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RESEARCH PAPER

Feedback stabilization of non-homogeneous bilinear systems with a finite time delay

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Summary

This paper investigates the feedback stabilization of non-homogeneous delayed bilinear systems, evolving in Hilbert state space. More precisely, under observability like assumption, we prove the exponential and strong stability of the solution by using a bounded feedback control. The partial stabilization is discussed as well. The proof of the main results is based on the decomposition method. The decay estimates of the corresponding solution are obtained. Finally, some examples are presented.

KEYWORDS:
feedback stabilization, non-homogeneous delayed bilinear systems, delay feedback control

1 | INTRODUCTION

Bilinear systems appear in the control of various real problems which frequently appear in engineering, biological, nuclear (see, e.g. [9,20,23,25]), and the references therein). In the modeling process, one should take into account different parameters to be closer to reality, such as time delay (see e.g. [16,15]). The presence of the delay in a system can be a source of instability. Indeed, even arbitrarily small delays in the feedback may destabilize the system at hand (see, e.g., [24,10]). Therefore, it is important to understand the effect of the delay on the system's stability. Non-homogeneous delayed bilinear systems like (1) are found in many areas of engineering and modeling several problems in the real world (see e.g. [11,29]). Stability results for the delayed bilinear systems have been recently obtained in [18,19]. This work mainly investigates the feedback stabilization question of the non-homogeneous delayed bilinear system, given by:

$$\begin{cases} \dfrac{dy(t)}{dt} = Ay(t) + u(t)(By(t-r) + b), \quad \forall t > 0 \\ y_0 = \varphi \in \mathcal{C} = C([-r, 0], H), \end{cases} \tag{1}$$

where A is the infinitesimal generator of a linear C_0-semigroup of contractions $S(t)$ on a real Hilbert space H, with inner product $\langle ., . \rangle$, and corresponding norm $\|.\|$, $B : H \to H$ is a linear bounded operator of H, the positive real number r denotes the delay which is assumed to be constant, $b \in H$ is a fixed vector, and the real-valued function $u(t)$ is the control.

In the sequel of this paper, we consider $\mathcal{C} := C([-r, 0], H)$, the Banach space of continuous H-valued functions on $[-r, 0]$, equipped with the supremum norm:

$$\|\psi\|_{\mathcal{C}} = \sup_{-r\leq\theta\leq 0} \|\psi(\theta)\|, \text{ for all } \psi \in \mathcal{C}.$$

If ψ is a continuous function from $[-r, +\infty)$ to X and $t \in [-r, +\infty)$, then the history function ψ_t denotes the element of $\mathcal{C}$ defined by:

$$\psi_t(\theta) = \psi(t + \theta) \text{ for all } \theta \in [-r, 0].$$

The stabilization question of the homogeneous undelayed version of system (1) (i.e. the case where $b = r = 0$) has been studied in various works (see e.g.[3,4,26,14,13,12]). Moreover, there are some works that have been dedicated to the case of the non-homogeneous (i.e. $b \neq 0$) bilinear system (1) (see e.g.[5,7,8,1,6,17], and the references therein).

This work's main contribution is to provide sufficient conditions for exponential, strong, and partial stabilization of the non-homogeneous, delayed bilinear system (1).
The paper is organized as follows. The next section deals with the existence, uniqueness, and regularity of the system's solution (1). In the third section, we give sufficient conditions that ensure the full state strong stabilization, leading to an explicit asymptotic decay rate. In the fourth section, we consider partial stabilization, which consists of stabilizing only a part of the state. Also, we provide sufficient conditions for exponential stabilization. Finally, some illustrating examples are presented in section five.

2 | WELL-POSSEDNESS

We consider the following delayed bounded feedback control:

$$u(t) = -\rho \frac{\langle y(t), By(t-r) + b\rangle}{|\langle y(t), By(t-r) + b\rangle| + 1}, \tag{2}$$

Where $\rho > 0$ is the gain control. This leads to the closed loop system:

$$\begin{cases} \dfrac{dy(t)}{dt} = Ay(t) + F(y_t), \quad \forall t > 0 \\ y_0 = \varphi \in C, \end{cases} \tag{3}$$

where

$$F(\psi) = -\rho \frac{\langle B\psi(-r) + b, \psi(0)\rangle}{|\langle B\psi(-r) + b, \psi(0)\rangle| + 1}(B\psi(-r) + b), \text{ for all } \psi \in C := C([-r, 0], H). \tag{4}$$

In this section, we aim to study the existence, uniqueness, and regularity of the system's solution (3)). We start our study by the well-posedness result.

Proposition 1. Assume that A generates a C_0-semigroup $S(t)$ of contractions. Then, the system (3) admits a unique global mild solution. Moreover, we have the following estimate:

$$\|y(\tau)\|^2 - \|y(t)\|^2 \geq 2\rho \int_\tau^t \frac{\langle By(s-r) + b, y(s)\rangle^2}{|\langle By(s-r) + b, y(s)\rangle| + 1} ds, \text{ for all } 0 \leq \tau \leq t. \tag{5}$$

In particular, we have

$$\|y(t)\| \leq \|\varphi\|_C \text{ , for all } t \geq -r \tag{6}$$

Proof. Firstly let us show that F, defined by (4), is a locally Lipschitz continuous function from C to H. To this end, let R be such that for all $\psi_1, \psi_2 \in C$, $\|\psi_1\|, \|\psi_2\| \leq R$, we have:

$$\begin{aligned} \|F(\psi_1) - F(\psi_2)\| \leq {} & \rho \|B\| \|\psi_1(-r) - \psi_2(-r)\| \\ & + \rho G(\psi_1, \psi_2) \|B\psi_2(-r) + b\| \end{aligned} \tag{7}$$

where

$$G(\psi_1, \psi_2) = \left| \frac{\langle B\psi_1(-r) + b, \psi_1(0)\rangle}{|\langle B\psi_1(-r) + b, \psi_1(0)\rangle| + 1} - \frac{\langle B\psi_2(-r) + b, \psi_2(0)\rangle}{|\langle B\psi_2(-r) + b, \psi_2(0)\rangle| + 1} \right|$$

By definition of $\|.\|_C$, we have $\|\psi_1(-r) - \psi_2(-r)\| \leq \|\psi_1 - \psi_2\|_C$. By making use of the function $f(x) = \frac{x}{|x|+1}$, we get

$$G(\psi_1, \psi_2) \leq |\langle B\psi_1(-r) + b, \psi_1(0)\rangle - \langle B\psi_2(-r) + b, \psi_2(0)\rangle| \tag{8}$$

We also have

$$\begin{aligned} |\langle B\psi_1(-r) + b, \psi_1(0)\rangle - \langle B\psi_2(-r) + b, \psi_2(0)\rangle| \leq {} & |\langle \psi_1(0) - \psi_2(0), B\psi_1(-r) + b\rangle| \\ & + |\langle \psi_2(0), B\psi_1(-r) - B\psi_2(-r)\rangle| \end{aligned}$$

It follows that

$$\begin{aligned} |\langle B\psi_1(-r) + b, \psi_1(0)\rangle - \langle B\psi_2(-r) + b, \psi_2(0)\rangle| \leq {} & (\|B\| \|\psi_1\|_C + \|b\|) \|\psi_1 - \psi_2\|_C \\ & + \|B\| \|\psi_2\|_C \|\psi_1 - \psi_2\|_C \end{aligned} \tag{9}$$

then, using (8) and (9), we get

$$G(\psi_1,\psi_2) \leq (\|B\| \left\|\psi_1\right\|_C + \|B\| \left\|\psi_2\right\|_C + \|b\|) \left\|\psi_1 - \psi_2\right\|_C$$

It follows from the above inequality and (7) that

$$\begin{aligned} \left\|F(\psi_1) - F(\psi_2)\right\| &\leq \rho \|B\| \left\|\psi_1 - \psi_2\right\|_C \\ &+\rho(\|B\| \left\|\psi_2\right\|_C + \|b\|)(\|B\| \left\|\psi_1\right\|_C + \|B\| \left\|\psi_2\right\|_C + \|b\|) \left\|\psi_1 - \psi_2\right\|_C \end{aligned}$$

we deduce that F is locally Lipschitz. According to Theorem 2.6 ([30], p. 51), the system (3) admits a unique mild solution defined on a maximal interval $\left[0, T_\varphi\right))$, which is continuous with respect to the initial state given by the following variation of constants formula:

$$y(t) = S(t)\varphi(0) - \rho \int_0^t \frac{\langle By(s-r) + b, y(s)\rangle}{|\langle By(s-r) + b, y(s)\rangle| + 1} S(t-s)(By(s-r) + b)ds \tag{10}$$

Consider the following function:

$$\begin{aligned} f : \left[0, T_\varphi\right) &\to \quad H \\ t &\mapsto F(y_t) \end{aligned}$$

Let $0 < T < T_\varphi$. We have that $f \in C([0,T], H)$, so there exist a sequence $(f_n) \subset C^1([0,T], H)$ such that

$$f_n \to f \text{ uniformly in } C([0,T], H).$$

Let $(\xi_n) \subset D(A)$ such that (ξ_n) converges to $\varphi(0)$ in H, and define the function:

$$y_n(t) = S(t)\xi_n + \int_0^t S(t-s) f_n(s)ds, \text{ for all } t \in [0,T] \tag{11}$$

Then, $y_n(t) \in D(A)$ and $y_n \in C^1([0,T], H)$ (see e.g.,[28], p. 187) and we have

$$\dot{y}_n(t) = Ay_n(t) + f_n(t),\ t \in (0,T),\ y_n(0) = \xi_n.$$

Using the dissipativity of the operator A, we obtain:

$$\frac{d}{dt} \left\|y_n(t)\right\|^2 \leq 2\left\langle f_n(t), y_n(t)\right\rangle, \text{ for all } t \in (0,T)$$

Integrating this inequality and using the continuity of y_n with respect to t, we derive:

$$\left\|y_n(t)\right\|^2 - \left\|y_n(\tau)\right\|^2 \leq 2\int_\tau^t \left\langle f_n(s), y_n(s)\right\rangle ds, \text{ for all } 0 \leq \tau \leq t \leq T.$$

Using (11) and the fact that (f_n) is bounded with respect to n, we can see that (y_n) is bounded as well. Then, by passing to the limit in the last estimation, we get via the dominated convergence theorem,

$$\left\|y_n(t)\right\|^2 - \left\|y_n(\tau)\right\|^2 \leq -2\rho \int_\tau^t \frac{\langle By(s-r) + b, y(s)\rangle^2}{|\langle By(s-r) + b, y(s)\rangle| + 1} ds, \text{ for all } 0 \leq \tau \leq t \leq T.$$

Since T is arbitrary, the inequality (5) holds for all $t < T_\varphi$. We deduce from Theorem 2.6 ([30], p. 51) that $T_\varphi = \infty$. In other words, the system (3) possesses a unique global solution $y(t)$, moreover, we have

$$\|y(t)\| \leq \|\varphi\|_C, \text{ for all } t \geq -r.$$

□

3 | STRONG STABILIZATION WITH DECAY ESTIMATE

In the sequel we suppose that $H = H^u \oplus H^s$ where H^u and H^s are two closed subspace of H, which are orthogonal to each one and invariant under $S(t)$. Thus, $S(t)$ induces a C_0-semigroup $S^u(t)$ (resp. $S^s(t)$) on H^u (resp. H^s). We also consider the following assumptions:

$$(\mathbf{H}_1)\ BH^u \subseteq H^u, \text{ and } (\mathbf{H}_2)\ BH^s \subseteq H^s.$$

Under the hypothesis $(\mathbf{H}_1)$ and $(\mathbf{H}_2)$, the systems (1) can be decomposed into the two following systems:

$$\begin{cases} \dot{y}^u(t) = A^u y^u(t) + v(t)(B^u y^u(t-r) + b^u), \quad \forall t > 0 \\ y_0^u = \varphi^u \in C^u =: C([-r,0], H^u), \end{cases} \tag{12}$$

and

$$\begin{cases} \dot{y}^s(t) = A^s y^s(t) + v(t)(B^s y^s(t-r) + b^s), \quad \forall t > 0 \\ y_0^s = \varphi^s \in C^s =: C([-r,0], H^s), \end{cases} \tag{13}$$

where A^u and B^u are respectively the restrictions of A and B in H^u, A^s and B^s are respectively the restrictions of A and B in H^s. z^u and z^s are the components of the solution $z \in H$ on H^u and H^s respectively. We further suppose that A^u generates a C_0-semigroup of contractions $S^u(t)$.

In the rest of the paper, we consider the following feedback control:

$$v^u(t) = -\rho \frac{\langle y^u(t), By^u(t-r) + b^u \rangle}{|\langle y^u(t), By^u(t-r) + b^u \rangle| + 1} \tag{14}$$

Using Proposition 1, we deduce that the system (12) admits a unique global mild solution given by

$$\begin{cases} y^u(t) = S^u(t)\varphi^u(0) + \int_0^t S^u(t-\tau) v^u(\tau) (B^u y^u(\tau - r) + b^u) \, d\tau, \; t \geq 0 \\ y_0^u = \varphi^u \in C^u \end{cases} \tag{15}$$

and verifies the following estimations

$$\|y^u(\tau)\|^2 - \|y^u(t)\|^2 \geq 2\rho \int_\tau^t \frac{\langle B^u y^u(s-r) + b^u, y^u(s) \rangle^2}{|\langle B^u y^u(s-r) + b^u, y^u(s) \rangle| + 1} ds, \text{ for all } 0 \leq \tau \leq t. \tag{16}$$

Then, we have

$$\|y^u(t)\| \leq \|\varphi^u\|_C, \text{ for all } t \geq -r \tag{17}$$

Theorem 1. Let A generates a C_0-semigroup $S(t)$, and suppose that the following conditions holds:

1. $S^u(t)$ is a contraction semigroup,
2. there exist $\delta, T > 0$ such that:

$$\int_r^T |\langle B^u S^u(s-r)\xi + b^u, S^u(s)\xi \rangle| \, ds \geq \delta \, \|\xi\|^{\gamma_{b^u}}, \quad \text{for all } \xi \in H^u \tag{18}$$

where

$$\gamma_{b^u} = \begin{cases} 2 & \text{if } b^u = 0 \\ 1 & \text{if } b^u \neq 0 \end{cases}$$

- if $\gamma_{b^u} = 2$, then the feedback (14) strongly stabilizes the system (12), and we have the following decay estimate:

$$\|y^u(t)\| = O\left(\frac{1}{\sqrt{t-r}}\right), \text{ as } t \longrightarrow +\infty.$$

- if $\gamma_{b^u} = 1$, then the feedback (14) exponentially stabilizes the system (12). More precisely, there exists $\beta > 0$, such that

$$\|y^u(t)\| \leq e^{-\beta} \|\varphi^u\|_C \, e^{-\frac{\beta}{T}(t-r)}, \text{ for all } \; t \geq r$$

Proof. According to the discussion, at the beginning of this section, the system (12) has a unique global mild solution which is given by:

$$\begin{cases} y^u(t) = S^u(t)\varphi^u(0) + \int_0^t S^u(t-\tau) v^u(\tau) (B^u y^u(\tau - r) + b^u) \, d\tau, \; t \geq 0 \\ y_0^u = \varphi^u \in C^u \end{cases}$$

by using the fact that

$$|v^u(t)| \leq \rho \left|\left\langle y^u(t), B^u y^u(t-r) + b^u \right\rangle\right|,$$

we get the following estimation:

$$\left\| y^u(t) - S^u(t)\varphi^u(0)\right\| \leq \rho\left(\|B^u\|\,\|\varphi^u\|_C + \|b^u\|\right)\int_0^t \left|\left\langle y^u(\tau), B^u y^u(\tau-r) + b^u\right\rangle\right| d\tau, \quad \forall t \geq 0. \tag{19}$$

On the other hand, let $s \geq r$, we have

$$\begin{aligned} \left|\left\langle B^u S^u(s-r) y^u(0) + b^u, S^u(s) y^u(0)\right\rangle\right| &\leq \left|\left\langle B^u S^u(s-r) y^u(0) - B^u y^u(s-r), S^u(s) y^u(0)\right\rangle\right| \\ &+ \left|\left\langle B^u y^u(s-r) + b^u, S^u(s) y^u(0) - y^u(s)\right\rangle\right| \\ &+ \left|\left\langle B^u y^u(s-r) + b^u, y^u(s)\right\rangle\right|. \end{aligned}$$

using the fact that the semigroup $S^u(t)$ is of contraction, it comes:

$$\begin{aligned} \left|\left\langle B^u S^u(s-r) y^u(0) + b^u, S^u(s) y^u(0)\right\rangle\right| &\leq \|B^u\|\,\|\varphi^u\|_C \left\|S^u(s-r) y^u(0) - y^u(s-r)\right\| \\ &+ \left(\|B^u\|\,\|\varphi^u\|_C + \|b^u\|\right)\left\|S^u(s) y^u(0) - y^u(s)\right\| \\ &+ \left|\left\langle B^u y^u(s-r) + b^u, y^u(s)\right\rangle\right|. \end{aligned}$$

It follows from the last inequality and (19) that,

$$\begin{aligned} \left|\left\langle B^u S^u(s-r) y^u(0) + b^u, S^u(s) y^u(0)\right\rangle\right| &\leq \rho\left(\|B^u\|\,\|\varphi^u\|_C + \|b^u\|\right)^2 \int_0^{s-r} \left|\left\langle y^u(\tau), B^u y^u(\tau-r) + b^u\right\rangle\right| d\tau \\ &+\rho\left(\|B^u\|\,\|\varphi^u\|_C + \|b^u\|\right)^2 \int_0^{s} \left|\left\langle y^u(\tau), B^u y^u(\tau-r) + b^u\right\rangle\right| d\tau \\ &+ \left|\left\langle B^u y^u(s-r) + b^u, y^u(s)\right\rangle\right| \\ &\leq 2\rho\left(\|B^u\|\,\|\varphi^u\|_C + \|b^u\|\right)^2 \int_0^{s} \left|\left\langle y^u(\tau), B^u y^u(\tau-r) + b^u\right\rangle\right| d\tau \\ &+ \left|\left\langle y^u(s), B^u y^u(s-r) + b^u\right\rangle\right|. \end{aligned}$$

Using the superposition property of the solution of the system (12), then for all $t \geq 0$, we get

$$\begin{aligned} \left|\left\langle B^u S^u(s-r) y^u(t) + b^u, S^u(s) y^u(t)\right\rangle\right| &\leq 2\rho\left(\|B^u\|\,\|\varphi^u\| + \|b^u\|\right)^2 \int_t^{s+t} \left|\left\langle y^u(\tau), B^u y^u(\tau-r) + b^u\right\rangle\right| d\tau \\ &+ \left|\left\langle y^u(t+s), B^u y^u(t+s-r) + b^u\right\rangle\right|, \end{aligned}$$

which by integrating over $[r, T+r]$ with respect to s gives

$$\begin{aligned} \int_r^{T+r} \left|\left\langle B^u S^u(s-r) y^u(t) + b^u, S^u(s) y^u(t)\right\rangle\right| ds &\leq R^u \int_r^{T+r}\int_0^s \left|\left\langle y^u(t+\tau), B^u y^u(t+\tau-r) + b^u\right\rangle\right| d\tau ds \\ &+ \int_r^{T+r} \left|\left\langle y^u(t+s), B^u y^u(t+s-r) + b^u\right\rangle\right| ds \\ &\leq \left(R^u(T+r) + 1\right) I^u \end{aligned}$$

where

$$R^u = 2\rho\left(\|B^u\|\,\|\varphi^u\|_C + \|b^u\|\right)^2 \text{ and } I^u = \int_r^{T+r} \left|\left\langle y^u(t+s), B^u y^u(t+s-r) + b^u\right\rangle\right| ds.$$

This, together with (18), gives:

$$\delta\left\|y^u(t)\right\|^{\gamma_{b^u}} \leq ((T+r)R^u + 1)\int_{t+r}^{t+T+r} \left|\left\langle y^u(\tau), B^u y^u(\tau-r) + b^u\right\rangle\right| d\tau$$

Hölder's inequality gives

$$\delta\left\|y^u(t)\right\|^{\gamma_{b^u}} \leq K\left(\int_{t+r}^{t+T+r} \left|\left\langle y^u(\tau), B^u y^u(\tau-r) + b^u\right\rangle\right|^2 d\tau\right)^{\frac{1}{2}}$$

with $K = \sqrt{T}\,((T+r)R^u + 1)$. Using (16) and the fact that $\|y^u(t)\|$ is decreasing, we obtain

$$\delta\left\|y^u(t+r+T)\right\|^{\gamma_{b^u}} \leq \frac{1}{\sqrt{2}} K_1 \left(\left\|y^u(t+r)\right\|^2 - \left\|y^u(t+r+T)\right\|^2\right)^{\frac{1}{2}}$$

with $K_1 = K\sqrt{\frac{(\|B^u\|\|\varphi^u\|_C + \|b^u\|)\|\varphi^u\|_C + 1}{\rho}}$, then

$$\left\|y^u(t+r+T)\right\|^2 \leq \left\|y^u(t+r)\right\|^2 - 2\frac{\delta^2}{K_1^2}\left\|y^u(t+r+T)\right\|^{2\gamma_{b^u}}$$

Letting $s_k = \|y^u(kT+r)\|^2$ with k is a positive integer number, we derive

$$s_{k+1} \leq s_k - 2\frac{\delta^2}{K_1^2} s_{k+1}^{\gamma_{b^u}}. \tag{20}$$

Let us examine the two following cases:

Case 1: If $\gamma_{b^u} = 2$ i.e $b^u = 0$, by using ([2], Lemma 5.2), we deduce that there exists a positive constant M (depending on γ, δ and K_1) such that

$$s_k \leq \frac{M}{(k+1)^{\frac{1}{2}}}, \text{ for all } k \geq 0,$$

then, we have the following estimate:

$$\|y^u(t)\| = O\left(\frac{1}{\sqrt{t-r}}\right), \text{ as } t \longrightarrow +\infty. \tag{21}$$

Case 2: If $\gamma_{b^u} = 1$ i.e $b^u \neq 0$, from (20)

$$s_{k+1} \leq \frac{1}{C} s_k$$

where $C = 1 + 2\frac{\delta^2}{K_1^2}$. Therefore, we have

$$s_k \leq e^{-k\ln C} s_0,$$

which gives

$$\|y^u(kT+r)\| \leq e^{-k\frac{ln(C)}{2}} \|\varphi^u\|_C.$$

We deduce that

$$\|y^u(t)\| \leq e^{-\frac{ln(C)}{2}} \|\varphi^u\|_C \, e^{-\frac{ln(C)}{2T}(t-r)}, \text{ for all } t \geq r. \tag{22}$$

Hence $y^u(t)$ decays exponentially with the decay rate $\beta^* = \frac{\ln C}{2T}$. □

Letting $H^u = H$ in Theorem 1, we deduce the following corollary:

Corollary 1. Let A generates a C_0-semigroup of contractions $S(t)$:

- If there exist $\delta, T > 0$ such that:

$$\int_r^T |\langle BS(s-r)\xi, S(s)\xi\rangle| \, ds \geq \delta \|\xi\|^2, \quad \text{for all } \xi \in H,$$

  then the feedback $v(t) = -\rho\frac{\langle y(t), By(t-r)\rangle}{|\langle y(t), By(t-r)\rangle| + 1}$, strongly stabilizes the system (1), and we have the following decay estimate:

$$\|y(t)\| = O\left(\frac{1}{\sqrt{t-r}}\right), \text{ as } t \longrightarrow +\infty.$$

- If there exists $\delta, T > 0$ such that:

$$\int_r^T |\langle BS(s-r)\xi + b, S(s)\xi\rangle| \, ds \geq \delta \|\xi\|, \quad \text{for all } \xi \in H,$$

  then the feedback $v(t) = -\rho\frac{\langle y(t), By(t-r)+b\rangle}{|\langle y(t), By(t-r)+b\rangle| + 1}$ exponentially stabilizes the system (1).

Let us recall the following lemma that will be used in the sequel of the paper

Lemma 1 (see,[21]). Let $x(t)$ be continuous and non-negative on $[0, h]$ and satisfy

$$x(t) \leq a(t) + \int_0^t \left(a_1(s)x(s) + b(s)\right) ds,$$

where $a_1(t)$ and $b(t)$ are non-negative integrable functions on the interval $[0,h]$, with $a(t)$ bounded there. Then, we have

$$x(t) \leq \int_0^t b(s)\,ds + \sup_{0\leq t\leq h} |a(t)| \exp\left(\int_0^t a_1(s)\,ds\right), \ \forall t \in [0,h].$$

Now, we are ready to establish the strong stabilization result concerning the full system (3).

Theorem 2. Suppose that assumptions 1. and 2. of Theorem 1 holds. Furthermore, suppose that $S^s(t)$ is an exponentially stable semigroup, i.e. there exists $M>0$ and $\alpha>0$ such that

$$\|S^s(t)\| \leq M e^{-\alpha t}, \ \text{for all } t \geq 0$$

. Then, we have the following results:

1. If $\gamma_{b^u} = 2$ and $\rho \leq \frac{\alpha}{M\|B^s\|e^{\alpha r}}$. the feedback control (14) Strongly stabilizes the system (3), and we have the following estimate:
$$\|y(t)\| = O\left(\frac{1}{\sqrt{t}}\right), \ \text{as } t \longrightarrow +\infty.$$
2. If $\gamma_{b^u} = 1$ and $\rho \leq \frac{\alpha}{M\|B^s\|e^{\alpha r}}$, then the corresponding solution of the system (3) is exponentially stable by using the feedback control (14).

Proof. The corresponding solution of (13) is given by the following variations of constants formula:

$$\begin{cases} y^s(t) = S^s(t)\varphi^s(0) + \int_0^t S^s(t-\tau)\,v^u(\tau)\,(B^s y^s(\tau-r) + b^s)\,d\tau, \ t\geq 0 \\ y_0^s = \varphi^s \in C^s \end{cases}$$

Then, we have

$$\|y^s(t)\| \ \leq \ M\,\|\varphi^s\|\,e^{-\alpha t} + \int_0^r |v^u(\tau)|\,M e^{-\alpha(t-\tau)}\,(\|B^s\|\,\|y^s(\tau-r)\| + \|b^s\|)\,d\tau$$
$$+ \int_r^t |v^u(\tau)|\,M e^{-\alpha(t-\tau)}\,(\|B^s\|\,\|y^s(\tau-r)\| + \|b^s\|)\,d\tau$$

then,

$$\|y^s(t)\| \ \leq \ M\,\|\varphi^s\|\,e^{-\alpha t} + \int_0^r |v^u(\tau)|\,M e^{-\alpha(t-\tau)}\,(\|B^s\|\,\|y^s(\tau-r)\| + \|b^s\|)\,d\tau$$
$$+ e^{-\alpha(t-r)} \int_0^{t-r} |v^u(\tau+r)|\,M e^{\alpha\tau}\,(\|B^s\|\,\|y^s(\tau)\| + \|b^s\|)\,d\tau.$$

Let $t \geq r$ and $\omega^s(t) = \|y^s(t)\|\,e^{\alpha t}$, then we have

$$\omega^s(t) \leq \xi + e^{\alpha r} \int_0^{t-r} |v^u(\tau+r)|\,M e^{\alpha\tau}\,(\|B^s\|\,\|y^s(\tau)\|\,e^{\alpha\tau} + \|b^s\|\,e^{\alpha\tau})\,d\tau$$

with $\xi = M\,\|\varphi^s\| + \int_0^r |v^u(\tau)|\,M e^{\alpha\tau}\,(\|B^s\|\,\|y^s(\tau-r)\| + \|b^s\|)\,d\tau$, and then

$$\omega^s(t) \leq \xi + \int_0^t M e^{\alpha r}\,|v^u(\tau+r)|\,(\|B^s\|\,\omega^s(\tau) + \|b^s\|\,e^{\alpha\tau})\,d\tau$$

By applying Lemma 1, we deduce that

$$\omega^s(t) \leq M e^{\alpha r}\,\|b^s\| \int_0^t |v^u(\tau+r)|\,e^{\alpha\tau} d\tau + \xi \exp\left(M\,\|B^s\|\,e^{\alpha r} \int_0^t |v^u(\tau+r)|\,ds\right), \ \forall t \geq 0.$$

from which, it comes for all $t \geq 0$

$$\|y^s(t)\| \leq Me^{-\alpha(t-r)}\|b^s\| \int_0^t |v^u(\tau+r)| e^{\alpha\tau} d\tau + \xi \exp\left(M\|B^s\| e^{\alpha r} \int_0^t |v^u(\tau+r)| ds - \alpha t \right). \tag{23}$$

Two cases will be considered

Case 1: $\gamma_{b^u} = 2$ i.e. $b^u = 0$. Using (21), we get

$$|v^u(t)| \leq \frac{\rho\|B^u\|}{\sqrt{t^2 - rt}}$$

Thus, from (23) we have

$$\begin{aligned} \|y^s(t)\| \leq\ & M\rho\|B^u\|\|b^s\| e^{-\alpha(t-r)} \int_0^t \frac{e^{\alpha\tau}}{\sqrt{\tau^2+r\tau}} d\tau \\ & + \xi \exp\left(M\|B^s\| e^{\alpha r} \int_0^t |v^u(\tau+r)| d\tau - \alpha t \right). \end{aligned}$$

Since, the function $f(t) = \frac{e^{\alpha t}}{\sqrt{t^2+tr}}$ is decreasing on $[0,\epsilon]$ and increasing on $[\epsilon, +\infty[$, with $\epsilon = \frac{-(\alpha r-1)+\sqrt{\alpha^2 r^2+1}}{2\alpha}$. Thus, for any $t \geq \epsilon$, we have

$$\begin{aligned} \|y^s(t)\| \leq\ & M\rho\|B^u\|\|b^s\| e^{-\alpha(t-r-\epsilon)} \int_0^\epsilon \frac{d\tau}{\sqrt{\tau^2+r\tau}} \\ & + M\rho\|B^u\|\|b^s\| e^{-\alpha(t-r)} \int_\epsilon^t \frac{e^{\alpha\tau}}{\sqrt{\tau^2+r\tau}} d\tau \\ & + \xi \exp\left(M\|B^s\| e^{\alpha r} \int_0^t |v^u(\tau+r)| d\tau - \alpha t \right) \end{aligned} \tag{24}$$

Hence, we get

$$\begin{aligned} \|y^s(t)\| \leq\ & \rho M\|B^u\|\|b^s\| e^{-\alpha(t-r-\epsilon)} \ln\left(\frac{2}{r}\epsilon + 1 + \frac{2}{r}\sqrt{\epsilon^2 + r\epsilon} \right) \\ & + \frac{M\rho\|B^u\|\|b^s\|}{\sqrt{t^2+rt}} e^{\alpha r} \\ & + \xi \exp\left(M\|B^s\| e^{\alpha r} \int_0^t |v^u(\tau+r)| d\tau - \alpha t \right) \end{aligned}$$

Taking into account the fact that

$$|v^u(t)| \leq \rho \tag{25}$$

Hence,

$$\begin{aligned} \|y^s(t)\| \leq\ & \rho M\|B^u\|\|b^s\| e^{-\alpha(t-r)} M_\epsilon + \frac{\rho M\|B^u\|\|b^s\|}{\sqrt{t^2+rt}} e^{\alpha r} \\ & + \xi \exp\left((\rho M\|B^s\| e^{\alpha r} - \alpha) t \right) \end{aligned} \tag{26}$$

with $M_\epsilon = e^{\alpha\epsilon} \ln\left(\frac{2}{r}\epsilon + 1 + \frac{2}{r}\sqrt{\epsilon^2 + r\epsilon} \right)$.
If we choose $\rho \leq \frac{\alpha}{M\|B^s\|e^{\alpha r}}$, we can see that the system (13) is strongly stable. Moreover, we have the following estimates:

$$\|y^s(t)\| = O\left(\frac{1}{\sqrt{t^2+rt}} \right), \text{ as } t \longrightarrow +\infty.$$

Then, by using Theorem 1, we obtain

$$\|y(t)\| = O\left(\frac{1}{\sqrt{t}}\right), \text{ as } t \longrightarrow +\infty.$$

Case 2: If $\gamma_{b^u} = 1$ i.e. $\gamma_{b^u} \neq 0$ from Theorem 1, we have

$$\begin{aligned}
|v^u(t)| &= \rho \left| \frac{\langle y^u(t), B^u y^u(t-r) + b^u \rangle}{|\langle y^u(t), B y^u(t-r) + b^u \rangle| + 1} \right| \\
&\leq \rho \left| \langle y^u(t), B^u y^u(t-r) + b^u \rangle \right| \\
&\leq \rho \left| \langle y^u(t), B^u y^u(t-r) \rangle \right| + \rho \left| \langle y^u(t), b^u \rangle \right| \\
&\leq \rho\, e^{-2\beta} \|\varphi^u\|_C\, e^{-\frac{\beta}{T}(t-r)} \|B^u\|\, e^{-\frac{\beta}{T}(t-2r)} \|\varphi^u\|_C + \rho \|b^u\|\, e^{-\beta} e^{-\frac{\beta}{T}(t-r)} \|\varphi^u\|_C,
\end{aligned}$$

where β is defined in (22). According to (23) , and (25), we obtain:

$$\begin{aligned}
\|y^s(t)\| \leq\ & \rho\, e^{-2\beta} M e^{-\alpha(t-r)} \|B^u\|\, \|b^s\|\, \|\varphi^u\|_C^2\, e^{\frac{\beta}{T} r} \int_0^t e^{\left(\alpha - \frac{2\beta}{T}\right)\tau} d\tau \\
& + \rho\, e^{-\beta} \|b^u\|\, \|\varphi^u\|_C\, M e^{-\alpha(t-r)} \|b^s\| \int_0^t e^{\left(\alpha - \frac{\beta}{T}\right)\tau} d\tau \\
& + \xi \exp\left(\left(\rho M \|B^s\|\, e^{\alpha r} - \alpha\right) t\right).
\end{aligned}$$

After calculation of the above integrals, we deduce that if $\rho \leq \frac{\alpha}{M\|B^s\|e^{\alpha r}}$ the system (13) is exponentially stable. Then, by using Theorem 1, we deduce that the control (14) ensures the exponential stability of the full state. □

Remark 1.

1. In the undelayed homogeneous case, with $\gamma_{b^u} = 2$, we can see that the component $y^s(t)$ of $y(t)$ decays exponentially, retrieving thus the result of[27].
2. In the delayed case, with $\gamma_{b^u} = 2$, we retrieve the result of [18].
3. It follows from (26) that if $\gamma_{b^u} = 1$, $\rho \leq \frac{\alpha}{M\|B^s\|e^{\alpha r}}$ and $b^s = 0$, then the system (1) is exponentially stable.
4. If $\gamma_{b^u} = 1$, and $B^s = 0$ then, the corresponding solution of system (1) is exponentially stabilizable.without any condition on ρ.

4 | PARTIAL STRONG STABILIZATION

Our main result concerning the partial strong stabilization can be stated as follows:

Theorem 3. Suppose that:

1. A is the infinitesimal generator of a linear C_0-semigroup of contractions $S(t)$ on H,
2. B is a linear bounded operator,
3. C is a linear operator from H to H,
4. there exist $\delta, T > 0$ such that,

$$\int_r^T |\langle BS(s-r)y + b, S(s)y \rangle|\, ds \geq \delta \|Cy\|^\gamma, \text{ for all } y \in H \text{ and } \gamma > 0 \tag{27}$$

Then, the feedback (2) partially strongly stabilize the system (1).

Proof. According to Proposition 1, the system (1) controlled by the feedback (2) possesses a unique global mild solution which is given by:

$$y(t) = S(t)\varphi(0) - \rho \int_0^t \frac{\langle By(s-r) + b, y(s) \rangle}{|\langle By(s-r) + b, y(s) \rangle| + 1} S(t-s)(By(s-r) + b)ds$$

Then, using (6), we get:

$$\|y(t)-S(t)\varphi(0)\| \leq \rho(\|B\|\,\|\varphi\|_C+\|b\|)\int_0^t \frac{|\langle By(s-r)+b,y(s)\rangle|}{|\langle By(s-r)+b,y(s)\rangle|+1}ds, \text{ for all } t\geq 0. \tag{28}$$

Furthermore, for all $s \geq r$ we have

$$\begin{aligned}|\langle BS(s-r)y(0)+b,S(s)y(0)\rangle| \;\leq\; & |\langle BS(s-r)y(0)-By(s-r),S(s)y(0)\rangle| \\ & +|\langle By(s-r)+b,S(s)y(0)-y(s)\rangle| \\ & +|\langle By(s-r)+b,y(s)\rangle|\,.\end{aligned}$$

Then, using (28) and the contraction property of $S(t)$, we deduce that for all $s \geq r$

$$\begin{aligned}|\langle BS(s-r)y(0)+b,S(s)y(0)\rangle| \;\leq\; & \rho\,\|B\|\,\|\varphi\|_C\,(\|B\|\,\|\varphi\|_C+\|b\|)\int_0^{s-r}\frac{|\langle By(\tau-r)+b,y(\tau)\rangle|}{|\langle By(\tau-r)+b,y(\tau)\rangle|+1}d\tau \\ & +\rho(\|B\|\,\|\varphi\|_C+\|b\|)^2\int_0^{s}\frac{|\langle By(\tau-r)+b,y(\tau)\rangle|}{|\langle By(\tau-r)+b,y(\tau)\rangle|+1}d\tau \\ & +|\langle By(s-r)+b,y(s)\rangle|\,.\end{aligned} \tag{29}$$

It follows that for all $T \geq s \geq r$, we have

$$\begin{aligned}|\langle BS(s-r)y(0)+b,S(s)y(0)\rangle| \;\leq\; & K_1\int_0^{T}\frac{|\langle By(\tau-r)+b,y(\tau)\rangle|}{|\langle By(\tau-r)+b,y(\tau)\rangle|+1}d\tau \\ & +|\langle By(s-r)+b,y(s)\rangle|\,,\end{aligned}$$

with $K_1=\rho\left(2\,\|B\|\,\|\varphi\|_C+\|b\|\right)(\|B\|\,\|\varphi\|_C+\|b\|)$. Then,

$$\begin{aligned}|\langle BS(s-r)y(0)+b,S(s)y(0)\rangle| \;\leq\; & K_1\int_0^{T}|\langle By(\tau-r)+b,y(\tau)\rangle|\,d\tau \\ & +|\langle By(s-r)+b,y(s)\rangle|\,.\end{aligned}$$

Now, using the superposition property of the solution $y(t)$, we obtain via the above inequality that, for all $(s,t)\in[r,T]\times\mathbb{R}^+$

$$\begin{aligned}|\langle BS(s-r)y(t)+b,S(s)y(t)\rangle| \;\leq\; & K_1\int_t^{t+T}\langle By(\tau-r)+b,y(\tau)\rangle\,d\tau \\ & +|\langle By(t+s-r)+b,y(t+s)\rangle|\end{aligned}$$

Thus, integrating the last inequality with respect to $s\in[r,T]$, we deduce that

$$\int_r^T|\langle BS(s-r)y(t)+b,S(s)y(t)\rangle|\,ds \leq (K_1(T-r)+1)\int_t^{t+T}|\langle By(s-r)+b,y(s)\rangle|\,ds$$

then, using Holder's inequality, we obtain

$$\int_r^T|\langle BS(s-r)y(t),S(s)y(t)\rangle|\,ds \leq K(\int_t^{t+T}|\langle By(s-r)+b,y(s)\rangle|^2\,ds)^{\frac{1}{2}}$$

with $K=(K_1(T-r)+1)\sqrt{T}$. This combined with the inequality (5) gives:

$$\delta\,\|Cy(t)\|^{\gamma_b} \leq (\frac{K}{K_2}(\|y(t)\|^2-\|y(t+T)\|^2))^{\frac{1}{2}} \tag{30}$$

with $K_2=\frac{2\rho}{(\|B\|\|\varphi\|_C+\|b\|)\|\varphi\|_C+1}$. Now, using the above inequality and (6), we conclude that the system (1) is strongly partially stabilizable by the feedback control (2). □

5 | APPLICATIONS

5.1 | Example 1

Consider the uncoupled system described by:

$$\begin{cases} y'(t) = -\alpha y(t) + u(t)\, y(t-r) + u(t)\, q & t \geq 0 \\ \dfrac{\partial^2 z(x,t)}{\partial t^2} = \dfrac{\partial^2 z(x,t)}{\partial x^2} + u(t)\dfrac{\partial z(x,t-r)}{\partial t} & x \in (0,1),\ t \geq 0 \\ z(x,t-r) = z_0(x),\quad \frac{\partial z(x,t-r)}{\partial t} = z_1(x),\quad y(t-r) = y_0 & x \in (0,1),\ t \in [0,r] \\ z(0,t) = z(1,t) = 0 & t \geq 0 \end{cases} \tag{31}$$

Where $\alpha > 0$ and $q \in \mathbb{R}$. The system (31) can be written in the form of (1) in the state-space $H = \mathbb{R} \times \mathcal{X}$, where the space $\mathcal{X} = H_0^1(0,1) \times L^2(0,1)$ endowed with the inner product

$$\langle (X_1, X_2), (Y_1, Y_2) \rangle_{\mathcal{X}} = \int_0^1 \frac{dX_1}{dx}(x)\frac{dY_1}{dx}(x)dx + \int_0^1 X_2(x)Y_2(x)dx,$$

The operators A, B, and the vector b are defined as follows.

$$A = \begin{pmatrix} -\alpha & 0 & 0 \\ 0 & 0 & Id \\ 0 & \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix} \text{ with } D(A) = \mathbb{R} \times (H_0^1(0,1) \cap H^2(0,1)) \times H_0^1(0,1).$$

and

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Id \end{pmatrix},\ b = \begin{pmatrix} q \\ 0 \\ 0 \end{pmatrix}$$

Here, we take $H^s = \mathbb{R} \times \{0\}_{\mathcal{X}}$ and $H^u = \{0\} \times \mathcal{X}$, we have $H = H^u \oplus H^s$. Then, we have

$$A^s = \begin{pmatrix} -\alpha & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } A^u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & Id \\ 0 & \frac{\partial^2}{\partial x^2} & 0 \end{pmatrix}$$

and

$$B^s = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \text{ and } B^u = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & Id \end{pmatrix}.$$

Here, $(\varphi_n := \sqrt{2}\sin(n\pi x))_{n\geq 1}$ is a complete orthonormal eigenfunctions system of $\frac{\partial^2}{\partial x^2}$ with Dirichlet boundary condition associated to the eigenvalues $(\lambda_n = -n^2\pi^2)_{n\geq 1}$. Moreover, it is well know that the contraction semigroup S^u is defined by

$$S^u(t)\begin{pmatrix} z_2 \\ z_3 \end{pmatrix} = \sum_{n=1}^{+\infty} \begin{pmatrix} \langle z_2, \varphi_n \rangle \cos(n\pi t) + \frac{1}{n\pi}\langle z_3, \varphi_n \rangle \sin(n\pi t) \\ -n\pi \langle z_2, \varphi_n \rangle \sin(n\pi t) + \frac{1}{n\pi}\langle z_3, \varphi_n \rangle \cos(n\pi t) \end{pmatrix} \varphi_n,\ \forall z = (z_1, z_2, z_3) \in H.$$

Then,

$$\begin{aligned}\langle B^u S^u(t-r)z, S^u(t)z\rangle &= \sum_{n=1}^{+\infty} n^2\pi^2 \langle z_2,\varphi_n\rangle^2 \sin(n\pi(t-r))\sin(n\pi t)\\ &\quad-\sum_{n=1}^{+\infty}\langle z_2,\varphi_n\rangle\langle z_3,\varphi_n\rangle \sin(n\pi(t-r))\cos(n\pi t)\\ &\quad-\sum_{n=1}^{+\infty}\langle z_3,\varphi_n\rangle\langle z_2,\varphi_n\rangle \cos(n\pi(t-r))\sin(n\pi t)\\ &\quad+\sum_{n=1}^{+\infty}\frac{1}{n^2\pi^2}\langle z_3,\varphi_n\rangle^2 \cos(n\pi(t-r))\cos(n\pi t)\end{aligned}$$

Then, integrating this formula over $[r,T]$ with $r=2k$ and $T=2k'$ where $k,k'\in\mathbb{N}$ such that $k\le k'$, we obtain

$$\begin{aligned}\int_r^T |\langle B^u S^u(s-r)z, S^u(s)z\rangle|\,ds &= \sum_{n=1}^{+\infty} n^2\pi^2\langle z_1,\varphi_n\rangle^2\int_r^T \sin^2(n\pi t)dt\\ &\quad+\sum_{n=1}^{+\infty}\frac{1}{n^2\pi^2}\langle z_2,\varphi_n\rangle^2\int_r^T\cos^2(n\pi t)dt\\ &\ge \sum_{n=1}^{+\infty}\Big(\frac{1}{n^2\pi^2}\langle z_2,\varphi_n\rangle^2+n^2\pi^2\langle z_1,\varphi_n\rangle^2\Big)\\ &\ge \gamma\,\|z\|^2\end{aligned}$$

with $\gamma=\dfrac{1}{\pi^2}$. According to Theorem 2, then for any $0<\rho\le\frac{\alpha}{M\|B^s\|e^{\alpha r}}$. the following feedback control

$$u(t)=-\rho\frac{\displaystyle\int_0^1\frac{\partial z(x,t)}{\partial t}\frac{\partial z(x,t-r)}{\partial t}dx}{\left|\displaystyle\int_0^1\frac{\partial z(x,t)}{\partial t}\frac{\partial z(x,t-r)}{\partial t}dx\right|+1},\ \rho>0$$

strongly stabilizes the system (31), and we have the following estimate:

$$\left\|\left(y(t),z(t),\frac{\partial z(t)}{\partial t}\right)\right\|_H = O\left(\frac{1}{\sqrt{t}}\right),\ \text{as } t\longrightarrow+\infty.$$

5.2 | Example 2

Consider the system defined in $\Omega=(0,1)$ by the following equation

$$\begin{cases}\dfrac{\partial z(x,t)}{\partial t}=\dfrac{\partial^2 z(x,t)}{\partial x^2}+v(t)\left(Bz(x,t-r)+b\right) & t\ge0,\ x\in(0,1)\\ z_t=\varphi,\ z'(0)=z'(1)=0 & t\in[-r,0]\end{cases}\tag{32}$$

Taking $H=L^2(\Omega)$ as a state space and

$$Az=\frac{\partial^2 z(x,t)}{\partial x^2}\ \text{ with }\ D(A)=\left\{z\in H^2(\Omega):\ z'(0)=z'(1)=0\right\}$$

The spectrum of A is given by the simple eigenvalues $\lambda_j=-\pi^2(j-1)^2$, for all $j\in\mathbb{N}^*$ associated with the eigenvectors $\psi_1(x)=1$ and $\psi_j(x)=\sqrt{2}\cos((j-1)\pi x)$, $\forall j\ge2$. The unstable subspace is given by $H^u=span\left(\psi_1\right)$, , and we have

$S^u(t)\xi = \langle \xi, \psi_1 \rangle \psi_1$. Furthermore, let $b \in L^2(\Omega)$ such that $\int_0^1 b(x)dx \neq 0$.

Let $Bz = \sum_{j=2}^{+\infty} \alpha_j \langle z, \psi_j \rangle \psi_j$, with α_j are real positif numbers and $\sum_{j=2}^{+\infty} \alpha_j$ is convergent.
Here we can see that $BH^u = (0) \subset H^u$ and $BH^s \subset H^s$. It is easy to see that

$$\begin{aligned}\int_r^T |\langle B^u S^u(s-r)\xi + b, S^u(s)\xi \rangle| \, ds &= \int_r^T |\langle b, S^u(s)\xi \rangle| \, ds \\ &\geq \delta \|\xi\|, \forall \xi \in H^u\end{aligned}$$

with $\delta = (T-r)\left|\int_0^1 b(x)dx\right|$. Then, by applying Theorem 1, we deduce that the following control:

$$v(t) = \frac{\int_0^1 y(t)dt \int_0^1 b(x)dx}{\left|\int_0^1 y(t)dt \int_0^1 b(x)dx\right| + 1}$$

exponentially stabilize the system (32).

6 | CONCLUSION

This paper has proposed a bounded feedback control that ensures the stability of non-homogeneous bilinear systems and compensates for the destabilizing effect of the delay time term r. The presence of the non-homogeneous term b in the feedback control directly affects the degree of stability of the system at hand. The present study does not cover other interesting situations; this is when the operator B acting on the delayed state is unbounded or if the delay depends on time.

References

1. Akkouchi, M. and Bounabat, A. (2003). Weak stabilizability of a non autonomous and non-linear system. *Mathematica Pannonica*, 14(1):56–61.

2. Ammari, K. and Tucsnak, M. (2000). Stabilization of Bernoulli-Euler beams by means of a pointwise feedback force. *SIAM Journal on Control and Optimization*, 39(4):1160–1181.

3. Ball, J. and Slemrod, M. (1979). Feedback stabilization of distributed semilinear control systems. *Applied Mathematics and Optimization*, 5(1):169–179.

4. Berrahmoune, L. (1999). Stabilization and decay estimate for distributed bilinear systems. *Systems & control letters*, 36(3):167–171.

5. Bounabat, A. and Gauthier, J. (1991). Weak stabilizability of infinite-dimensional non-linear systems. *Applied Mathematics Letters*, 4(1):95–98.

6. Bounit, H. (2003). Comments on the feedback stabilization for bilinear control systems. *A*pplied Mathematics Letters, 16(6):847–851.

7. Bounit, H. and Hammouri, H. (1998). Stabilization of infinite-dimensional semilinear systems with dissipative drift. *A*pplied Mathematics and Optimization, 37(2):225–242.

8. Bounit, H. and Hammouri, H. (1999). Feedback stabilization for a class of distributed semilinear control systems. *N*on-linear Analysis: Theory, Methods & Applications, 37(8):953–969.

9. Christoskov, I. D., & Petkov, P. T. (2002). A practical procedure of bilinear weighted core kinetics parameters computation for the purpose of experimental reactivity determination. Annals of Nuclear Energy, 29(9), 1041-1054.

10. Datko, R. (1988). Not all feedback stabilized hyperbolic systems are robust with respect to small time delays in their feedbacks. SIAM Journal on Control and Optimization, 26(3), 697-713.

11. Feingesicht, M., Raibaudo, C., Polyakov, A., Kerherve, F., & Richard, J. P. (2016, June). A bilinear input-output model with state-dependent delay for separated flow control. In 2016 European Control Conference (ECC) (pp. 1679-1684). IEEE.

12. El Ayadi, R., & Ouzahra, M. (2019). Feedback stabilization for unbounded bilinear systems using bounded control. IMA Journal of Mathematical Control and Information, 36(4), 1073-1087.

13. Boutoulout, A., El Ayadi, R., & Ouzahra, M. (2014). An unbounded stabilization problem of bilinear systems. Mathematics and Computers in Simulation, 102, 39-50.

14. Ayadi, R. E., Ouzahra, M., & Boutoulout, A. (2012). Strong stabilisation and decay estimate for unbounded bilinear systems. International journal of control, 85(10), 1497-1505. ISO 690

15. Fridman, E., Mondié, S., & Saldivar, B. (2010). Bounds on the response of a drilling pipe model. IMA Journal of Mathematical Control and Information, 27(4), 513-526.

16. Hale, J. K. (1993). Verduyn Lunel S M. Introduction to functional differential equations. *Appl. Math. Sciences, 99.*

17. Hamidi, Z. and Ouzahra, M. (2018). Partial stabilisation of non-homogeneous bilinear systems. *International Journal of Control*, 91(16):1251–1258.

18. Hamidi, Z., Ouzahra, M., & Elazzouzi, A. (2020). Strong Stabilization of Distributed Bilinear Systems with Time Delay. Journal of Dynamical and Control Systems, 26(2), 243-254.

19. Hamidi, Z., Elazzouzi, A., & Ouzahra, M. (2021). Feedback Stabilization of Delayed Bilinear Systems. Differential Equations and Dynamical Systems, 1-14

20. Marrero-Ponce, Y., Contreras-Torres, E., Garcia-Jacas, C. R., Barigye, S. J., CubillÃ¡n, N., & Alvarado, Y. J. (2015). Novel 3D bio-macromolecular bilinear descriptors for protein science: Predicting protein structural classes. Journal of theoretical biology, 374, 125-137.

21. Mitrinović, D. S., Pečarić, J. E., & Fink, A. M. (1991). Inequalities of gronwall type of a single variable. In Inequalities Involving Functions and Their Integrals and Derivatives (pp. 353-400). Springer, Dordrecht.

22. Mohler, R. R. (1970). Natural bilinear control processes. IEEE Transactions on Systems Science and Cybernetics, 6(3), 192-197.

23. Mohler, R. R., (1973). Bilinear control processes: with applications to engineering, ecology, and medicine. Academic Press, Inc.

24. Nicaise, S., & Pignotti, C. (2006). Stability and instability results of the wave equation with a delay term in the boundary or internal feedbacks. SIAM Journal on Control and Optimization, 45(5), 1561-1585.

25. Pisarski, D., Bajer, C. I., Dyniewicz, B., & Bajkowski, J. M. (2016). Vibration control in smart coupled beams subjected to pulse excitations. Journal of Sound and Vibration, 380, 37-50.

26. Ouzahra, M. (2008). Strong stabilization with decay estimate of semilinear systems. *Systems & Control Letters*, 57(10):813–815.

27. Ouzahra, M. (2009). Stabilisation of infinite-dimensional bilinear systems using quadratic feedback control. *International Journal of Control*, 82(9), 1657-1664.

28. Pazy, A. (1983). *Semigroups of linear operators and applications to partial differential equations*. Springer-Verlag New York.

29. Sanchez, T., Polyakov, A., Hetel, L., & Fridman, E. (2018, December). A switching controller for a class of MIMO bilinear time-delay systems. In 2018 IEEE International Conference on the Science of Electrical Engineering in Israel (ICSEE) (pp. 1-4). IEEE.

30. Wu, J. (2012). *Theory and applications of partial functional differential equations*, volume 119. Springer Science & Business Media.

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