COEFFICIENT BOUNDS IN THE CLASS OF FUNCTIONS ASSOCIATED WITH q-FUNCTION THEORY

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ABSTRACT. In this paper, we use the concept of q-calculus in geometric function theory. For some α, $\alpha \in [0,1)$, we consider normalized analytic functions f such that $f'(z)/\mathrm{d}_q f(z)$ lies in half-plane $\{w : \mathfrak{Re}\ w > \alpha\}$ for all z, $|z| < 1$. Here d_q is the Jackson q-derivative operator well-known in the q-calculus theory. The paper is devoted to the coefficient problems of such functions for real and for complex numbers q. Coefficient bounds are of particular interest, because of them some geometrical properties of the function can be obtained.

1. Introduction

Quantum calculus (q-calculus) began with Frank Hilton Jackson in the early twentieth century and aroused the interest of many researchers as the relationship between mathematics and physics. It has many applications in various areas of mathematics such as number theory, combinatorics, orthogonal polynomials or basic hypergeometric functions. In geometric function theory, q-calculus is used to study q-analogs of subclasses of analytic functions. Researchers use operators of q-calculus to define and analyze subclasses of analytic functions. Finding estimates of function coefficients is particularly important because it reveals the geometric properties of these functions. For example, estimating a second function coefficient in the class of univalent functions gives the growth and the distortion properties. Coefficient problems for functions belonging to classes related to q-theory of functions are discussed, for instance, in [1, 2, 6, 7, 8, 11, 12, 13, 14, 15].

In the sequel we will use the following well-known definitions and notations. Let $\mathbb{C}$ be the open complex plane and $\mathbb{D}$ be the unit disc, $\mathbb{D} := \{z \in \mathbb{C} : \ |z| < 1\}$. Let $\mathcal{H}$ denote the class of analytic functions in $\mathbb{D}$. Let $\mathcal{A}$ be the subclass of $\mathcal{H}$ consisting of functions normalized by $f(0) = 0$, $f'(0) = 1$, i.e.

$$f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in \mathbb{D}. \tag{1.1}$$

Jackson in [4, 5] introduced and studied the q-derivative (q-difference operator), $0 \le q \le 1$, as

$$\mathrm{d}_q f(z) = \begin{cases} \left(f(qz) - f(z)\right)/(qz - z) & \text{when} \quad z \neq 0,\ 0 \le q < 1, \\ f'(0) & \text{when} \quad z = 0, \\ f'(z) & \text{when} \quad q = 1. \end{cases} \tag{1.2}$$

Thus, from (1.2) for a function f given by (1.1) we have

$$\mathrm{d}_q f(z) = 1 + \sum_{n=2}^{\infty} [n]_q a_n z^{n-1}, \quad \text{where} \quad [n]_q = \sum_{k=0}^{n-1} q^k, \quad n = 2, 3, \dots\ . \tag{1.3}$$

We have also

$$\mathrm{d}_q f(z) = \frac{1}{z}\left\{f(z) * h_q(z)\right\} = \frac{1}{z}\left\{f(z) * \frac{z}{(1-qz)(1-z)}\right\}, \tag{1.4}$$

where $*$ denotes the convolution or Hadamard product of power series. For the generalization of Jackson's q-derivative, instead of a real number q we may consider a complex number ζ, $|\zeta| \le 1$. Namely, in [9] the following generalization of (1.4) for $\zeta \in \mathbb{C}$, $|\zeta| \le 1$ was defined

$$\mathrm{d}_\zeta f(z) = \frac{1}{z}\left\{f(z) * h_\zeta(z)\right\} = \frac{1}{z}\left\{f(z) * \frac{z}{(1-\zeta z)(1-z)}\right\} \tag{1.5}$$

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(see also [10]). The above function h_ζ has the form

$$h_\zeta(z) = \frac{z}{(1-\zeta z)(1-z)} = \sum_{n=1}^{\infty}[n]_\zeta z^n, \quad z \in \mathbb{D}, \quad \text{where}$$

$$[n]_\zeta = \sum_{k=0}^{n-1}\zeta^k = \frac{1-\zeta^n}{1-\zeta}, \quad n = 2, 3, \ldots$$

and it is starlike for all complex numbers ζ, $|\zeta| \le 1$. It is easy to check that if $\zeta = 1$, then the function h_ζ becomes the well-known Koebe function

$$h_1(z) = \frac{z}{(1-z)^2} = \sum_{n=1}^{\infty} nz^n, \quad z \in \mathbb{D}.$$

Making use of the operator d_ζ defined in (1.5), we introduce the subclass $\mathcal{R}(\zeta,\alpha)$ of the class $\mathcal{A}$ for $0 \le \alpha < 1$.

Definition. Let $f \in \mathcal{A}$. For a given complex number ζ, $|\zeta| \le 1$, we say that f is in the class $\mathcal{R}(\zeta,\alpha)$, $0 \le \alpha < 1$, if

$$\mathfrak{Re}\left\{\frac{f'(z)}{\mathrm{d}_\zeta f(z)}\right\} > \alpha, \quad z \in \mathbb{D}, \tag{1.6}$$

where the operator d_ζ is defined in (1.5).

Remark 1. It is easy to see that $\mathcal{R}(1,\alpha) = \mathcal{A}$.

Remark 2. For the function $h(z) = z/(1-z)$, we have

$$\mathfrak{Re}\left\{\frac{h'(z)}{\mathrm{d}_\zeta h(z)}\right\} = \mathfrak{Re}\left\{\frac{1-\zeta z}{1-z}\right\} > \frac{1+|\zeta|}{2}, \quad z \in \mathbb{D},$$

so for a given ζ, $|\zeta| \le 1$, the function $h \in \mathcal{R}(\zeta,(1+|\zeta|)/2)$.

Remark 3. For f given by (1.1) condition (1.6) becomes

$$\mathfrak{Re}\left\{\frac{1+\sum_{n=2}^{\infty} na_n z^{n-1}}{1+\sum_{n=2}^{\infty}[n]_\zeta a_n z^{n-1}}\right\} > \alpha, \quad z \in \mathbb{D}. \tag{1.7}$$

In this paper we discuss the coefficients problems in the class $\mathcal{R}(\zeta,\alpha)$ for real and for complex ζ.

2. Coefficient bounds in the class $\mathcal{R}(\zeta,\alpha)$

Theorem 2.1. *Assume that $|\zeta| \le 1$ and $\alpha \in [0,1)$. If f is given by (1.1) and belongs to the class $\mathcal{R}(\zeta,\alpha)$, then*

$$|a_k|^2 \le \frac{1}{|k-[k]_\zeta|^2}\sum_{n=1}^{k-1}\left\{|(1-2\alpha)[n]_\zeta + n|^2 - |n-[n]_\zeta|^2\right\}|a_n|^2 \tag{2.1}$$

for all $k = 2, 3, \ldots$.

Proof. We have

$$\frac{f'(z)}{\mathrm{d}_\zeta f(z)} = \frac{1+(1-2\alpha)w(z)}{1-w(z)}$$

for some $w \in \mathcal{H}$ such that $|w(z)| < 1$ in $\mathbb{D}$, $w(0) = 0$. This gives

$$\sum_{n=1}^{\infty} na_n z^{n-1} - \sum_{n=1}^{\infty}[n]_\zeta a_n z^{n-1} = w(z)\left\{(1-2\alpha)\sum_{n=1}^{\infty}[n]_\zeta a_n z^{n-1} + \sum_{n=1}^{\infty} na_n z^{n-1}\right\}$$

with $a_1 = 1$. Therefore, we can write

$$\sum_{n=1}^{k}(n-[n]_\zeta)a_n z^{n-1} + \sum_{n=k+1}^{\infty} b_n z^{n-1} = w(z)\sum_{n=1}^{k-1}\{(1-2\alpha)[n]_\zeta + n\}a_n z^{n-1}$$

for some b_n, $k+1 \le n$, where b_n can be expressed in terms of the coefficients of f and w. This gives

$$\begin{aligned}\left|\sum_{n=1}^{k}(n-[n]_\zeta)a_n z^{n-1} + \sum_{n=k+1}^{\infty} b_n z^{n-1}\right|^2 &= \left|w(z)\sum_{n=1}^{k-1}\{(1-2\alpha)[n]_\zeta + n\}a_n z^{n-1}\right|^2 \\ &\le \left|\sum_{n=1}^{k-1}\{(1-2\alpha)[n]_\zeta + n\}a_n z^{n-1}\right|^2, \end{aligned} \tag{2.2}$$

where

$$\sum_{n=1}^{k}(n-[n]_\zeta)a_n z^{n-1} + \sum_{n=k+1}^{\infty} b_n z^{n-1} := \sum_{n=1}^{\infty} d_n z^n$$

is an analytic function in the unit disc. Making use of the known formula (see, for instance [3])

$$\int_0^{2\pi} \left| \sum_{n=1}^{\infty} d_n (re^{i\theta})^n \right|^2 d\theta = 2\pi \sum_{n=1}^{\infty} |d_n|^2 r^{2n}$$

and integrating on $z = re^{i\theta}$, $0 < r < 1$, $0 \leq \theta < 2\pi$, both sides of (2.2), we obtain

$$\sum_{n=1}^{k} |n - [n]_\zeta|^2 |a_n|^2 r^{2(n-1)} + \sum_{n=k+1}^{\infty} |b_n|^2 r^{2(n-1)} \leq \sum_{n=1}^{k-1} |(1-2\alpha)[n]_\zeta + n|^2 |a_n|^2 r^{2(n-1)}$$

which upon letting $r \to 1$ gives

$$\sum_{n=1}^{k} |n - [n]_\zeta|^2 |a_n|^2 \leq \sum_{n=1}^{k-1} |(1-2\alpha)[n]_\zeta + n|^2 |a_n|^2$$

and this leads to the desired result (2.1). □

Corollary 2.2. *Assume that* $|\zeta| \leq 1$ *and* $\alpha \in [0, 1)$*. If* f *is given by* (1.1) *and belongs to the class* $\mathcal{R}(\zeta, \alpha)$*, then*

$$|a_2| \leq \frac{2(1-\alpha)}{|1-\zeta|}. \tag{2.3}$$

The result is sharp.

Proof. From Theorem 2.1, we have (2.1) for $k = 2$:

$$|a_2|^2 \leq \frac{1}{|2 - [2]_\zeta|^2} \left\{ |(1-2\alpha)[1]_\zeta + 1|^2 - |1 - [1]_\zeta|^2 \right\} |a_1|^2 = \frac{|2-2\alpha|^2}{|1-\zeta|^2}.$$

To show that (2.3) is sharp we will show that the function

$$f_\zeta(z) = z + \frac{2(1-\alpha)}{1-\zeta} z^2 + \cdots = \sum_{k=1}^{\infty} c_k z^k, \tag{2.4}$$

where

$$c_1 = 1, \quad c_k = \frac{\prod_{n=1}^{k-1} \{(1-2\alpha)[n]_\zeta + n\}}{\prod_{n=2}^{k} \{n - [n]_\zeta\}}, \quad k = 2, 3, \ldots \tag{2.5}$$

is in the class $\mathcal{R}(\zeta, \alpha)$. From (2.5), we obtain

$$c_k = c_{k-1} \frac{(1-2\alpha)[k-1]_\zeta + (k-1)}{k - [k]_\zeta}, \quad k = 2, 3, \ldots. \tag{2.6}$$

Therefore

$$\lim_{k\to\infty} \left| \frac{c_k}{c_{k-1}} \right| = \lim_{k\to\infty} \left| \frac{(1-2\alpha)[k-1]_\zeta + k - 1}{k - [k]_\zeta} \right| = 1$$

hence the series (2.4) converges in $\mathbb{D}$. Relation (2.6) implies that

$$c_k z^{k-1} = c_{k-1} z^{k-1} \frac{(1-2\alpha)[k-1]_\zeta + (k-1)}{k - [k]_\zeta}, \quad k = 2, 3, \ldots$$

or that

$$\{k - [k]_\zeta\} c_k z^{k-1} = c_{k-1} z^{k-1} \{(1-2\alpha)[k-1]_\zeta + (k-1)\}, \quad k = 2, 3, \ldots.$$

This gives

$$\sum_{k=1}^{\infty} \{k - [k]_\zeta\} c_k z^{k-1} = \sum_{k=2}^{\infty} c_{k-1} \{(1-2\alpha)[k-1]_\zeta + (k-1)\} z^{k-1}$$

or

$$\sum_{k=1}^{\infty} k c_k z^{k-1} - \sum_{k=1}^{\infty} [k]_\zeta c_k z^{k-1} = z \left\{ (1-2\alpha) \sum_{k=1}^{\infty} [k]_\zeta c_k z^{k-1} + \sum_{k=1}^{\infty} k c_k z^{k-1} \right\}$$

which is equivalent to

$$f'_\zeta(z) - \mathrm{d}_\zeta f_\zeta(z) = z \left\{ (1-2\alpha) \mathrm{d}_\zeta f_\zeta(z) + f'_\zeta(z) \right\}.$$

This may be rewritten in the form

$$\frac{f'_\zeta(z)}{\mathrm{d}_\zeta f_\zeta(z)} = \frac{1 + (1-2\alpha)z}{1-z},$$

which proves that $f_\zeta \in \mathcal{R}(\zeta, \alpha)$. □

If we try to use Theorem 2.1 to find bounds for $|a_3|$, there is a problem. This is because applying formula (2.1) gives

$$
\begin{aligned}
|a_3|^2 &\le \frac{\left\{|(1-2\alpha)[1]_\zeta+1|^2-|1-[1]_\zeta|^2\right\}|a_1|^2}{|3-[3]_\zeta|^2}+\frac{\left\{|(1-2\alpha)[2]_\zeta+2|^2-|2-[2]_\zeta|^2\right\}|a_2|^2}{|3-[3]_\zeta|^2}\\
&= \frac{4(1-\alpha)^2}{|2-\zeta-\zeta^2|^2}+\frac{\left\{|(1-2\alpha)(1+\zeta)+2|^2-|1-\zeta|^2\right\}|a_2|^2}{|2-\zeta-\zeta^2|^2} \qquad (2.7)
\end{aligned}
$$

and the expression $\left\{|(1-2\alpha)(1+\zeta)+2|^2-|1-\zeta|^2\right\}$ has not established sign for complex numbers ζ. In general, the formula (2.1) contains

$$\left\{|(1-2\alpha)[n]_\zeta+n|^2-|n-[n]_\zeta|^2\right\}$$

which my be negative or nonnegative. Some difficulties follows also from the complicated formulas for real and imaginary parts of $[n]_\zeta$. Namely, for $\zeta=\rho e^{i\alpha}$, we have

$$\mathfrak{Re}\{[n]_\zeta\}=1+\rho\cos\alpha+\ldots+\rho^{n-1}\cos(n-1)\alpha=\frac{1-\rho\cos\alpha-\rho^n\cos n\alpha+\rho^{n+1}\cos(n-1)\alpha}{1-2\rho\cos\alpha+\rho^2}$$

and

$$\mathfrak{Im}\{[n]_\zeta\}=\rho\sin\alpha+\ldots+\rho^{n-1}\sin(n-1)\alpha=\frac{\rho\sin\alpha-\rho^n\sin n\alpha+\rho^{n+1}\sin(n-1)\alpha}{1-2\rho\cos\alpha+\rho^2}.$$

These difficulties does not occur in the case when ζ is a real number. For this reason, in the next results, we consider real q instead of complex ζ.

Lemma 2.3. *Assume that $q\in(0,1]$ and $\alpha\in[0,1)$. If f is given by (1.1) and belongs to the class $\mathcal{R}(q,\alpha)$, then we have the following coefficient relations*

$$\frac{1}{|k-[k]_q|^2}\sum_{n=1}^{k-1}\left\{|(1-2\alpha)[n]_q+n|^2-|n-[n]_q|^2\right\}|a_n|^2 \;\le\; \frac{\prod_{n=1}^{k-1}|(1-2\alpha)[n]_q+n|^2}{\prod_{n=2}^{k}|n-[n]_q|^2} \qquad (2.8)$$

for all $k=2,3,\ldots$.

Proof. For $k=2$ we have

$$
\begin{aligned}
\frac{1}{|2-[2]_q|^2}\sum_{n=1}^{1}\left\{|(1-2\alpha)[n]_q+n|^2-|n-[n]_q|^2\right\}|a_n|^2 &= \frac{|(1-2\alpha)[1]_q+1|^2}{|2-[2]_q|^2}\\
&= \frac{\prod_{n=1}^{1}|(1-2\alpha)[n]_q+n|^2}{\prod_{n=2}^{2}|n-[n]_q|^2}.
\end{aligned}
$$

Now, assume that (2.8) holds for integer s, $s\ge 2$. It is easy to see that

$$|(1-2\alpha)[s]_q+s|^2-|s-[s]_q|^2\ge 0$$

for all integer s, $s \geq 2$. From this and from (2.1), we obtain

$$
\begin{aligned}
&\frac{1}{|(s+1)-[s+1]_q|^2}\sum_{n=1}^{s}\left\{|(1-2\alpha)[n]_q+n|^2-|n-[n]_q|^2\right\}|a_n|^2\\
=\ &\frac{\sum_{n=1}^{s-1}\left\{|(1-2\alpha)[n]_q+n|^2-|n-[n]_q|^2\right\}|a_n|^2+\left\{|(1-2\alpha)[s]_q+s|^2-|s-[s]_q|^2\right\}|a_s|^2}{|(s+1)-[s+1]_q|^2}\\
\leq\ &\frac{\sum_{n=1}^{s-1}\left\{|(1-2\alpha)[n]_q+n|^2-|n-[n]_q|^2\right\}|a_n|^2}{|(s+1)-[s+1]_q|^2}\\
&+\frac{\left\{|(1-2\alpha)[s]_q+s|^2-|s-[s]_q|^2\right\}}{|(s+1)-[s+1]_q|^2}\ \frac{\sum_{n=1}^{s-1}\left\{|(1-2\alpha)[n]_q+n|^2-|n-[n]_q|^2\right\}|a_n|^2}{|s-[s]_q|^2}\\
=\ &\frac{|(1-2\alpha)[s]_q+s|^2}{|(s+1)-[s+1]_q|^2}\ \frac{\sum_{n=1}^{s-1}\left\{|(1-2\alpha)[n]_q+n|^2-|n-[n]_q|^2\right\}|a_n|^2}{|s-[s]_q|^2}\\
\leq\ &\frac{|(1-2\alpha)[s]_q+s|^2}{|(s+1)-[s+1]_q|^2}\ \frac{\prod_{n=1}^{s-1}|(1-2\alpha)[n]_q+n|^2}{\prod_{n=2}^{s}|n-[n]_q|^2}\\
=\ &\frac{\prod_{n=1}^{s}|(1-2\alpha)[n]_q+n|^2}{\prod_{n=2}^{s+1}|n-[n]_q|^2}\,.
\end{aligned}
$$

Therefore, by the induction, (2.8) holds for all $k = 2, 3, \ldots$. ☐

Theorem 2.4. *Assume that $q \in (0, 1]$ and $\alpha \in [0, 1)$. If f is given by (1.1) and belongs to the class $\mathcal{R}(q, \alpha)$, then*

$$|a_k| \leq \frac{\prod_{n=1}^{k-1}((1-2\alpha)[n]_q+n)}{\prod_{n=2}^{k}(n-[n]_q)} \tag{2.9}$$

for all $k = 2, 3, \ldots$. The result is sharp.

Proof. From inequalities (2.1) and (2.8), we immediately get that (2.9) holds in the class $\mathcal{R}(q, \alpha)$. To show that (2.9) is sharp, it is enough to prove that the function g_q such that

$$g_q(z)=\sum_{k=1}^{\infty}b_kz^k,\quad b_1=1,\quad b_k=\frac{\prod_{n=1}^{k-1}\{(1-2\alpha)[n]_q+n\}}{\prod_{n=2}^{k}\{n-[n]_q\}},\quad k=2,3,\ldots \tag{2.10}$$

is in the class $\mathcal{R}(q, \alpha)$. From (2.10), we obtain recurrence relation

$$(k-[k]_q)b_k=\{(1-2\alpha)[k-1]_q+k-1\}\,b_{k-1},\quad k=2,3,\ldots\ . \tag{2.11}$$

Because

$$\lim_{k\to\infty}\left|\frac{b_k}{b_{k-1}}\right|=\lim_{k\to\infty}\left|\frac{(1-2\alpha)[k-1]_q+k-1}{k-[k]_q}\right|=1,$$

hence the series (2.10) converges in $\mathbb{D}$. Condition (2.11) implies that

$$\sum_{n=1}^{\infty}nb_nz^{n-1}-\sum_{n=1}^{\infty}[n]_qb_nz^{n-1}=\left\{(1-2\alpha)\sum_{n=1}^{\infty}[n]_qb_nz^n+\sum_{n=1}^{\infty}nb_nz^n\right\}$$

or that

$$g_q'(z)-\mathrm{d}_qg_q(z)=z(1-2\alpha)\mathrm{d}_qg_q(z)+zg_q'(z).$$

From this, we have

$$\frac{g_q'(z)}{\mathrm{d}_qg_q(z)}=\frac{1+(1-2\alpha)z}{1-z},$$

which proves that $g_q \in \mathcal{R}(q, \alpha)$. ☐

We have proved Theorem 2.1 for a complex parameter ζ, while Theorem 2.4 only for a real parameter q. It is because of difficulties with a complex version of Lemma 2.3.

Open problem. *Does the bound (2.9) hold when q is a complex number?*

This question can be written in the following conjecture.

Conjecture. *Assume that $|\zeta| \leq 1$ and $\alpha \in [0, 1)$. If f is given by (1.1) and belongs to the class $\mathcal{R}(\zeta, \alpha)$, then*

$$|a_k|\leq\frac{\prod_{n=1}^{k-1}|(1-2\alpha)[n]_\zeta+n|}{\prod_{n=2}^{k}|n-[n]_\zeta|},\quad k=2,3,\ldots\ . \tag{2.12}$$

From Corollary 2.2, for $k = 2$ the bound (2.12) holds and is sharp. If ζ is a real number, then the bound (2.12) holds and is sharp by Theorem 2.4.

3. Conclusion

In the geometric function theory, the q-derivative allows us to extend the well-known classes of analytic functions. If in the definition of the class of functions we replace the derivative f' of the function f by the q-derivative $\mathrm{d}_q f$, we get the generalization of this class. In this way, many new classes of functions have been created recently. In this paper, using convolution and taking complex numbers q, we have defined a certain operator that can be called a generalization of Jackson's derivative. Then using this operator, we have defined the new class of functions. For this newly-defined class, we have discussed the coefficient problems and we have set an open problem to solve.

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References

[1] O. Altinta, N. Mustafa, Coefficient bounds and distortion theorems for the certain analytic functions, Turk. J. Math. 43 (2019) 985–997; doi:10.3906/mat-1812-85.
[2] L. L. Fan, Z. G. Wang, S. Khan, S. Hussain, M. Naeem, T. Mahmood, Coefficient bounds for certain subclasses of q-starlike functions, Mathematics 2019, 7, 969 (2019); doi:10.3390/math7100969.
[3] A. W. Goodman, Univalent Functions, Mariner Publishing Co.: Tampa, Florida (1983).
[4] F. H. Jackson, On q-functions and certain difference operator, Trans. Royal Society of Edinburgh 46(1908) 253–281.
[5] F. H. Jackson, On q-definite integrals, Quarterly J. Pure and Appl. Math. 41(1910) 193–203.
[6] S. Mahmood, M. Jabeen, S. N. Malik, H. M. Srivastava, R. Manzoor, S. M. J. Riaz, Some coefficient inequalities of q-starlike functions associated with conic domain defined by q-derivative, J. Funct. Spaces, 2018, Article ID 8492072 (2018); https://doi.org/10.1155/2018/8492072.
[7] S. Mahmood, Q. Z. Ahmad, H. M. Srivastava, N. Khan, B. Khan, M. Tahir, A certain subclass of meromorphically q-starlike functions associated with the Janowski functions, J. Inequal. Appl. 2019:88 (2019); https://doi.org/10.1186/s13660-019-2020-z.
[8] P. Nandini, S. Latha, Coefficient inequalities for classes of q-starlike and q-convex functions using q-derivative, J. Rajasthan Academy of Physical Sciences, 15(4)(2016) 291–297.
[9] K. Piejko, J. Sokół, On convolution and q-calculus, Boletin de la Sociedad Matematica Mexicana (2019); https://doi.org/10.1007/s40590-019-00258-y
[10] K. Piejko, J. Sokół, K. Trąbka-Więcław, On q-calculus and starlike functions, Iran. J. Sci. Technol. Trans. A Sci. 43(6)(2019) 2879–2883.
[11] S. K. Sahoo, N. L. Sharma, On a generalization of close-to-convex functions, Ann. Polon. Math., 113(2015) 93–108.
[12] T. M. Seoudy, M. K. Aouf, Coefficient estimates of new classes of q-starlike and q-convex functiond of complex order, J. Math. Inequal. 10(1)(2016) 135–145; doi:10.7153/jmi-10-11.
[13] L. Shi, Q. Khan, G. Srivastava, J. L. Liu, M. Arif, A study of multivalent q-starlike functions connected with circular domain, Mathematics 2019, 7, 670 (2019); doi:10.3390/math7080670.
[14] H. M. Srivastava, Q. Z. Ahmad, N. Khan, N. Khan, B. Khan, Hankel and Toeplitz determinants for a subclass of q-starlike functions associated with a general conic domain, Mathematics 2019, 7, 181 (2019); doi:10.3390/math7020181.
[15] H. M. Srivastava, B. Khan, N. Khan, Q. Z. Ahmad, Coefficient inequalities for q-starlike functions associated with the Janowski functions, Hokkaido Math. J., 48 (2019) 407–425.

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