A generalized SEIR epidemic model for transmission dynamics of SARS-CoV-2 and its dynamically consistent discrete model

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Abstract

Although Coronavirus disease 2019 (COVID-19) pandemic caused by severe acute respiratory syndrome coronavirus 2 (SARS-CoV-2) has been controlled and prevented, mathematical modeling and analysis of transmission dynamics of COVID-19 still plays an essential role not only in the post COVID-19 era but also in the study of infectious diseases. This is an important foundation to propose effective strategies and measures for controlling diseases and projecting public health. This work is devoted to proposing and analyzing a new mathematical study for transmission dynamics of COVID-19. We first introduce a generalized SEIR epidemic model that use general nonlinear incidence rates to describe the "psychological" effect. After that, a rigorous mathematical analysis for the proposed COVID-19 model is performed. We establish the positivity and boundedness, calculate the basic reproduction number, determine possible (disease-free and endemic-disease) equilibrium points and investigate their asymptotic stability properties of the SEIR model. The obtained results improve and extend an SEIR model constructed in a recent work. For the purpose of numerical simulation, the Mickens' methodology is applied to construct a dynamically consistent nonstandard finite difference (NSFD) model for the proposed SEIR epidemic model. The constructed NSFD scheme has the ability to provide reliable approximations that not only preserve the dynamical properties of the SEIR model for all the values of the step size but also are easy to be implemented. Finally, a set of illustrative numerical experiments is conducted to support the theoretical findings and to confirm advantages of the NSFD scheme over some well-known standard ones.

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1. Introduction

Mathematical modeling and analysis of infectious diseases has become a fundamental and essential approach for discovering characteristics and mechanisms of epidemics as well as for predicting possible scenarios in reality [3, 11, 12, 24, 41]. Studying mathematical models of infectious diseases can provide us with appropriate strategies for controlling and preventing diseases. This is very useful for public health and preventive healthcare. For this reason, in efforts to prevent the COVID-19 epidemic, many mathematicians and epidemiologists have proposed

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and analyzed a great number of mathematical models describing transmission dynamics of the COVID-19 (see, for example, [6, 22, 33, 34, 39, 40, 49, 50, 51, 54, 59, 60] and references therein). As an important consequence, mitigation and prevention measures of COVID-19 outbreaks were suggested. Recently, we have performed a mathematical study for transmission dynamics of SARS-CoV-2 with waning immunity [30].

We now revisit a recognized mathematical model of the COVID-19 proposed by Rohith and Devika in [50]. The model is represented by a system of nonlinear differential equations:

$$\begin{aligned}
\dot{S} &= \mu - \frac{\beta_0 SI}{1+\alpha I^2} - \mu S,\\
\dot{E} &= \frac{\beta_0 SI}{1+\alpha I^2} - (\sigma+\mu)E,\\
\dot{I} &= \sigma E - (\gamma+\mu)I,\\
\dot{R} &= \gamma I - \mu R.
\end{aligned} \tag{1}$$

In this model, the total population is divided four classes depending the status of individuals with respect to the COVID-19, that are susceptible (S), exposed (E), infected (I), and removed (R) classes; the birth/death rate is represented by μ; γ is the the recovery rate and σ is the measure of rate at which the exposed individuals become infected. We refer the readers to [50] for more details of the model (1). In [50], bifurcation analysis and control problem for the model (1) have been studied rigorously.

If in the nonlinear incidence rate $\dfrac{\beta_0 SI}{1+\alpha I^2}$ we set $\psi(I) = 1+\alpha I^2$, then $\psi(I)$ satisfies the following properties

(H1): $\psi(0) = 1$;

(H2): $\psi(I) > 0$ for $I > 0$;

(H3): $\psi'(I) \geq 0$ for $I \geq 0$.

The family of nonlinear incidence rates satisfying the conditions **(H1)-(H3)** was proposed in [35]. These functions are not only biologically motivated, can be used to interpret the "psychological" effect but also include many famous incidence functions [23, 35, 37].

Motivated and inspired by the recognized works considering nonlinear incidence rates [23, 35, 37] as well as the importance of mathematical models of the COVID-19, in this work we consider a generalized version of the model (1) by replacing the function $1+\alpha I^2$ by general ones satisfying **(H1)-(H3)**. More precisely, we propose the following model

$$\begin{aligned}
\dot{S} &= \mu - \frac{\beta_0 SI}{\psi(I)} - \mu S,\\
\dot{E} &= \frac{\beta_0 SI}{\psi(I)} - (\sigma+\mu)E,\\
\dot{I} &= \sigma E - (\gamma+\mu)I,\\
\dot{R} &= \gamma I - \mu R,
\end{aligned} \tag{2}$$

where $\psi(I)$ is any function satisfying **(H1)-(H3)**. Note that if setting $f(I) = \dfrac{I}{\psi(I)}$, then

(i) $f(0) = 0$, $f(I) > 0$ for $I > 0$;

(ii) $f(I)/I$ is continuous and monotonely non-increasing for $I > 0$, and $\lim_{I\to 0^+} f(I)/I$ exists, denoted by

$\beta (0 < \beta < \infty)$;

(iii) $\int_{0+}^{1} (1/f(u))du \leq \int_{0+}^{1} (1/u)du = \infty$.

So, the function $f(I)$ also satisfies properties given in [37]. This means that the model (2) is not only a generalization of the model (1) but also can provide with us more epidemic scenarios. This is very useful in both theory and practice. This is why we consider the model (1) the context of the general incidence rates.

In the first part of this work, positivity, boundedness, the basic reproduction number, possible equilibria and asymptotic stability properties of the model (2) are analyzed rigorously. It is proved, by the Lyapunov stability theory, that a unique disease-free equilibrium (DFE) point is globally asymptotically stable if the basic reproduction number $\mathcal{R}_0$ satisfies $\mathcal{R}_0 < 1$; and the disease-endemic equilibrium (DEE) point exists and is locally asymptotically stable if $\mathcal{R}_0 > 1$. Consequently, qualitative dynamical properties of the model (2) are determined fully and mitigation and prevention measures of COVID-19 outbreaks are suggested. Also, the obtained results improve and extend the ones presented in the benchmark work [50]

In the second part, we construct a reliable numerical scheme for the purpose of numerical simulation as well as construction of scientific computation programs for predicting COVID-19 epidemic. To achieve this objective, we utilize the Mickens' methodology [42, 43, 44, 45, 46] to formulate a dynamically consistent nonstandard finite difference (NSFD) scheme for the model (2). It is well-known that the main advantage of the NSFD schemes over standard one is that they can preserve essential mathematical features of differential equations regardless of the values of the step size [42, 43, 44, 45, 46]. Therefore, they are efficient and appropriate to simulate behaviour of dynamical differential equation models over long time periods. Nowadays, NSFD schemes have become an efficient approach for numerically solving real-world problems (see, for example, [1, 2, 4, 5, 10, 13, 14, 15, 31, 32, 47, 48, 57, 58]). Recently, we have developed the Mickens' methodology to construct NSFD schemes for mathematical models of phenomena and processes coming from sciences and technology like biology, ecology, or other natural sciences [16, 17, 18, 19, 20, 21, 25, 26, 27, 28, 29].

In the third part, a set of illustrative numerical experiments is conducted to support the theoretical results and to demonstrate advantages of the constructed NSFD scheme over some standard ones. The numerical examples provide strong evidence that confirms the validity of the main results of this work. It is proved that the standard Euler and second-order Runge-Kutta (RK2) schemes can generate numerical approximations which are negative and unstable for some given step sizes. This means that the dynamics of the model (2) cannot be preserved. However, the NSFD scheme correctly preserve the dynamics of (2) for the same step sizes.

The plan of this work is as follows:

Dynamics of the model (2) is studied in Section 2. The NSFD scheme is formulated and analyzed in Section 3. Some numerical experiments are conducted in Section 4. Some conclusions and open problems are discussed in the last section.

2. Dynamics of the generalized SEIR model

We first establish the positivity and boundedness of the model (2).

Lemma 1. *The set* $\Omega = \{(S,E,I,R) \in \mathbb{R}^4 | S,E,I,R \geq 0,\ S+E+I+R = 1\}$ *is a positively invariant set of the model* (2)*, that is,* $(S(t),E(t),I(t),R(t)) \in \Omega$ *for* $t > 0$ *if* $(S(0),E(0),I(0),R(0)) \in \Omega$.

Proof. First, it follows from the system (2) that

$$\begin{aligned} \dot{S}\big|_{S=0} &= \mu, \\ \dot{E}\big|_{E=0} &= \frac{\beta_0 SI}{\psi(I)}, \\ \dot{I}\big|_{I=0} &= \sigma E, \\ \dot{R}\big|_{R=0} &= \gamma I. \end{aligned}$$

Therefore, from Theorem [55, Theorem B.7] we conclude that $S(t),E(t),I(t),R(t) \geq 0$ for $t > 0$ whenever $S(0),E(0),I(0),R(0) \geq 0$.

Next, setting $N(t) = S(t)+E(t)+I(t)+R(t)$ for $t \geq 0$. Then, from (2) we obtain

$$\dot{N} = \mu - \mu N, \quad N(0) = 1,$$

which follows that $N(t) = 1$ for $t \geq 0$. This is the desired conclusion. The proof is complete. □ □

As a direct consequence of Lemma 2, it is sufficient to consider the following sub-model of (2)

$$\begin{aligned} \dot{S} &= \mu - \frac{\beta_0 SI}{\psi(I)} - \mu S, \\ \dot{E} &= \frac{\beta_0 SI}{\psi(I)} - (\sigma+\mu)E, \\ \dot{I} &= \sigma E - (\gamma+\mu)I \end{aligned} \tag{3}$$

on its feasible set given by

$$\Omega^* = \{(S,I,E) \in \mathbb{R}^3 | S,E,I \geq 0,\ S+E+I \leq 1\}. \tag{4}$$

We now determine possible equilibrium points and compute the basic reproduction number of the model (3).

Theorem 1 (Equilibria and basic reproduction number)**.** *(i) The model* (3) *always possesses a disease-free equilibrium (DFE) point* $P_f = (S_f, E_f, I_f) = (1,0,0)$ *for all the values of the parameters.*
(ii) The basic reproduction number of the model (3) *can be computed as*

$$\mathcal{R}_0 = \frac{\beta_0 \sigma}{(\sigma+\mu)(\gamma+\mu)}.$$

(iii) The model (3) *has a unique disease-endemic equilibrium (DEE) point* $P_e = (S_e, E_e, I_e)$ *if and only if* $\mathcal{R}_0 > 1$. *Moreover, if existing* P_e *it is given by*

$$\begin{aligned} E_e &= \frac{\gamma+\mu}{\sigma} I_e, \\ S_e &= \frac{(\sigma+\mu)(\gamma+\mu)\psi(I_e)}{\sigma\beta_0}, \end{aligned}$$

where I_e *is the unique positive solution of the equation*

$$F(I) = \mu - \mu\frac{(\sigma+\mu)(\gamma+\mu)\psi(I)}{\sigma} - \frac{(\sigma+\mu)(\gamma+\mu)}{\sigma} I = 0.$$

Proof. **Proof of Part (i)**. To determine equilibrium points, we consider the following system of algebraic equations

$$
\begin{aligned}
&\mu - \frac{\beta_0 SI}{\psi(I)} - \mu S = 0,\\
&\frac{\beta_0 SI}{\psi(I)} - (\sigma+\mu)E = 0,\\
&\sigma E - (\gamma+\mu)I = 0.
\end{aligned} \tag{5}
$$

It follows from the third equation of (5) that

$$E = \frac{\gamma+\mu}{\sigma} I.$$

Combining this with the second equation we obtain

$$I\left[\frac{\beta_0 S}{\psi(I)} - \frac{(\sigma+\mu)(\gamma+\mu)}{\sigma}\right] = 0. \tag{6}$$

Hence, the system (5) always possesses a trivial solution $(S, E, I) = (1, 0, 0)$, which corresponds to a disease-free equilibrium point of the model (3).

Proof of Part (ii). We apply the method in [56] to compute the basic reproduction number. If reordering the variables in (3) as $x = (E, I, S)$, then the DFE point is transformed to $x_f = (E_f, I_f, S_f)$ and (3) can be written in the matrix form

$$\dot{x} = \mathcal{F}(x) - \mathcal{V}(x),$$

where

$$\mathcal{F}(x) = \begin{pmatrix} \dfrac{\beta_0 SI}{\psi(I)} \\ \\ 0 \\ \\ \mu \end{pmatrix}, \quad \mathcal{V}(x) = \begin{pmatrix} (\sigma+\mu)E, \\ \\ -\sigma E + (\gamma+\mu)I, \\ \\ \dfrac{\beta_0 SI}{\psi(I)} + \mu S \end{pmatrix}.$$

Consequently,

$$D\mathcal{F}(x_f) = \begin{pmatrix} 0 & \beta_0 & 0 \\ \\ 0 & 0 & 0 \\ \\ 0 & 0 & 0 \end{pmatrix}, \quad D\mathcal{V}(x_f) = \begin{pmatrix} \sigma+\mu & 0 & 0 \\ \\ -\sigma & \gamma+\mu & 0 \\ \\ 0 & 0 & \mu \end{pmatrix}.$$

Hence,

$$\mathcal{R}_0 = \rho(FV^{-1}) = \frac{\beta_0 \sigma}{(\sigma+\mu)(\gamma+\mu)}.$$

Proof of Part (iii). Note that a DEE point is a trivial solution of (5). From the 1st and 2nd equations of (5), we obtain

$$\frac{\beta_0 SI}{\psi(I)} = (\sigma+\mu)E = \frac{(\sigma+\mu)(\gamma+\mu)}{\sigma} I. \tag{7}$$

On the other hand, it follows from (6) that

$$S = \frac{(\sigma+\mu)(\gamma+\mu)\psi(I)}{\sigma\beta_0}. \tag{8}$$

Combining (7) and (8) with the first equation (5) leads to an equation for I

$$F(I) = \mu - \mu\frac{(\sigma+\mu)(\gamma+\mu)\psi(I)}{\sigma} - \frac{(\sigma+\mu)(\gamma+\mu)}{\sigma}I = 0.$$

It is easy to verify that

$$\begin{aligned} &F(0) = \mu\Big(1 - \frac{1}{\mathcal{R}_0}\Big),\\ &F(1) < 0,\\ &F'(I) < 0. \end{aligned}$$

Therefore, if $\mathcal{R}_0 > 1$ then (2) has a unique positive solution $I_e \in (0,1)$, which corresponds to a unique DEE point. The proof is complete. □ □

We now analyze local and global asymptotic stability of the model (3).

Theorem 2 (Local asymptotic stability)**.** *(i) The DFE point P_f is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

(ii) The DEE point P_e is locally asymptotically stable if it exists ($\mathcal{R}_0 > 1$).

Proof. **Proof of Part (i).** The Jacobian matrix of the system (3) evaluating at P_f is given by

$$J(P_f) = \begin{pmatrix} -\mu & 0 & -\beta_0 \\ 0 & -(\sigma+\mu) & \beta_0 \\ 0 & \sigma & -(\gamma+\mu) \end{pmatrix}.$$

Hence, one of the three eigenvalues of $J(P_f)$ is $\lambda_1 = -\mu < 0$ and the two remaining eigenvalues are the ones of the sub-matrix

$$J^0(P_f) = \begin{pmatrix} -(\sigma+\mu) & \beta_0 \\ \sigma & -(\gamma+\mu) \end{pmatrix}.$$

It is clear that

$$Trace(J^0) < 0, \quad \det(J^0) = (\sigma+\mu)(\gamma+\mu) - \beta_0\sigma = (\sigma+\mu)(\gamma+\mu)(1-\mathcal{R}_0) > 0.$$

By Routh-Hurwitz criteria (see [7, Theorem 4.4]), we conclude that all of the eigenvalues of $J(P_f)$ are negative or have negative real part. Hence, the local asymptotic stability of P_f is confirmed. Otherwise, if $\mathcal{R}_0 > 1$ then $\det(J^0) < 0$, and hence, P_f is unstable.

Proof of Part (ii). Note that P_f exists if and only if $\mathcal{R}_0 > 1$. For the sake of convenience, we denote $f(I) = \beta_0 I/\psi(I)$. Then, the Jacobian matrix of the system (3) evaluating at P_e is

$$J(P_e) = \begin{pmatrix} -(\mu + f(I_e)) & 0 & -S_e f'(I_e) \\ \\ f(I_e) & -(\sigma+\mu) & S_e f'(I_e) \\ \\ 0 & \sigma & -(\gamma+\mu) \end{pmatrix}.$$

Consequently, the characteristic polynomial of $J(P_e)$ is given by

$$P_J(\lambda) = \lambda^3 + a_2\lambda^2 + a_1\lambda + a_0,$$

where

$$\begin{aligned}
a_1 &= f(I_e) + \gamma + 3\mu + \sigma, \\
a_2 &= (\gamma+\mu)(f(I_e) + 2\mu + \sigma) + (f(I_e)+\mu)(\mu+\sigma) - S_e f'(I_e)\sigma, \\
a_3 &= (f(I_e)+\mu)(\gamma+\mu)(\mu+\sigma) - S f'(I_e)\mu\sigma.
\end{aligned}$$

It follows from $f(I_e) > 0$ and $f'(I_e) < 0$ that

$$\begin{aligned}
&a_1 > 0, \quad a_2 > 0, \\
&a_1 a_2 - a_3 = f^2(I_e)\gamma + 2f^2(I_e)\mu + f^2(I_e)\sigma + f(I_e)\gamma^2 + 6f(I_e)\gamma\mu + 2f(I_e)\gamma\sigma \\
&\ + 8f(I_e)\mu^2 + 6f(I_e)\mu\sigma + f(I_e)\sigma^2 - S_e f'(I_e) f(I_e)\sigma + 2\gamma^2\mu + \gamma^2\sigma + 8\gamma\mu^2 \\
&\ + 6\gamma\mu\sigma + \gamma\sigma^2 - S_e f'(I_e)\gamma\sigma + 8\mu^3 + 8\mu^2\sigma + 2\mu\sigma^2 - 2S_e f'(I_e)\mu\sigma - S_e f'(I_e)\sigma^2 > 0.
\end{aligned}$$

Therefore, we infer from Routh-Hurwitz criteria ([7, Theorem 4.4]) that P_f is locally asymptotically stable. The proof is completed. $\square$ $\square$

Theorem 3 (Global asymptotic stability of the DFE point)**.** *The DFE point P_f is not only locally asymptotically stable but also globally asymptotically stable with respect to Ω^* when $\mathcal{R}_0 < 1$.*

Proof. Consider a candidate Lyapunov function $V : \Omega^* \to \mathbb{R}_+$ given by

$$V(S, E, I) = \left(S - S_f - S_f \ln\frac{S}{S_f}\right) + E + \frac{\mu+\sigma}{\sigma} I.$$

The derivative of V along with solutions of the system (3) is

$$\begin{aligned}
\frac{dV}{dt} &= \frac{dV}{dS}\frac{dS}{dt} + \frac{dV}{dE}\frac{dE}{dt} + \frac{dV}{dI}\frac{dI}{dt} \\
&= \left(\mu - \frac{\beta_0 SI}{\psi(I)} - \mu S\right)\frac{S - S_f}{S} + \left[\frac{\beta_0 SI}{\psi(I)} - (\sigma+\mu)E\right] + \frac{\sigma+\mu}{\sigma}\big[\sigma E - (\gamma+\mu)I\big] \\
&= -\frac{\mu}{S}(S - S_f)^2 + I\left[\frac{\beta_0}{\psi(I)} - \frac{(\sigma+\mu)(\gamma+\mu)}{\sigma}\right] \\
&\le -\frac{\mu}{S}(S - S_f)^2 + I\left[\beta_0 - \frac{(\sigma+\mu)(\gamma+\mu)}{\sigma}\right] \\
&\le -\frac{\mu}{S}(S - S_f)^2 + \frac{(\sigma+\mu)(\gamma+\mu)}{\sigma}(\mathcal{R}_0 - 1)I.
\end{aligned}$$

Since $\mathcal{R}_0 < 1$, $dV/dt \leq 0$ for all $S, E, I \geq 0$ and $dV/dt = 0$ if and only if $S = S_f$ and $I = I_f$. Consequently, it follows from the LaSalle's invariant principle [36] that P_e is globally asymptotically stable. The proof is completed. $\square$ $\square$

In [38], Li and Muldowney proposed a general criterion for the orbital stability of periodic orbits associated with higher-dimensional nonlinear autonomous systems as well as the theory of competitive systems of differential equations to study the global stability of an SEIR model that is similar to the model (3). So, by applying the approach in [38], we can obtain the global asymptotic stability of P_e.

Proposition 1. *The DEE point P_e is not only locally asymptotically stable but also globally asymptotically stable with respect to the interior of Ω^* whenever $\mathcal{R}_0 > 1$.*

Remark 1. *In [50], only the local asymptotic stability of the DFE point of the model* (1) *was investigated. Therefore, the stability analysis of the model* (2) *presented in this section provides an important improvement for the results constructed in [50]. On the other hand, the global stability of the DFE points of the models* (1) *and* (2) *is very important because it means that the COVID-19 epidemic can be extinguished (when $\mathcal{R}_0 < 1$), and hence, some mitigation and prevention measures of COVID-19 outbreaks can be suggested (see [30]).*

3. Construction of dynamically consistent NSFD model

Our main objective in this section is to formulate an NSFD model which is dynamically consistent with the model (2). For this purpose, we first consider the model (2) on a time interval $[0, T]$ and partition this interval by a uniform mesh

$$0 = t_0 < t_1 < \ldots < t_{N-1} < T_N = T,$$

where $t_n - t_{n-1} = \Delta t$ for $n \geq 1$. Let us denote by $\big(S_n, E_n, I_n, R_n\big)$ the intended approximation for $\big(S(t_n), E(t_n), I(t_n), R(t_n)\big)$, respectively. By using the Mickens' methodology [42, 43, 44, 45, 46], we replace the differential equation model (2) by a difference equation one as follows:

$$\begin{aligned} \dot{S}(t_n) &\approx \frac{S_{n+1} - S_n}{\phi(\Delta t)}, \\ \dot{E}(t_n) &\approx \frac{E_{n+1} - E_n}{\phi(\Delta t)}, \\ \dot{I}(t_n) &\approx \frac{I_{n+1} - I_n}{\phi(\Delta t)}, \\ \dot{R}(t_n) &\approx \frac{R_{n+1} - R_n}{\phi(\Delta t)}, \end{aligned} \tag{9}$$

and

$$\begin{aligned} &\mu - \frac{\beta_0 S(t_n) I(t_n)}{\psi(I_n)} - \mu S(t_n) \approx \mu - \frac{\beta_0 S_{n+1} I_n}{\psi(I_n)} - \mu S_{n+1}, \\ &\frac{\beta_0 S(t_n) I(t_n)}{\psi(I_n)} - (\sigma + \mu) E(t_n) \approx \frac{\beta_0 S_{n+1} I_n}{\psi(I_n)} - (\sigma + \mu) E_{n+1}, \\ &\sigma E(t_n) - (\gamma + \mu) I(t_n) \approx \sigma E_{n+1} - (\gamma + \mu) I_{n+1}, \\ &\gamma I - \mu R \approx \gamma I_{n+1} - \mu R_{n+1}, \end{aligned} \tag{10}$$

where $\phi(\Delta t) = \Delta t + \mathcal{O}(\Delta t^2)$ as $\Delta t \to 0$, which is called a nonstandard denominator function. The approximations (9) and (10) lead to the following NSFD scheme

$$\begin{aligned}
\frac{S_{n+1} - S_n}{\phi(\Delta t)} &= \mu - \frac{\beta_0 S_{n+1} I_n}{\psi(I_n)} - \mu S_{n+1},\\
\frac{E_{n+1} - E_n}{\phi(\Delta t)} &= \frac{\beta_0 S_{n+1} I_n}{\psi(I_n)} - (\sigma + \mu) E_{n+1},\\
\frac{I_{n+1} - I_n}{\phi(\Delta t)} &= \sigma E_{n+1} - (\gamma + \mu) I_{n+1},\\
\frac{R_{n+1} - R_n}{\phi(\Delta t)} &= \gamma I_{n+1} - \mu R_{n+1}.
\end{aligned} \tag{11}$$

Our task is to analyze dynamics of the NSFD model (11).

Lemma 2. *The set* $\Omega = \big\{(S, E, I, R) \in \mathbb{R}^4 | S, E, I, R \geq 0,\ S + E + I + R = 1\big\}$ *is a positively invariant set of the NSFD model* (11)*, that is,* $\big(S_n, E_n, I_n, R_n\big) \in \Omega$ *for* $n \geq 1$ *if* $\big(S(0), E(0), I(0), R(0)\big) \in \Omega$.

Proof. The lemma is proved by mathematical induction. First, it is easy to transform the NSFD scheme (11) to explicit form as follows:

$$\begin{aligned}
S_{n+1} &= \frac{S_n + \phi\mu}{1 + \phi\dfrac{\beta_0 I_n}{\psi(I_n)} + \phi\mu},\\
E_{n+1} &= \frac{E_n + \phi\dfrac{\beta_0 I_n S_{n+1}}{\psi(I_n)}}{1 + \phi(\sigma + \mu)},\\
I_{n+1} &= \frac{I_n + \phi\sigma E_{n+1}}{1 + \phi(\gamma + \mu)},\\
R_{n+1} &= \frac{R_n + \phi\gamma I_{n+1}}{1 + \phi\mu},
\end{aligned}$$

which implies that $S_{n+1}, E_{n+1}, I_{n+1}, R_{n+1} \geq 0$ if $S_n, E_n, I_n, R_n \geq 0$.

Next, setting $P_n = S_n + E_N + I_n + R_n$ for $n \geq 0$. From (11) we obtain

$$\frac{P_{n+1} - P_n}{\phi} = \mu - \mu P_{n+1}, \quad P_0 = 1.$$

or equivalently

$$P_{n+1} = \frac{P_n + \phi\mu}{1 + \phi\mu}, \quad P_0 = 1.$$

It is easy to verify that $\{P_n\}$ with $P_n = 1$ is the unique solution of this difference equation. The proof is complete. $\square$ $\square$

As a direct consequence of Lemma 2, it suffices to consider the following sub-model of (11)

$$\begin{aligned}
\frac{S_{n+1} - S_n}{\phi(\Delta t)} &= \mu - \frac{\beta_0 S_{n+1} I_n}{\psi(I_n)} - \mu S_{n+1},\\
\frac{E_{n+1} - E_n}{\phi(\Delta t)} &= \frac{\beta_0 S_{n+1} I_n}{\psi(I_n)} - (\sigma + \mu) E_{n+1},\\
\frac{I_{n+1} - I_n}{\phi(\Delta t)} &= \sigma E_{n+1} - (\gamma + \mu) I_{n+1}
\end{aligned} \tag{12}$$

on the set Ω^* given by (4).

We now compute the basic reproduction number for the discrete model (12) by the next generation matrix approach [8]. It is easy to verify that (12) always has a unique DFE point $P_f^* = (S_f^*, E_f^*, I_f^*) = (1, 0, 0)$ for all the values of the parameters. If reordering the variables in (12) as (E_n, I_n, S_n), then the DFE point is transformed to $(0, 0, 1)$. The Jacobian matrix of (12) at P_f^* is given by

$$J(P_f^*) = \begin{pmatrix} \dfrac{1}{1+\phi(\sigma+\mu)} & \dfrac{\phi\beta_0}{1+\phi(\sigma+\mu)} & 0 \\[2ex] \dfrac{\phi\sigma}{[1+\phi(\gamma+\mu)][1+\phi(\sigma+\mu)]} & \dfrac{1}{1+\phi(\gamma+\mu)} + \dfrac{\phi\sigma}{1+\phi(\gamma+\mu)} & 0 \\[2ex] 0 & \dfrac{\phi\beta_0}{1+\phi\mu} & \dfrac{1}{1+\phi\mu} \end{pmatrix}.$$

Following the method in [4], we write $J(P_f^*)$ in the form

$$J(P_f^*) = \begin{pmatrix} F+T & 0 \\[2ex] A & C \end{pmatrix},$$

where

$$F = \begin{pmatrix} 0 & 0 \\[2ex] \dfrac{\phi\sigma}{[1+\phi(\gamma+\mu)][1+\phi(\sigma+\mu)]} & \dfrac{\phi\sigma}{1+\phi(\gamma+\mu)} \end{pmatrix},$$

$$T = \begin{pmatrix} \dfrac{1}{1+\phi(\sigma+\mu)} & \dfrac{\phi\beta_0}{1+\phi(\sigma+\mu)} \\[2ex] 0 & \dfrac{1}{1+\phi(\gamma+\mu)} \end{pmatrix},$$

$$A = 0,$$

$$C = \frac{1}{1+\phi\mu}.$$

It is easy to verify that F and T are non-negative, $F+T$ is irreducible, and the matrix C and T satisfy

$$\rho(C) < 1, \quad \rho(T) < 1.$$

Therefore, the basic reproduction number of the discrete model (12) can be computed as

$$\mathcal{R}_0 = \rho\big(F(I-T)^{-1}\big) = \frac{\beta_0\sigma}{(\sigma+\mu)(\gamma+\mu)}.$$

This means that the basic reproduction numbers of (12) and (3) are identical. The following assertion is a direct consequence of Theorem 2.1 in [4].

Corollary 1. *The DFE point P_f^* of the NSFD scheme* (12) *is locally asymptotically stable if $\mathcal{R}_0 < 1$ and unstable if $\mathcal{R}_0 > 1$.*

Similarly to Theorem 2, it is easy to verify that the NSFD model (12) has a unique DEE point $P_e^* = (S_e^*, E_e^*, I_e^*)$ if and only if $\mathcal{R}_0 > 1$, where $P_e^* = P_e$. By using the approach proposed in [27], the local asymptotic stability of P_e^* is established as follows.

Proposition 2. *Suppose that $\mathcal{R}_0 > 1$. Then, there exists a positive number $\phi^* > 0$ that plays as a stability threshold for the NSFD model* (12)*, that is, P_e^* locally asymptotically stable whenever*

$$\phi(\Delta t) < \phi^* \quad \textit{for all} \quad \Delta t > 0.$$

Remark 2. *Similarly to Theorem 3 in [27], we can verify that the NSFD scheme* (12) *is convergent of order* 1.

The results constructed in this section lead to the following theorem.

Theorem 4. *The NSFD scheme* (11) *is dynamically consistent with respect to the positivity, boundedness and asymptotic stability of the SEIR model* (2) *if*

$$\phi(\Delta t) < \tau^* \quad \textit{for all} \quad \Delta t > 0,$$

where

$$\tau^* = \begin{cases} \infty \quad \textit{if} \quad \mathcal{R}_0 < 1, \\ \\ \phi^* \quad \textit{if} \quad \mathcal{R}_0 > 1. \end{cases}$$

Remark 3. *In numerical experiments performed in the next section, we will use the following denominator function for the NSFD scheme* (12)

$$\phi(\Delta t) = \begin{cases} \Delta t \quad \textit{if} \quad \mathcal{R}_0 < 1, \\ \\ \dfrac{1 - e^{-(1/\phi^*)\Delta t}}{(1/\phi^*)} \quad \textit{if} \quad \mathcal{R}_0 > 1. \end{cases}$$

4. Numerical experiments

In this section, we report some numerical examples to support the theoretical findings. In all numerical examples, the function $\psi(I) = 1 + \alpha I^2$ (see [50]) and the following data will be used.

Table 1: The parameters used in numerical simulations

Case	μ	γ	σ	β_0	α	Source	$\mathcal{R}_0$	GAS equilibrium point
1	0.2	0.8	0.8	0.25	0.75	Assumed	0.4	$P_f = (1, 0, 0)$
2	0.1	1/7	1/5	0.2	0.75	[50]	0.5490	$P_f = (1, 0, 0)$
3	0.1	1/7	1/5	0.75	0.5	[50]	2.0588	$P_e = (0.4904, 0.1699, 0.1399)$

First, we compare the NSFD scheme (12) with two well-known standard Runge-Kutta schemes, namely, the Euler scheme and the second-order Runge-Kutta (RK2) scheme (see [9]). The numerical solutions obtained by these schemes are depicted in Figures 1-3. From these figures, we observe that the Euler and RK2 schemes generate the approximations that are not only negative but also unstable for the step size $\Delta t = 1.5$. Hence, the dynamical properties of the SEIR model are destroyed. Conversely, the NSFD scheme correctly preserve the dynamics of the SEIR model for the same step size. Even when using a larger step size, namely, $\Delta t = 2.0$, the dynamics of the SEIR model is still preserved by the NSFD scheme (see Figure 3). This is evidence supporting the claim that the NSFD scheme preserves the dynamics of the SEIR model for all finite step sizes.

We now confirm the global asymptotic stability of the SEIR model by numerical solutions. Figures 4 and 5 sketch numerical solutions obtained by the NSFD scheme over the time interval $[0, 150]$ with the step size $\Delta t = 10^{-5}$. It is clear that the DFE point is globally asymptotically stable when $\mathcal{R}_0 < 1$ and the DEE point is globally stable when $\mathcal{R}_0 > 1$. This result support the ones constructed in Section 2.

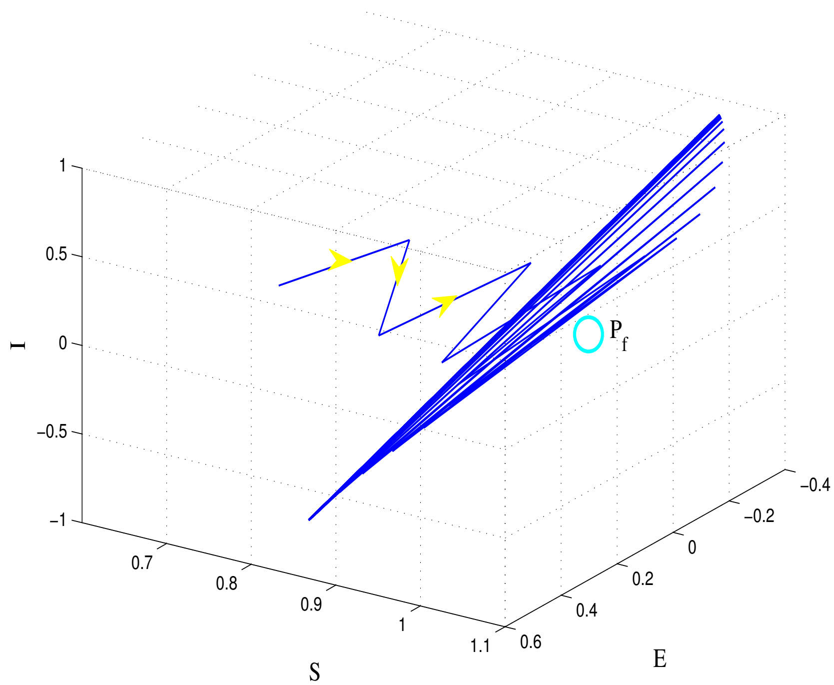

Figure 1: The phase space generated by the Euler scheme in Case 1.

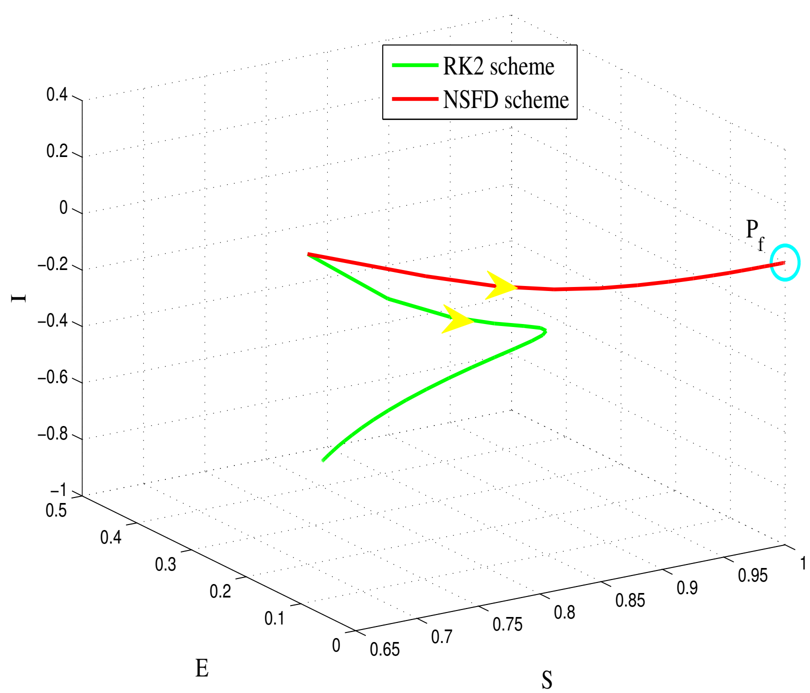

Figure 2: The phase spaces generated by the RK2 and NSFD schemes in Case 1.

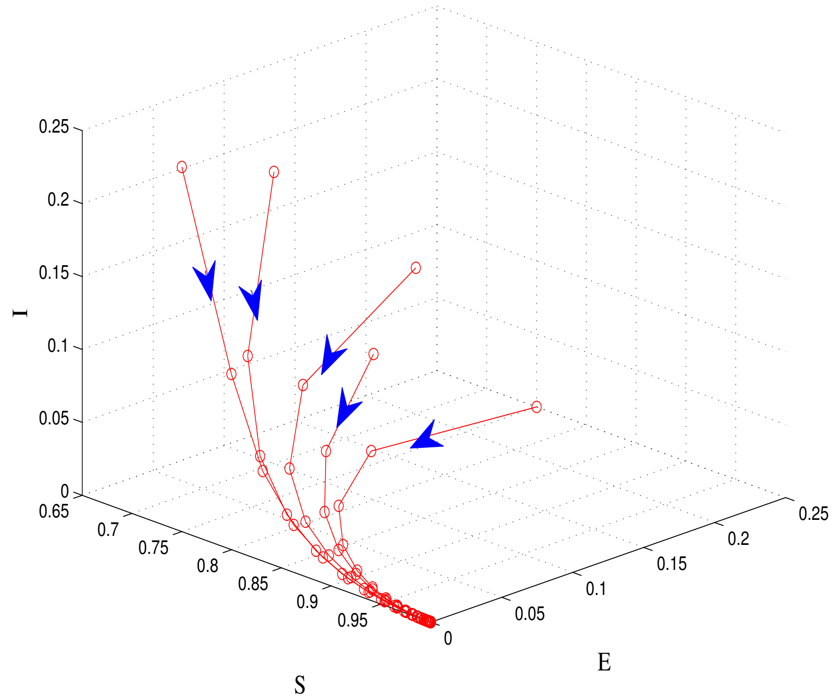

Figure 3: The phase spaces generated by the NSFD scheme in Case 1 with $\Delta t = 2.0$.

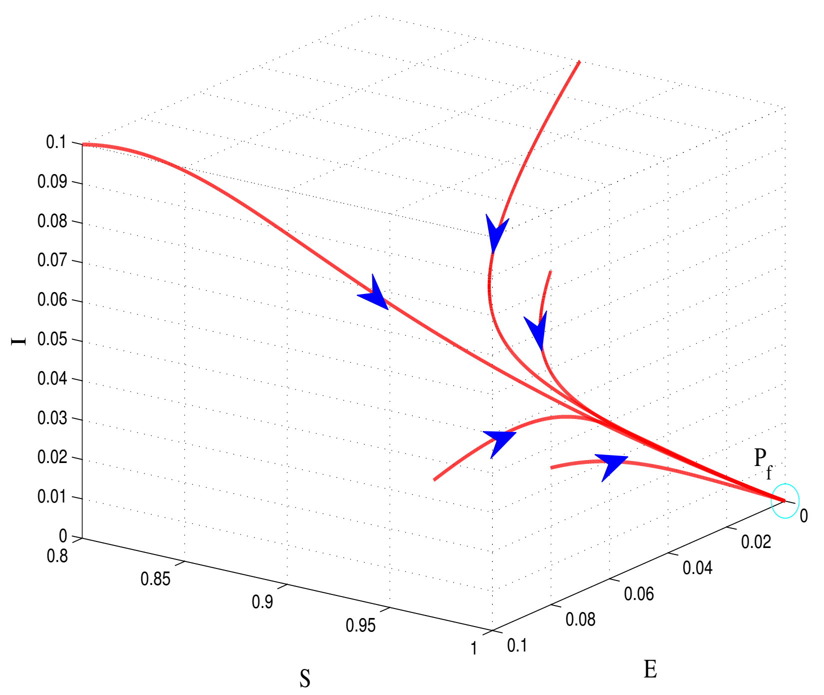

Figure 4: The phase spaces generated by the NSFD scheme in Case 2 with $\phi(\Delta t) = \Delta t$.

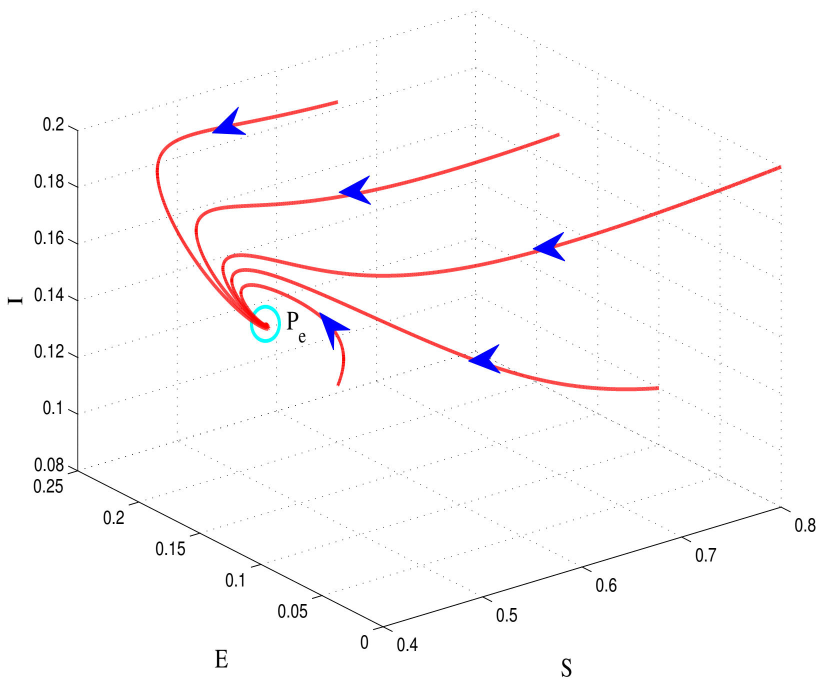

Figure 5: The phase spaces generated by the NSFD scheme in Case 3 with $\phi(\Delta t) = 1 - e^{-\Delta t}$.

5. Discussions and conclusions

As the main conclusion of this work, we have provided a new mathematical study for transmission dynamics of COVID-19 model. The obtained results improved and extended the ones presented in the benchmark work [50]. On the other hand, we have constructed and analyzed an nonstandard numerical scheme that has the ability to generate reliable approximations preserving the dynamical properties of the COVID-19 model regardless of the chosen step sizes. Finally, a set of illustrate numerical experiments has been also conducted to support and illustrate the theoretical findings. The numerical results provided evidence that confirms not only the validity of the theoretical findings but also the advantages of the NSFD scheme over some well-known standard ones.

In the near future, we will consider the SEIR model with vaccination to discover effects of vaccines. Fractional-order versions and the parameter estimation problem with applications will be also studied. On the other hand, the construction of high-order dynamically consistent NSFD schemes for the SEIR model will be paid attention to.

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