Traveling waves for a nonlocal dispersal SIR epidemic model with the mass action infection mechanism☆

Xin Wu[a], Zhaohai Ma[b]*

[a] *School of Sciences, East China Jiao Tong University, Nanchang, 330013, People's Republic of China*

[b] *School of Science, China University of Geosciences, Beijing 100083, People's Republic of China*

Abstract

This paper is concerned with a nonlocal dispersal susceptible–infected–recovered (SIR) epidemic model adopted with the mass action infection mechanism. We mainly study the existence and non-existence of traveling waves connecting the infection-free equilibrium state and the endemic equilibrium state. The main difficulties lie in the fact that the semiflow generated here does not admit the order-preserving property. Meanwhile, this new model brings some new challenges due to the unboundedness of the nonlinear term. We overcome these difficulties to obtain the boundedness of traveling waves with the speed $c > c_{\min}$ by some analysis techniques firstly and then prove the existence of traveling waves by employing Lyapunov–LaSalle theorem and Lebesgue dominated convergence theorem. By utilizing a approximating method, we study the existence of traveling waves with the critical wave speed $c_{\min}$. Our results on this new model may provide some implications on disease modelling and controls.

Keywords: Mass action infection mechanism; SIR model; Nonlocal dispersal; Traveling waves; Lyapunov–LaSalle theorem
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1. Introduction

In this paper, we investigate the following nonlocal dispersal SIR epidemic model with the mass action infection mechanism:

$$\frac{\partial}{\partial t}S(x,t) = d_S(J_1 * S(x,t) - S(x,t)) + \Lambda - \mu S(x,t) - \beta S(x,t)I(x,t), \tag{1.1}$$

$$\frac{\partial}{\partial t}I(x,t) = d_I(J_2 * I(x,t) - I(x,t)) - \mu I(x,t) + \beta S(x,t)I(x,t) - \gamma I(x,t), \tag{1.2}$$

$$\frac{\partial}{\partial t}R(x,t) = d_R(J_3 * R(x,t) - R(x,t)) + \gamma I(x,t) - \mu R(x,t), \tag{1.3}$$

where $x \in \mathbb{R}, t > 0$ and $d_S,\ d_I,\ d_R,\ \Lambda,\ \beta,\ \mu,\ \gamma$ are the positive constants. Here $S(x,t),\ I(x,t)$ and $R(x,t)$ stand for the densities of susceptible, infective and removed individuals at position x and time t, respectively. The parameters $d_S,\ d_I,\ d_R$ describe the spatial motility of each class; the constant $\Lambda > 0$ represents the entering flux of the susceptible; $\gamma > 0$ is the recovery

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*Corresponding author
Email address: zhaohaima@cugb.edu.cn (Zhaohai Ma[b])

rate of the infective population; μ is a positive parameter representing the death rates for all the susceptible, the infective, and the removed population. Moreover, $J_i(y)(i=1,2,3)$ denotes the probability distribution of dispersal over distance y, $J_1 * S(x,t) = \int_{\mathbb{R}} J_1(x-y)S(y,t)\mathrm{d}y$ is the rate at which the susceptible individuals are arriving at position x from all other places and $-S(x,t) = \int_{\mathbb{R}} J_1(x-y)S(x,t)\mathrm{d}y$ is the rate at which they are leaving location x to travel to all other sites. Thus, $J_1 * S(x,t) - S(x,t)$ can be interpreted as that the rate of susceptible individuals at position x at time t depends on the influence of neighboring $S(x,t)$ at all other positions y. Simultaneously, $J_2 * I(x,t) - I(x,t)$ and $J_3 * R(x,t) - R(x,t)$ describe that the rates of infected and removed individuals at position x at time t depend on the influence of neighboring $I(x,t)$ and $R(x,t)$ at all other positions y, respectively. Meanwhile, throughout this paper, we give the following assumptions on the kernel functions J_i:

(J) $J_i \in C^1(\mathbb{R}),\ J_i(y) = J_i(-y) \geq 0,\ \int_{\mathbb{R}} J_i(y)\mathrm{d}y = 1$ and J_i is compactly supported, $i = 1,2,3$.

In the study of population dynamics and disease propagation, the reaction-diffusion equations are often used to describe biological or physical evolution process. Since traveling wave solutions can successfully predict several diseases spread, for example, see [2], the investigations on traveling wave solutions for epidemic models with the random diffusion are attracting more and more attention, see [4, 15, 19]. Nevertheless, these disease models may underestimate speeds of disease propagation. Lutscher et al. [25] introduced integral operators instead of Laplacian operators to solve such problems. Also, the nonlocal dispersal is better described as a long range process rather than as a local one in many situations such as in population ecology, materials science, phase transition, genetics, neurology and epidemiology. We can refer to [6, 8, 9, 42, 43, 44] for more results on traveling wave solutions of the nonlocal dispersal problems. In modelling disease dynamics, an infection mechanism needs to be assumed and adopted. There are various infection mechanisms, such as the mass action βSI, see [26, 31], standard incidence mechanisms $\beta SI/N$, see [14] and saturated incidence mechanisms $\frac{\beta IS}{1+\zeta I}$, see [5, 15, 16, 39, 40], where $\beta,\ \zeta$ are positive constants. Here, in this paper, we mainly consider the mass action infection mechanisms. We can refer to [1, 3, 18, 20, 23, 24, 30, 41] for more details about the other infection mechanisms and the nonlinear incidence rates.

Hosono and Ilyas [17] considered the Kermack-McKendrick equations

$$\frac{\partial}{\partial t}S(x,t) = d_1 \frac{\partial^2}{\partial x^2}S(x,t) - \beta S(x,t)I(x,t), \tag{1.4}$$

$$\frac{\partial}{\partial t}I(x,t) = d_2 \frac{\partial^2}{\partial x^2}I(x,t) + \beta S(x,t)I(x,t) - \gamma I(x,t), \tag{1.5}$$

$$\frac{\partial}{\partial t}R(x,t) = d_3 \frac{\partial^2}{\partial x^2}R(x,t) + \gamma I(x,t), \tag{1.6}$$

where $x \in \mathbb{R}, t > 0$ and d_i is the rate of diffusion of each sub-population, $i = 1,2,3$. By using the Schauder fixed point theorem, they proved the existence of traveling wave solution of this system when $d_1 = 1$. Moreover, they verified that for $\beta S_0 > \gamma$ and $c > c_{\min} := 2\sqrt{d_2(\beta S_0 - \gamma)}$, the system (1.4)–(1.5) admits a traveling wave solution $(S(x+ct), I(x+ct))$ satisfying the boundary conditions $S(+\infty) = S_\infty$, $S(-\infty) = S_0$, $S_0 > S_\infty$ and $I(\pm\infty) = 0$. More works, we refer to [5, 27, 32, 34, 35, 44].

Recently, Chen et al. [7] considered the following lattice dynamical system

$$\frac{dS_n}{dt} = (S_{n+1} - 2S_n + S_{n-1}) + \mu - \mu S_n - \beta S_n I_n,\ n \in \mathbb{Z}, \tag{1.7}$$

$$\frac{dI_n}{dt} = (I_{n+1} - 2I_n + I_{n-1}) - \mu I_n + \beta S_n I_n - \gamma I_n,\ n \in \mathbb{Z}. \tag{1.8}$$

They proved that for $c \geq c_{\min}$, there exists a bounded traveling wave solution $(S(x+ct), I(x+ct))$ of the system (1.7)–(1.8) such that $0 < S(\cdot) < 1$ and $I(\cdot) > 0$ in $\mathbb{R}$. Moreover, it holds

$$0 < \liminf_{\xi\to+\infty} S(\xi) \leq s^* \leq \limsup_{\xi\to+\infty} S(\xi) < 1 \text{ and } 0 < \liminf_{\xi\to+\infty} I(\xi) \leq i^* \leq \limsup_{\xi\to+\infty} I(\xi) < +\infty, \tag{1.9}$$

where (s^*, i^*) the endemic equilibrium state. On the other hand, if $c < c_{\min}$, there exists no bounded traveling wave solution $(S(x+ct), I(x+ct))$ of the system (1.7)–(1.8). For more discrete epidemic model, we refer to [12, 36].

Li et al. [22] investigated a nonlocal dispersal delayed SIR model with constant external supplies and Holling-II incidence rate

$$\frac{\partial}{\partial t}S(x,t) = d_1(J * S(x,t) - S(x,t)) + B - \sigma S(x,t) - \frac{\beta S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)}, \tag{1.10}$$

$$\frac{\partial}{\partial t}I(x,t) = d_2(J * I(x,t) - I(x,t)) + \frac{\beta S(x,t)I(x,t-\tau)}{1+\alpha I(x,t-\tau)} - (\mu+\gamma)I(x,t), \tag{1.11}$$

$$\frac{\partial}{\partial t}R(x,t) = d_3(J * R(x,t) - R(x,t)) + \gamma I(x,t) - \mu R(x,t), \tag{1.12}$$

where $x \in \mathbb{R}, t > 0$ and $\tau > 0$ is the time delay. By using Schauder fixed point theorem with upper-lower solutions and employing the Lyapunov method, they obtained the existence of traveling waves of the subsystem (1.10)–(1.11) for $\beta > \mu + \gamma$ and $c > c_{\min}$. Meanwhile, when $c = c_{\min}$, by a limiting approach, the existence of traveling waves with exact asymptotic boundary behavior is obtained. The non-existence of traveling wave solution for the subsystem (1.10)–(1.11) is also established when $0 < c < c_{\min}$ or $\beta \leq \mu + \gamma$. More studies on SIR model of the same structure as (1.10)–(1.12), we refer to [11, 22, 38].

Since equation (1.3) is decoupled with equations (1.1)–(1.2), we only consider subsystem (1.1) and (1.2). Note that the corresponding ordinary differential equations of subsystem (1.1)–(1.2) always has a disease-free equilibrium $E^0 = (\Lambda/\mu, 0)$ and a unique endemic equilibrium $E^* = (s^*, i^*)$ if the basic reproduction number $\mathfrak{R}_0 = \frac{\beta\Lambda}{\mu(\mu+\gamma)} > 1$, where

$$s^* = \frac{\mu+\gamma}{\beta} \quad \text{and} \quad i^* = \frac{\beta\Lambda - \mu(\mu+\gamma)}{\beta(\mu+\gamma)}.$$

A traveling wave solution of subsystem (1.1) and (1.2) is a special solution with the type

$$(S(x+ct), I(x+ct)) = (S(\xi), I(\xi)),\ \xi = x + ct, \tag{1.13}$$

where the parameter c is called the wave speed. Our aim is to study the traveling wave solutions for subsystem (1.1) and (1.2) connecting E^0 and E^*. Substituting (1.13) into (1.1) and (1.2), we deduce the following wave profile system

$$cS'(\xi) = d_S\left(J_1 * S(\xi) - S(\xi)\right) + \Lambda - \mu S(\xi) - \beta S(\xi)I(\xi), \tag{1.14}$$

$$cI'(\xi) = d_I\left(J_2 * I(\xi) - I(\xi)\right) + \beta S(\xi)I(\xi) - (\mu+\gamma)I(\xi). \tag{1.15}$$

Thus, the traveling wave solution of subsystem (1.1) and (1.2) connecting E^0 and E^* is a special solution $(S(\xi), I(\xi))$ satisfying (1.14)–(1.15) and the asymptotic boundary conditions

$$\lim_{\xi\to-\infty}(S(\xi), I(\xi)) = (\Lambda/\mu, 0), \quad \lim_{\xi\to+\infty}(S(\xi), I(\xi)) = (s^*, i^*). \tag{1.16}$$

Now, we state our main result on the existence and non-existence of traveling waves for the system (1.1) and (1.2) in the following theorem.

Theorem 1.1. *There exists a positive constant $c_{\min}$ such that if $\mathfrak{R}_0 > 1$ and $c \geq c_{\min}$, then the system* (1.14)–(1.15) *has a nonnegative solution* (S, I) *satisfying the asymptotic boundary conditions* (1.16). *Furthermore,*

(*i*) $0 < S(\cdot) < \Lambda/\mu$ *and* $0 < I(\cdot) < +\infty$ *in* $\mathbb{R}$.

(*ii*) *As* $\xi \to -\infty$, $I(\xi) = \mathcal{O}(\mathrm{e}^{\lambda_1 \xi})$ *if* $c > c_{\min}$ *and* $I(\xi) = \mathcal{O}(\mathrm{e}^{\lambda_0 \xi})$ *if* $c = c_{\min}$.

If $\mathfrak{R}_0 > 1$ *and* $c \in (-\infty, 0) \cup (0, c_{\min})$, *then there exists no nonnegative solution* (S, I) *of the system* (1.14)–(1.15) *satisfying the asymptotic boundary conditions* (1.16).

We point out that unlike in (1.10)–(1.12), the adoption of the mass action in (1.10)–(1.12) makes the analysis more difficult and challenging than the analysis of (1.10)–(1.12) on the above mentioned topics. For example, it can be easily to obtain the uniformly boundedness of I by constructing a bounded upper solution of (1.10)–(1.12) while the upper solution for I in our new model is unbounded and make the problem more complicated. Moreover, the Holling-II incidence term $I/(1+\alpha I)$ assumes bounded infection force while the mass action term βSI implies a unbounded infection force. Compared with the results in [7], we successfully obtain the existence of the limits $\lim_{\xi\to+\infty} S(\xi)$ and $\lim_{\xi\to+\infty} I(\xi)$. On the other hand, our model (1.1)–(1.2) is more complicated due to the effect of the nonlocal dispersal. Let us mention some main methods used in the investigation of the existence of traveling wave solutions, such as, monotone iteration coupled with upper-lower solutions [37], the general theory of traveling waves for monotone semiflows [21], the geometric singular perturbation method [28], the shooting method [10] and connection index theory [13]. Unfortunately, the lost of order-preserving property for system (1.1)–(1.2) makes these classic methods fail to apply. Inspired by the work of [7, 36, 45], we construct an invariant cone in a large but bounded domain with the initial functions being defined well and apply Schauder fixed point theorem on this cone. Then, by passing to the unbounded domain with a limiting argument, we have successfully obtained the existence of nontrivial traveling wave solutions of the system (1.1)–(1.2). Here, we overcome some difficulties to give the boundedness of I by some technical analysis. We should point out that the exact boundary behavior of $S(\xi)$ and $I(\xi)$ at $\xi = +\infty$ is obtained by Lyapunov–LaSalle invariance principle. Generally, the critical wave plays a more important role than the non-critical waves in determining the evolution dynamics of the system (1.1)–(1.2). By the approximating method, we also obtain the existence of traveling wave solutions with the critical wave speed $c_{\min}$ for the system (1.1)–(1.2). Here, the difficulty is how to prove the asymptotic boundary behavior of I at $-\infty$. Finally, we make full use of the structure of the system (1.1)–(1.2) and give the proof of the non-existence of traveling wave solutions when $c \in (-\infty, 0) \cup (0, c_{\min})$.

The organization of this paper is as follows. In Section 2, we first prove some useful lemmas, which will be used in the proof of our main result. Then, we consider the boundedness of component I. Finally, we establish the existence of traveling wave solutions of the system (1.1)–(1.2) for $c > c_{\min}$. Section 3 is devoted to the existence of critical waves of the system (1.1)–(1.2). In Section 4, we obtain the nonexistence of traveling wave solutions. The paper ends with the simulation and a brief discussion in Section 5.

2. Existence of traveling wave solutions for $c > c_{\min}$

Throughout this section, we always assume that $\mathfrak{R}_0 := \frac{\beta\Lambda}{\mu(\mu+\gamma)} > 1$. Linearizing (1.15) at the disease-free equilibrium $E^0 = (\Lambda/\mu, 0)$ yields a corresponding characteristic equation as follows

$$f(\lambda, c) = d_I \left(\int_{\mathbb{R}} J_2(y) \mathrm{e}^{-\lambda y} \mathrm{d}y - 1 \right) - c\lambda + \beta \frac{\Lambda}{\mu} - \mu - \gamma.$$

By a direct calculation, utilizing (J), we obtain

$$f(0, c) = \beta\Lambda/\mu - \mu - \gamma > 0, \quad f(\lambda, +\infty) = -\infty \quad \text{for} \quad \lambda > 0,$$

$$\frac{\partial f(\lambda,c)}{\partial c} = -\lambda < 0 \text{ for } \lambda > 0, \quad \frac{\partial f(\lambda,c)}{\partial \lambda}\Big|_{\lambda=0} = -c < 0, \quad \frac{\partial^2 f(\lambda,c)}{\partial \lambda^2} = d_I \int_{\mathbb{R}} J_2(y) y^2 \mathrm{e}^{-\lambda y} \mathrm{d}y > 0.$$

Lemma 2.1. *Suppose that* $\mathfrak{R}_0 := \frac{\beta\Lambda}{\mu(\mu+\gamma)} > 1$. *Then there exist two positive constants* $c_{\min}$ *and* λ_0 *such that*

$$f(\lambda_0, c_{\min}) = 0 \quad \text{and} \quad \frac{\partial}{\partial \lambda} f(\lambda_0, c_{\min}) = 0.$$

Furthermore, if $0 < c < c_{\min}$, *then* $f(\lambda, c) > 0$ *for all* $\lambda > 0$; *if* $c > c_{\min}$, *then* $f(\lambda, c) = 0$ *has two positive real roots* $\lambda_1 := \lambda_1(c)$ *and* $\lambda_2 := \lambda_2(c)$ *with* $0 < \lambda_1 < \lambda_0 < \lambda_2$ *and* $f(\lambda, c) > 0$ *for* $\lambda \in (0, \lambda_1) \cup (\lambda_2, +\infty)$; $f(\lambda, c) < 0$ *for* $\lambda \in (\lambda_1, \lambda_2)$.

Next, we always assume that $c > c_{\min}$ in this section. Define

$$\begin{aligned} \overline{S}(\xi) :&= \frac{\Lambda}{\mu}, \quad \underline{S}(\xi) := \max\left\{\frac{\Lambda}{\mu} - \sigma \mathrm{e}^{\varepsilon\xi}, 0\right\}, \\ \overline{I}(\xi) :&= \mathrm{e}^{\lambda_1 \xi}, \quad \underline{I}(\xi) := \max\left\{\mathrm{e}^{\lambda_1\xi} - M\mathrm{e}^{(\lambda_1+\eta)\xi}, 0\right\}, \end{aligned}$$

where $\xi \in \mathbb{R}$ and M, σ, ε and η are four positive constants to be determined in the following lemma.

Lemma 2.2. *It holds that*

$$c\overline{S}'(\xi) \geq d_S(J_1 * \overline{S}(\xi) - \overline{S}(\xi)) + \Lambda - \mu\overline{S}(\xi) - \beta\overline{S}(\xi)\underline{I}(\xi). \tag{2.1}$$

The proof is trivial and omitted.

Lemma 2.3. *The function* $\overline{I}(\xi) = \mathrm{e}^{\lambda_1\xi}$ *satisfies*

$$c\overline{I}'(\xi) \geq d_I(J_2 * \overline{I}(\xi) - \overline{I}(\xi)) + \beta\overline{S}(\xi)\overline{I}(\xi) - (\mu+\gamma)\overline{I}(\xi) \tag{2.2}$$

for any $\xi \in \mathbb{R}$.

Proof. It follows from the definition of $\overline{I}(\xi)$ that

$$\begin{aligned} &d_I(J_2 * \overline{I}(\xi) - \overline{I}(\xi)) - c\overline{I}'(\xi) + \beta\overline{S}(\xi)\overline{I}(\xi) - (\mu+\gamma)\overline{I}(\xi) \\ =&\mathrm{e}^{\lambda_1\xi}\left[d_I\left(\int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\lambda_1 y}\mathrm{d}y - 1\right) - c\lambda_1 + \frac{\beta\Lambda}{\mu} - \mu - \gamma\right] \\ =&\mathrm{e}^{\lambda_1\xi} f(\lambda_1, c) = 0. \end{aligned}$$

This finishes the proof. $\square$

Lemma 2.4. *Let* $\varepsilon \in (0, \lambda_1)$ *be sufficiently small. Then, the function* $\underline{S}(\xi) = \max\left\{\frac{\Lambda}{\mu} - \sigma\mathrm{e}^{\varepsilon\xi}, 0\right\}$ *satisfies*

$$c\underline{S}'(\xi) \leq d_S(J_1 * \underline{S}(\xi) - \underline{S}(\xi)) + \Lambda - \mu\underline{S}(\xi) - \beta\underline{S}(\xi)\overline{I}(\xi) \tag{2.3}$$

for any $\xi \neq \xi_1 := \frac{1}{\varepsilon}\ln\frac{\Lambda}{\sigma\mu}$ *and* $\sigma > \frac{\Lambda}{\mu}$ *large enough.*

Proof. When $\xi \geq \xi_1$, then the inequality (2.3) holds immediately because $\underline{S}(\xi) = 0$ on $[\xi_1, +\infty)$. When $\xi < \xi_1$, then

$$\underline{S}(\xi) = \frac{\Lambda}{\mu} - \sigma\mathrm{e}^{\varepsilon\xi} \quad \text{and} \quad \overline{I}(\xi) = \mathrm{e}^{\lambda_1\xi}. \tag{2.4}$$

It follows that

$$d_S\left(J_1 * \underline{S}(\xi) - \underline{S}(\xi)\right) - c\underline{S}'(\xi) + \Lambda - \mu\underline{S}(\xi) - \beta\underline{S}(\xi)\overline{I}(\xi)$$

$$
\begin{aligned}
&\geq d_S\left(-\int_{\mathbb{R}} J_1(y)\sigma \mathrm{e}^{\varepsilon(\xi-y)}\mathrm{d}y+\sigma\mathrm{e}^{\varepsilon\xi}\right)+c\sigma\varepsilon\mathrm{e}^{\varepsilon\xi}+\mu\sigma\mathrm{e}^{\varepsilon\xi}-\beta\left(\frac{\Lambda}{\mu}-\sigma\mathrm{e}^{\varepsilon\xi}\right)\mathrm{e}^{\lambda_1\xi}\\
&\geq \mathrm{e}^{\varepsilon\xi}\left\{\sigma\left[-d_S\left(\int_{\mathbb{R}} J_1(y)\mathrm{e}^{-\varepsilon y}\mathrm{d}y-1\right)+c\varepsilon+\mu\right]-\frac{\beta\Lambda}{\mu}\mathrm{e}^{(\lambda_1-\varepsilon)\xi}\right\}. \qquad (2.5)
\end{aligned}
$$

Noting that

$$
\lim_{\varepsilon\to 0}\left[-d_S\left(\int_{\mathbb{R}} J_1(y)\mathrm{e}^{-\varepsilon y}\mathrm{d}y-1\right)+c\varepsilon+\mu\right]=\mu>0,
$$

we have for sufficiently small $\varepsilon\in(0,\lambda_1)$ that

$$
-d_S\left(\int_{\mathbb{R}} J_1(y)\mathrm{e}^{-\varepsilon y}\mathrm{d}y-1\right)+c\varepsilon+\mu>0. \qquad (2.6)
$$

As a consequence, by choosing

$$
\sigma>\frac{\beta\Lambda}{\mu\left[-d_S\left(\int_{\mathbb{R}} J_1(y)\mathrm{e}^{-\varepsilon y}\mathrm{d}y-1\right)+c\varepsilon+\mu\right]},
$$

we deduce from (2.5) and (2.6) that

$$
\begin{aligned}
&d_S\left(J_1*\underline{S}(\xi)-\underline{S}(\xi)\right)-c\underline{S}'(\xi)+\Lambda-\mu\underline{S}(\xi)-\beta\underline{S}(\xi)\overline{I}(\xi)\\
\geq&\mathrm{e}^{\varepsilon\xi}\left\{\sigma\left[-d_S\left(\int_{\mathbb{R}} J_1(y)\mathrm{e}^{-\varepsilon y}\mathrm{d}y-1\right)+c\varepsilon+\mu\right]-\frac{\beta\Lambda}{\mu}\right\}\\
\geq&0.
\end{aligned}
$$

The proof is completed. □

Lemma 2.5. *Assume that $\eta\in(0,\min\{\varepsilon,\lambda_2-\lambda_1\})$. Then, the function $\underline{I}(\xi)$ satisfies*

$$
c\underline{I}'(\xi)\leq d_I(J_2*\underline{I}(\xi)-\underline{I}(\xi))+\beta\underline{S}(\xi)\underline{I}(\xi)-(\mu+\gamma)\underline{I}(\xi) \qquad (2.7)
$$

for any $\xi\neq\xi_2:=-\frac{1}{\eta}\ln M$ and large enough $M>1$.

Proof. For $\xi>\xi_2$, then the inequality (2.7) holds immediately since $\underline{I}(\xi)=0$ on $(\xi_2,+\infty)$.

For $\xi<\xi_2$, choose $M_1>1$ so large that $-\frac{1}{\eta}\ln M_1+1=\frac{1}{\varepsilon}\ln\frac{\Lambda}{\sigma\mu}$. Take $M\geq M_1$, then $\underline{S}(\xi)=\frac{\Lambda}{\mu}-\sigma\mathrm{e}^{\varepsilon\xi}$, $\underline{I}(\xi)=\mathrm{e}^{\lambda_1\xi}-M\mathrm{e}^{(\lambda_1+\eta)\xi}$. It follows that

$$
\begin{aligned}
&d_I(J_2*\underline{I}(\xi)-\underline{I}(\xi))-c\underline{I}'(\xi)+\beta\underline{S}(\xi)\underline{I}(\xi)-(\mu+\gamma)\underline{I}(\xi)\\
\geq&d_I\left[\int_{\mathbb{R}} J_2(y)\left(\mathrm{e}^{\lambda_1(\xi-y)}-M\mathrm{e}^{(\lambda_1+\eta)(\xi-y)}\right)\mathrm{d}y-\left(\mathrm{e}^{\lambda_1\xi}-M\mathrm{e}^{(\lambda_1+\eta)\xi}\right)\right]-c\Big[\lambda_1\mathrm{e}^{\lambda_1\xi}\\
&-M(\lambda_1+\eta)\mathrm{e}^{(\lambda_1+\eta)\xi}\Big]+\beta\left(\frac{\Lambda}{\mu}-\sigma\mathrm{e}^{\varepsilon\xi}\right)\left(\mathrm{e}^{\lambda_1\xi}-M\mathrm{e}^{(\lambda_1+\eta)\xi}\right)-(\mu+\gamma)\left(\mathrm{e}^{\lambda_1\xi}-M\mathrm{e}^{(\lambda_1+\eta)\xi}\right)\\
=&-d_IM\int_{\mathbb{R}} J_2(y)\mathrm{e}^{(\lambda_1+\eta)(\xi-y)}\mathrm{d}y+d_IM\mathrm{e}^{(\lambda_1+\eta)\xi}+cM(\lambda_1+\eta)\mathrm{e}^{(\lambda_1+\eta)\xi}-\beta M\frac{\Lambda}{\mu}\mathrm{e}^{(\lambda_1+\eta)\xi}\\
&-\beta\sigma\mathrm{e}^{(\lambda_1+\varepsilon)\xi}+M\beta\sigma\mathrm{e}^{(\lambda_1+\eta+\varepsilon)\xi}+(\mu+\gamma)M\mathrm{e}^{(\lambda_1+\eta)\xi}\\
\geq&\mathrm{e}^{(\lambda_1+\eta)\xi}\left[-Mf(\lambda_1+\eta,c)-\beta\sigma\mathrm{e}^{(\varepsilon-\eta)\xi}\right].
\end{aligned}
$$

Let $M>\max\left\{\frac{\beta\sigma}{-f(\lambda_1+\eta,c)},M_1\right\}$. Then

$$
d_I(J_2*\underline{I}(\xi)-\underline{I}(\xi))-c\underline{I}'(\xi)+\beta\underline{S}(\xi)\underline{I}(\xi)-(\mu+\gamma)\underline{I}(\xi)\geq 0 \quad \text{for} \quad \xi\leq\xi_2.
$$

This completes the proof. □

Let $A > \max\left\{-\frac{1}{\varepsilon}\ln\frac{\Lambda}{\sigma\mu}, \frac{1}{\eta}\ln M\right\}$ and

$$\Sigma_A = \left\{ (\phi(\cdot),\psi(\cdot)) \in C([-A,A],\mathbb{R}^2) \middle| \begin{array}{l} \phi(-A) = \underline{S}(-A),\ \psi(-A) = \underline{I}(-A), \\ \underline{S}(\xi) \le \phi(\xi) \le \overline{S}(\xi) \text{ for } \xi \in [-A,A], \\ \underline{I}(\xi) \le \psi(\xi) \le \overline{I}(\xi) \text{ for } \xi \in [-A,A]. \end{array} \right\}.$$

It is easy to see that Σ_A is a closed, convex subset of $C([-A,A],\mathbb{R}^2)$. For any $(\phi(\cdot),\psi(\cdot)) \in \Sigma_A$, define

$$\widehat{\phi}(\xi) = \begin{cases} \phi(A), \xi > A, \\ \phi(\xi),\ |\xi| \le A, \\ \underline{S}(\xi),\ \xi < -A, \end{cases} \qquad \widehat{\psi}(\xi) = \begin{cases} \psi(A), \xi > A, \\ \psi(\xi),\ |\xi| \le A, \\ \underline{I}(\xi),\ \xi < -A. \end{cases}$$

Thus, we consider the following initial value problem

$$cS'(\xi) = d_S\int_{\mathbb{R}} J_1(y)\widehat{\phi}(\xi-y)\mathrm{d}y + \Lambda - (d_S+\mu)S(\xi) - \beta S(\xi)\psi(\xi), \tag{2.8}$$

$$cI'(\xi) = d_I\int_{\mathbb{R}} J_2(y)\widehat{\psi}(\xi-y)\mathrm{d}y + \beta\phi(\xi)\psi(\xi) - (d_I+\mu+\gamma)I(\xi), \tag{2.9}$$

$$S(-A) = \underline{S}(-A),\ I(-A) = \underline{I}(-A). \tag{2.10}$$

According to the standard ODE theory, we can obtain that (2.8)–(2.10) admit a unique solution $(S_A(\xi), I_A(\xi))$ satisfying $(S_A(\cdot), I_A(\cdot)) \in C^1([-A,A]) \times C^1([-A,A])$. Now, we define an operator $\mathcal{G} := (\mathcal{G}_1,\mathcal{G}_2) : \Sigma_A \longrightarrow C([-A,A])$ by

$$\mathcal{G}_1[\phi(\cdot),\psi(\cdot)](\xi) = S_A(\xi),\quad \mathcal{G}_2[\phi(\cdot),\psi(\cdot)](\xi) = I_A(\xi),\ \xi \in [-A,A].$$

Now, we prove that the operator $\mathcal{G}$ is completely continuous.

Lemma 2.6. *The operator $\mathcal{G} : \Sigma_A \longrightarrow \Sigma_A$ is completely continuous.*

Proof. We can easily obtain that $\mathcal{G}(\Sigma_A) \subseteq \Sigma_A$ from Lemmas 2.2–2.5. Since $S_A(\xi), I_A(\xi) \in C^1([-A,A])$, the compactness of the operator $\mathcal{G} = (\mathcal{G}_1,\mathcal{G}_2)$ follows from the definition of Σ_A and the Arzela-Ascoli theorem.

Now, we prove the continuity of $\mathcal{G} = (\mathcal{G}_1,\mathcal{G}_2)$. Let $(\phi^i(\cdot),\psi^i(\cdot)) \in \Sigma_A,\ i = 1,2$ and assume that

$$\mathcal{G}_1[\phi^i(\cdot),\psi^i(\cdot)](\xi) = S_A^i(\xi),\quad \mathcal{G}_2[\phi^i(\cdot),\psi^i(\cdot)](\xi) = I_A^i(\xi).$$

Since

$$\int_{\mathbb{R}} J_1(y)\widehat{\phi^i}(\xi-y)\mathrm{d}y = \int_{-\infty}^{-A} J_1(\xi-y)\underline{S}(y)\mathrm{d}y + \int_{-A}^{A} J_1(\xi-y)\phi^i(y)\mathrm{d}y + \int_{A}^{+\infty} J_1(\xi-y)\phi^i(A)\mathrm{d}y, i = 1,2,$$

we can get that

$$\left|\int_{\mathbb{R}} J_1(y)\widehat{\phi}^1(\xi-y)\mathrm{d}y - \int_{\mathbb{R}} J_1(y)\widehat{\phi}^2(\xi-y)\mathrm{d}y\right| \le 2\max_{\tau\in[-A,A]}\left|\phi^1(\tau)-\phi^2(\tau)\right|.$$

Similarly, we have that

$$\left|\int_{\mathbb{R}} J_2(y)\widehat{\psi}^1(\xi-y)\mathrm{d}y - \int_{\mathbb{R}} J_2(y)\widehat{\psi}^2(\xi-y)\mathrm{d}y\right| \le 2\max_{\tau\in[-A,A]}\left|\psi^1(\tau)-\psi^2(\tau)\right|.$$

Additionally, we know that

$$S_A(\xi) = \underline{S}(-A)\exp\left\{-\frac{1}{c}\int_{-A}^{\xi}\left(d_S+\mu+\beta\psi(\tau)\right)\mathrm{d}\tau\right\}$$

$$+\frac{1}{c}\int_{-A}^{\xi}\exp\left\{-\int_{\zeta}^{\xi}\left(d_S+\mu+\beta\psi(\tau)\right)\mathrm{d}\tau\right\}\left[d_SJ_1*\widehat{\psi}(\zeta)+\Lambda\right]\mathrm{d}\zeta, \tag{2.11}$$

and

$$\begin{aligned}I_A(\xi)=&\underline{I}(-A)\exp\left\{-\frac{d_I+\mu+\gamma}{c}(\xi+A)\right\}\\&+\frac{1}{c}\int_{-A}^{\xi}\exp\left\{-\frac{d_I+\mu+\gamma}{c}(\xi-\zeta)\right\}\left[d_IJ_2*\widehat{\psi}(\zeta)+\beta\phi(\zeta)\psi(\zeta)\right]\mathrm{d}\zeta.\end{aligned} \tag{2.12}$$

Moreover,

$$\begin{aligned}&\left|\left(d_IJ_2*\widehat{\psi^1}(\zeta)+\beta\phi^1(\zeta)\psi^1(\zeta)\right)-\left(d_IJ_2*\widehat{\psi^2}(\zeta)+\beta\phi^2(\zeta)\psi^2(\zeta)\right)\right|\\\le&2d_I\max_{\xi\in[-A,A]}\left|\psi^1(\xi)-\psi^2(\xi)\right|+\beta\left|\phi^1(\xi)\right|\left|\psi^1(\xi)-\psi^2(\xi)\right|+\beta\left|\psi^2(\xi)\right|\left|\phi^1(\xi)-\phi^2(\xi)\right|\\\le&\beta\mathrm{e}^{\lambda_1A}\max_{\xi\in[-A,A]}\left|\phi^1(\xi)-\phi^2(\xi)\right|+\left(2d_I+\frac{\beta\Lambda}{\mu}\right)\max_{\xi\in[-A,A]}\left|\psi^1(\xi)-\psi^2(\xi)\right|.\end{aligned} \tag{2.13}$$

Consequently, we infer from (2.11)–(2.13) that $\mathcal{G}$ is continuous. This finishes the proof. □

Based on the previous discussion, Schauder fixed point theorem implies that there exists $(S_A,I_A)\in\Sigma_A$ such that $(S_A(\xi),I_A(\xi))=\mathcal{G}[S_A,I_A](\xi)$ for all $\xi\in[-A,A]$. Next, our aim is to obtain the existence of traveling waves of (1.1)–(1.2) by letting A tend to infinity. Before doing this, we must give some prior estimates for $S_A(\xi)$ and $I_A(\xi)$ in the space $C^{1,1}([-B,B])$, where

$$C^{1,1}([-B,B])=\left\{\omega\in C^1([-B,B])|\omega\text{ and }\omega'\text{are Lipschitz continuous}\right\}$$

with the norm

$$||\omega||_{C^{1,1}([-B,B])}=\max_{\zeta\in[-B,B]}|\omega(\zeta)|+\max_{\zeta\in[-B,B]}|\omega'(\zeta)|+\sup_{\zeta^1,\zeta^2\in[-B,B],\zeta^1\neq\zeta^2}\frac{|\omega'(\zeta^1)-\omega'(\zeta^2)|}{\zeta^1-\zeta^2}.$$

Then, we have the following theorem.

Theorem 2.1. *For given $B>0$, there exists a constant $C^\sharp(B)$ such that*

$$||S_A||_{C^{1,1}([-B,B])}\le C^\sharp(B),\quad ||I_A||_{C^{1,1}([-B,B])}\le C^\sharp(B),$$

for any $A>\max\left\{-\frac{1}{\varepsilon}\ln\frac{\Lambda}{\sigma\mu},\frac{1}{\eta}\ln M,B+\varrho_1,B+\varrho_2\right\}$, where ϱ_i is the radius of suppJ_i, $i=1,2$.

Proof. From the above discussion, we know that (S_A,I_A) satisfies

$$cS_A'(\xi)=d_SJ_1*\widehat{S_A}(\xi)+\Lambda-(d_S+\mu)S_A(\xi)-\beta S_A(\xi)I_A(\xi), \tag{2.14}$$
$$cI_A'(\xi)=d_IJ_2*\widehat{I_A}(\xi)+\beta S_A(\xi)I_A(\xi)-(d_I+\mu+\gamma)I_A(\xi) \tag{2.15}$$

for $\xi\in[-B,B]$, where

$$\widehat{S_A}(\xi)=\begin{cases}S_A(A),\xi>A,\\S_A(\xi),\ |\xi|\le A,\\\underline{S}(\xi),\quad\xi<-A,\end{cases}\qquad\widehat{I_A}(\xi)=\begin{cases}I_A(A),\xi>A,\\I_A(\xi),\ |\xi|\le A,\\\underline{I}(\xi),\quad\xi<-A.\end{cases}$$

Since $0 \le S_A(\xi) \le \Lambda/\mu,\ 0 \le I_A(\xi) \le \mathrm{e}^{\lambda_1 \xi}$, then from (2.14) and (2.15), we get

$$|S_A'(\xi)| \le \frac{\left(2d_S + \mu + \beta \overline{I}(B)\right)\Lambda/\mu + \Lambda}{c},$$

and

$$|I_A'(\xi)| \le \frac{\left(d_I \int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\lambda_1 y}\mathrm{d}y + d_I + \mu + \gamma + \beta\Lambda/\mu\right)\overline{I}(B)}{c},$$

for $\xi \in [-B, B]$. Thus, there exists a positive constant $C_1(B)$ such that

$$||S_A||_{C^1([-B,B])} \le C_1(B) \quad \text{and} \quad ||I_A||_{C^1([-B,B])} \le C_1(B). \tag{2.16}$$

Next, we will prove that $S_A'(\xi)$ and $I_A'(\xi)$ are Lipschitz continuous. For any $\xi,\ \zeta \in [-B, B]$, it follows from (2.16) that

$$|S_A(\xi) - S_A(\zeta)| \le C_1(B)|\xi - \zeta| \ \text{ and } \ |I_A(\xi) - I_A(\zeta)| \le C_1(B)|\xi - \zeta|. \tag{2.17}$$

From (2.14), we obtain that

$$\begin{aligned} c\left(S_A'(\xi) - S_A'(\zeta)\right) =& d_S\left(J_1 * \widehat{S_A}(\xi) - J_1 * \widehat{S_A}(\zeta)\right) \\ & - (d_S + \mu)\left(S_A(\xi) - S_A(\zeta)\right) - \left(\beta S_A(\xi)I_A(\xi) - \beta S_A(\zeta)I_A(\zeta)\right). \end{aligned} \tag{2.18}$$

Let L_{J_i} be the Lipschitz constant of J_i, $i = 1, 2$. Since

$$\begin{aligned} & \left|J_1 * \widehat{S_A}(\xi) - J_1 * \widehat{S_A}(\zeta)\right| \\ =& \left|\int_{\xi-\varrho_1}^{\xi+\varrho_1} J_1(\xi - y)\widehat{S_A}(y)\mathrm{d}y - \int_{\zeta-\varrho_1}^{\zeta+\varrho_1} J_1(\zeta - y)\widehat{S_A}(y)\mathrm{d}y\right| \\ =& \left|\int_{\zeta+\varrho_1}^{\xi+\varrho_1} J_1(\xi - y)\widehat{S_A}(y)\mathrm{d}y - \int_{\zeta-\varrho_1}^{\xi-\varrho_1} J_1(\zeta - y)\widehat{S_A}(y)\mathrm{d}y + \int_{\xi-\varrho_1}^{\zeta+\varrho_1} \left(J_1(\xi - y) - J_1(\zeta - y)\right)\widehat{S_A}(y)\mathrm{d}y\right| \\ \le& \frac{2\Lambda}{\mu}||J_1||_{L^\infty(\mathbb{R})}|\xi - \zeta| + \left|\left(\int_{\xi-\varrho_1}^{\zeta-\varrho_1} + \int_{\zeta-\varrho_1}^{\zeta+\varrho_1}\right)\left(J_1(\xi - y) - J_1(\zeta - y)\right)\widehat{S_A}(y)\mathrm{d}y\right| \\ \le& \frac{2\Lambda}{\mu}\left(2||J_1||_{L^\infty(\mathbb{R})} + \varrho_1 L_{J_1}\right)|\xi - \zeta|, \end{aligned} \tag{2.19}$$

and

$$\begin{aligned} |\beta S_A(\xi)I_A(\xi) - \beta S_A(\zeta)I_A(\zeta)| =& \beta\left|S_A(\xi)\left(I_A(\xi) - I_A(\zeta)\right) + \left(S_A(\xi) - S_A(\zeta)\right)I_A(\zeta)\right| \\ =& \beta\left(\Lambda/\mu + \overline{I}(B)\right)C_1(B)|\xi - \zeta|, \end{aligned} \tag{2.20}$$

we infer that

$$|S_A'(\xi) - S_A'(\zeta)| \le \frac{1}{c}\left\{\frac{2d_S\Lambda}{\mu}\left(2||J_1||_{L^\infty(\mathbb{R})} + \varrho_1 L_{J_1}\right) + \left[\mu + d_S + \beta\left(\Lambda/\mu + \overline{I}(B)\right)\right]C_1(B)\right\}|\xi - \zeta|.$$

Utilizing (2.15), we have

$$c\left(I_A'(\xi) - I_A'(\zeta)\right) = d_I\left(J_2 * \widehat{I_A}(\xi) - J_2 * \widehat{I_A}(\zeta)\right)$$

$$-\left(d_I+\mu+\gamma\right)\left(I_A(\xi)-I_A(\zeta)\right)+\left(\beta S_A(\xi)I_A(\xi)-\beta S_A(\zeta)I_A(\zeta)\right). \tag{2.21}$$

In view of

$$\begin{aligned}
&\left|J_2*\widehat{I_A}(\xi)-J_2*\widehat{I_A}(\zeta)\right|\\
=&\left|\int_{\xi-\varrho_2}^{\xi+\varrho_2}J_2(\xi-y)\widehat{I_A}(y)\mathrm{d}y-\int_{\zeta-\varrho_2}^{\zeta+\varrho_2}J_2(\zeta-y)\widehat{I_A}(y)\mathrm{d}y\right|\\
=&\left|\int_{\zeta+\varrho_2}^{\xi+\varrho_2}J_2(\xi-y)\widehat{I_A}(y)\mathrm{d}y-\int_{\zeta-\varrho_2}^{\xi-\varrho_2}J_2(\zeta-y)\widehat{I_A}(y)\mathrm{d}y+\int_{\xi-\varrho_2}^{\zeta+\varrho_2}\left(J_2(\xi-y)-J_2(\zeta-y)\right)\widehat{I_A}(y)\mathrm{d}y\right|\\
\leq&\left[3\overline{I}(B+\varrho_2)||J_2||_{L^\infty(\mathbb{R})}+\overline{I}(B-\varrho_2)||J_2||_{L^\infty(\mathbb{R})}\right]|\xi-\zeta|+\left|\int_{\xi-\varrho_2}^{\xi+\varrho_2}\left(J_2(\xi-y)-J_2(\zeta-y)\right)\widehat{I_A}(y)\mathrm{d}y\right|\\
\leq&\left[3\overline{I}(B+\varrho_2)||J_2||_{L^\infty(\mathbb{R})}+\overline{I}(B-\varrho_2)||J_2||_{L^\infty(\mathbb{R})}+2\varrho_2\overline{I}(B+\varrho_2)L_{J_2}\right]|\xi-\zeta|,
\end{aligned} \tag{2.22}$$

it follows from (2.17) and (2.20)–(2.22) that

$$\begin{aligned}
|I_A'(\xi)-I_A'(\zeta)|\leq&\frac{C_1(B)}{c}\left[\beta\left(\Lambda/\mu+\overline{I}(B)\right)+(d_I+\mu+\gamma)\right]|\xi-\zeta|\\
&+\frac{1}{c}\left[3\overline{I}(B+\varrho_2)||J_2||_{L^\infty(\mathbb{R})}+\overline{I}(B-\varrho_2)||J_2||_{L^\infty(\mathbb{R})}+2\varrho_2\overline{I}(B+\varrho_2)L_{J_2}\right]|\xi-\zeta|.
\end{aligned}$$

Consequently, the proof of this theorem is completed. □

By Theorem 2.1, we have the following theorem.

Theorem 2.2. *If $\mathfrak{R}_0>1$, then for any $c>c_{\min}$, there exists a pair of functions $(S(\xi),I(\xi))$ which satisfy* (1.14)–(1.15) *on $\mathbb{R}$. Moreover,*

$$0<S(\xi)<\frac{\Lambda}{\mu}\quad\text{and}\quad I(\xi)>0\quad\text{on}\quad\mathbb{R}. \tag{2.23}$$

Proof. We choose a sequence $\{A_n\}_{n=1}^{+\infty}$ satisfying $A_n>\max\left\{-\frac{1}{\varepsilon}\ln\frac{\Lambda}{\sigma\mu},\frac{1}{\eta}\ln M,B+\varrho_1,B+\varrho_2\right\}$ such that $A_n\to+\infty$ as $n\to+\infty$. Obviously, there exists some $(S_{A_n},I_{A_n})\in\Sigma_{A_n}$ for each n such that Theorem 2.1 holds. According to the estimates in Theorem 2.1, there exists a subsequence $\{A_{n_k}\}$ by diagonal extraction such that $\lim_{k\to+\infty}A_{n_k}=+\infty$ and

$$S_{A_{n_k}}(\xi)\to S(\xi)\quad\text{and}\quad I_{A_{n_k}}(\xi)\to I(\xi)\quad\text{in}\quad C^1_{loc}(\mathbb{R})\quad\text{as}\quad k\to+\infty, \tag{2.24}$$

satisfying

$$cS'_{A_{n_k}}(\xi)=d_SJ_1*\widehat{S_{A_{n_k}}}(\xi)+\Lambda-(d_S+\mu)S_{A_{n_k}}(\xi)-\beta S_{A_{n_k}}(\xi)I_{A_{n_k}}(\xi), \tag{2.25}$$

$$cI'_{A_{n_k}}(\xi)=d_IJ_2*\widehat{I_{A_{n_k}}}(\xi)+\beta S_{A_{n_k}}(\xi)I_{A_{n_k}}(\xi)-(d_I+\mu+\gamma)I_{A_{n_k}}(\xi), \tag{2.26}$$

and

$$\underline{S}(\xi)\leq S_{A_{n_k}}(\xi)\leq\overline{S}(\xi),\quad\underline{I}(\xi)\leq I_{A_{n_k}}(\xi)\leq\overline{I}(\xi)\quad\text{for}\quad\xi\in(-A_{n_k},A_{n_k}), \tag{2.27}$$

where $(S(\cdot),I(\cdot))\in C^1(\mathbb{R})\times C^1(\mathbb{R})$. It follows from (J) and the Lebesgue dominated convergence theorem that

$$\lim_{k\to+\infty}\int_{\mathbb{R}}J_1(\xi-y)\widehat{S_{A_{n_k}}}(y)\mathrm{d}y=\int_{\mathbb{R}}J_1(\xi-y)S(y)\mathrm{d}y=J_1*S(\xi), \tag{2.28}$$

and

$$\lim_{k\to+\infty}\int_{\mathbb{R}} J_2(\xi-y)\widehat{I_{A_{n_k}}}(y)\mathrm{d}y=\int_{\mathbb{R}} J_2(\xi-y)I(y)\mathrm{d}y=J_2*I(\xi). \tag{2.29}$$

Furthermore, passing to limits in (2.25) and (2.26), we can obtain from (2.24), (2.28) and (2.29) that (S,I) satisfies (1.14) and (1.15).

To obtain (2.23), we employ the contradiction argument. Assume that there is a constant $\xi_0\in\mathbb{R}$ such that $I(\xi_0)=0$. Due to $I(\xi)=\underline{I}(\xi)\geq 0,\ \xi\in\mathbb{R}$, then ξ_0 is the minimal point of $I(\xi)$, which implies that $I'(\xi_0)=0$. It follows that $\int_{\mathbb{R}} J_2(y)I(\xi_0-y)dy=0$, which contradicts to that $I(\xi)>0$ for $\xi<\xi_2$. Thus, $I(\xi)>0$ in $\mathbb{R}$.

Suppose that there exists some $\overline{\xi}$ so that $S(\overline{\xi})=0$. Then, $S'(\overline{\xi})=0$. Meanwhile, $S(\xi)$ at $\xi=\overline{\xi}$ satisfies

$$0=cS'(\overline{\xi})=d_S\left(\int_{\mathbb{R}} J_1(y)S(\overline{\xi}-y)dy-S(\overline{\xi})\right)+\Lambda-\mu S(\overline{\xi})-\beta S(\overline{\xi})I(\overline{\xi})\geq\Lambda.$$

This contradiction leads to the inequality $S(\xi)>0$ in $\mathbb{R}$. Also, if there exists $\widetilde{\xi}\in\mathbb{R}$ such that $S(\widetilde{\xi})=\Lambda/\mu$, then

$$0=cS'(\widetilde{\xi})=d_S\left(\int_{\mathbb{R}} J_1(y)S(\widetilde{\xi}-y)dy-S(\widetilde{\xi})\right)+\Lambda-\mu S(\widetilde{\xi})-\beta S(\widetilde{\xi})I(\widetilde{\xi})\leq-\beta\Lambda I(\widetilde{\xi})/\mu.$$

This is impossible since $I(\xi)>0$ in $\mathbb{R}$. The proof of this theorem is finished. □

In order to show the convergence of the traveling wave solutions toward the endemic equilibrium $E^*=(s^*,i^*)$ at $\xi=+\infty$, we construct a suitable Lyapunov functional. First, we need to derive the boundedness property of the solution $I(\xi)$ of system (1.1)–(1.2).

Lemma 2.7. *Let $M>0$ and $N>0$ be real numbers. Then, there exists a constant $C^\dagger=C^\dagger(M,N)>0$ such that, for any continuous functions $P(\cdot)$ and $Q(\cdot)$ with $P(\cdot)\geq M$ and $Q(\cdot)\geq -N$ for all $\xi\in\mathbb{R}$ and for any positive function $W(\cdot)\in C^1(\mathbb{R})$ satisfying*

$$W'(\xi)\geq P(\xi)\int_{\mathbb{R}} J_i(y)W(\xi-y)\mathrm{d}y+Q(\xi)W(\xi)\quad \text{for all}\ \ \xi\in\mathbb{R},\quad i=1,2, \tag{2.30}$$

it holds

$$\int_{\mathbb{R}} J_i(y)W(\xi-y)\mathrm{d}y\leq C^\dagger(M,N)W(\xi)\quad \text{for all}\ \ \xi\in\mathbb{R},\quad i=1,2, \tag{2.31}$$

Proof. We derive from (2.30) that for $i=1,2$

$$W'(\xi)+NW(\xi)\geq P(\xi)\int_{\mathbb{R}} J_i(y)W(\xi-y)\mathrm{d}y+Q(\xi)W(\xi)+NW(\xi)\geq 0\ \text{ for all }\ \xi\in\mathbb{R}. \tag{2.32}$$

It follows that $W(\xi)\mathrm{e}^{N\xi}$ is increasing in $\mathbb{R}$ and so

$$W(\xi-y)\leq W(\xi)\mathrm{e}^{Ny}\ \text{ for any }\ y\in[0,+\infty). \tag{2.33}$$

From (2.32), we deduce that

$$\left(\mathrm{e}^{N\xi}W(\xi)\right)'\geq P(\xi)\mathrm{e}^{N\xi}\int_{\mathbb{R}} J_i(y)W(\xi-y)\mathrm{d}y\geq M\mathrm{e}^{N\xi}\int_{\mathbb{R}} J_i(y)W(\xi-y)\mathrm{d}y \tag{2.34}$$

for all $\xi \in \mathbb{R}$. Choose $\tilde{\varrho}_i > 0$ satisfies $2\tilde{\varrho}_i < \varrho_i$. Integrating the inequality (2.34) from $-\infty$ to ξ and using the fact that $W(\xi)\mathrm{e}^{N\xi}$ is increasing, we obtain that

$$\begin{aligned}
\mathrm{e}^{N\xi}W(\xi) \geq & M\int_{-\infty}^{\xi}\mathrm{e}^{N\zeta}\int_{\mathbb{R}}J_i(y)W(\zeta-y)\mathrm{d}y\mathrm{d}\zeta = M\int_{\mathbb{R}}\mathrm{e}^{Ny}J_i(y)\int_{-\infty}^{\xi}\mathrm{e}^{N(\zeta-y)}W(\zeta-y)\mathrm{d}\zeta\mathrm{d}y \\
\geq & M\int_{\mathbb{R}}\mathrm{e}^{Ny}J_i(y)\int_{\xi-\tilde{\varrho}_i}^{\xi}\mathrm{e}^{N(\zeta-y)}W(\zeta-y)\mathrm{d}\zeta\mathrm{d}y \\
\geq & M\tilde{\varrho}_i\int_{\mathbb{R}}J_i(y)\mathrm{e}^{N(\xi-\tilde{\varrho}_i)}W(\xi-\tilde{\varrho}_i-y)\mathrm{d}y, \quad \xi \in \mathbb{R}
\end{aligned} \tag{2.35}$$

which implies that

$$\int_{-\infty}^{0}J_i(y)W(\xi-\tilde{\varrho}_i-y)\mathrm{d}y \leq \frac{\mathrm{e}^{N\tilde{\varrho}_i}}{M\tilde{\varrho}_i}W(\xi), \quad \xi \in \mathbb{R}. \tag{2.36}$$

Furthermore, we can conclude from (2.35) that

$$\begin{aligned}
\mathrm{e}^{N\xi}W(\xi) \geq & M\tilde{\varrho}_i\int_{\mathbb{R}}\mathrm{e}^{Ny}J_i(y)\mathrm{e}^{N(\xi-\tilde{\varrho}_i-y)}W(\xi-\tilde{\varrho}_i-y)\mathrm{d}y \\
\geq & M\tilde{\varrho}_i\int_{-\infty}^{-2\tilde{\varrho}_i}\mathrm{e}^{Ny}J_i(y)\mathrm{e}^{N(\xi-\tilde{\varrho}_i-y)}W(\xi-\tilde{\varrho}_i-y)\mathrm{d}y \\
\geq & M\tilde{\varrho}_i\int_{-\infty}^{-2\tilde{\varrho}_i}\mathrm{e}^{Ny}J_i(y)\mathrm{e}^{N(\xi+\tilde{\varrho}_i)}W(\xi+\tilde{\varrho}_i)\mathrm{d}y, \quad \xi \in \mathbb{R}
\end{aligned} \tag{2.37}$$

which means that

$$W(\xi+\tilde{\varrho}_i) \leq \frac{1}{M\tilde{\varrho}_i\mathrm{e}^{N\tilde{\varrho}_i}\int_{-\infty}^{-2\tilde{\varrho}_i}\mathrm{e}^{Ny}J_i(y)\mathrm{d}y}W(\xi), \quad \xi \in \mathbb{R}. \tag{2.38}$$

It follows from (2.36) and (2.38) that

$$\begin{aligned}
\int_{\mathbb{R}}J_i(y)W(\xi-y)\mathrm{d}y \leq & \int_{-\infty}^{0}J_i(y)W(\xi-y)\mathrm{d}y + \int_{0}^{+\infty}J_i(y)W(\xi-y)\mathrm{d}y \\
\leq & \frac{1}{M\tilde{\varrho}_i\mathrm{e}^{N\tilde{\varrho}_i}\int_{-\infty}^{-2\tilde{\varrho}_i}\mathrm{e}^{Ny}J_i(y)\mathrm{d}y}\int_{-\infty}^{0}J_i(y)W(\xi-\tilde{\varrho}_i-y)\mathrm{d}y + W(\xi)\int_{0}^{+\infty}J_i(y)\mathrm{e}^{Ny}\mathrm{d}y \\
\leq & \left(\frac{1}{M^2\tilde{\varrho}_i^2\int_{-\infty}^{-2\tilde{\varrho}_i}\mathrm{e}^{Ny}J_i(y)\mathrm{d}y} + \int_{0}^{+\infty}J_i(y)\mathrm{e}^{Ny}\mathrm{d}y\right)W(\xi), \quad \xi \in \mathbb{R}.
\end{aligned}$$

If we choose $C^{\dagger}(M,N) = \frac{1}{M^2\tilde{\varrho}_i^2\int_{-\infty}^{-2\tilde{\varrho}_i}\mathrm{e}^{Ny}J_i(y)\mathrm{d}y} + \int_0^{+\infty}J_i(y)\mathrm{e}^{Ny}\mathrm{d}y$, then, the proof of this lemma is finished. □

Lemma 2.8. *Let* $\{(c_n, S_n, I_n)\}$ *be a sequence of traveling wave solution of system* (1.1)–(1.2) *and* $c_n \in (c_{\min}, c_{\min}+1)$. *If* $\{\xi_n\}_{n=1}^{+\infty}$ *is a sequence of real numbers such that* $I_n(\xi_n) \to +\infty$ *as* $n \to +\infty$, *then* $\lim_{n\to+\infty} S_n(\xi_n) = 0$.

Proof. Assume that there is a subsequence of $\{\xi_n\}_{n=1}^{+\infty}$ (still denoted by $\{\xi_n\}_{n=1}^{+\infty}$) such that $I_n(\xi_n) \to +\infty$ as $n \to +\infty$ and $S_n(\xi_n) \geq \delta$ for any $n \in \mathbb{N}$ and some constant $\delta > 0$. It follows from $0 < S_n < \Lambda/\mu$ and (1.14) that $S_n'(\xi_n) \leq (d_S+\mu)\Lambda/(\mu c_{\min})$ for any $\xi \in \mathbb{R}$. Thus,

$$S_n(\xi) = S_n(\xi_n) - S_n'(\tilde{\xi}_n)(\xi_n - \xi) \geq \frac{\delta}{2} \text{ for any } \xi \in [\xi_n - \ell, \xi_n],$$

where $\tilde{\xi}_n \in (\xi, \xi_n)$ and $\ell = \mu\delta c_{\min}/[2(d_S+\mu)\Lambda]$. On the other hand, it follows from Lemma 2.7 and (1.15) that there is a constant L independently of $n \in \mathbb{N}$ such that $|I_n'(\xi)/I_n(\xi)| \le L$ for any $\xi \in \mathbb{R}$. Thus,

$$\frac{I_n(\xi_n)}{I_n(\xi)} = \exp \int_{\xi}^{\xi_n} \frac{I_n'(s)}{I_n(s)} \mathrm{d}s \le \mathrm{e}^{L\ell} \quad \text{for any} \quad \xi \in [\xi_n - \ell, \xi_n].$$

Then, $\min_{\xi \in [\xi_n - \ell, \xi_n]} I_n(\xi) \ge \mathrm{e}^{-L\ell} I_n(\xi_n) \to +\infty$ as $n \to +\infty$. It follows from (1.14) and (2.23) that

$$\max_{\xi \in [\xi_n - \ell, \xi_n]} S_n'(\xi) \le \frac{(d_S+\mu)\Lambda}{\mu c_{\min}} - \frac{\beta\delta}{2(c_{\min}+1)} \min_{\xi \in [\xi_n - \ell, \xi_n]} I_n(\xi) \to -\infty \quad \text{as} \quad n \to +\infty.$$

We can conclude that there exists a constant $N_0 > 0$ such that

$$S_n'(\xi) < -\frac{\Lambda}{\mu\ell} \quad \text{for } \forall n \ge N_0, \quad \forall \xi \in [\xi_n - \ell, \xi_n]. \tag{2.39}$$

Then, we infer from (2.23) and (2.39) that $S_n(\xi_n) < 0$ for $n \ge N_0$, a contradiction to the fact that $S_n(\xi) > 0$ in $\mathbb{R}$. This completes the proof. □

Lemma 2.9. *If* $\limsup_{\xi \to +\infty} I(\xi) = +\infty$*, then* $\lim_{\xi \to +\infty} I(\xi) = +\infty$.

Proof. We claim that $\lim_{\xi \to +\infty} I(\xi) = +\infty$ by a contradiction argument. If $\liminf_{\xi \to +\infty} I(\xi) < +\infty$, there exists a point sequence $\{\xi_n\}_{n=1}^{+\infty}$ such that $\lim_{n \to +\infty} I(\xi_n) = \liminf_{\xi \to +\infty} I(\xi) = I_{inf} < +\infty$. Without loss of generality, we can suppose that $I(\xi_n) \in (0, I_{inf}+1)$ for all $n \in \mathbb{N}$. Now, for each n, we choose a point $\kappa_n \in [\xi_n, \xi_{n+1}]$ such that $I(\kappa_n) = \sup_{\xi \in [\xi_n, \xi_{n+1}]} I(\xi)$. It follows from $\limsup_{\xi \to +\infty} I(\xi) = +\infty$ that $\lim_{n \to +\infty} I(\kappa_n) = +\infty$. We may assume $I(\kappa_n) \ge (I_{inf}+1)\mathrm{e}^{L\varrho_2}$, where $\varrho_2 = \mathrm{diam supp} J_2$ and $L = \sup_{\xi \in \mathbb{R}} |I'(\xi)/I(\xi)|$. Since

$$\frac{I(\kappa_n)}{I(\xi)} = \exp\left\{ \int_{\xi}^{\kappa_n} \frac{I'(\zeta)}{I(\zeta)} \mathrm{d}\zeta \right\} \le \mathrm{e}^{L|\kappa_n - \xi|} \le \mathrm{e}^{L\varrho_2} \quad \text{if} \quad |\xi - \kappa_n| \le \varrho_2, \tag{2.40}$$

which means that $I(\xi) \ge I_{inf} + 1$ for all $\xi \in [\kappa_n - \varrho_2, \kappa_n + \varrho_2]$ and $I(\kappa_n - y) \le I(\kappa_n)$ for all $y \in suppJ_2$. Consequently, we deduce that $[\kappa_n - \varrho_2, \kappa_n + \varrho_2] \subseteq (\xi_n, \xi_{n+1})$ and $I'(\kappa_n) = 0$. Utilizing the equation (1.15), we obtain

$$0 = cI'(\kappa_n) = d_I\left(J_2 * I(\kappa_n) - I(\kappa_n)\right) + \beta S(\kappa_n) I(\kappa_n) - (\mu+\gamma) I(\kappa_n) \le \left[\beta S(\kappa_n) - (\mu+\gamma)\right] I(\kappa_n).$$

We infer from Lemma 2.8 that $\lim_{n \to +\infty} S(\kappa_n) = 0$, which leads to a contradiction. Hence, the claim of this lemma is shown. □

Proposition 2.1. *([46, Proposition 3.7]). Assume that* $c > 0$ *and* $B(\cdot)$ *is a continuous function and* $B(\pm\infty) := \lim_{\xi \to \pm\infty} B(\xi)$*. Let* $Z(\cdot)$ *be a measurable function satisfying*

$$cZ(\xi) = D_i \int_{\mathbb{R}} J_i(y) \mathrm{e}^{-\int_{\xi-y}^{\xi} Z(s)\mathrm{d}s} \mathrm{d}y + B(\xi), \quad \xi \in \mathbb{R}, \quad i = 1, 2.$$

Then, Z *is uniformly bounded and continuous. Moreover,* $\nu^{\pm} := \lim_{\xi \to \pm\infty} Z(\xi)$ *exist and are real number roots of the characteristic equation*

$$c\nu = D_i \int_{\mathbb{R}} J_i(y) \mathrm{e}^{-\nu y} \mathrm{d}y + B(\pm\infty), \quad i = 1, 2.$$

Theorem 2.3. *The function* I *is bounded.*

Proof. We assume on the contrary that $\limsup_{\xi\to+\infty} I(\xi) = +\infty$. Lemmas 2.8 and 2.9 yield $\lim_{\xi\to+\infty} I(\xi) = +\infty$ and $\lim_{\xi\to+\infty} S(\xi) = 0$. Dividing (1.15) by I and setting $Z(\xi) = I'(\xi)/I(\xi)$, we have

$$cZ(\xi) = d_I \int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\int_{\xi-y}^{\xi} Z(s)\mathrm{d}s}\mathrm{d}y + \beta S(\xi) - (d_I + \mu + \gamma). \tag{2.41}$$

Since $\lim_{\xi\to+\infty} S(\xi) = 0$, Proposition 2.1 implies that $\lim_{\xi\to+\infty} Z(\xi)$ exists and satisfies the following equation

$$c\nu = d_I \int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\nu y}\mathrm{d}y - (d_I + \mu + \gamma). \tag{2.42}$$

Set $f_1(\nu, c) = d_I \int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\nu y}\mathrm{d}y - c\nu - (d_I + \mu + \gamma)$. An easy calculation gives

$$f_1(0, c) = -\mu - \gamma < 0, \ \frac{\partial f_1}{\partial \nu}\,|_{\nu=0} = -c < 0, \text{ and } \frac{\partial^2 f_1}{\partial \nu^2}(\nu, c) = d_I \int_{\mathbb{R}} J_2(y)y^2\mathrm{e}^{-\lambda y}\mathrm{d}y > 0.$$

We infer from the above inequalities and $f_1(+\infty, c) = +\infty$ that equation (2.42) has a unique positive solution $\hat{\nu}$. In view of $\lim_{\xi\to+\infty} I(\xi) = +\infty$, we have $\lim_{\xi\to+\infty} Z(\xi) = \hat{\nu} > 0$. Noting that

$$d_I \int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\lambda_i y}\mathrm{d}y - c\lambda_i - d_I - \mu - \gamma = -\frac{\beta\Lambda}{\mu} < 0, \quad i = 1, 2,$$

then we get $\lambda_1 < \lambda_2 < \hat{\nu}$. Hence, there exists a positive constant $\hat{\xi}$ large enough and C_0 such that $I(\xi) \geq C_0 \exp\left(\frac{\lambda_2+\hat{\nu}}{2}\xi\right)$ for all $\xi \geq \hat{\xi}$. On the other hand, by the construction of S, we know that $I(\xi) \leq \mathrm{e}^{\lambda_1\xi}$. Thus, we get a contradiction and complete the proof. □

Now, we will utilize the Lyapunov–LaSalle theorem to prove the asymptotic behavior $(S(\xi), I(\xi))$ as $\xi \to +\infty$.

The proof of Theorem 1.1 for $c > c_{\min}$. From Theorems 2.2 and 2.3, it remains to show that $\lim_{\xi\to+\infty}(S(\xi), I(\xi)) = (s^*, i^*)$. Let

$$g(y) = y - 1 - \ln y \quad \text{in} \quad (0, +\infty),$$

and

$$\Psi_i^+(y) = \int_y^{+\infty} J_i(x)\mathrm{d}x \quad \text{in} \quad [0, +\infty), \quad \Psi_i^-(y) = \int_{-\infty}^{y} J_i(x)\mathrm{d}x \quad \text{in} \quad (-\infty, 0], \quad i = 1, 2.$$

Direct calculations yield

$$g(y) = y - 1 - \ln y \geq 0 \quad \text{in} \quad (0, +\infty), \quad g(y) = y - 1 - \ln y = 0 \quad \text{if and only if} \quad y = 1.$$

Moreover,

$$\Psi_i^+(0) = \Psi_i^-(0) = \frac{1}{2}, \quad \frac{\mathrm{d}}{\mathrm{d}y}\Psi_i^+(y) = -J_i(y), \quad \frac{\mathrm{d}}{\mathrm{d}y}\Psi_i^-(y) = J_i(y), \quad i = 1, 2. \tag{2.43}$$

Now, we consider the Lyapunov functional $\mathcal{V}(S, I) : \mathbb{R}^+ \longrightarrow \mathbb{R}$ as follows

$$\mathcal{V}(S, I)(\xi) := cs^* g\left(\frac{S(\xi)}{s^*}\right) + ci^* g\left(\frac{I(\xi)}{i^*}\right) + d_S s^* \mathcal{U}^S(S)(\xi) + d_I i^* \mathcal{U}^I(I)(\xi), \tag{2.44}$$

where

$$\mathcal{U}^S(S)(\xi) = \int_0^{+\infty} \Psi_1^+(y) g\left(\frac{S(\xi - y)}{s^*}\right)\mathrm{d}y - \int_{-\infty}^{0} \Psi_1^-(y) g\left(\frac{S(\xi - y)}{s^*}\right)\mathrm{d}y$$

$$\mathcal{U}^I(I)(\xi) = \int_0^{+\infty} \Psi_2^+(y) g\left(\frac{I(\xi-y)}{i^*}\right) \mathrm{d}y - \int_{-\infty}^0 \Psi_2^-(y) g\left(\frac{I(\xi-y)}{i^*}\right) \mathrm{d}y.$$

In view of $0 < S(\xi) < \Lambda/\mu$ and $0 < I(\xi) < K$ for some constant $K > 0$ and any $\xi \geq 0$, we infer that there exists $m \in \mathbb{R}$ such that $m \leq \mathcal{V}(S,I)(\xi) < +\infty$ for any $\xi \geq 0$. Differentiating the equality (2.44), we get

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{V}(S,I)(\xi) =& \left(1-\frac{s^*}{S(\xi)}\right) cS'(\xi) + \left(1-\frac{i^*}{I(\xi)}\right) cI'(\xi) + d_S s^* \frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{U}^S(S)(\xi) + d_I i^* \frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{U}^I(I)(\xi) \\
=& \underbrace{\left(1-\frac{s^*}{S(\xi)}\right)\left(d_S\int_{\mathbb{R}} J_1(y)S(\xi-y)\mathrm{d}y - d_S S(\xi)\right) + d_S s^* \frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{U}^S(S)(\xi)}_{\Theta_1} \\
&+ \underbrace{\left(1-\frac{i^*}{I(\xi)}\right)\left(d_I\int_{\mathbb{R}} J_2(y)I(\xi-y)\mathrm{d}y - d_I I(\xi)\right) + d_I i^* \frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{U}^I(I)(\xi)}_{\Theta_2} \qquad (2.45) \\
&+ \underbrace{\left(1-\frac{s^*}{S(\xi)}\right)(\Lambda - \mu S(\xi) - \beta S(\xi)I(\xi)) + \left(1-\frac{i^*}{I(\xi)}\right)(\beta S(\xi)I(\xi) - (\mu+\gamma)I(\xi))}_{\Theta_3}.
\end{aligned}$$

Along the solution $(S(\xi), I(\xi))$, we obtain that

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{U}^S(S)(\xi) =& \int_0^{+\infty} \Psi_1^+(y)\frac{\mathrm{d}}{\mathrm{d}\xi} g\left(\frac{S(\xi-y)}{s^*}\right)\mathrm{d}y - \int_{-\infty}^0 \Psi_1^-(y)\frac{\mathrm{d}}{\mathrm{d}\xi} g\left(\frac{S(\xi-y)}{s^*}\right)\mathrm{d}y \\
=& -\int_0^{+\infty} \Psi_1^+(y)\frac{\mathrm{d}}{\mathrm{d}y} g\left(\frac{S(\xi-y)}{s^*}\right)\mathrm{d}y + \int_{-\infty}^0 \Psi_1^-(y)\frac{\mathrm{d}}{\mathrm{d}y} g\left(\frac{S(\xi-y)}{s^*}\right)\mathrm{d}y \\
=& -\Psi_1^+(y) g\left(\frac{S(\xi-y)}{s^*}\right)\Big|_0^{+\infty} - \int_0^{+\infty} J_1(y) g\left(\frac{S(\xi-y)}{s^*}\right)\mathrm{d}y \\
&+ \Psi_1^-(y) g\left(\frac{S(\xi-y)}{s^*}\right)\Big|_{-\infty}^0 - \int_{-\infty}^0 J_1(y) g\left(\frac{S(\xi-y)}{s^*}\right)\mathrm{d}y \\
=& g\left(\frac{S(\xi)}{s^*}\right) - \int_{\mathbb{R}} J_1(y) g\left(\frac{S(\xi-y)}{s^*}\right)\mathrm{d}y. \qquad (2.46)
\end{aligned}$$

It follows from (2.46) that

$$\begin{aligned}
\Theta_1 =& \left(1-\frac{s^*}{S(\xi)}\right)\left(d_S\int_{\mathbb{R}} J_1(y)S(\xi-y)\mathrm{d}y - d_S S(\xi)\right) + d_S s^* \frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{U}^S(S)(\xi) \\
=& \left(1-\frac{s^*}{S(\xi)}\right)\left(d_S\int_{\mathbb{R}} J_1(y)S(\xi-y)\mathrm{d}y - d_S S(\xi)\right) + d_S s^* \left(g\left(\frac{S(\xi)}{s^*}\right) - \int_{\mathbb{R}} J_1(y) g\left(\frac{S(\xi-y)}{s^*}\right)\mathrm{d}y\right) \\
=& d_S\int_{\mathbb{R}} J_1(y)S(\xi-y)\mathrm{d}y - d_S S(\xi) - d_S s^* \int_{\mathbb{R}} J_1(y)\frac{S(\xi-y)}{S(\xi)}\mathrm{d}y + d_S s^* \\
&+ d_S s^* \left(\frac{S(\xi)}{s^*} - 1 - \ln\frac{S(\xi)}{s^*}\right) - d_S s^* \int_{\mathbb{R}} J_1(y)\left(\frac{S(\xi-y)}{s^*} - 1 - \ln\frac{S(\xi-y)}{s^*}\right)\mathrm{d}y \\
=& -d_S s^* \int_{\mathbb{R}} J_1(y) g\left(\frac{S(\xi-y)}{S(\xi)}\right)\mathrm{d}y. \qquad (2.47)
\end{aligned}$$

In a similar way, we get

$$\Theta_2 = -d_I i^* \int_{\mathbb{R}} J_2(y) g\left(\frac{I(\xi-y)}{I(\xi)}\right)\mathrm{d}y. \qquad (2.48)$$

Using the facts that $\Lambda = \mu s^* + \beta s^* i^*$ and $\beta s^* = \mu + \gamma$, we deduce that

$$
\begin{aligned}
\Theta_3 &= \left(1 - \frac{s^*}{S(\xi)}\right)(\mu s^* - \mu S(\xi) + \beta s^* i^* - \beta S(\xi) I(\xi)) + \left(1 - \frac{i^*}{I(\xi)}\right)(\beta S(\xi) I(\xi) - \beta s^* I(\xi)) \\
&= - \frac{(S(\xi) - s^*)^2}{S(\xi)}(\mu + \beta i^*) + \beta (S(\xi) - s^*)(i^* - I(\xi)) + \beta (I(\xi) - i^*)(S(\xi) - s^*) \\
&= - \frac{(S(\xi) - s^*)^2}{S(\xi)}(\mu + \beta i^*). \qquad (2.49)
\end{aligned}
$$

It follows from (2.45) and (2.47)-(2.49) that

$$
\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{V}(S, I)(\xi) = &- d_S s^* \int_{\mathbb{R}} J_1(y) g\left(\frac{S(\xi - y)}{S(\xi)}\right)\mathrm{d}y \\
&- d_I i^* \int_{\mathbb{R}} J_2(y) g\left(\frac{I(\xi - y)}{I(\xi)}\right)\mathrm{d}y - \frac{(S(\xi) - s^*)^2}{S(\xi)}(\mu + \beta i^*) \le 0 \quad \text{for} \quad \xi \ge 0, \qquad (2.50)
\end{aligned}
$$

which indicates that $\mathcal{V}(S, I)(\cdot)$ is non-increasing for $\xi \ge 0$. Now, choose an increasing sequence $\{\xi_n\}_{n=0}^{+\infty}$ such that $\xi_n \to \infty$ as $n \to +\infty$ and let $S_n(\xi) = S(\xi + \xi_n),\ I_n(\xi) = I(\xi + \xi_n)$. Since $\{S_n\}_{n=0}^{+\infty}$ and $\{I_n\}_{n=0}^{+\infty}$ are uniformly bounded in $C^{1,1}(\mathbb{R})$, up to extraction a subsequence, we can assume that there exist two nonnegative functions $\check{S}(\cdot)$ and $\check{I}(\cdot)$ such that $S_n(\cdot) \to \check{S}(\cdot),\ I_n(\cdot) \to \check{I}(\cdot)$ in $C^0_{loc}(\mathbb{R})$ as $n \to +\infty$. Furthermore, since $\mathcal{V}(S, I)$ is bounded from below and non-increasing in ξ, then for large n, $m \le \mathcal{V}(S_n, I_n)(\xi) \le \mathcal{V}(S, I)(\xi + \xi_n) \le \mathcal{V}(S, I)(\xi)$. Consequently, there exists some $\varpi \in \mathbb{R}^1$ such that $\lim_{n\to+\infty} \mathcal{V}(S_n, I_n)(\xi) = \lim_{\xi+\xi_n\to+\infty} \mathcal{V}(S, I)(\xi + \xi_n) = \varpi,\ \forall \xi \in \mathbb{R}$. We infer from Lebesgue dominated convergence theorem that $\varpi = \lim_{n\to+\infty} \mathcal{V}(S_n, I_n)(\xi) = \mathcal{V}(\check{S}, \check{I})(\xi),\ \forall \xi \in \mathbb{R}$. Therefore, $\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{V}(\check{S}, \check{I})(\xi) = 0,\ \forall \xi \ge 0$. By the classical Lyapunov–LaSalle invariance principle, solutions limit to $\mathcal{M}$, the largest invariant subset in $\left\{(S(\xi), I(\xi)) | \frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{V}(S, I)(\xi) = 0\right\}$. We note that $\frac{\mathrm{d}}{\mathrm{d}\xi}\mathcal{V}(S, I)(\xi)$ is only zero if $S(\xi) = s^*,\ I(\xi) \equiv \check{C}$ for some constant $\check{C}$. In particular, from (1.14)-(1.15), this requires that for any solution in $\mathcal{M}$ we have $S(\xi) = s^*,\ I(\xi) = i^*$ for all ξ, and so $\mathcal{M}$ consists of the single point E^*. It follows that $\check{S} = s^*,\ \check{I} = i^*$, which implies $\lim_{\xi\to+\infty}(S(\xi), I(\xi)) = E^* = (s^*, i^*)$. The proof is completed. □

3. Existence of travelling wave solutions for $c = c_{\min}$

In this section, we intend to prove the existence of traveling wave solution of system (1.1)-(1.2) with $c = c_{\min}$ by the approximating method. Choose a strictly decreasing sequence $\{c_n\}_{n=1}^{+\infty} \subseteq (c_{\min}, c_{\min} + 1)$ such that $\lim_{n\to+\infty} c_n = c_{\min}$. Let (c_n, S_n, I_n) be a traveling wave solution of system (1.1)-(1.2) satisfying the asymptotic boundary conditions (1.16). Then, we have the following result.

Lemma 3.1. *It holds* $\limsup_{n\to+\infty} ||I_n(\cdot)||_{L^\infty(\mathbb{R})} < +\infty$.

Proof. For contradiction, we may assume that $||I_n(\cdot)||_{L^\infty(\mathbb{R})} \to +\infty$ as $n \to +\infty$. Since $I_n(\cdot)$ is bounded in $\mathbb{R}$ for each n, then there exists $\xi_n \in \mathbb{R}$ such that $I_n(\xi_n) \ge n/(n+1)||I_n(\cdot)||_{L^\infty(\mathbb{R})}$. It is not difficult to see that $I_n(\xi_n) \to +\infty$ as $n \to +\infty$. Now, we define $\tilde{S}_n(\xi) = S_n(\xi + \xi_n),\ \tilde{I}_n(\xi) = I_n(\xi + \xi_n)/I_n(\xi_n)$. Then, for each $n \in \mathbb{N}$, $(\tilde{S}_n(\cdot), \tilde{I}_n(\cdot))$ satisfies

$$
c_n \tilde{S}_n'(\xi) = d_S \left(\int_{\mathbb{R}} J_1(y)\tilde{S}_n(\xi - y)\mathrm{d}y - \tilde{S}_n(\xi)\right) + \Lambda - \mu \tilde{S}_n(\xi) - \beta \tilde{S}_n(\xi)\tilde{I}_n(\xi) I_n(\xi_n), \qquad (3.1)
$$

$$
c_n \tilde{I}_n'(\xi) = d_I \left(\int_{\mathbb{R}} J_2(y)\tilde{I}_n(\xi - y)\mathrm{d}y - \tilde{I}_n(\xi)\right) + \beta \tilde{S}_n(\xi)\tilde{I}_n(\xi) - (\mu + \gamma)\tilde{I}_n(\xi). \qquad (3.2)
$$

From (1.15), we know that

$$I_n'(\xi) \geq \frac{d_I}{c_{\min}+1}\int_{\mathbb{R}} J_2(y)I_n(\xi-y)\mathrm{d}y - \frac{d_I+\mu+\gamma}{c_{\min}}I_n(\xi).$$

It derives from Lemma 2.7 that $|I_n'(\xi)/I_n(\xi)|$ is globally bounded in $\mathbb{R}$. In view of

$$\tilde{I}_n(\xi) = I_n(\xi+\xi_n)/I_n(\xi_n) = \exp\int_{\xi_n}^{\xi+\xi_n}\frac{I_n'(s)}{I_n(s)}\mathrm{d}s,$$

we deduce that $\sup_{n\in\mathbb{N}}||\tilde{I}_n(\cdot)||_{L^\infty(\mathcal{K})} < +\infty$ for any compact set $\mathcal{K} \subseteq \mathbb{R}$. Moreover, $I_n(\cdot+\xi_n) \to +\infty$ in $C^0_{loc}(\mathbb{R})$ as $n \to +\infty$. According to Lemma 2.8, we have $\tilde{S}_n(\cdot) \to 0$ in $C^0_{loc}(\mathbb{R})$ as $n \to +\infty$. On the other hand,

$$\tilde{I}_n'(\xi) = \frac{I_n'(\xi+\xi_n)}{I_n(\xi_n)} = \frac{I_n'(\xi+\xi_n)}{I_n(\xi+\xi_n)}\cdot \tilde{I}_n(\xi).$$

Therefore, $\sup_{n\in\mathbb{N}}||\tilde{I}_n'(\cdot)||_{L^\infty(\mathcal{K})} < +\infty$ for any compact set $\mathcal{K} \subseteq \mathbb{R}$. Furthermore, each $\tilde{I}_n(\cdot)$ satisfies

$$c_n\tilde{I}_n''(\xi) = d_I\left(\int_{\mathbb{R}} J_2(y)\tilde{I}_n'(\xi-y)\mathrm{d}y - \tilde{I}_n'(\xi)\right) + \beta\tilde{S}_n'(\xi)\tilde{I}_n(\xi) + \beta\tilde{S}_n(\xi)\tilde{I}_n'(\xi) - (\mu+\gamma)\tilde{I}_n'(\xi).$$

Thus, we also conclude that $\sup_{n\in\mathbb{N}}||\tilde{I}_n''(\cdot)||_{L^\infty(\mathcal{K})} < +\infty$ for any compact set $\mathcal{K} \subseteq \mathbb{R}$. Hence, from Arzela–Ascoli theorem, some sequence $\{\tilde{I}_{n_k}(\xi)\}$ exists, still denoted by $\{\tilde{I}_n(\xi)\}$, such that $\tilde{I}_n(\cdot) \to \tilde{I}_\infty(\cdot)$ in $C^1_{loc}(\mathbb{R})$ as $n \to +\infty$ and $\tilde{I}_\infty(\cdot)$ satisfies

$$c_{\min}\tilde{I}_\infty'(\xi) = d_I\left(\int_{\mathbb{R}} J_2(y)\tilde{I}_\infty(\xi-y)\mathrm{d}y - \tilde{I}_\infty(\xi)\right) - (\mu+\gamma)\tilde{I}_\infty(\xi) \quad \text{in } \mathbb{R}, \tag{3.3}$$

Here we use the fact that $\tilde{S}_n(\cdot) \to 0$ in $C^0_{loc}(\mathbb{R})$ as $n \to +\infty$. Now, we claim that $\tilde{I}_\infty(\xi) > 0$ for all $\xi \in \mathbb{R}$. Otherwise, there exists some $\xi_* \in \mathbb{R}$ such that $\tilde{I}_\infty(\xi_*) = 0$. Then, $\tilde{I}_\infty'(\xi_*) = 0$. It follows from (3.3) that $\int_{\mathbb{R}} J_2(y)\tilde{I}_\infty(\xi_*-y)\mathrm{d}y = 0$, which means $\tilde{I}_\infty(\cdot) \equiv 0$ in $\mathbb{R}$. Note that $\tilde{I}_\infty(0) = \lim_{n\to+\infty}\tilde{I}_n(0) = 1$, this leads to a contradiction. Thus, the claim holds. It is noticed that $I_n(\xi+\xi_n) \leq ||I_n(\cdot)||_{L^\infty(\mathbb{R})} \leq \left(1+\frac{1}{n}\right)I_n(\xi_n)$, i.e. $\tilde{I}_n(\xi) \leq 1+1/n$. Letting $n \to +\infty$, we obtain $\tilde{I}_\infty(\cdot) \leq 1$ in $\mathbb{R}$, which indicates that $\tilde{I}_\infty(0) = 1$ is the global maximum of $\tilde{I}_\infty$. Consequently, it holds $0 \leq -(\mu+\gamma) < 0$, this is impossible. Thus, we infer that $\limsup_{\xi\to+\infty}||I_n(\cdot)||_{L^\infty(\mathbb{R})} < +\infty$ and this lemma is proved. □

Lemma 3.2. *There exists $\sigma > 0$ such that for any traveling wave solution (c_n, S_n, I_n), it holds $I_n'(\xi) > 0$ if $I_n(\xi) \leq \sigma$ for any $\xi \in \mathbb{R}$.*

Proof. Without loss of generality, we assume that there exists a point sequence $\{\xi_n\}_{n=1}^{+\infty}$ such that $\lim_{n\to+\infty} I_n(\xi_n) = 0$ and $I_n'(\xi_n) \leq 0$ for all $n \in \mathbb{N}$. Denoting $\tilde{S}_n(\xi) = S_n(\xi+\xi_n)$ and $\tilde{I}_n(\xi) = I_n(\xi+\xi_n)$, then $(\tilde{S}_n(\cdot), \tilde{I}_n(\cdot))$ satisfies

$$c_n\tilde{S}_n'(\xi) = d_S\left(\int_{\mathbb{R}} J_1(y)\tilde{S}_n(\xi-y)\mathrm{d}y - \tilde{S}_n(\xi)\right) + \Lambda - \mu\tilde{S}_n(\xi) - \beta\tilde{S}_n(\xi)\tilde{I}_n(\xi), \tag{3.4}$$

$$c_n\tilde{I}_n'(\xi) = d_I\left(\int_{\mathbb{R}} J_2(y)\tilde{I}_n(\xi-y)\mathrm{d}y - \tilde{I}_n(\xi)\right) + \beta\tilde{S}_n(\xi)\tilde{I}_n(\xi) - (\mu+\gamma)\tilde{I}_n(\xi). \tag{3.5}$$

Applying Lemma 2.7 to (1.15), we obtain that $|I_n'(\xi)/I_n(\xi)|$ are globally bounded in $\mathbb{R}$. Therefore, $\tilde{I}_n(\cdot) \to 0$ in $C^0_{loc}(\mathbb{R})$ as $n \to +\infty$. Furthermore, we have $\tilde{I}_n(\cdot) \to 0$ in $C^1_{loc}(\mathbb{R})$ as $n \to +\infty$. On the other hand, by Lemma 3.1, we know that $||\tilde{I}_n(\cdot)||_{C^1(\mathbb{R})} < +\infty$. Moreover, we deduce from (3.4) that $\tilde{S}_n'(\cdot)$ and $\tilde{S}_n''(\cdot)$ are uniformly bounded in $\mathbb{R}$. Hence, up to extraction a sequence, there exists a function $\tilde{S}_\infty(\cdot) \in C^1(\mathbb{R})$ such that $\tilde{S}_n(\cdot) \to \tilde{S}_\infty(\cdot)$ in $C^1_{loc}(\mathbb{R})$ as $n \to +\infty$, we have

$$c_{\min}\tilde{S}_\infty'(\xi) = d_S\left(\int_{\mathbb{R}} J_1(y)\tilde{S}_\infty(\xi - y)\mathrm{d}y - \tilde{S}_\infty(\xi)\right) + \Lambda - \mu\tilde{S}_\infty(\xi). \tag{3.6}$$

Let $S_{inf} = \inf_{\xi\in\mathbb{R}} \tilde{S}_\infty(\xi)$ and choose a sequence $\{\zeta_n\}$ such that $\tilde{S}_\infty(\zeta_n) \to S_{\inf}$ as $n \to +\infty$. Setting $\hat{S}_n(\xi) = \tilde{S}_\infty(\xi + \zeta_n)$, we may assume that $\hat{S}_n(\cdot) \to \hat{S}_\infty(\cdot)$ in $C^1_{loc}(\mathbb{R})$ as $n \to +\infty$ for a function $\hat{S}_\infty(\cdot) \in C^1(\mathbb{R})$ solving

$$c_{\min}\hat{S}_\infty'(\xi) = d_S\int_{\mathbb{R}} J_1(y)\left(\hat{S}_\infty(\xi - y) - \hat{S}_\infty(\xi)\right)\mathrm{d}y + \Lambda - \mu\hat{S}_\infty(\xi). \tag{3.7}$$

Note that $S_{inf} \le \hat{S}_\infty(\cdot) \le \Lambda/\mu$ for $\xi \in \mathbb{R}$. It follows from $\hat{S}_\infty(0) = S_{inf}$ that

$$\Lambda - \mu S_{inf} = -d_S\int_{\mathbb{R}} J_1(y)\left(\hat{S}_\infty(-y) - \hat{S}_\infty(0)\right)\mathrm{d}y \le 0,$$

which implies $S_{inf} \ge \Lambda/\mu$. As a consequence, we infer that $\tilde{S}_\infty(\cdot) \equiv \Lambda/\mu$ for $\xi \in \mathbb{R}$. Now, define $\hat{I}_n(\xi) = I_n(\xi + \xi_n)/I_n(\xi_n)$, we know from the proof of Lemma 3.1 that

$$\sup_{n\in\mathbb{N}} ||\hat{I}_n(\cdot)||_{L^\infty(\mathcal{K})} < +\infty,\ \sup_{n\in\mathbb{N}} ||\hat{I}_n'(\cdot)||_{L^\infty(\mathcal{K})} < +\infty,\ \sup_{n\in\mathbb{N}} ||\hat{I}_n''(\cdot)||_{L^\infty(\mathcal{K})} < +\infty$$

for any compact set $\mathcal{K} \subseteq \mathbb{R}$ and

$$c_n\hat{I}_n'(\xi) = d_I\int_{\mathbb{R}} J_2(y)\left(\hat{I}_n(\xi - y) - \hat{I}_n(\xi)\right)\mathrm{d}y + \beta\tilde{S}_n(\xi)\hat{I}_n(\xi) - (\mu + \gamma)\,\hat{I}_n(\xi). \tag{3.8}$$

Then, from Arzela–Ascoli theorem, up to extraction a subsequence, we have $\hat{I}_n(\cdot) \to \hat{I}_\infty(\cdot)$ in $C^1_{loc}(\mathbb{R})$, which satisfies

$$c_{\min}\hat{I}_\infty'(\xi) = d_I\int_{\mathbb{R}} J_2(y)\left(\hat{I}_\infty(\xi - y) - \hat{I}_\infty(\xi)\right)\mathrm{d}y + \left(\frac{\beta\Lambda}{\mu} - \mu - \gamma\right)\hat{I}_\infty(\xi). \tag{3.9}$$

Moreover, $\hat{I}_\infty(\cdot) > 0$ in $\mathbb{R}$. Letting $Z(\xi) = \hat{I}_\infty'(\xi)/\hat{I}_\infty(\xi)$, it follows from (3.9) that

$$c_{\min}Z(\xi) = d_I\int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\int_{\xi-y}^{\xi} Z(s)\mathrm{d}s}\mathrm{d}y - d_I + \frac{\beta\Lambda}{\mu} - \mu - \gamma. \tag{3.10}$$

We infer from Proposition 2.1 that $\lim_{\xi\to\pm\infty} Z(\xi)$ exists and satisfies the equation

$$c_{\min}\lambda = d_I\int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\lambda y}\mathrm{d}y - d_I + \frac{\beta\Lambda}{\mu} - \mu - \gamma. \tag{3.11}$$

It is easy to see that (3.11) has a unique positive root, then we conclude that $\lim_{\xi\to\pm\infty} Z(\xi) > 0$. Since $Z(0) = \hat{I}_\infty'(0)/\hat{I}_\infty(0) = \lim_{n\to+\infty} I_n'(\xi_n)/I_n(\xi_n) \le 0$. Thus, the continuous function $Z(\cdot)$ has a minimum $Z(\xi^*) = Z_{min}$ in $\mathbb{R}$. By differentiating (3.10), one gets

$$c_{\min}Z'(\xi) = d_I\int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\int_{\xi-y}^{\xi} Z(s)\mathrm{d}s}\left(Z(\xi - y) - Z(\xi)\right)\mathrm{d}y.$$

It follows that $Z(\xi) \equiv Z(\xi^*)$. Hence, $\lim_{\xi\to\pm\infty} Z(\xi) = Z(\xi^*) \le Z(0) \le 0$, which leads to a contradiction. The claim of this lemma is holds. $\square$

The proof of Theorem 1.1 for $c = c_{\min}$. For $\sigma > 0$ in Lemma 3.2, we can choose $\sigma > 0$ small enough such that $0 < \sigma < i^*$. In view of $\lim_{\xi \to -\infty} I_n(\xi) = 0$ and $I_n(\cdot) > 0$ in $\mathbb{R}$ for any $n \in \mathbb{N}$, there exists $\xi_n \in \mathbb{R}$ such that $I_n(\xi_n) = \sigma$ for each n. Set $\breve{S}_n(\xi) = S_n(\xi + \xi_n)$, $\breve{I}_n(\xi) = I_n(\xi + \xi_n)$. It follows from Lemma 3.1 that $\breve{I}_n(\cdot)$ are uniformly bounded in $\mathbb{R}$ and so $||\breve{S}_n(\cdot)||_{C^2(\mathbb{R})}$ and $||\breve{I}_n(\cdot)||_{C^2(\mathbb{R})}$ are all uniformly bounded in $\mathbb{R}$. Thus, by Arzela-Ascoli theorem, there exists $(S_*(\cdot), I_*(\cdot)) \in C^1(\mathbb{R}) \times C^1(\mathbb{R})$ such that $\breve{S}_n(\cdot) \to S_*(\cdot)$ and $\breve{I}_n(\cdot) \to I_*(\cdot)$ in $C^1_{loc}(\mathbb{R})$ as $n \to +\infty$. Moreover, $(S_*(\cdot), I_*(\cdot))$ satisfies

$$c_{\min} S_*'(\xi) = d_S \int_{\mathbb{R}} J_1(y) \left(S_*(\xi - y) - S_*(y)\right) \mathrm{d}y + \Lambda - \mu S_*(\xi) - \beta S_*(\xi) I_*(\xi), \tag{3.12}$$

$$c_{\min} I_*'(\xi) = d_I \int_{\mathbb{R}} J_2(y) \left(I_*(\xi - y) - I_*(\xi)\right) \mathrm{d}y + \beta S_*(\xi) I_*(\xi) - (\mu + \gamma) I_*(\xi). \tag{3.13}$$

Furthermore, $0 \le S_*(\cdot) \le \Lambda/\mu$, $0 \le I_*(\cdot) < +\infty$ in $\mathbb{R}$ and $I_*(0) = \sigma$.

Below, we divide three steps to complete the proof.

Step 1. $0 < S_*(\cdot) < \Lambda/\mu$ and $I_*(\cdot) > 0$ in $\mathbb{R}$.

Assume for contrary that $\tilde{\xi}_*$ such that $I_*(\tilde{\xi}_*) = 0$. Then, $I_*'(\tilde{\xi}_*) = 0$. It follows from (3.13) that $\int_{\mathbb{R}} J_2(y) \left(I_*(\tilde{\xi}_* - y) - I_*(\tilde{\xi}_*)\right) \mathrm{d}y = 0$. Hence, $I_*(\xi) \equiv I_*(\tilde{\xi}_*) = 0$. This contradicts to $I_*(0) = \sigma > 0$. Consequently, $I_*(\cdot) > 0$ in $\mathbb{R}$. As in the proof of Theorem 2.2, we can show that $0 < S_*(\cdot) < \Lambda/\mu$ in $\mathbb{R}$.

Step 2. The asymptotic behavior of $S_*(\cdot)$ and $I_*(\cdot)$ as $\xi \to -\infty$.

Since $I_*(0) = \sigma$, Lemma 3.2 implies that $I_*'(\cdot) > 0$ in $(-\infty, 0]$. Therefore, the limit $\lim_{\xi \to -\infty} I_*(\xi)$ exists and $\underline{i} := \lim_{\xi \to -\infty} I_*(\xi)$. It follows that $\underline{i} \in [0, \sigma)$. If $\underline{i} > 0$, set $(S_*)_n(\xi) = S_*(\xi + \xi_n^*)$ and $(I_*)_n(\xi) = I_*(\xi + \xi_n^*)$ for any $\xi_n^* \to -\infty$ as $n \to +\infty$. Up to extraction a subsequence, we have $(I_*)_n(\cdot) \to \underline{i}$ and $(S_*)_n(\cdot) \to (S_*)_\infty(\cdot)$ for some function $(S_*)_\infty(\cdot) \in C^1(\mathbb{R})$ in $C^1_{loc}(\mathbb{R})$. Moreover, $(S_*)_\infty(\cdot)$ satisfies $0 = \beta (S_*)_\infty(\xi) \underline{i} - (\mu + \gamma) \underline{i}$, which indicates $(S_*)_\infty(\xi) = (\mu + \gamma)/\beta$. In a similar way, we infer that $\underline{i} = \lim_{\xi \to -\infty} I_*(\xi) = \left(\Lambda - \mu (S_*)_\infty(\xi)\right) / \beta (S_*)_\infty(\xi) = i^* < \sigma$, which leads to a contradiction. Consequently, $\lim_{\xi \to -\infty} I_*(\xi) = 0$. As in the proof of Lemma 3.2, we get $\lim_{\xi \to -\infty} S_*(\xi) = \Lambda/\mu$. Finally, from (3.13), we obtain that

$$c_{\min} \frac{I_*'(\xi)}{I_*(\xi)} = d_I \int_{\mathbb{R}} J_2(y) \frac{I_*(\xi - y)}{I_*(\xi)} \mathrm{d}y + \beta S_*(\xi) - (d_I + \mu + \gamma).$$

In view of $\lim_{\xi \to -\infty} S_*(\xi) = \Lambda/\mu$, it follows from Proposition 2.1 that $\lim_{\xi \to -\infty} I_*'(\xi)/I_*(\xi) = \nu_*$, where ν_* satisfies

$$c_{\min} \nu_* = d_I \int_{\mathbb{R}} J_2(y) \mathrm{e}^{-\nu_* y} \mathrm{d}y + \beta \Lambda/\mu - (d_I + \mu + \gamma).$$

According to Lemma 2.1, $\nu_* = \lambda_0$. Thus, we infer that $I_*(\xi) = \mathcal{O}(\mathrm{e}^{\lambda_0 \xi})$ as $\xi \to -\infty$.

Step 3. The asymptotic behavior of $S_*(\cdot)$ and $I_*(\cdot)$ as $\xi \to +\infty$.

By replacing c to $c_{\min}$ in the proof $c > c_{\min}$, we obtain the desire result. □

4. Non-existence of traveling wave solutions

Theorem 4.1. *Suppose $\mathfrak{R}_0 > 1$. Then, the system (1.14)–(1.15) has no non-trivial and non-negative solution $(S(x+ct), I(x+ct))$ satisfying the asymptotic boundary conditions (1.16) for any $c < c_{\min}$ and $c \neq 0$.*

Proof. Suppose for the contrary that $(S(x + ct), I(x + ct))$ is a pair of positive solution of (1.14)–(1.15). Moreover, condition (1.16) holds. We infer from Lemma 2.7 and

$$c \frac{I'(\xi)}{I(\xi)} \ge d_I \int_{\mathbb{R}} J_2(\xi) \mathrm{e}^{\int_{\xi}^{\xi - y} \frac{I'(s)}{I(s)} \mathrm{d}s} \mathrm{d}y - (d_I + \mu + \gamma)$$

that $\int_{\mathbb{R}} J_2(\xi)\mathrm{e}^{\int_\xi^{\xi-y} I'(s)/I(s)\mathrm{d}s}\mathrm{d}y$ is bounded in $\mathbb{R}$ and so is $I'(\xi)/I(\xi)$. For any sequence $\{\xi_n\}$ converging to $-\infty$, define $S_n(\xi) = S(\xi+\xi_n),\ I_n(\xi) = I(\xi+\xi_n)/I(\xi_n)$. Obviously, (S_n, I_n) satisfies

$$cI_n'(\xi) = d_I \int_{\mathbb{R}} J_2(y)\left(I_n(\xi-y) - I_n(\xi)\right)\mathrm{d}y + \beta S_n(\xi)I_n(\xi) - (\mu+\gamma)I_n(\xi).$$

It follows from $S(-\infty) = \Lambda/\mu$ that $\lim_{n\to+\infty} S_n(\xi) = \Lambda/\mu$ locally uniformly in $\mathbb{R}$. In view of $I_n'(\xi)/I_n(\xi)$ is bounded in $L^\infty(\mathbb{R})$, we obtain that $I_n(\xi)$ are locally uniformly bounded in $\mathbb{R}$. Therefore, $I_n'(\xi)$ and $I_n''(\xi)$ are locally uniformly bounded in $\mathbb{R}$ too. By Arzela–Ascoli theorem, we infer that, up to extraction a subsequence, $I_n(\xi) \to I_{-\infty}(\xi)$ in $C^1_{loc}(\mathbb{R})$ as $n\to+\infty$, where $I_{-\infty}(\xi)$ is a nonnegative function and satisfies

$$cI_{-\infty}'(\xi) = d_I \int_{\mathbb{R}} J_2(y)\left(I_{-\infty}(\xi-y) - I_{-\infty}(\xi)\right)\mathrm{d}y + \left(\frac{\beta\Lambda}{\mu} - \mu - \gamma\right) I_{-\infty}(\xi). \tag{4.1}$$

Moreover, $I_{-\infty}(\cdot) \geq 0$ in $\mathbb{R}$ and $I_{-\infty}(0) = 1$. As proof in Lemma 3.1, we have $I_{-\infty}(\cdot) > 0$ in $\mathbb{R}$. Let $Z(\xi) = I_{-\infty}'(\xi)/I_{-\infty}(\xi)$. By Proposition 2.1, the limit $\lim_{\xi\to\pm\infty} Z(\xi)$ exist and are the roots of the equation

$$c\lambda = d_I \int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\lambda y}\mathrm{d}y - d_I + \frac{\beta\Lambda}{\mu} - \gamma - \mu.$$

But when $0 < c < c_{\min}$, the above equation has no root in $\mathbb{R}$, which leads to a contradiction.

For the case $c < 0$. Denote $\hat{S}(\xi) = S(-\xi),\ \hat{I}(\xi) = I(-\xi)$. Then, $(\hat{S}(+\infty), \hat{I}(+\infty)) = (\Lambda/\mu, 0)$ and $\hat{I}$ satisfies

$$\hat{I}'(\xi) \geq \frac{d_I}{|c|}\int_{\mathbb{R}} J_2(y)\hat{I}(\xi-y)\mathrm{d}y - \frac{d_I+\mu+\gamma}{|c|}\hat{I}(\xi).$$

Applying Lemma 2.7 again, we infer that $\hat{I}'(\xi)/\hat{I}(\xi)$ is bounded in $\mathbb{R}$. Since $\hat{I}(\cdot) > 0$ in $\mathbb{R}$ and $\hat{I}(+\infty) = 0$, we can choose a sequence $\{\hat{\xi}_n\}_{n=1}^{+\infty}$ satisfying $\hat{\xi}_n \to +\infty$ as $n\to+\infty$ such that $\hat{I}'(\hat{\xi}_n) \leq 0$. Set $\hat{S}_n(\xi) = \hat{S}(\xi+\hat{\xi}_n),\ \hat{I}_n(\xi) = \hat{I}(\xi+\hat{\xi}_n)/\hat{I}(\hat{\xi}_n)$. By Arzela–Ascoli theorem, there exists some $\hat{I}_\infty \in C^1(\mathbb{R})$ such that $\hat{I}_n(\xi) \to \hat{I}_\infty(\xi)$ in $C^1_{loc}(\mathbb{R})$. Moreover, $\hat{I}_\infty$ satisfies (4.1) with $|c|$ and $I_{-\infty}$ instead of c and $\hat{I}_\infty$, respectively. It is not difficult to get that $\hat{I}_\infty(0) = 1$ and $\hat{I}_\infty'(0) \leq 0$. Denote $\hat{Z}(\xi) = \hat{I}_\infty'(\xi)/\hat{I}_\infty(\xi)$. Then $\hat{Z}(\xi)$ satisfies

$$|c|\hat{Z}(\xi) = d_I \int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\int_{\xi-y}^{\xi}\hat{Z}(s)\mathrm{d}s}\mathrm{d}y - d_I + \frac{\beta\Lambda}{\mu} - \mu - \gamma. \tag{4.2}$$

It follows from Proposition 2.1 that $\hat{Z}(\pm\infty)$ exist and satisfy

$$|c|\hat{Z}(\pm\infty) = d_I \int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\int_{\xi-y}^{\xi}\hat{Z}(\pm\infty)\mathrm{d}s}\mathrm{d}y - d_I + \frac{\beta\Lambda}{\mu} - \mu - \gamma. \tag{4.3}$$

We only consider the case $c < -c_{\min}$, for this case, (4.3) has two positive roots. Consequently, $\lim_{\xi\to\pm\infty}\hat{Z}(\xi) > 0$. By differentiating the equation (4.2), one gets that

$$|c|\hat{Z}'(\xi) = d_I \int_{\mathbb{R}} J_2(y)\mathrm{e}^{-\int_{\xi-y}^{\xi}\hat{Z}(s)\mathrm{d}s}\left(\hat{Z}(\xi-y) - \hat{Z}(\xi)\right)\mathrm{d}y. \tag{4.4}$$

If $\hat{Z}(\xi)$ has a minimum $\hat{Z}(\hat{\xi}_0)$ in $\mathbb{R}$, then $\hat{Z}(\xi) \equiv \hat{Z}(\hat{\xi}_0)$. Thus,

$$\inf_{\xi\in\mathbb{R}} \hat{Z}(\xi) \geq \min\left\{\lim_{\xi\to-\infty}\hat{Z}(\xi), \lim_{\xi\to+\infty}\hat{Z}(\xi)\right\} > 0.$$

In particularly, $\hat{I}_\infty'(0) > 0$. This contradicts to $\hat{I}_\infty'(0) \leq 0$. The proof is completed. □

5. Simulation and discussion

Recently, great attention has been paid to the existence and non-existence of traveling wave solutions in epidemic models with spatial diffusion. The basic idea is that epidemic models described by reaction diffusion systems can give rise to a moving zone of transition from an infective state to a disease-free state. The existence and non-existence of non-trivial traveling wave solutions indicate whether or not the disease can spread. In this paper, we have investigated the existence and nonexistence of traveling wave solutions for a nonlocal dispersal SIR model with mass action infection mechanism. From Theorem 1.1, we infer that whether the disease can spread or not depends on $\Re_0$ and $c_{\min}$. Here the minimal wave speed $c_{\min}$ is determined by the following equations

$$f(\lambda, c) = 0, \quad \frac{\partial f(\lambda, c)}{\partial \lambda} = 0,$$

where

$$f(\lambda, c) = d_I \left(\int_{\mathbb{R}} J_2(y) \mathrm{e}^{-\lambda y} \mathrm{d}y - 1 \right) - c\lambda + \frac{\beta \Lambda}{\mu} - \mu - \gamma.$$

For simplicity, we choose the nonlocal kernel functions $J_i(x) = J_\rho(x) = \frac{1}{\rho} J(\frac{x}{\rho}), i = 1, 2$, where

$$J(x) = \begin{cases} C \frac{1}{\sqrt{4\pi\rho^2}} \mathrm{e}^{-\frac{x^2}{4\rho^2}}, & x \in [-3\rho^2, 3\rho^2], \\ 0, & x \in \mathbb{R} \backslash [-3\rho^2, 3\rho^2], \end{cases} \tag{5.1}$$

where C is a constant to ensure that $\int_{\mathbb{R}} J(x) \mathrm{d}x = 1$. In the biological environment, the parameter ρ in (5.1) measures the nonlocal dispersal distance and reflects the diffusive ability of the individuals.

Note that the function

$$\mathcal{H}(\rho) := d_I \int_{\mathbb{R}} J_\rho(y) \mathrm{e}^{-\lambda y} \mathrm{d}y = d_I \int_{\mathbb{R}_+} J(y) (\mathrm{e}^{\lambda \rho y} + \mathrm{e}^{-\lambda \rho y}) \mathrm{d}y$$

is strictly increasing on $\mathbb{R}^+$. Then, for any $\rho_1, \rho_2 \in \mathbb{R}_+$ with $\rho_1 < \rho_2$,

$$\begin{aligned} f_{\rho_1}(\lambda, c) =& d_I \int_{\mathbb{R}} J_{\rho_1}(x) (\mathrm{e}^{-\lambda x_1} - 1) \mathrm{d}x - c\lambda + \frac{\beta \Lambda}{\mu} - \mu - \gamma \\ <& d_I \int_{\mathbb{R}} J_{\rho_2}(x) (\mathrm{e}^{-\lambda x_1} - 1) \mathrm{d}x - c\lambda + \frac{\beta \Lambda}{\mu} - \mu - \gamma = f_{\rho_2}(\lambda, c), \end{aligned}$$

which implies that the graph of $\lambda \to f(\lambda, c)$ moves upwards as ρ increasing. Thus, the minimal wave speed $c_{\min}$ is an increasing function of the diffusion distance ρ. Similarly, we can obtain that the minimal $c_{\min}$ is also an increasing function of the diffusion rate d_I and transmission rate β, and is a decreasing function of the recovery rate γ.

Biologically speaking, it implies that the stronger diffusive ability, larger diffusion rate and bigger transmission rate will increase the minimal wave speed. However, the bigger recovery rate will decrease the minimal wave speed.

Here, we should note that $c_{\min}$ is an increasing function of the diffusion rate d_I and diffusion ability ρ, but, these two parameters have some different effect to the minimal wave speed. With the help of the software MATLAB, we can plot the picture of $c_{\min}(d_I, \rho)$ with the two parameters d_I and ρ (see Figure 1). Here, d_I is the diffusion rate and ρ is the diffusion ability of the kernel function ($3\rho^2$ can be explained as the diffusion distance due to the Gaussian kernel function). From the Figure 1, we see that $c_{\min}(d_I, \rho)$ is an increasing function with respect to the parameters d_I, ρ. However, $c_{\min}(d_I, \rho)$ is growing much faster with the diffusion ability ρ than the diffusion rate d_I.

Biologically speaking, it implies that if we want to get a small minimal wave speed, making the diffusion ability weaker is a much more efficient way.

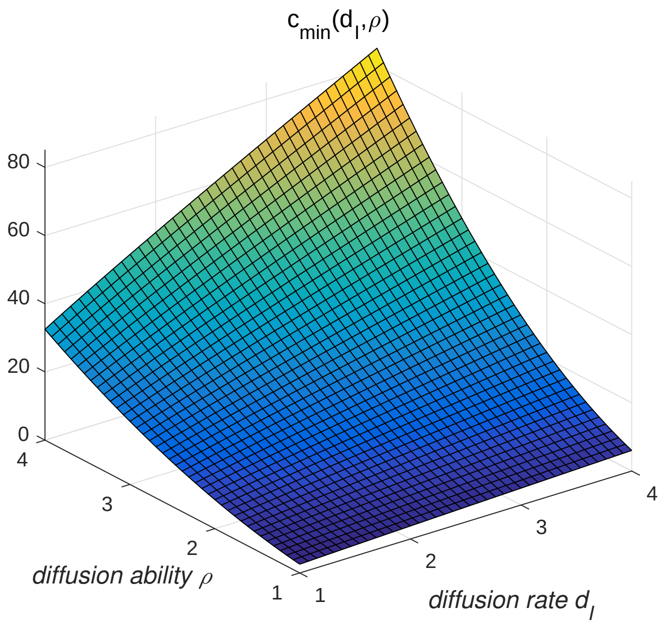

Figure 1: Minimal wave speed $c_{\min}(d,\rho)$ with the parameters $\frac{\beta\Lambda}{\mu}-\mu-\gamma=1$.

Next, we consider system (1.1) and (1.2) with the Neumann boundary condition and initial data,

$$\begin{cases} S(x,0)=\frac{\Lambda}{\mu}\left[1-\frac{1}{4}\left(1+\tanh\frac{x}{6}\right)^2\right]+s^*, x\in\Omega, \\ I(x,0)=i^*\left[1-\frac{1}{4}\left(1-\tanh\frac{x}{6}\right)^2\right], \qquad x\in\Omega, \end{cases} \tag{5.2}$$

where we assume $\Omega=[-300,300]$ for simplicity. To demonstrate the existence of travelling wave solutions of the system (1.1) and (1.2), we choose the parameters as $d_S=d_I=1,\rho=1,\lambda=10^4,\beta=5*10^{-6},\gamma=0.2,\mu=0.1$. It is easy to show that system (1.1) and (1.2) has two steady states $E^0=(10^5,0)$ and $E^*=(60000,13333)$ and the basic reproduction number $\mathfrak{R}_0=\frac{\beta\Lambda}{\mu(\mu+\gamma)}=\frac{5}{3}>1$. From the Lemmas 2.1, we can obtain that the minimal wave speed $c_{\min}=1.421$. It follows from Theorem 1.1 that if $\mathfrak{R}_0>1$ and $c>c_{\min}$, the system (1.1) and (1.2) has a traveling wave $(S(x+ct),I(x+ct))$ connecting the E^0 and E^*. The existence of the travelling wave solutions of the system (1.1) and (1.2) can be observed in Figure $2-3$.

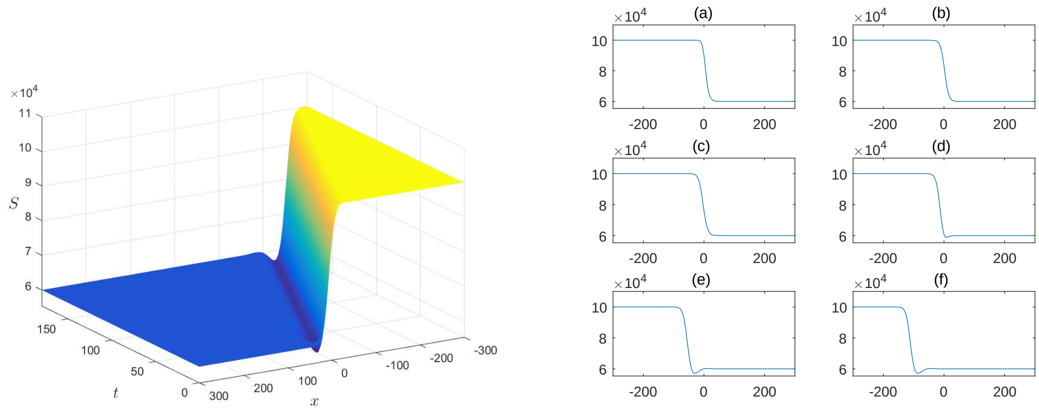

Figure 2: The left picture denotes the solution S of the system (1.1)–(1.2) with Neumann boundary conditions and initial data (5.2). From (a) to (f), the solution $S(x,t)$ plots at times $t = 0, 2, 4, 10, 40, 80$ and behaves as a traveling wave solution and travels from right to left.

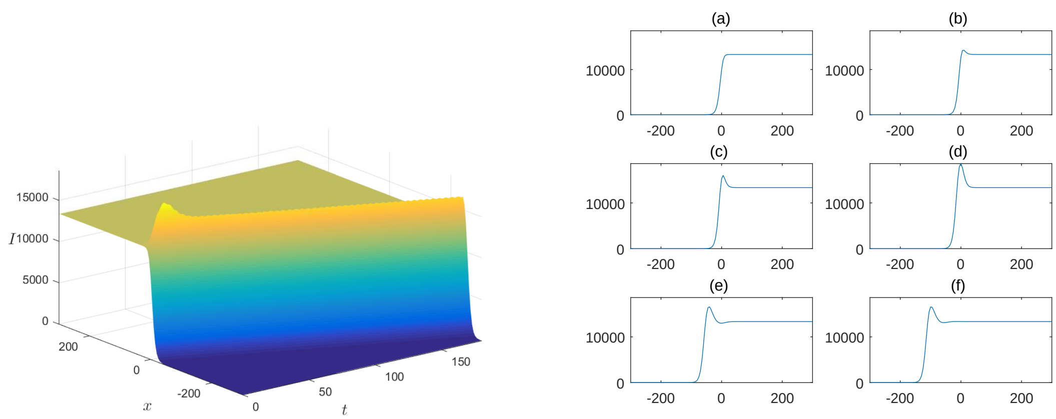

Figure 3: The left picture denotes the solution I of the system (1.1)–(1.2) with Neumann boundary conditions and initial data (5.2). From (a) to (f), the solution $I(x,t)$ plots at times $t = 0, 2, 4, 10, 40, 80$ and behaves as a traveling wave solution and travels from right to left.

From the biological considerations, threshold dynamics play an important role in the control strategies of the disease transmission. Here, the minimal wave speed $c_{\min}$ and the basic reproduction number $\mathfrak{R}_0$ are two extremely important parameters in the model (1.1)–(1.2), which describes the disease transmission qualitatively. Biological speaking, the existence and non-existence of the travelling wave solutions reveal whether the disease can spread or not. From Theorem 1.1, it can be seen that if $\mathfrak{R}_0 > 1$ and $c \geq c_{\min}$, then system (1.1)–(1.2) has a traveling wave solution connecting the infection-free steady state E^0 and endemic steady state E^*, which implies that the disease can spread from the infection-free state to endemic steady state. If $\mathfrak{R}_0 > 1$ and $c \in (0, c_{\min})$, there exists no traveling wave solution of the system (1.1)–(1.2), which implies that even if the reproduction number $\mathfrak{R}_0 > 1$, the disease cannot spread from the infection-free state to endemic steady state as long as the wave speed $c \in (0, c_{\min})$.

Finally, we remark that there are quite a few spaces to deserve further investigations. For example, we can study the asymptotic speed of propagation, the uniqueness and stability of traveling wave solutions in such model. We leave these as our further study.

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