Research Article

Debdulal Ghosh*, Lukas Pflug, and Jitendra Kumar

# Asymptotic analysis of time dependent solutions for the coagulation equation with source and efflux

**Abstract:** This article provides mathematical proof of the existence of stationary solutions for the coagulation equation including source and efflux terms. We demonstrate the convergence of time dependent solutions to these stationary solutions and highlight the exponential rate of convergence. These properties are analyzed for affine linear coagulation kernels, non-negative source terms and positive efflux rates. Numerical examples are included to demonstrate the predicted convergence behaviour.

**Keywords:** Coagulation equation; Affine linear kernel; Stationary solution; Stability

## 1 Introduction

The integro-differential equation modeling coagulation processes was originally proposed by Smoluchowski in 1917 [24] and the continuous form of the coagulation equation was introduced in [19]. These equations appear in various scientific fields, including astrophysics [7], chemical and process engineering [21] and aerosol science [22]. As in many application a source of new particles and a efflux of particles exists we will takes these phenomena into account in the contribution. The mentioned coagulation equation including a source and efflux term is presented in e.g. [4, 18] as the following

***Corresponding Author: Debdulal Ghosh:** Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur, West Bengal - 721302, India. email: debdulal.email@gmail.com
**Jitendra Kumar:** Department of Mathematics, Indian Institute of Technology Kharagpur, Kharagpur, West Bengal - 721302, India. email: jkumar@maths.iitkgp.ac.in
**Lukas Pflug:** Chair of Applied Mathematics (Continuous Optimization), Department Mathematics, Friedrich-Alexander-Universität Erlangen-Nürnberg, Cauerstr. 11, 91058 Erlangen, Germany; Central Institute for Scientific Computing, Friedrich-Alexander-Universität Erlangen-Nürnberg, Martensstr. 5a, 91058 Erlangen, Germany

integro-partial differential equation:

$$\begin{aligned}\partial_t c(x,t) &= \tfrac{1}{2}\int_0^x K(x-y,y)c(x-y,t)c(y,t)\,\mathrm{d}y - c(x,t)\int_0^\infty K(x,y)c(y,t)\,\mathrm{d}y \\ &\quad + q(x) - a(x)c(x,t) \qquad (1)\\ c(x,0) &= c_0(x),\end{aligned}$$

for all $(x,t) \in \mathbb{R}_{>0} \times \mathbb{R}_{>0}$. In this equation $c(x,t)$ is the *concentration* of particles of size $x$ at time $t$ and the function $K$ is the so called *coagulation kernel*. $K(x,y)$ gives the rate at which a particle of size $x$ and particles of size $y$ form a new particle of size $x+y$. The term $q$ is the *source function* and $a(x)c(x,t)$ is the *efflux term*. A brief physical interpretation of the model described in Equation (1) can be found in [18].

**Definition 1.1** (Stationary solution)**.** *A solution $\bar{c}$ to Equation* (1) *is said to be a stationary or equilibrium solution if it satisfies*

$$\tfrac{1}{2}\int_0^x K(x-y,y)\bar{c}(x-y)\bar{c}(y)\,\mathrm{d}y - \bar{c}(x)\int_0^\infty K(x,y)\bar{c}(y)\,\mathrm{d}y = a(x)\bar{c}(x) - q(x) \qquad (2)$$

$\forall x \in \mathbb{R}_{>0}$. *That is, $c(\cdot,t) \equiv \bar{c}$ is a solution of* (1) *with – by construction – vanishing time derivative.*

To further characterize the behaviour of the solution $c$ to the initial value problem (IVP) defined in Equation (1) for $t \to \infty$, we define the following:

**Definition 1.2** (Stability of the system)**.** *If the time dependent solution $c$ to the IVP defined in Equation* (1) *tends to the equilibrium solution as $t \to \infty$, then the system is said to be stable. If the rate of this convergence is proportional to* $\exp(-k_0 t)$ *for $k_0 \in \mathbb{R}_{>0}$, then we call the system exponentially or asymptotically stable.*

In the literature, the coagulation equation with a source and efflux term have also been studied in a discrete setting including a spatial coordinate $z \in \mathbb{R}^d$ with $d \in \mathbb{N}_{\geq 1}$. This results in the following equation:

$$\begin{aligned}\partial_t c_i(z,t) + \mathrm{div}_z(v_i(z,t)c_i) &= \frac{1}{2}\sum_{j=1}^{i-1} K_{i-j,j}c_{i-j}(z,t)c_j(z,t) \qquad (3)\\ &\quad - c_i(z,t)\sum_{j=1}^{\infty} K_{i,j}c_i(z,t) \qquad (4)\\ &\quad + q_i(z,t) - a_i(z,t)c_i(z,t)\\ c_i(z,0) &= c_i^0(z),\end{aligned}$$

where $c_i(z,t)$ is the concentration of the $i^{\text{th}}$ particle size at spatial location $z$ and time $t$, $v_i$ is the spatial velocity of particles of size $i$ and $K_{i,j}$ is the coagulation kernel of particles of size $i$ and $j$ forming a new particle of size $i+j$. The function $c_i^0(z)$ denotes the concentration of particles of size $i$ at the location $z$ and initial time $t=0$. A brief study on this can be found in [1, 11].

In this contribution we prove the following for the coagulation equation (1) with a linear coagulation kernel:

(i) the existence and uniqueness of a stationary solution and
(ii) the convergence of time dependent solutions to this stationary solution.

It is important to note that, prior to this article, these two problems have only been studied for a constant coagulation kernel. Moreover, it is worth noting that this article is the first to analyze the problems for *any* continuous source function. To the best of the authors knowledge, in all earlier works the source function was assumed to have a special form, such as an exponential function [4].

This work is organized as follows: In Section 2, we prove the existence and uniqueness of a stationary solution. Section 3 centres on proving convergence of time dependent solution to their respective stationary solution. A numerical example, showing the proved convergence behaviour, is shown in Section 4. In the final Section 5, we provide a brief conclusion and propose some future directions for this study.

### 1.1 Literature

For the discrete case, in [26] the authors analyze the existence of solutions of the coagulation-fragmentation equation with a source function and efflux term, i.e. they consider equation (4) without the term $\mathrm{div}_z(v_i(z,t)c_i)$. In [11] the authors prove the existence of a stationary solution and the convergence of the time dependent solution to a stationary solution. The first study to focus on the coagulation equation with a source term, but no efflux term, was [4]. In this study, the authors proved the existence of stationary solution and determined properties of the stationary solution, such as boundedness of the moments of the equilibrium solution. In the case when the source term is of the form $q(x) = \exp(-ax)\ \forall x \in \mathbb{R}_{>0}$, with $a \in \mathbb{R}_{>0}$, the existence of an equilibrium solution is shown. Importantly, an explicit expression of an equilibrium solution is provided. However, the authors only consider constant coagulation kernels.

### 1.2 Definitions, Notation and Assumptions

In line with [5, Page 4] and [4, Chapter 2] we define the following function spaces:

**Definition 1.3** (Function spaces)**.** *For $\lambda \in \mathbb{R}_{>0}$ we define:*

$$\Omega_\lambda(T) := \left\{ c \in L^\infty([0,T]; C(\mathbb{R}_{>0})) \ : \ \|c\|_\lambda < \infty \right\}$$

*with the norm:*

$$\|c\|_\lambda := \sup_{t \in (0,T)} \int_{\mathbb{R}_{>0}} \exp(\lambda x)|c(x,t)| \, \mathrm{d}x.$$

*In addition we define $\Omega(T) := \bigcup_{\lambda>0} \Omega_\lambda(T)$ and the cone of non-negative functions in $\Omega(T)$, i.e. $\Omega^+(T) := \{c \in \Omega(T) \ : \ c \geqq 0\}$.*

In this paper, we assume the coagulation kernel to be affine linear, i.e.

**Assumption 1.4** (Affine linearity of the coagulation kernel)**.** *The coagulation kernel $K$ in the present manuscript is restricted to symmetric affine linear functions, i.e.*

$$K(x,y) = \mathcal{K}_0 + \mathcal{K}_1(x+y),$$

*with $\mathcal{K}_0, \mathcal{K}_1 \in \mathbb{R}_{\geq 0}$.*

The kernels under consideration include well known kernels, such as Kapur's kernel (granulation kernel, see e.g. [14]), Smoluchowski's kernel (linear velocity profile, see e.g. [25]), non-linear velocity profile (see e.g. [23]) and Friedlander kernels (aerosol dynamics kernel, see e.g. [10]).

Throughout the rest of the paper we use the following *notation*: The solution $c$ to Equation (1) will be called the **time dependent solution** or **instationary solution**, the solution $\bar{c}$ to equation Equation (2) will be termed the **equilibrium solution** or **stationary solution**, $q$ will be called the **source term** and $a$ the **efflux rate**. For convenience, for $k \in \mathbb{N}$ we define the $k-$th moments of the instationary and the stationary solution, as well as the source term, as follows:

$$\mathcal{N}_k := \int_{\mathbb{R}_{>0}} x^k \bar{c}(x) \, \mathrm{d}x, \quad \mathcal{M}_k(t) := \int_{\mathbb{R}_{>0}} x^k c(x,t) \, \mathrm{d}x \ \text{ and } \ \mathcal{Q}_k := \int_{\mathbb{R}_{>0}} x^k q(x) \, \mathrm{d}x. \quad (5)$$

For two functions $f, g : \mathbb{R}_{>0} \mapsto \mathbb{R}$, their convolution $f * g : \mathbb{R}_{>0} \mapsto \mathbb{R}$ is defined as

$$(f * g)(x) := \int_0^x f(y) g(x-y) \, \mathrm{d}y.$$

## 2 Existence and uniqueness of an equilibrium solution

In this section we will demonstrate the existence and uniqueness of an equilibrium solution as defined in Definition 1.1 of the IVP defined in Equation (1).

**Lemma 2.1** (Condition for the stationary solution)**.** *The stationary solution $\bar{c}$ to Equation* (1) *satisfies*

$$\bar{c}(x) = \mathcal{A}(\bar{c})(x), \quad \textit{with} \quad \mathcal{A}(\bar{c})(x) := \frac{1}{2}\frac{\big(\mathcal{K}_0 + \mathcal{K}_1 x\big)(\bar{c} * \bar{c})(x) + 2q(x)}{\big(\mathcal{K}_0 + \mathcal{K}_1 x\big)\mathcal{N}_0 + \mathcal{K}_1\mathcal{N}_1 + a(x)}. \tag{6}$$

*Proof.* By Equation (2) and the linearity of the coagulation kernel (see Assumption 1.4) for the stationary solution $\bar{c}$ we obtain:

$$\begin{aligned} 0 &= \tfrac{1}{2}\textstyle\int_0^x (\mathcal{K}_0 + \mathcal{K}_1 x)\bar{c}(x-y)\bar{c}(y)\,\mathrm{d}y - \bar{c}(x)\int_0^\infty \big(\mathcal{K}_0 + \mathcal{K}_1(x+y)\big)\bar{c}(y)\,\mathrm{d}y \\ &\quad + q(x) - a(x)\bar{c}(x) \\ &= \tfrac{1}{2}\textstyle\int_0^x \big(\mathcal{K}_0 + \mathcal{K}_1 x\big)\bar{c}(x-y)\bar{c}(y)\,\mathrm{d}y - \bar{c}(x)\Big(\big(\mathcal{K}_0 + \mathcal{K}_1 x\big)\mathcal{N}_0 + \mathcal{K}_1\mathcal{N}_1\Big) \\ &\quad + q(x) - a(x)\bar{c}(x) \\ &= \tfrac{1}{2}\big(\mathcal{K}_0 + \mathcal{K}_1 x\big)(\bar{c} * \bar{c})(x) + q(x) - \bar{c}(x)\Big(\big(\mathcal{K}_0 + \mathcal{K}_1 x\big)\mathcal{N}_0 + \mathcal{K}_1\mathcal{N}_1 + a(x)\Big), \end{aligned}$$

which directly results in the claimed equality. □

To show the existence of a stationary solution, for $\alpha \in \mathbb{R}_{>0}$ we apply the Banach fixed-point theorem [27, Theorem 1.A] to the space of continuous functions on $[0,\alpha]$, i.e. $C([0,\alpha])$. To do this, we require the following contraction property:

**Lemma 2.2** (Contraction property of $\mathcal{A}$)**.** *Under the maximum norm, the operator $\mathcal{A}$ : $C([0,\alpha] \mapsto C([0,\alpha])$ - as defined in Lemma 2.1 - is contractive on*

$$\mathfrak{B}(R_\alpha) := \{u \in C([0,\alpha]) \,:\, \|u\|_{L^\infty((0,\alpha))} < R_\alpha\} \ \textit{with} \ R_\alpha := \frac{(\mathcal{K}_0\mathcal{N}_0 + \mathcal{K}_1\mathcal{N}_1)}{\alpha(\mathcal{K}_0 + \alpha\mathcal{K}_1)}$$

.

*Proof.* Observe that

$$\begin{aligned} \|\mathcal{A}(\bar{c_1}) - \mathcal{A}(\bar{c_2})\|_{L^\infty((0,\alpha))} &= \tfrac{1}{2}\tfrac{|(\mathcal{K}_0+\mathcal{K}_1 x)((\bar{c}_1+\bar{c}_2)*(\bar{c}_1-\bar{c}_2))(x)|}{2(\mathcal{K}_0+\mathcal{K}_1 x)\mathcal{N}_0+\mathcal{K}_1\mathcal{N}_1+a(x)} \\ &\le \tfrac{1}{2}\tfrac{(\mathcal{K}_0+\mathcal{K}_1\alpha)\|\bar{c}_1+\bar{c}_2\|_{L^\infty((0,\alpha))}\|\bar{c}_1-\bar{c}_2\|_{L^\infty((0,\alpha))}\alpha}{2\mathcal{K}_0\mathcal{N}_0+\mathcal{K}_1\mathcal{N}_1} \\ &= \tfrac{\alpha(\mathcal{K}_0+\alpha\mathcal{K}_1)}{2(\mathcal{K}_0\mathcal{N}_0+\mathcal{K}_1\mathcal{N}_1)}\|\bar{c_1} - \bar{c_2}\|_{L^\infty((0,\alpha))}\|\bar{c_1} + \bar{c_2}\|_{L^\infty((0,\alpha))}. \end{aligned} \tag{7}$$

It is clear that $\mathcal{A}$ is a mapping from the Banach space $L^\infty((0,\alpha))$ onto itself. Therefore, from (7) we obtain that $\mathcal{A}$ is *contractive* on $M \subset L^\infty((0,\alpha))$ if there exists $\varepsilon > 0$ s.t.

$$\tfrac{\alpha(\mathcal{K}_0+\alpha\mathcal{K}_1)}{(\mathcal{K}_0\mathcal{N}_0+\mathcal{K}_1\mathcal{N}_1)}\|\bar{c}\| \le 1 - \varepsilon$$

is satisfied for all $c \in M$. This holds by construction for $M = \mathfrak{B}(R_\alpha)$ for sufficiently small $\alpha > 0$. □

Now, we show the *invariance property* of the ball $\mathfrak{B}(R_\alpha)$ with respect to the operator $\mathcal{A}$.

**Lemma 2.3** (Self mapping property of $\mathcal{A}$ on $\mathfrak{B}(R_\alpha)$)**.** *There exists $\alpha \in \mathbb{R}_{>0}$ s.t. $\mathcal{A}$ is a self mapping on $\mathfrak{B}(R_\alpha)$, i.e. $\|\mathcal{A}(\bar{c})\|_{L^\infty((0,\alpha))} < R_\alpha$ for all $\bar{c} \in \mathfrak{B}(R_\alpha)$.*

*Proof.* We see that

$$\begin{aligned}\|\mathcal{A}(\bar{c})\|_{L^\infty((0,\alpha))} &= \sup_{x\in(0,\alpha)} \left| \tfrac{1}{2} \tfrac{(\mathcal{K}_0+\mathcal{K}_1 x)(\bar{c}*\bar{c})(x)+2q(x)}{(\mathcal{K}_0+\mathcal{K}_1 x)\mathcal{N}_0+\mathcal{K}_1\mathcal{N}_1+a(x)} \right| \\ &\le \tfrac{1}{2} \tfrac{(\mathcal{K}_0+\mathcal{K}_1\alpha)\alpha\|\bar{c}\|^2_{L^\infty((0,\alpha))}+2\|q\|_{L^\infty((0,\alpha))}}{\mathcal{K}_0\mathcal{N}_0+\mathcal{K}_1\mathcal{N}_1}.\end{aligned}$$

As by construction $\mathcal{A}(c)$ is continuous, $\mathcal{A}$ is a self mapping on $\mathfrak{B}(R_\alpha)$ for $\alpha \in \mathbb{R}_{>0}$ if

$$\tfrac{1}{2} \tfrac{\alpha(\mathcal{K}_0+\mathcal{K}_1\alpha)\|\bar{c}\|^2_{L^\infty((0,\alpha))}+2\|q\|_{L^\infty((0,\alpha))}}{\mathcal{K}_0\mathcal{N}_0+\mathcal{K}_1\mathcal{N}_1} \le \tfrac{(\mathcal{K}_0\mathcal{N}_0+\mathcal{K}_1\mathcal{N}_1)}{\alpha(\mathcal{K}_0+\alpha\mathcal{K}_1)},$$

which, as $\bar{c} \in \mathfrak{B}(R_\alpha)$, is equivalent to

$$\alpha\big(\mathcal{K}_0 + \mathcal{K}_1\alpha\big)\|q\|_{L^\infty((0,\alpha))} \le \tfrac{1}{2}\big(\mathcal{K}_0\mathcal{N}_0 + \mathcal{K}_1\mathcal{N}_1\big)^2. \tag{8}$$

Therefore, the ball $\mathfrak{B}(R_\alpha)$ remains invariant with respect to $\mathcal{A}$, i.e., $\|\mathcal{A}(\bar{c})\|_{L^\infty((0,\alpha))} < R_\alpha$ for all $\bar{c} \in \mathfrak{B}(R_\alpha)$, whenever Equation (8) is satisfied for $\alpha$ positive but sufficiently small. Through estimates, we obtain that Equation (8) is satisfied for

$$\alpha \in \big(0\,,\,\min\big\{1\,,\,\tfrac{\mathcal{K}_0\mathcal{N}_0+\mathcal{K}_1\mathcal{N}_1}{(\mathcal{K}_0+\mathcal{K}_1)\|q\|_{L^\infty((0,1))}}\big\}\big].$$

□

Using the Banach fixed-point theorem [27, Theorem 1.A], we can now show existence of a unique solution for Equation (2).

**Lemma 2.4** (Existence of a unique solution on a small time horizon)**.** *Let $\alpha \in \mathbb{R}_{>0}$ be chosen s.t. $\mathcal{A}$ is a self mapping on $\mathfrak{B}(R_\alpha)$ (see Lemma 2.3). Then, there exists a unique solution $\bar{c} \in \mathfrak{B}(R_\alpha)$ for Equation* (2)*.*

*Proof.* By Lemma 2.2 and Lemma 2.3, the existence and uniqueness of the solution $\bar{c} \in C([0,\alpha])$ to Equation (6) is a direct consequence of the Banach fixed-point theorem [27, Theorem 1.A]. □

The latter result can be extended as follows:

**Lemma 2.5** (Existence of a unique solution on a semi-infinite time horizon)**.** *There exists a unique solution to Equation* (2) *on $\mathbb{R}_{>0}$.*

*Proof.* For the value of $\alpha$ that satisfies the restrictions in Lemma 2.4, Equation (6) has been shown to have a unique solution $\bar{c}(x)$ on $[0,\alpha]$ in Lemma 2.4. To show that Equation (6) admits a unique solution on $\mathbb{R}_{>0}$, we now iteratively extend the domain of existence by a constant steplength $\alpha$.
Note that for any $x$ in $[\alpha, 2\alpha]$, Equation (6) can be written as:

$$\mathcal{A}(c)(x) = \frac{1}{2}\frac{(\mathcal{K}_0+\mathcal{K}_1 x)\int_0^x c(x-y)c(y)\,\mathrm{d}y + q(x)}{(\mathcal{K}_0+\mathcal{K}_1 x)\mathcal{N}_0+\mathcal{K}_1\mathcal{N}_1+a(x)},$$

with $c(x) = \hat{c}(x)$ for all $x \in (0,\alpha]$ and where $\hat{c}$ is denoted by $Lemma$ 2.4

$$\begin{aligned}
&= \frac{1}{2}\frac{(\mathcal{K}_0+\mathcal{K}_1 x)\Big(\int_0^{x-\alpha} c(x-y)\bar{c}(y)\,\mathrm{d}y + \int_{x-\alpha}^{\alpha}\bar{c}(x-y)\bar{c}(y)\,\mathrm{d}y + \int_\alpha^x \bar{c}(x-y)c(y)\,\mathrm{d}y\Big) + q(x)}{(\mathcal{K}_0+\mathcal{K}_1 x)\mathcal{N}_0+\mathcal{K}_1\mathcal{N}_1+a(x)}\\
&= \frac{(\mathcal{K}_0+\mathcal{K}_1 x)\Big(\int_\alpha^x c(y)\bar{c}(x-y)\,\mathrm{d}y + \frac{1}{2}\int_{x-\alpha}^{\alpha}\bar{c}(x-y)\bar{c}(y)\,\mathrm{d}y\Big) + q(x)}{(\mathcal{K}_0+\mathcal{K}_1 x)\mathcal{N}_0+\mathcal{K}_1\mathcal{N}_1+a(x)}\\
&= \int_\alpha^x c(y)\underbrace{\frac{(\mathcal{K}_0+\mathcal{K}_1 x)\bar{c}(x-y)}{(\mathcal{K}_0+\mathcal{K}_1 x)\mathcal{N}_0+\mathcal{K}_1\mathcal{N}_1+a(x)}}_{:=h(x,y)}\,\mathrm{d}y + \underbrace{\frac{(\mathcal{K}_0+\mathcal{K}_1 x)\frac{1}{2}\int_{x-\alpha}^{\alpha}\bar{c}(x-y)\bar{c}(y)\,\mathrm{d}y + q(x)}{(\mathcal{K}_0+\mathcal{K}_1 x)\mathcal{N}_0+\mathcal{K}_1\mathcal{N}_1+a(x)}}_{:=g(x)}.
\end{aligned}$$

Thus, the solution of the fixed-point problem $\mathcal{A}_1(c)(x) = c(x)$ for $x \in [\alpha, 2\alpha]$ can be written in the following form:

$$c(x) = \int_\alpha^x c(y)h(x,y)\,\mathrm{d}y + g(x). \tag{9}$$

As $h, g$ are continuous functions, by standard methods (see e.g. [8]) this linear Volterra equation of the second kind has a unique solution $\hat{c}(x) \in C([\alpha, 2\alpha])$. The function

$$\tilde{c}(x) = \begin{cases} \bar{c}(x) & \text{if } x \in (0,\alpha] \\ \hat{c}(x) & \text{if } x \in (\alpha, 2\alpha] \end{cases}$$

satisfies Equation (6) for $x \in [0, 2\alpha]$ and is continuous as $\hat{c}(\alpha) = g(\alpha) = \mathcal{A}(\hat{c})(\alpha) = \hat{c}(\alpha)$. We can now analogously extend the solution to the interval $(0, 3\alpha)$, and so on. Hence the result follows. □

## 3 Exponential stability

In this section, we demonstrate convergence of time dependent solutions to the stationary solution for the coagulation equation (1) with if the effluc function is constant, i.e. $a(x) \equiv \mathfrak{a}\ \forall x \in \mathbb{R}$ with $\mathfrak{a} \in \mathbb{R}_{>0}$, assumption 1.4 is satisfied and the following assumption holds:

**Assumption 3.1** (Condition on source term)**.** *The source term in* (1) *is a non-negative continuous function on* $\mathbb{R}_{\geq 0}$*, i.e.* $q \in C(\mathbb{R}_{\geq 0}; \mathbb{R}_{\geq 0})$.

With the help of this assumption we can derive the following estimates for the zeroth and first moment of the instationary solution:

**Corollary 3.2** (Estimates of $\mathcal{M}_0$ and exponential stability of $\mathcal{M}_1$)**.** *The function* $\mathcal{M}_0$ *as defined in* (5) *satisfies*

$$\tfrac{d}{dt}\mathcal{M}_0(t) = -\tfrac{1}{2}\mathcal{K}_0\mathcal{M}_0^2(t) - \mathcal{K}_1\mathcal{M}_0(t)\mathcal{M}_1(t) + \mathcal{Q}_0 - \mathfrak{a}\mathcal{M}_0(t) \tag{10}$$

*and can be estimated as follows*

$$\mathcal{M}_0(t) \leq \exp(-\mathfrak{a}t)\mathcal{M}_0(0) + (1-\exp(-\mathfrak{a}t))\tfrac{\mathcal{Q}_0}{\mathfrak{a}} \tag{11}$$

$\forall t \in \mathbb{R}_{\geq 0}$. *Moreover* $\forall t \in \mathbb{R}_{\geq 0}$, $\mathcal{M}_1$ *as defined in* (5) *is denoted by:*

$$\mathcal{M}_1(t) = \exp(-\mathfrak{a}t)\mathcal{M}_1(0) + \big(1-\exp(-\mathfrak{a}t)\big)\tfrac{\mathcal{Q}_1}{\mathfrak{a}}. \tag{12}$$

*Proof.* By integrating (1) over $[0,\infty)$ with respect to $x$, we obtain

$$\begin{aligned}
\tfrac{d}{dt}\mathcal{M}_0(t) &= \tfrac{1}{2}\int_0^\infty\int_0^x \big(\mathcal{K}_0 + x\mathcal{K}_1\big)c(x-y,t)c(y,t)\,\mathrm{d}y\,\mathrm{d}x - \mathfrak{a}\int_0^\infty c(x,t) \\
&\quad - \int_0^\infty\int_0^\infty \big(\mathcal{K}_0 + \mathcal{K}_1(x+y)\big)c(x,t)c(y,t)\,\mathrm{d}y\,\mathrm{d}x + \int_0^\infty q(x)\,\mathrm{d}x\,\mathrm{d}x \\
&= \tfrac{1}{2}\int_0^\infty\int_0^\infty \big(\mathcal{K}_0 + \mathcal{K}_1(x+y)\big)c(x,t)c(y,t)\,\mathrm{d}x\,\mathrm{d}y \\
&\quad - \int_0^\infty\int_0^\infty \big(\mathcal{K}_0 + \mathcal{K}_1(x+y)\big)c(x,t)c(y,t)\,\mathrm{d}y\,\mathrm{d}x + \mathcal{Q} - \mathfrak{a}\mathcal{M}_0(t) \\
&= -\tfrac{1}{2}\big(\mathcal{K}_0\mathcal{M}_0^2(t) + 2\mathcal{K}_1\mathcal{M}_0(t)\mathcal{M}_1(t)\big) + \mathcal{Q}_0 - \mathfrak{a}\mathcal{M}_0(t) \\
&\leq \mathcal{Q}_0 - \mathfrak{a}\mathcal{M}_0(t).
\end{aligned}$$

The first postulated inequality in the corollary is a trivial consequence of the latter formula. The second equality is a direct consequence of the fact that $\tfrac{d}{dt}\mathcal{M}_1(t) = \mathcal{Q}_1 - \mathfrak{a}\mathcal{M}_1(t)$. □

For the difference between the first moment of the stationary solution and the instationary solution, we have the following:

**Corollary 3.3** (Equality for difference between $\mathcal{N}_1$ and $\mathcal{M}_1$)**.** *For the first moments, i.e.* $\mathcal{M}_1, \mathcal{N}_1$ *as defined in* (5)*, we obtain the following estimate:*

$$|\mathcal{M}_1(t) - \mathcal{N}_1| = \exp(-\mathfrak{a}t)|\mathcal{M}_1(0) - \mathcal{N}_1|. \tag{13}$$

*Proof.* As $\mathcal{M}_1(t) = \exp(-\mathfrak{a}t)\mathcal{M}_1(0) + \big(1-\exp(-\mathfrak{a}t)\big)\tfrac{\mathcal{Q}_1}{\mathfrak{a}}$ and $\mathcal{N}_1 = \tfrac{\mathcal{Q}_1}{\mathfrak{a}}$, we obtain $\mathcal{M}_1(t) = \exp(-\mathfrak{a}t)\mathcal{M}_1(0) + \big(1-\exp(-\mathfrak{a}t)\big)\mathcal{N}_1$. Thus, the claim holds. □

The zeroth moment of the stationary solution can be estimated as follows:

**Corollary 3.4** (Equalities for $\mathcal{N}_0$)**.** *The zeroth moment as defined in Equation* (5) *satisfies*

$$\mathcal{N}_0 = \frac{\sqrt{2\mathfrak{a}^2\mathcal{K}_0\mathcal{Q}_0 + (\mathfrak{a}^2 + \mathcal{K}_1\mathcal{Q}_1)^2} - \mathfrak{a}^2 - \mathcal{K}_1\mathcal{Q}_1}{\mathfrak{a}\mathcal{K}_0}, \tag{14}$$

*for* $\mathcal{K}_0 > 0$*, and for* $\mathcal{K}_0 = 0$

$$\mathcal{N}_0 = \frac{\mathfrak{a}\mathcal{Q}_0}{\mathfrak{a}^2 + \mathcal{K}_1\mathcal{Q}_1}. \tag{15}$$

To demonstrate exponential stability, the following assumption is crucial:

**Assumption 3.5** (Implicit condition on moments)**.** *There exists* $T \in \mathbb{R}_{\geq 0}$ *s.t. for all* $t > T$ *the zeroth and first moment as defined in* (5) *satisfy* $\mathcal{M}_0(t) \neq \mathcal{N}_0$ *and*

$$\frac{\mathcal{M}_1(t) - \mathcal{N}_1}{\mathcal{M}_0(t) - \mathcal{N}_0} \geq 0.$$

Without loss of generality, in the following we assume that the latter assumption is satisfied for $T = 0$.

In the following remark, we exemplary state an setting that satisfies this assumption.

**Remark 3.1** (Data satisfying Assumption 3.1)**.** *Let* $\mathfrak{a} = 1, q(x) = \exp(-x),\ c(x,0) = \exp(-x), \mathcal{K}_0 = 1$*, and* $\mathcal{K}_1 = 1$*. By construction* $\mathcal{Q}_0 = 1,\ \mathcal{Q}_1 = 1, \mathcal{N}_0 = \sqrt{6} - 2$ *and* $\mathcal{N}_1 = 1$*, and thus* $\mathcal{M}_1(t) = \exp(-t)\mathcal{M}_1(0) + (1 - \exp(-t)) = 1,\ \mathcal{M}_1(0) = 1$ *and* $\frac{d}{dt}\mathcal{M}_0(t) = -\frac{1}{2}(\mathcal{M}_0^2(t) + 2\mathcal{M}_0(t)) + 1 - \mathcal{M}_0(t),\ \mathcal{M}_0(0) = 1$*. Consequently* $\mathcal{M}_1(t) - \mathcal{N}_1 = 0$ *and* $\mathcal{M}_0(t) = \frac{(5\sqrt{6}-12)\exp(-t\sqrt{2})+\sqrt{6}}{1-(5-2\sqrt{6})\exp(-t\sqrt{2})} - 2$*. This leads to* $\mathcal{M}_0(t) - \mathcal{N}_0 = \frac{2\sqrt{6}}{(5+2\sqrt{6})\exp(t\sqrt{2})-1}$*. Hence,* $\mathcal{M}_0(t) - \mathcal{N}_0 > 0$ *and Assumption 3.5 is satisfied.*

Now, we can show exponential convergence of the zeroth moment of the instationary solution.

**Lemma 3.6** (Exponential convergence of $\mathcal{M}_0$ to $\mathcal{N}_0$)**.** *Let Assumption 3.5 be satisfied, then we obtain*

$$|\mathcal{M}_0(t) - \mathcal{N}_0| \leq |\mathcal{M}_0(0) - \mathcal{N}_0| \exp(-\psi_0(t)), \tag{16}$$

$$\text{where } \psi_0(t) := -t\big(\mathfrak{a} + \tfrac{1}{2}\mathcal{K}_0\mathcal{N}_0\big) - \int_0^t \mathcal{K}_1\mathcal{M}_1(s) + \tfrac{1}{2}\mathcal{K}_0\mathcal{M}_0(s)\,\mathrm{d}s.$$

*Proof.* Integrating (2) on $[0, \infty)$ with respect to $x$, *for an equilibrium solution* we obtain

$$\tfrac{1}{2}\mathcal{K}_0\mathcal{N}_0^2 + \mathcal{K}_1\mathcal{N}_0\mathcal{N}_1 + \mathfrak{a}\mathcal{N}_0 = \mathcal{Q}.$$

Together with (10), this yields

$$
\begin{aligned}
&\tfrac{d}{dt}\big(\mathcal{M}_0(t)-\mathcal{N}_0\big)\\
&= -\tfrac{1}{2}\mathcal{K}_0\mathcal{M}_0^2(t) - \mathcal{K}_1\mathcal{M}_0(t)\mathcal{M}_1(t) + \mathcal{Q} - \mathfrak{a}\mathcal{M}_0(t)\\
&\qquad + \tfrac{1}{2}\mathcal{K}_0\mathcal{N}_0^2 + \mathcal{K}_1\mathcal{N}_0\mathcal{N}_1 - \mathcal{Q} + \mathfrak{a}\mathcal{N}_0\\
&= -\tfrac{1}{2}\mathcal{K}_0\big(\mathcal{M}_0^2(t)-\mathcal{N}_0^2\big) - \mathcal{K}_1\big(\mathcal{M}_0(t)\mathcal{M}_1(t)-\mathcal{N}_0\mathcal{N}_1\big) - \mathfrak{a}(\mathcal{M}_0(t)-\mathcal{N}_0)
\end{aligned}
$$

by the assumption 3.5 $\mathcal{M}_0(t)\neq\mathcal{N}_0$ and we obtain

$$
= \Big(\tfrac{1}{2}\mathcal{K}_0\big(\mathcal{M}_0(t)+\mathcal{N}_0\big) + \mathcal{K}_1\mathcal{M}_1(t) + \mathcal{K}_1\mathcal{N}_0\tfrac{\mathcal{M}_1(t)-\mathcal{N}_1}{\mathcal{M}_0(t)-\mathcal{N}_0} + \mathfrak{a}\Big)\big(\mathcal{N}_0-\mathcal{M}_0(t)\big).
$$

The non negativity assumption on the fraction in the latter formula in Assumption 3.5 concludes the proof.

□

To prove convergence we first define an auxiliary initial value problem for the difference between the stationary and the instationary solutions.

**Lemma 3.7** (Initial value problem for $c-\bar{c}$)**.** *We define* $f(x,t) := c(x,t)-\bar{c}(x)$ *for all* $(x,t)\in\mathbb{R}_{\geq 0}\times\mathbb{R}_{>0}$. *Then* $f$ *satisfies the following initial value problem:*

$$
\begin{aligned}
\partial_t f(x,t) + \mathfrak{a}f(x,t) &= (\mathcal{K}_0+\mathcal{K}_1x)\Big(\tfrac{1}{2}\big(f(\cdot,t)*f(\cdot,t)\big)(x) + \big(f(\cdot,t)*\bar{c}\big)(x)\Big)\\
&\quad - \big(\mathcal{K}_0+\mathcal{K}_1x\big)\Big(\bar{c}(x)\big(\mathcal{M}_0(t)-\mathcal{N}_0\big) + f(x,t)\mathcal{M}_0(t)\Big)\\
&\quad - \mathcal{K}_1\Big(\bar{c}(x)\big(\mathcal{M}_1(t)-\mathcal{N}_1\big) + f(x,t)\mathcal{M}_1(t)\Big) \qquad (17)\\
f(x,0) &= c_0(x)-\bar{c}(x).
\end{aligned}
$$

*Proof.* Equations (1) and (2) result in

$$
\begin{aligned}
\partial_t f(x,t) + \mathfrak{a}f(x,t) &= \tfrac{1}{2}\int_0^x K(x-y,y)\big(c(x-y,t)c(y,t)-\bar{c}(x-y)\bar{c}(y)\big)\,\mathrm{d}y \qquad (18)\\
&\quad + \int_0^\infty K(x,y)\big(\bar{c}(x)\bar{c}(y)-c(x,t)c(y,t)\big)\,\mathrm{d}y. \qquad (19)
\end{aligned}
$$

For the right hand side of (18), we obtain

$$
\begin{aligned}
&\tfrac{1}{2}\int_0^x K(x-y,y)\big(c(x-y,t)c(y,t)-\bar{c}(x-y)\bar{c}(y)\big)\,\mathrm{d}y\\
&\quad = \tfrac{1}{2}(\mathcal{K}_0+\mathcal{K}_1x)\int_0^x \big(c(x-y,t)c(y,t) - \bar{c}(x-y)c(y,t)\\
&\qquad + \bar{c}(x-y)c(y,t) - \bar{c}(x-y)\bar{c}(y)\big)\,\mathrm{d}y.
\end{aligned}
$$

By definition of $f$ this results in

$$
\begin{aligned}
&= \tfrac{1}{2}(\mathcal{K}_0+\mathcal{K}_1x)\int_0^x \big(f(x-y,t)c(y,t) + \bar{c}(x-y)f(y,t)\big)\,\mathrm{d}y\\
&= \tfrac{1}{2}(\mathcal{K}_0+\mathcal{K}_1x)\int_0^x \big(f(x-y,t)c(y,t) - f(x-y,t)\bar{c}(y)\\
&\qquad + f(x-y,t)\bar{c}(y) + \bar{c}(x-y)f(y,t)\big)\,\mathrm{d}y.
\end{aligned}
$$

Again, by definition of $f$ this results in

$$
\begin{aligned}
&= \tfrac{1}{2}(\mathcal{K}_0 + \mathcal{K}_1 x) \int_0^x \big(f(x-y,t)f(y,t) + f(x-y,t)\bar{c}(y) + \bar{c}(x-y)f(y,t)\big)\,\mathrm{d}y \\
&= \tfrac{1}{2}(\mathcal{K}_0 + \mathcal{K}_1 x)\Big(\big(f(\cdot,t) * f(\cdot,t)\big)(y) + 2\big(f(\cdot,t) * \bar{c}\big)(x)\Big).
\end{aligned}
$$

Similarly from (19) we have

$$
\begin{aligned}
&\int_0^\infty K(x,y)\big(\bar{c}(x)\bar{c}(y) - c(x,t)c(y,t)\big)\,\mathrm{d}y \\
&\quad = -\int_0^\infty (\mathcal{K}_0 + \mathcal{K}_1(x+y)) \\
&\qquad\qquad \cdot \big(c(x,t)c(y,t) - \bar{c}(x)c(y,t) + \bar{c}(x)c(y,t) - \bar{c}(x)\bar{c}(y)\big)\,\mathrm{d}y \\
&\quad = -\int_0^\infty (\mathcal{K}_0 + \mathcal{K}_1(x+y))\big(f(x,t)c(y,t) + \bar{c}(x)f(y,t)\big)\,\mathrm{d}y \\
&\quad = -\big(\mathcal{K}_0 + \mathcal{K}_1 x\big) f(x,t) \int_0^\infty c(y,t)\,\mathrm{d}y - \mathcal{K}_1 f(x,y) \int_0^\infty y c(y,t)\,\mathrm{d}y \\
&\qquad - \big(\mathcal{K}_0 + \mathcal{K}_1 x\big)\bar{c}(x) \int_0^\infty c(y,t) - \bar{c}(y)\,\mathrm{d}y - \mathcal{K}_1 \bar{c}(x) \int_0^\infty y\big(c(y,t) - \bar{c}(y)\big)\,\mathrm{d}y.
\end{aligned}
$$

By definition of $\mathcal{M}_0.\mathcal{M}_1, \mathcal{N}_0, \mathcal{N}_1$ in (5), this results in

$$
\begin{aligned}
&= -(\mathcal{K}_0 + \mathcal{K}_1 x) f(x,t)\mathcal{M}_0(t) - \mathcal{K}_1 f(x,t)\mathcal{M}_1(t) \\
&\quad - (\mathcal{K}_0 + \mathcal{K}_1 x)\bar{c}(x)(\mathcal{M}_0(t) - \mathcal{N}_0) - \mathcal{K}_1 \bar{c}(x)(\mathcal{M}_1(t) - \mathcal{N}_1).
\end{aligned}
$$

Thus, for $f$ we obtain the claimed integro-differential equation, which concludes the proof. □

To prove pointwise convergence of the instationary solution to its stationary counterpart, we use the following two lemmata:

**Lemma 3.8.** *For the IVP (1) with the assumptions $a \equiv \mathfrak{a}$, continuous sources function $q$ and Assumption (1.4), the solution $u$ of a linear part of (17) converge uniformly on $[0, m]$, for any $0 < m < \infty$ to zero as $t \to \infty$ with exponential rate, i.e. there exists $\rho \in \mathbb{R}_{>0}$ s.t. $\forall t \in \mathbb{R}_{>0}$ the following holds:*

$$
\|u(\cdot,t)\|_{L^\infty((0,m))} \le \rho \|u(\cdot,0)\|_{L^\infty((0,m))} \exp\left(-t(\mathcal{K}_0\mathcal{N}_0 + \mathcal{K}_1\mathcal{N}_1 + \mathfrak{a})\right),
$$

*Proof.* Equation (17) can be rewritten as

$$
\begin{aligned}
&\partial_t f(x,t) + \mathfrak{a} f(x,t) && (20)\\
&= (\mathcal{K}_0 + \mathcal{K}_1 x)\big(f(\cdot,t) * \bar{c}\big)(x) - (\mathcal{K}_0 + \mathcal{K}_1 x) f(x,t)\mathcal{N}_0 - \mathcal{K}_1 f(x,t)\mathcal{N}_1 && (21)\\
&\quad + (\mathcal{K}_0 + \mathcal{K}_1 x)\tfrac{1}{2}\big(f(\cdot,t) * f(\cdot,t)\big)(x) - \big(\mathcal{K}_0 + \mathcal{K}_1 x\big)\bar{c}(x)\big(\mathcal{M}_0(t) - \mathcal{N}_0\big) && (22)\\
&\quad - \mathcal{K}_1 \bar{c}(x)\big(\mathcal{M}_1(t) - \mathcal{N}_1\big) - (\mathcal{K}_0 + \mathcal{K}_1 x) f(x,t)\big(\mathcal{M}_0(t) - \mathcal{N}_0\big) \\
&\quad - \mathcal{K}_1 f(x,t)\big(\mathcal{M}_1(t) - \mathcal{N}_1\big) && (23)
\end{aligned}
$$

We will first neglect (22) and (23) and investigate (20) and (21). The full equation, i.e. (20) - (23) will be considered in lemma 3.9. Based on the following linear auxiliary problem, i.e. with $u_0 : \mathbb{R} \mapsto \mathbb{R}$

$$\partial_t u(x,t) + \mathfrak{a} u(x,t) = (\mathcal{K}_0 + \mathcal{K}_1 x)\big(u(\cdot,t) * \bar{c}\big)(x) \\ - u(x,t)\big(\mathcal{K}_0\mathcal{N}_0 + x\mathcal{K}_1\mathcal{N}_0 + \mathcal{K}_1\mathcal{N}_1\big) \tag{24}$$

$$u(x,0) = u_0(x) \tag{25}$$

Let $U(\cdot,t), C(\cdot,t), \bar{C}, U_0$ be the Laplace transform w.r.t to the spatial variable of $u(\cdot,t)$, $c(\cdot,t), \bar{c}$ and $u_0$, respectively for all $t \in \mathbb{R}_{\geq 0}$. By taking the Laplace transform of (24) with respect to $x$, we obtain the following partial differential equation:

$$U_t(p,t) + \mathfrak{a} U(p,t) = \mathcal{K}_0 U(p,t)\bar{C}(p) - \mathcal{K}_1\big(U_p(p,t)\bar{C}(p) + U(p,t)\bar{C}_p(p)\big) \\ - \big(\mathcal{K}_0\mathcal{N}_0 + \mathcal{K}_1\mathcal{N}_1\big)U(p,t) + \mathcal{K}_1\mathcal{N}_0 U_p(p,t)$$

$$U(p,0) = U_0(p)$$

for $U$ which is equivalent to

$$U_t(p,t) + \mathcal{K}_1\big(\bar{C}(p) - \mathcal{N}_0\big)U_p(p,t) = \big(\mathcal{K}_0(\bar{C}(p) - \mathcal{N}_0) \\ - \mathcal{K}_1(\bar{C}_p(p) - \mathcal{K}_1) - \mathfrak{a}\big)U(p,t) \tag{26}$$

$$U(p,0) = U_0(p). \tag{27}$$

As $\mathcal{K}_0\mathcal{N}_0 + \mathcal{K}_1\mathcal{N}_1 + \mathfrak{a}$ is independent of $p$, we can substitute $U$ by $W$ defined as

$$W(\cdot,*) :\equiv \exp\left(\psi(*)\right) U(\cdot,*) \qquad \forall t \in \mathbb{R}_{\geq 0}. \tag{28}$$

where $\psi(*) :\equiv \int_0^* (\mathcal{K}_0\mathcal{N}_0 + \mathcal{K}_1\mathcal{N}_1 + \mathfrak{a}\,)ds$ and obtain from (26) and (27) the following initial value problem for $W$:

$$W_t(p,t) + \mathcal{K}_1(\bar{C}(p) - \mathcal{N}_0)W_p(p,t) = (\mathcal{K}_0\bar{C}(p) - \mathcal{K}_1\bar{C}_p(p))W(p,t) \tag{29}$$

$$W(p,0) = U_0(p). \tag{30}$$

The characteristic equation (see e.g. [2, 13, 9]) of (29) is given by

$$\mathrm{d}t = \frac{\mathrm{d}p}{\mathcal{K}_1(\bar{C}(p) - \mathcal{N}_0)} = \frac{\mathrm{d}(W(p,t))}{(\mathcal{K}_0\bar{C}(p) - \mathcal{K}_1\bar{C}_p(p))W(p,t)}. \tag{31}$$

By analysis similar to that in [6] (see (3.10) to (3.12), with $b = 0$) and from (28) we can conclude that there exists $\rho \in \mathbb{R}_{>0}$ s.t. the claim holds. $\square$

**Theorem 3.9** (Exponential rate of convergence from $c$ to $\bar{c}$ ). *For the IVP (1) with the assumptions $a(x) = \mathfrak{a}$, a continuous source function $q$ and Assumption (1.4), further* *suppose that condition (3.5) holds. Then, for any* $0 < m < \infty$ *the solution* $c$ *of (17) uniformly converges to* $\bar{c}$ *on* $[0, m]$ *as* $t \to \infty$ *with exponential rate, i.e. for every* $m \in \mathbb{R}_{>0}$ *there exists* $\gamma \in \mathbb{R}_{>0}$ *s.t.*

$$\|c(\cdot,t) - \bar{c}\|_{L^\infty((0,m))} \leq \gamma \exp(-t\nu)$$

*for all* $t \in \mathbb{R}_{>0}$ *and* $\nu < \mathfrak{a}$.

*Proof.* Here we recall the IVP

$$\begin{aligned}
&\partial_t f(x,t) + \mathfrak{a} f(x,t) \\
&= (\mathcal{K}_0 + \mathcal{K}_1 x)\big(f(\cdot,t) * \bar{c}\big)(x) - (\mathcal{K}_0 + \mathcal{K}_1 x) f(x,t)\mathcal{N}_0 - \mathcal{K}_1 f(x,t)\mathcal{N}_1 \\
&\quad + (\mathcal{K}_0 + \mathcal{K}_1 x)\tfrac{1}{2}\big(f(\cdot,t) * f(\cdot,t)\big)(x) - \big(\mathcal{K}_0 + \mathcal{K}_1 x\big)\bar{c}(x)\big(\mathcal{M}_0(t) - \mathcal{N}_0\big) \\
&\quad - \mathcal{K}_1 \bar{c}(x)\big(\mathcal{M}_1(t) - \mathcal{N}_1\big) - (\mathcal{K}_0 + \mathcal{K}_1 x) f(x,t)\big(\mathcal{M}_0(t) - \mathcal{N}_0\big) \\
&\quad - \mathcal{K}_1 f(x,t)\big(\mathcal{M}_1(t) - \mathcal{N}_1\big).
\end{aligned} \tag{32}$$

Let us denote $u(x,t) = T_t u_0(x)$, where $u(x,t)$ is the solution of equation (24) and $T_t$ is the resulting semigroup operator. From Lemma 3.8 for the usual semigroup norm $\|T_t\| := \sup_{\|u_0\|_C \leq 1} \|T_t u_0\|_{L^\infty((0,m))}$ we obtain the following: There exists $\rho \in \mathbb{R}$ s.t. for $\nu := \frac{\mathfrak{a}}{2}$ the following holds:

$$\|T_t\| \leq \rho \exp(-\nu t), \tag{33}$$

for all $t \in [0, T]$. With the help of the semigroup operator, the nonlinear initial value problem (32) can be written in integral form as (see e.g. [20, p. 110] and [17, Thm 6.5])

$$\begin{aligned}
f(x,t) = T_t f_0 + \textstyle\int_0^t T_{t-s}\Big(&(\mathcal{K}_0 + \mathcal{K}_1 x)\tfrac{1}{2}\big(f(\cdot,s) * f(\cdot,s)\big)(x)\Big) \\
&- \big(\mathcal{K}_0 + \mathcal{K}_1 x\big)\bar{c}(x)\big(\mathcal{M}_0(s) - \mathcal{N}_0\big) \\
&- \mathcal{K}_1 \bar{c}(x)\big(\mathcal{M}_1(s) - \mathcal{N}_1\big) - (\mathcal{K}_0 + \mathcal{K}_1 x) f(x,s)\big(\mathcal{M}_0(s) - \mathcal{N}_0\big) \\
&- \mathcal{K}_1 f(x,s)\big(\mathcal{M}_1(s) - \mathcal{N}_1\big)\, \mathrm{d}s.
\end{aligned} \tag{34}$$

Let the right hand side of (34) be defined as $D(f(\cdot,t))$. Then it can easily be checked that for any fixed $t \geq 0$, $D$ maps $C([0,m])$ onto itself. Expressions (33) and (34) yield

$$\begin{aligned}
&\|D(f(\cdot,t))\|_{L^\infty((0,m))} \\
&\leq \rho \exp(-\nu t)\Big(\|f_0\|_{L^\infty((0,m))} + \textstyle\int_0^t \exp(\nu s)\Big(\tfrac{1}{2}(\mathcal{K}_0 + \mathcal{K}_1 m)\|f(\cdot,s)\|^2_{L^\infty((0,m))} \\
&\quad + \big((\mathcal{K}_0 + \mathcal{K}_1 m)|\mathcal{M}_0(s) - \mathcal{N}_0| + \mathcal{K}_1|\mathcal{M}_1(s) - \mathcal{N}_1|\big)\|f(\cdot,s)\|_{L^\infty((0,m))} \\
&\quad + \big((\mathcal{K}_0 + \mathcal{K}_1 m)|\mathcal{M}_0(s) - \mathcal{N}_0| + \mathcal{K}_1|\mathcal{M}_1(s) - \mathcal{N}_1|\big)\|\bar{c}\|_{L^\infty((0,m))}\Big)\, \mathrm{d}s\Big).
\end{aligned} \tag{35}$$

Multiplying (35) by $\exp(\nu t)$ and taking the supremum over $t \in (0,T)$, we can – with the norm as defined in definition 1.3 – establish the following condition

$$\|D(f)\|_\nu \leq \rho\|f_0\|_{L^\infty((0,m))} + \tfrac{\rho m}{2\nu}(\mathcal{K}_0 + \mathcal{K}_1 m)\|f\|_\nu^2 + \rho a_1 + a_2\|f\|_\nu,$$

where

$$\begin{aligned} a_1 &:= \big\|(\mathcal{K}_0 + \mathcal{K}_1 \mathrm{Id}(\cdot))|\bar{c}(\cdot)|\big\|_{L^\infty((0,m))} \textstyle\int_0^\infty \exp(\nu s)|\mathcal{M}_0(s) - \mathcal{N}_0|\,\mathrm{d}s \\ &\quad + \|\bar{c}\|_{L^\infty((0,m))} \textstyle\int_0^\infty \exp(\nu s)|\mathcal{M}_1(s) - \mathcal{N}_1|\,\mathrm{d}s \\ a_2 &:= (\mathcal{K}_0 + \mathcal{K}_1 m) \textstyle\int_0^\infty |\mathcal{M}_0(s) - \mathcal{N}_0|\,\mathrm{d}s + \mathcal{K}_1 \int_0^\infty |\mathcal{M}_1(s) - \mathcal{N}_1|\,\mathrm{d}s. \end{aligned} \tag{36}$$

From (35) it is possible to show that if

$$\|f_0\|_{L^\infty((0,m))} + a_1 \leq \tfrac{\nu(1-a_2)^2}{2\rho^2 m(\mathcal{K}_0+\mathcal{K}_1 m)} \quad \text{and} \quad a_2 < 1, \tag{37}$$

then the mapping $D$ has an invariant ball in $C([0,m])$ with radius $\eta$ satisfying $\eta_1 \leq \eta \leq \eta_2$ where $\eta_1$ and $\eta_2$ are the real positive roots of the quadratic equation

$$\tfrac{\rho m}{2\nu}(\mathcal{K}_0 + \mathcal{K}_1 m)z^2 - (1 - a_2)z + \rho(\|f_0\|_C + a_1) = 0. \tag{38}$$

In fact, if $\|f\|_\nu \leq \eta$ for some $\eta \in [\eta_1, \eta_2]$, then from (36) we obtain

$$\|D(f)\|_\nu \leq \rho\|f_0\|_C + \tfrac{\rho m}{2\nu}(\alpha + \delta m)\eta^2 + \rho a_1 + a_2\eta \leq \eta, \tag{39}$$

which follows from the fact that $\eta_1 \leq \eta_2$ and conditions (37) hold. We will now derive a conditions for $D$ to be a contraction on $C([0,m])$. For any $f_1$ and $f_2$, it follows from (33) and (34) that

$$\|D(f_1) - D(f_2)\|_{L^\infty((0,m))} \tag{40}$$

$$\begin{aligned} &\leq \tfrac{1}{2}\rho(\mathcal{K}_0 + \mathcal{K}_1 m) \textstyle\int_0^t \exp(-\nu(t-s))\|(f_1 - f_2) * (f_1 + f_2)\|_{L^\infty((0,m))}\,\mathrm{d}s \\ &\quad + (\mathcal{K}_0 + \mathcal{K}_1 m) \textstyle\int_0^t \exp(-\nu(t-s))|\mathcal{M}_0(s) - \mathcal{N}_0|\|f_1 - f_2\|_{L^\infty((0,m))}\,\mathrm{d}s \\ &\quad + \mathcal{K}_1 \textstyle\int_0^t \exp(-\nu(t-s))|\mathcal{M}_1(s) - \mathcal{N}_1|\|f_1 - f_2\|_{L^\infty((0,m))}\,\mathrm{d}s \\ &\leq \tfrac{\rho m}{2\nu}(\mathcal{K}_0 + \mathcal{K}_1 m)\exp(-\nu t)\|f_1 - f_2\|_\nu(\|f_1\|_\nu + \|f_2\|_\nu) \\ &\quad + \exp(-\nu t)a_2\|f_1 - f_2\|_\nu. \end{aligned} \tag{41}$$

If the functions $f_1$ and $f_2$ belong to a ball with radius $\eta$, that is $\|f_1\| \leq \eta$ and $\|f_2\| \leq \eta$, then from (39) we obtain

$$\|D(f_1) - D(f_2)\|_\nu \leq \left(\tfrac{\rho m}{\nu}(\mathcal{K}_0 + \mathcal{K}_1 m) + a_2\right)\|f_1 - f_2\|_\nu. \tag{42}$$

Thus the mapping $D$ is a contraction mapping in the ball with radius

$$\eta < \frac{(1 - a_2)}{m\rho(\mathcal{K}_0 + \mathcal{K}_1 m)}. \tag{43}$$

From equation (38), $\eta_1$ and $\eta_2$ are given by

$$\eta_{1,2} = \tfrac{(1-a_2)\nu}{\rho m(\mathcal{K}_0+\mathcal{K}_1 m)} \left(1 \pm \sqrt{1 - \tfrac{2\rho^2 m(\alpha+\delta m)(\|f_0\|_C + a_1)}{\nu(1-a_2)^2}}\right) \tag{44}$$

and hence the bound of contraction belongs to the closed interval $[\eta_1, \eta_2]$. From Banach's fixed-point theorem [27, Theorem 1.a], we see that there exists a solution of IVP (32) that is unique in the ball of radius $\|f\|_\nu \leq \eta_0$ and belongs to the ball of radius $\|f\|_\nu \leq \eta_1 < \eta_0$. Moreover, this solution tends to zero no slower than $\exp(-\nu t)$. □

The latter result can be extended to exponential convergence in $L^1((0,\infty))$ as follows:

**Lemma 3.10** ( Convergence in $L^1(0,\infty)$)**.** *Consider the problem (1) with $a(x) = \mathfrak{a}$ and continuous source function $q$. Further, suppose that Assumption (3.5) is satisfied. Then, the solution of the problem (1) converges in $L^1((0,\infty))$ to the equilibrium solution as $t \to \infty$ in $C([a,b])$ for all $0 \leq a < b < \infty$. And the rate of convergence is proportional to* $\exp(-\nu t)$.

*Proof.* In order to prove convergence in the space $L^1((0,\infty))$, using Theorem 3.9, we note that

$$\begin{aligned} \int_0^\infty |c(x,t) - \bar{c}(x)|\,\mathrm{d}x &= \int_0^m |c(x,t) - \bar{c}(x)|\,\mathrm{d}x + \int_m^\infty |c(x,t) - \bar{c}(x)|\,\mathrm{d}x \\ &\leq \eta_0 \exp(-\nu t) m + \int_m^\infty c(x,t) + \bar{c}(x)\ \mathrm{d}x. \end{aligned} \tag{45}$$

The latter integral can be estimated as follows:

$$\int_m^\infty c(t,x) + \bar{c}(x)\,\mathrm{d}x \leq \tfrac{1}{m}\int_m^\infty x\big(c(t,x) + \bar{c}(x)\big)\,\mathrm{d}x \leq \tfrac{\hat{\mathcal{M}}_1 + \mathcal{N}_1}{m}, \tag{46}$$

with $\hat{\mathcal{M}}_1 \in \mathbb{R}$ s.t. $\mathcal{M}_1(t) \leq \hat{\mathcal{M}}_1\ \forall t$ (see (13)). From (45), we thus have the following

$$\int_0^\infty |c(x,t) - \bar{c}(x)|\,\mathrm{d}x \leq \eta_0 \exp(-\nu t) m + \tfrac{\hat{\mathcal{M}}_1 + \mathcal{N}_1}{m}. \tag{47}$$

Therefore, for every $\varepsilon \in \mathbb{R}_{>0}$ choose $m_\varepsilon = \frac{2(\hat{\mathcal{M}}_1 + \mathcal{N}_1)}{\varepsilon}$ and $t_\varepsilon \in \mathbb{R}_{>0}$ sufficiently large s.t. $\exp(-\nu t_\varepsilon) \leq \frac{\varepsilon}{2\eta_0 m_\varepsilon}$. Then $\|c(\cdot,t) - \bar{c}\|_{L^\infty(\mathbb{R}_{>0})} \leq \varepsilon\ \forall t > t_\varepsilon$ and as $\varepsilon$ was arbitrarily chosen we have the claimed convergence. □

## 4 A numerical example

This section provides a numerical example pertaining to the analysis conducted in the previous section.

We consider the problem (1) with the coagulation kernel $K(x,y) = x + y$, i.e. $\mathcal{K}_0 = 0, \mathcal{K}_1 = 1$, the efflux term $a \equiv 1$, the source term $q(x) = \exp(-x)\ \forall x \in \mathbb{R}_{>0}$ and the initial condition $c_0(x) = \exp(-x)\ \forall x \in \mathbb{R}_{>0}$.

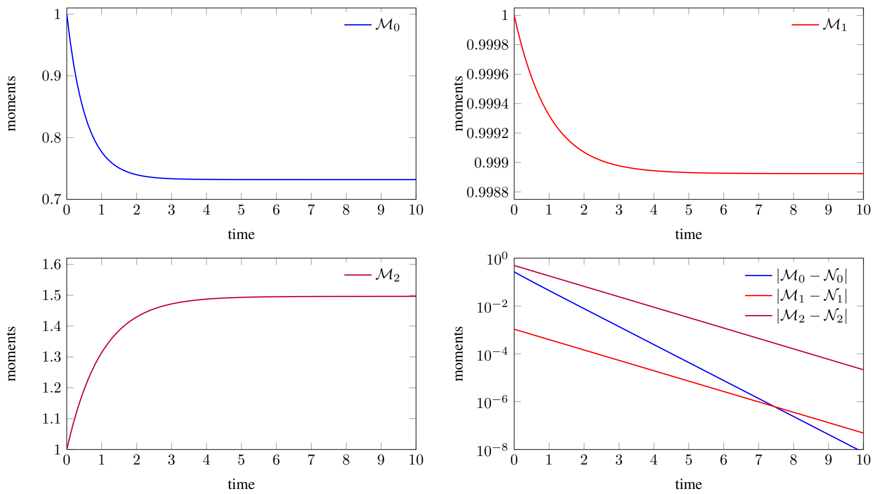


**Fig. 1.** Zeroth (**top left**), First (**top right**) and Second (**bottom left**) order moment of the time dependent solution $c(\cdot, t)$ of Equation $(1)$ for $t \in [0, 10]$. In the plot **bottom right**, the difference of the moments to their respective equilibrium counterpart is shown.

The finite volume scheme of [16] is applied to obtain numerical results with a high degree of accuracy. The computational domain under consideration is $[10^{-6}, 800]$ and it is discretized into 200 non-uniform subintervals

$$\Lambda_i := [x_{i-1/2}, x_{i+1/2}] \qquad i = 1, 2, \ldots, 200.$$

The end points of $\Lambda_i$ satisfy $x_{i+1/2} = r x_{i-1/2}$, where $r > 1$ is the geometric ratio. The mid-point of each $\Lambda_i$ is considered to be the *cell representative* or the *pivot*. The system of ODEs is solved using the adaptive Runge-Kutta 4(5) solver in MATLAB.

The number density function $c$, and the moments $\mathcal{M}_0$, $\mathcal{M}_1$ and $\mathcal{M}_2$ are calculated to observe the equilibrium of the system. An almost constant value of $\mathcal{M}_0$ after a certain time indicates that the total number of particles in the system remains approximately constant. Further, constant values of $\mathcal{M}_1$ and $\mathcal{M}_2$ after a certain amount of time support the conclusion that the system has achieved the equilibrium. The graphs of the functions $\mathcal{M}_0$, $\mathcal{M}_1$ and $\mathcal{M}_2$ (see Figure 1) are discussed in the following.

In the upper left plot in Figure 1, the *zeroth order moment* of $c$, i.e. $\mathcal{M}_0$, is shown with respect to time. From the figure we observe that after a short time, the zeroth moment remains almost constant. This can also be seen in the lower right plot of Figure 1, which shows the convergence of $\mathcal{M}_0$ towards its stationary counterpart $\mathcal{N}$. The blue curve clearly depicts an exponential convergence. This suggests that the total number of particles in the system approaches a constant value at exponential rate. In the upper right plot in Figure 1 the *first order moment* of $c$, i.e. $\mathcal{M}_1$, is shown with respect to time.

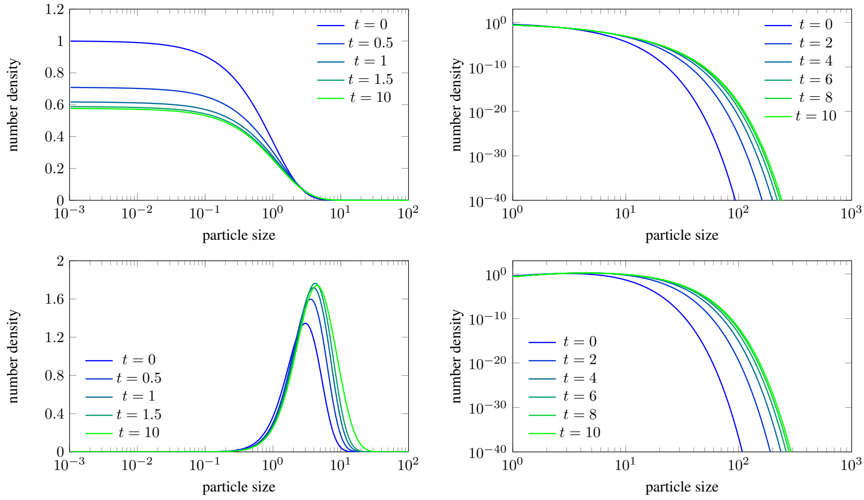


**Fig. 2.** Time dependent solution $c(x, t)$ at various time steps (**top row**) and their respective volume weighted distribution $x^3 c(x, t)$ (**bottom row**)

From the figure we observe that the mass of the system appears to be almost constant after a short time. Again, the lower right plot of the same figure depicts an exponential convergence of $\mathcal{M}_1$ to $\mathcal{N}_1$.

Finally, in the lower left plot of Figure 1 the *second order moment* of $c$, i.e. $\mathcal{M}_2$, is plotted and, again, convergence to $\mathcal{N}_2$ can be observed. This is further validated in the lower right plot of the same figure.

In the upper row of Figure 2, the density function $c$ is shown at various times $t$ over the particle size $x$. A clear trend of $c(t, \cdot)$ towards its stationary counterpart $\bar{c} \approx c(10, \cdot)$ can be observed. In the lower row of the figure, the same is shown for the volume weighted distribution i.e. $x \mapsto x^3 c(t, x)$ for various $t$.

From fig. 1 and fig. 2, it is evident that all moments are almost constant after $t = 10$. Moreover, the solution $c$ itself seems to have converged and thus the system has almost reached equilibrium after $t = 10$.

## 5 Conclusion

This article has examined the existence of stationary solutions and the convergence of instationary solutions to their respective stationary solution. The entire analysis was performed with the help of the Banach contraction mapping theorem, Laplace transforms and the method of variation of constants. The convergence proved in the earlier sections of the article is demonstrated by a numerical example. Explicit formulas for stationary solutions remain a subject of further research.

In terms of future research, the discussed coagulation kernel will be generalized [12] and kinetics both independent of growth size and size dependent growth kinetics (see e.g. [3] for a modeling perspective and [15] for a theoretical perspective) will be added to the model equation. Studying the asymptotic behaviour of these cases in addition to the case presented here will provide further insights into the system's behaviour.

**Acknowledgment:** L. Pflug is grateful for funding provided by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – Project-ID 416229255 – SFB 1411.

## References

[1] D Chae and PB Dubovskiĭ. Existence and uniqueness for spatially inhomogeneous coagulation equation with sources and effluxes. *Zeitschrift für Angewandte Mathematik und Physik*, 46(4):580–594, 1995.
[2] M Delgado. Classroom note: The lagrange–charpit method. *SIAM Review*, 39(2):298–304, 1997.
[3] JA Dirksen and TA Ring. Fundamentals of crystallization: Kinetic effects on particle size distributions and morphology. *Chemical Engineering Science*, 46(10):2389 – 2427, 1991.
[4] PB Dubovskiĭ. *Mathematical theory of coagulation*. Seoul National University, Research Institute of Mathematics, 1994.
[5] PB Dubovskiĭ and IW Stewart. Existence, uniqueness and mass conservation for the coagulation-fragmentation equation. *Mathematical Methods in the Applied Sciences*, 19(7):571–591, 1996.
[6] PB Dubovskiĭ and IW Stewart. Trend to equilibrium for the coagulation–fragmentation equation. *Mathematical Methods in Applied Sciences*, 19(10):761–772, 1996.
[7] CP Dullemond and C Dominik. Dust coagulation in protoplanetary disks: A rapid depletion of small grains. *Astronomy & Astrophysics*, 434(3):971–986, 2005.
[8] GC Evans. Volterra's integral equation of the second kind, with discontinuous kernel, second paper. *Transactions of the American Mathematical Society*, 12(4):429–472, 1911.
[9] LC Evans. *Partial Differential Equations*. American Mathematical Society, 1997.
[10] SK Friedlander. *Smoke, dust, and haze*, volume 198. Oxford University Press New York, 2000.
[11] H Gajewski. On a first order partial differential equation with nonlocal nonlinearity. *Mathematische Nachrichten*, 111(1):289–300, 1983.
[12] D Ghosh and J Kumar. Existence of mass conserving solution for the coagulation–fragmentation equation with singular kernel. *Japan Journal of Industrial and Applied Mathematics*, 35(3):1283–1302, 2018.
[13] F John. *Partial Differential Equations*. Springer US, 1978.
[14] PC Kapur. Kinetics of granulation by non-random coalescence mechanism. *Chemical Engineering Science*, 27(10):1863–1869, 1972.
[15] A Keimer and L Pflug. Existence, uniqueness and regularity results on nonlocal balance laws. *Journal of Differential Equations*, 263(7):4023 – 4069, 2017.

[16] J Kumar, G Kaur, and E Tsotsas. An accurate and efficient discrete formulation of aggregation population balance equation. *Kinetic and Related Models*, 9(2):373–391, 2016.

[17] DJ McLaughlin, W Lamb, and AC McBride. A semigroup approach to fragmentation models. *SIAM Journal on Mathematical Analysis*, 28(5):1158–1172, 1997.

[18] KW Min and WH Ray. On the mathematical modeling of emulsion polymerization reactors. *Journal of Macromolecular Science, Part C*, 11(2):177–255, 1974.

[19] H Muller. To the general theory of rapid coagulation. *Progress Reports on Colloids and Polymers*, 27(6):223–250, 1928.

[20] A Pazy. *Semigroups of linear operators and applications to partial differential equations*, volume 44. Springer Science & Business Media, 2012.

[21] D Ramkrishna. *Population balances: Theory and applications to particulate systems in engineering*. Elsevier, 2000.

[22] JH Seinfeld and SN Pandis. *Atmospheric chemistry and physics: from air pollution to climate change*. John Wiley & Sons, 2016.

[23] K Shiloh, S Sideman, and W Resnick. Coalescence and break-up in dilute polydispersions. *The Canadian Journal of Chemical Engineering*, 51(5):542–549, 1973.

[24] MV Smoluchowski. An experiment on mathematical theorization of coagulation kinetics of the colloidal solutions. *Zeitschrift für Physikalisch Chemie*, 92:129–168, 1917.

[25] MV Smoluchowski. Versuch einer mathematischen theorie der koagulationskinetik kolloider lösungen. *Zeitschrift für Physikalische Chemie*, 92(1):129–168, 1918.

[26] JL Spouge. An existence theorem for the discrete coagulation-fragmentation equations. *Mathematical Proceedings of the Cambridge Philosophical Society*, 96(2):351–357, 1984.

[27] E Zeidler. *Nonlinear Functional Analysis, I. Fixed-Point Theorems*. Springer-Verlag, New York, 1986.