A(T) + B(T)\left(\|a(t)\|_{C[0,T]} + \|b(t)\|_{C[0,T]}\right)\|u(x,y,t)\|_{B_{2,T}^{3,2}}, \tag{38}$$

where

$$A(T) = A_1(T) + A_2(T) + A_3(T), \quad B(T) = B_1(T) + B_2(T) + B_3(T).$$

Theorem 2. Let the conditions C_1-C_4 and the condition

$$(A(T)+2)^2 B(T) < 1, \tag{39}$$

be fulfilled. Then, problem (1)–(4), (9), (10) has a unique solution in the ball $K = K_R(\|z\|_{E_T^{3,2}} \le R = A(T)+2)$ of the space $E_T^{3,2}$.

Remark 1. Inequality (39) is satisfied for sufficiently small values of T.

Proof. Let us denote $z = [u(x,y,t), a(t), b(t)]^T$ and rewrite the system of equations (24),(29),(30) in the following operator equation

$$z = \Phi z, \tag{40}$$

where $\Phi = [\Phi_1, \Phi_2, \Phi_3]^T$, $\Phi_1(z), \Phi_2(z)$, and $\Phi_3(z)$ defined by the right sides of (24),(29), and (30), respectively.

Analogously to (33) we obtain that for any $z, z_1, z_2 \in K_R$ the following estimates hold:

$$\|\Phi z\|_{E_T^{3,2}} \le A(T) + B(T)\left(\|a(t)\|_{C[0,T]} + \|b(t)\|_{C[0,T]}\right)\|u(x,y,t)\|_{B_{2,T}^{3,2}} \le A(T) + B(T)(A(T)+2)^2, \tag{41}$$

$$\left\|\Phi z_1 - \Phi z_2\right\|_{E_T^{3,2}} \leq B(T)R(\left\|a_1(t) - a_2(t)\right\|_{C[0,T]} + \left\|b_1(t) - b_2(t)\right\|_{C[0,T]} + \left\|u_1(x,y,t) - u_2(x,y,t)\right\|_{B_{2,T}^{3,2}}). \tag{42}$$

Then taking into account (39) in (41) and (42), it follows that the operator Φ acts in the ball $K = K_R$ and is contractive. Therefore, in the ball $K = K_R$ the operator Φ has a unique fixed point $\{u, a, b\}$ that is a unique solution of (40), i.e. it is a unique solution of system (24), (29), (30), in the ball $K = K_R$.

Thus, we obtain that the function $u(x,y,t)$ as an element of the space $B_{2,T}^{3,2}$ is continuous and has continuous derivatives $u_x(x,y,t), u_{xx}(x,y,t), u_y(x,y,t), u_{xy}(x,y,t), u_{yy}(x,y,t), u_t(x,y,t), u_{tx}(x,y,t)$, and $u_{ty}(x,y,t)$ in D_T.

Hence Equation (13) enables us to obtain

$$u''_{k,n}(t) + \mu_{k,n}^2 u_{k,n}(t) = F_{k,n}(t;u,a,b), \quad k,n = 1,2,..., \quad 0 \leq t \leq T,$$

$$\left\{\sum_{n=1}^{\infty}\sum_{k=1}^{\infty}(\mu_{k,n}\left\|u''_{k,n}(t)\right\|_{C[0,T]})^2\right\}^{\frac{1}{2}} \leq 2\left[\|u(x,y,t)\|_{B_{2,T}^{3,2}} + \left\|\left\|f_x(x,y,t) + f_y(x,y,t)\right\|_{C[0,T]}\right\|_{L_2(Q_{xy})}\right.$$
$$\left. + \left\|\left\|a(t)(u_x(x,y,t) + u_y(x,y,t))\right\|_{C[0,T]}\right\|_{L_2(Q_{xy})} + \left\|\left\|b(t)(u_{tx}(x,y,t) + u_{ty}(x,y,t))\right\|_{C[0,T]}\right\|_{L_2(Q_{xy})}\right].$$

Thus it is clear that the derivative $u_{tt}(x,y,t)$ is continuous in the region D_T.

It is easy to verify that Equation (1) and conditions (2)–(4), (9), and (10) are satisfied in the ordinary sense. Consequently, $\{u(x,y,t), a(t), b(t)\}$ is a solution of problem (1)–(4), (9), (10) and by Lemma 1 this solution is unique in the ball $K = K_R$. The theorem is proved. □

Finally, from Theorem 1 and Theorem 2 immediately implies that the original problem (1)–(6) has a unique classical solution.

Theorem 3. Assume that all the conditions of Theorem 2 are satisfied and

$$\varphi(0,1) = h_1(0), \quad \psi(0,1) = h'_1(0),$$

$$\int_0^1\int_0^1 \varphi(x,y)dxdy = h_2(0), \quad \int_0^1\int_0^1 \psi(x,y)dxdy = h'_2(0).$$

Then problem (1)–(6) has a unique classical solution in the ball $K = K_R$ of the space $E_T^{3,2}$.

3 | NUMERICAL EXPERIMENTS

In this section, we discuss the numerical methods and results for the inverse problem (1)-(6). According to equation (19) in Section2, the exact solution $u(x,t)$ can be expressed as $u(x,y,t) = \sum_{n=1}^{\infty}\sum_{k=1}^{\infty} u_{k,n}(t)\cos\lambda_k x \sin\gamma_n y$, where $u_{k,n}(t)$ for $k,n = 1,2,\ldots$ are the unknown coefficients, and λ_k and γ_n are constants given by $\lambda_k = \frac{\pi}{2}(2k-1)$ and $\gamma_n = \frac{\pi}{2}(2n-1)$. Here we denote the numerical solution by $U(x,y,t)$, and let it be in the following form

$$U(x,y,t) = \sum_{i=1}^{M}\sum_{j=1}^{N} U_{i,j}(t)\cos\lambda_i x \sin\gamma_j y. \tag{43}$$

Here $U_{i,j}(t)$ is the unknown coefficient satisfying

$$U_{i,j}(t) = 4\int_0^1\int_0^1 U(x,y,t)\cos\lambda_i x \sin\gamma_j y dxdy.$$

Let $U(x,y,t)$ defined in (43) satisfy equations (1)-(2), we can show that

$$U''_{i,j}(t) + \mu_{i,j}^2 U_{i,j}(t) = f_{i,j}(t) + a(t)U_{i,j}(t) + b(t)U'_{i,j}(t), \quad 1 \leq i \leq M, 1 \leq j \leq N, \quad 0 \leq t \leq T, \tag{44}$$

$$U_{i,j}(0) = \varphi_{i,j}, \quad U'_{i,j}(0) = \psi_{i,j}, \quad 1 \leq i \leq M, 1 \leq j \leq N, \tag{45}$$

where

$$\mu_{i,j}^2 = \lambda_i^2 + \gamma_j^2,$$

$$f_{i,j}(t) = 4\int_0^1\int_0^1 f(x,y,t)\cos\lambda_i x\sin\gamma_j y dxdy,$$

$$\varphi_{i,j} = 4\int_0^1\int_0^1 \varphi(x,y)\cos\lambda_i x\sin\gamma_j y dxdy,$$

$$\psi_{i,j} = 4\int_0^1\int_0^1 \psi(x,y)\cos\lambda_i x\sin\gamma_j y dxdy.$$

Next, we take integral of both sides of equation (1) with respect to x and y, and apply (6) to the resulting formulation to get

$$h_2''(t) = \int_0^1\int_0^1 (u_{xx}+u_{yy})dxdy + a(t)h_2(t) + b(t)h_2'(t) + \int_0^1\int_0^1 f(x,y,t)dxdy.$$

We then substitute $u(x,y,t) = U(x,y,t)$ (where $U(x,y,t)$ is defined in (43)) into the equation above, and we can get the following equation

$$a(t)h_2(t) + b(t)h_2'(t) = h_2''(t) - \int_0^1\int_0^1 f(x,y,t)dxdy + \sum_{i=1}^{M}\sum_{j=1}^{N}(-1)^{i+1}\frac{\mu_{i,j}^2}{\lambda_i\gamma_j}U_{i,j}(t),\ \ 0\le t\le T. \tag{46}$$

Note that equation (46) can be regarded as an approximation of (26) using truncated series. Since both $a(t)$ and $b(t)$ are unknown functions, we need another equation to solve $a(t)$ and $b(t)$. We apply the overdetermination condition (5) to (1), and get

$$h_1''(t) = u_{xx}(0,1,t) + u_{yy}(0,1,t) + a(t)h_1(t) + b(t)h_1'(t) + f(0,1,t).$$

Substituting $u(x,y,t) = U(x,y,t)$ into the equation above leads to the following equation

$$a(t)h_1(t) + b(t)h_1'(t) = h_1''(t) - f(0,1,t) + \sum_{i=1}^{M}\sum_{j=1}^{N}(-1)^{j+1}\mu_{i,j}^2 U_{i,j}(t),\ \ 0\le t\le T. \tag{47}$$

We now need to further discretize the ODE systems (44)-(46) to solve for $a(t)$, $b(t)$ and $U_{i,j}(t)$ with $1\le i\le M$ and $1\le j\le N$. We partition the temporal interval $[0,T]$ into N_T subintervals: $[0,t_1],[t_1,t_2],\dots,[t_{N_T-1},t_{N_T}]$ where $t_n = n\Delta t$ and $\Delta t = T/N_T$ for $n = 1,2,\dots,N_T$. We also let $U_{i,j}^n$, a^n and b^n be the numerical approximation of $U_{i,j}(t_n)$, $a(t_n)$ and $b(t_n)$, respectively. Then we can apply the central difference finite difference method to approximate equation (44). That is,

$$\frac{U_{i,j}^{n+1} - 2U_{i,j}^n + U_{i,j}^{n-1}}{(\Delta t)^2} + \mu_{i,j}^2 U_{i,j}^n = f_{i,j}(t_n) + a^n U_{i,j}^n + b^n\frac{U_{i,j}^{n+1} - U_{i,j}^{n-1}}{2\Delta t},\ \ 1\le i\le M, 1\le j\le N. \tag{48}$$

Equation (48) shows that the computation of $U_{i,j}^{n+1}$ depends on $U_{i,j}^n$, a^n and b^n. Let $t = t_n$ in (46) and (47), we can get

$$a^n h_1(t_n) + b^n h_1'(t_n) = h_1''(t_n) - f(0,1,t_n) + \sum_{i=1}^{M}\sum_{j=1}^{N}(-1)^{j+1}\mu_{i,j}^2 U_{i,j}^n, \tag{49}$$

and

$$a^n h_2(t_n) + b^n h_2'(t_n) = h_2''(t_n) - \int_0^1\int_0^1 f(x,y,t_n)dxdy + \sum_{i=1}^{M}\sum_{j=1}^{N}(-1)^{i+1}\frac{\mu_{i,j}^2}{\lambda_i\gamma_j}U_{i,j}^n. \tag{50}$$

Equations (49) and (50) indicate that a^n and b^n can be updated using $U_{i,j}^n$.

The numerical method for the inverse problem (1)-(6) is as follows. We first compute a^0 and b^0 by solving the following equations

$$\begin{aligned} a^0h_1(0) + b^0h_1'(0) &= h_1''(0) - \varphi_{xx}(0,1) - \varphi_{yy}(0,1) - f(0,1,0),\\ a^0h_2(0) + b^0h_2'(0) &= h_2''(0) - \int_0^1\int_0^1 f(x,y,0)dxdy - \int_0^1 \varphi_x(1,y)dy + \int_0^1 \varphi_y(x,0)dx. \end{aligned}$$

With the condition that $h_1(t)h_2'(t) - h_2(t)h_1'(t) \ne 0$, the system above is always solvable. Then we compute $U_{i,j}^0$ for $1\le i\le M$ and $1\le j\le N$ using the Fourier transformation of $\varphi(x,y)$, and compute $U_{i,j}^1$ using the Fourier transformation of

$\varphi(x,y)+\Delta t\,\psi(x,y)+(\Delta t)^2/2\left(\varphi_{xx}+\varphi_{yy}+a^0\varphi+b^0\psi+f(x,y,0)\right)$. Next, for $n=1,2,\ldots,N_T$, we compute a^n, b^n and $U_{i,j}^{n+1}$ in alternating order. In particular, we first update a^n and b^n by solving the linear system (49)-(50), and then we update $U_{i,j}^{n+1}$ using the following equation

$$\left(1-\frac{b^n\Delta t}{2}\right)U_{i,j}^{n+1}=\left[2-(\Delta t)^2\mu_{i,j}^2+(\Delta t)^2a^n\right]U_{i,j}^n-\left(1+\frac{b^n\Delta t}{2}\right)U_{i,j}^{n-1}+(\Delta t)^2f_{i,j}(t_n),\quad 1\le i\le M,1\le j\le N. \tag{51}$$

Example 1

In this example, we consider the inverse problem (1)-(6) with the following functions

$$\begin{cases}\varphi(x,y)=\cos(\frac{\pi x}{2})\sin(\frac{\pi y}{2})+\cos(\frac{\pi x}{2})\sin(\frac{3\pi y}{2}),\\ \psi(x,y)=-\cos(\frac{\pi x}{2})\sin(\frac{3\pi y}{2}),\\ h_1(t)=1-e^{-t},\quad h_2(t)=\frac{4}{\pi^2}+\frac{4}{3\pi^2}e^{-t},\\ f(x,y,t)=-e^{-t}\cos(\frac{\pi x}{2})\sin(\frac{\pi y}{2})+(1+2\pi^2)\cos(\frac{\pi x}{2})\sin(\frac{3\pi y}{2})e^{-t}+(t-1)\cos(\frac{\pi x}{2})\sin(\frac{3\pi y}{2})e^{-2t}.\end{cases} \tag{52}$$

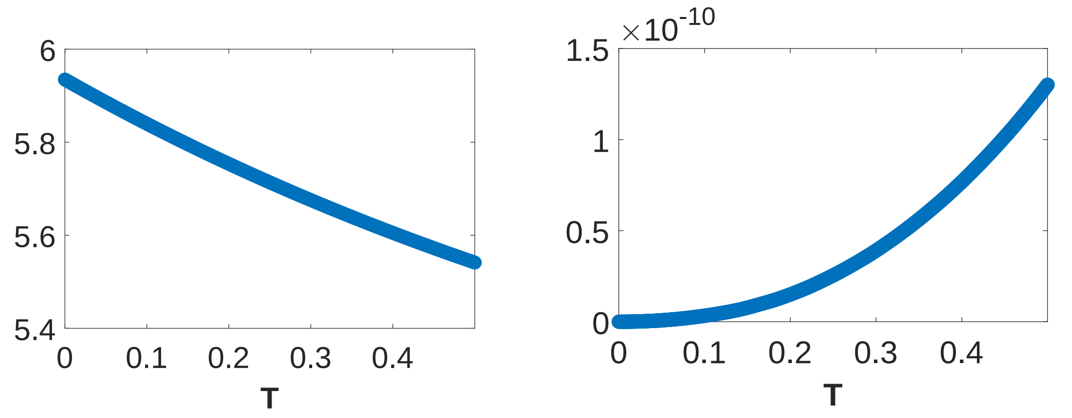
Due to the condition that $h_1(t)h_2'(t)-h_2(t)h_1'(t)\neq 0$, we can derive that the final time T for this example must satisfy that $T<\ln(7/3)$. In particular, we can calculate that $h_1(t)h_2'(t)-h_2(t)h_1'(t)=4/\pi^2(1-7/3e^{-t})$, and it is easy to show that $h_1h_2'-h_2h_1'<0$ for $t\in(0,\ln(7/3))$. We thus choose $T=0.5$ to make sure the solvability of this example. The exact solution of the problem is $a(t)=\pi^2/2+e^{-t}$, $b(t)=te^{-t}$ and $u(x,y,t)=\cos(\pi x/2)\sin(\pi y/2)+\cos(\pi x/2)\sin(3\pi y/2)e^{-t}$. We choose the time step size $\Delta t=10^{-3}$ in the simulation. The numerical solution and the absolute error of $a(t)$ for this example are given in Figure 1 . We can observe from Figure 1 (b) that the absolute error of $a(t)$ increases as t increases. The absolute maximum error of $a(t)$ for $t\in[0,0.5]$ is 1.3013×10^{-10}. The results for the other unknown coefficient $b(t)$ are shown in Figure 2 . We can see that the absolute error of b follows the same trend as a. Numerical simulation shows that the absolute maximum error of $b(t)$ is 3.8839×10^{-6}. The numerical solution of $u(x,y,T)$ with $T=0.5$ is given in Figure 3 (a). Since we have used the basis $\{\cos(\lambda_i x)\sin(\gamma_j y)\}_{i,j}$ in our numerical method, the numerical solution of u should satisfy the boundary conditions exactly. As we can observe from Figure 3 (a), the numerical solution of u is equal to zero at two boundaries $y=0$ and $x=1$, which is consistent with the Dirichlet boundary conditions in (3) and (4). To better visualize the performance of our numerical method, we further compute the pointwise absolute error of u at $T=0.5$ (see Figure 3 (b)). We find that the absolute maximum error of $u(\cdot,\cdot,T)$ occurs at $x=0$, $y=1$, and its value is 9.5497×10^{-8}. This example indicates that our numerical method can lead to accurate recovery of the unknown coefficients as well as the unknown function $u(x,y,t)$ for the inverse problem (1)-(6).

Example 2

In this numerical test, we consider the inverse problem (1)-(6) with the following functions

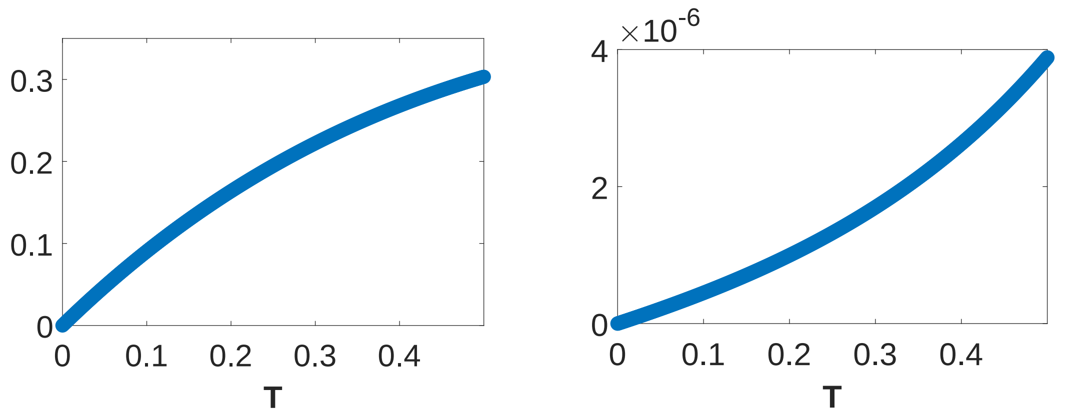
$$\begin{cases}\varphi(x,y)=2(1-x^2)(2y-y^2)+\cos(\frac{3\pi x}{2})\sin(\frac{\pi y}{2}),\\ \psi(x,y)=-(1-x^2)(2y-y^2),\\ h_1(t)=1+e^{-t}+\cos(t),\quad h_2(t)=\frac{4}{9}(1+e^{-t})-\frac{4}{3\pi^2}\cos(t),\\ f(x,y,t)=2(2y-y^2+1-x^2)(1+e^{-t})+(\frac{5\pi^2}{2}-1)\cos(\frac{3\pi x}{2})\sin(\frac{\pi y}{2})\cos(t)\\ \qquad+(1-x^2)(2y-y^2)\left(e^{-t}+e^{-t}\cos(t)-e^{-t}\sin(t)-\sin(t)\right).\end{cases} \tag{53}$$

Unlike the previous example where the exact solution can be expressed using finite Fourier modes, the exact solution of u in this example is an infinite series of the Fourier modes. In particular, the exact solution to the problem with the given data in (53) is $a(t)=\sin(t)$, $b(t)=\cos(t)$ and $u(x,y,t)=(1-x^2)(2y-y^2)(1+e^{-t})+\cos(\frac{3\pi x}{2})\sin(\frac{\pi y}{2})\cos(t)$. In order to determine the time interval such that the solution of the inverse problem exists, we first show that $h_1'(t)h_2(t)-h_1(t)h_2'(t)=(4/9+4/(3\pi^2))(\cos(t)\exp(-t)-\sin(t)\exp(-t)-\sin(t))$. To ensure $h_1'(t)h_2(t)-h_1(t)h_2'(t)\neq0$ for certain interval of t, we further calculate its derivative which is equal to $-(4/9+4/(3\pi^2))\cos(t)(1+2\exp(-t))$. Therefore, the function $h_1'(t)h_2(t)-h_1(t)h_2'(t)$ is decreasing when $t\in[0,\pi/2]$. Since $h_1'(0)h_2(0)-h_1(0)h_2'(0)=4/9+4/(3\pi^2)>0$ and $h_1'(t)h_2(t)-h_1(t)h_2'(t)=7.223\times10^{-8}$ when $t=0.3847861$, we



(a) Numerical solution of $a(t)$ **(b)** Absolute error of $a(t)$

FIGURE 1 Numerical solution and the absolute error of $a(t)$ for $t \in [0, T]$ in example 1. $T = 0.5$ and $\Delta t = 10^{-3}$ are used for the simulation.



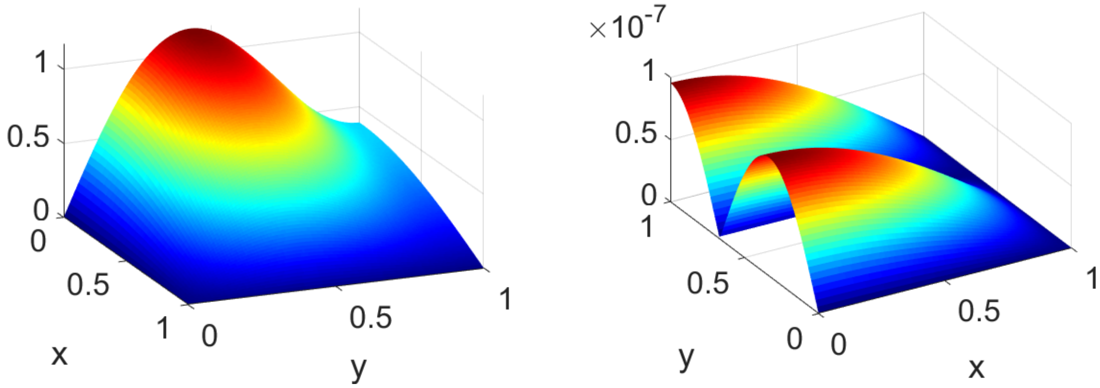
(a) Numerical solution of $b(t)$ **(b)** Absolute error of $b(t)$

FIGURE 2 Numerical solution and the absolute error of $b(t)$ for $t \in [0, T]$ in example 1. $T = 0.5$ and $\Delta t = 10^{-3}$ are used for the simulation.

can draw the conclusion that $h_1'(t)h_2(t) - h_1(t)h_2'(t) > 0$ for $t \in [0, 0.3847861]$. For the numerical test, we take the $T = 0.1$, $\Delta t = 10^{-4}$, $M = 30$ and $N = 30$. The numerical solution of $u(x, y, T)$ and its pointwise absolute error are shown in Figure 4 . The absolute maximum error of $u(\cdot, \cdot, T)$ occurs at $x = 0$, $y = 1$, and its value is 4.3590×10^{-4}. This example also shows that our numerical method can be used to solve the inverse problem and get accurate solutions.

4 | CONCLUSIONS

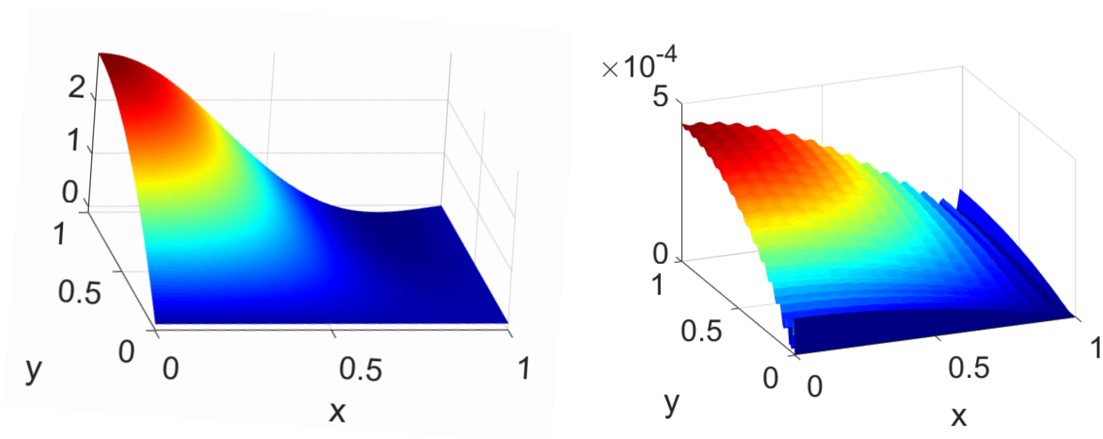
In the work, the classical solvability of a nonlinear inverse boundary value problem for a two-dimensional hyperbolic equation with nonlocal conditions was studied. First, the considered problem was reduced to an auxiliary inverse boundary value problem in a certain sense and its equivalence to the original problem is shown. Then using the Fourier method and contraction mappings principle, the existence and uniqueness theorem for auxiliary problem is proved. Further, on the basis of the equivalency of



(a) Numerical solution of $u(x, y, T)$

(b) Absolute error of $u(x, y, T)$

FIGURE 3 Numerical solution and the absolute error of $u(x, y, T)$ in example 1. $T = 0.5$ and $\Delta t = 10^{-3}$ are used for the simulation.



(a) Numerical solution of $u(x, y, T)$

(b) Absolute error of $u(x, y, T)$

FIGURE 4 Numerical solution and the absolute error of $u(x, y, T)$ in example 2. $T = 0.1$ and $\Delta t = 10^{-4}$ are used for the simulation.

these problems, the existence and uniqueness theorem for the classical solution of the original inverse coefficient problem is established for the smaller value of time. In addition, the numerical method for solving the inverse problem is proposed, and two numerical tests are performed to demonstrate the effectiveness of the numerical method.

5 | ACKNOWLEDGEMENTS

The authors sincerely thanks the ....

References

1. Aliev ZS, Mehraliev YT. An inverse boundary value problem for a second-order hyperbolic equation with nonclassical boundary conditions. *Dokl. Math.* 2014;90(1):513-517.

2. Alosaimi M, Lesnic D, Hào DN. Identification of the forcing term in hyperbolic equations. *Int. J. Comput. Math.* 2020;:DOI:10.1080/00207160.2020.1854744.

3. Azizbayov EI. On the unique recovery of time-dependent coefficient in a hyperbolic equation from nonlocal data. *Politehn. Univ. Bucharest Sci. Bull. Ser. A Appl. Math. Phys.* 2020;82(4):171-182.

4. Cannon JR, Dunninger DR. Determination of an unknown forcing function in a hyperbolic equation from overspecified data. *Ann. Mat. Pura Appl.* 1970;85(1):49-62.

5. Eskin G. Inverse hyperbolic problems with time-dependent coefficients. *Commun. Part. Differ. Eq.* 2007;32(11):1737-1758.

6. Hussein SO, Lesnic D. Determination of forcing functions in the wave equation. Part I: the space-dependent case. *J. Engrg. Math.* 2016;96(1):115-133.

7. Janno J, Šeletski A. Reconstruction of coefficients of higher order nonlinear wave equations by measuring solitary waves. *Wave Motion.* 2015;52:15-25.

8. Jiang DJ, Liu YK, Yamamoto M. Inverse source problem for the hyperbolic equation with a time-dependent principal part. *J. Differ. Equations.* 2017;262(1):653-681.

9. Kozhanov AI, Safiullova RR. Linear inverse problems for parabolic and hyperbolic equations. *J. Inverse Ill-pose. P.* 2010;18(1):1-24.

10. YT Mehraliyev, EI Azizbayov. A time-nonlocal inverse problem for a hyperbolic equation with an integral overdetermination condition. *Electron. J. Qual. Theory Differ. Equ.* 2021;(29):1-12.

11. Qi GB. An inverse problem for a one-dimensional semilinear hyperbolic equation with an unknown source. *J. Harbin Inst. Tech.* 1989;(4):14-20.

12. Salazar R. Determination of time-dependent coefficients for a hyperbolic inverse problem. *Inverse Probl.* 2013;29(9):095015.

13. Shcheglov AY. Inverse coefficient problem for a quasilinear hyperbolic equation with final overdetermination. *Comp. Math. Math. Phys.* 2006;46(4):616-635.

14. Slodička M Šeliga L. Determination of a time-dependent convolution kernel in a non-linear hyperbolic equation. *Inverse Probl. Sci. En.* 2016;24(6):1011-1029.

15. Tekin I, Mehraliyev YT, Ismailov MI. Existence and uniqueness of an inverse problem for nonlinear Klein-Gordon equation. *Math. Method. Appl. Sci.* 2019;42(10):3739-3753.

16. Uhlmann G, Zhai J. Inverse problems for nonlinear hyperbolic equations. *Discrete Contin. Dyn. Syst.* 2021;41(1):455-469.

17. Azizbayov EI, Mehraliyev YT. Nonlocal inverse boundary-value problem for a 2D parabolic equation with integral overdetermination condition. *Carpathian Math. Publ.* 2020;12:23-33.

18. Ismailov MI, Erkovan S, Huseynova AA. Fourier series analysis of a time-dependent perfusion coefficient determination in a 2D bioheat transfer process. *Trans. Natl. Acad. Sci. Azerb. Ser. Phys.-Tech. Math. Sci.* 2018;38(4):70-78.

19. Ivanchov MI, Vlasov V. Inverse problem for a two-dimensional strongly degenerate heat equation. *Electron. J. Differ. Eq.* 2018;2018(77):1-17.

20. Ivanchov MI, Kinash NE. Inverse problem for the heat-conduction equation in a rectangular domain. *Ukr. Math. J.* 2018;69(12):1865-1876.

21. Kinash NY. An inverse problem for a 2D parabolic equation with nonlocal overdetermination condition. *Carpathian Math. Publ.* 2016;8(1):107-117.

22. Zaynullov AR. An inverse problem for two-dimensional equations of finding the thermal conductivity of the initial distribution. *J. Samara State Tech. Univ, Ser. Phys. Math. Sci.*. 2015;19(4):667-679.

23. Dezin AA. The simplest solvable extensions of ultrahyperbolic and pseudoparabolic operators. *Dokl. Akad. Nauk SSSR.* 1963;148(5):1013-1016 (in Russian).

24. Bitsadze AV, Samarskii AA. Some elementary generalizations of linear elliptic boundary value problems. *Dokl. Akad. Nauk SSSR.* 1969;185(4):739-740 (in Russian).

25. Gordeziani DG, Avalishvili GA. On the constructing of solutions of the nonlocal initial boundary value problems for one-dimensional medium oscillation equations. *Mat. Model.* 2000;12(1):94-103 (in Russian).

26. Kerefov AA. Nonlocal boundary value problems for parabolic equations. *Differ. Equations.* 1979;15(1):74-78 (in Russian).

27. Kirichenko SV, Pul'kina LS. A problem with nonlocal initial data for one-dimensional hyperbolic equation. *Russian Math.* 2014;58(9):13-21.

28. Vabishchevich PN. Nonlocal parabolic problems and the inverse heat-conduction problem. *Differ. Equations.* 1981;17(7):761-765 (in Russian).

29. Pul'kina LS, Savenkova AE. A problem with a nonlocal with respect to time condition for multidimensional hyperbolic equations. *Russian Math.* 2016;60(10):33-43.

30. Ragusa MA, Scapellato A. Mixed Morrey spaces and their applications to partial differential equations. *Nonlinear Anal.* 2017;151:51-65.

31. Ragusa MA. Necessary and sufficient condition for a VMO function. *Appl. Math. Comput.* 2012;218(24):11952-11958.

32. Khudaverdiyev KI, Veliyev AA. *nvestigation of a one-dimensional mixed problem for a class of pseudohyperbolic equations of third order with non-linear operator right hand side*. Baku: Chashyoghly; 2010 (in Russian).

AUTHOR BIOGRAPHY

Yashar T. Mehraliyev. 1966-1976, Secondary education, Western Azerbaijan;
1977-1982, Undergraduate, Mechanics & Mathematics faculty, Baku State University, Baku, Azerbaijan;
1988-1993, Postgraduate, Mechanics & Mathematics faculty, Baku State University, Baku, Azerbaijan;
1994, Ph.D in Physical and Mathematical Sciences, Baku State University, Baku, Azerbaijan;
2015, Doctor of Sciences in Mathematics, Baku State University, Baku, Azerbaijan;
Degree: Doctor of Sciences in Mathematics;
Position: Full professor and Head of the Department of Differential and Integral Equations, Baku State University, Baku, Azerbaijan;
Present research interest: Direct and inverse problems with the local, non-local, and non-classical boundary conditions for the PDEs;
Publications: Over 200 publications.

He Yang. 2001-2004, Secondary education, China;
2004-2008, Undergraduate, Information & Computational Science, University of Science & Technology of China, Hefei, China;
2008-2010, Postgraduate, Mathematics, The State University of New York, Buffalo, USA;
2014, Ph.D in Mathematics, Rensselaer Polytechnic Institute, Troy, USA;
Degree: Ph.D in Mathematics;
Position: Assistant Professor of Mathematics, Augusta University, Augusta, USA;
Present research interest: Numerical methods of partial differential equations, inverse scattering problems, medical imaging.
Publications:. Over 20 publications.

Elvin I. Azizbayov. 1981-1992, Secondary education, Republic of Georgia;
1992-1997, Undergraduate, Mechanics & Mathematics faculty, Baku State University, Baku, Azerbaijan;
1997-2000, Postgraduate, Mechanics & Mathematics faculty, Baku State University, Baku, Azerbaijan;
2005, Ph.D in Physical and Math. Sciences, Institute of Mathematics & Mechanics, Azerbaijan National Academy of Sciences, Baku, Azerbaijan;
Degree: Ph.D in Physical and Mathematical Sciences;
Position: Head of the Department of Control for Intelligent Systems, The Academy of Public Administration under the President of the Republic of Azerbaijan, Baku, Azerbaijan;
Present research interest: Direct and inverse boundary-value problems for PDEs, Spectral theory of differential equations, Non-linear functional analysis, Numerical methods for partial differential equations.
Publications: Over 90 publications.

How to cite this article: Azizbayov EI , Yang H and Mehraliyev YT. Recovery of unknown coefficients in a two-dimensional hyperbolic equation with additional conditions. *Math Meth Appl Sci*. 2021; **: