A Numerical Method for Pricing Discrete Double Barrier Option by Lagrange Interpolation on Jacobi Nodes

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Abstract

In this paper, a rapid and high accurate numerical method for pricing discrete single and double barrier knock-out call options is presented. With regard to the well-known Black-Scholes model, the price of an option in each monitoring date could be calculated by computing a recursive integral formula that is based on the heat equation solution. We have approximated these recursive solutions with the aid of Lagrange interpolation on Jacobi polynomial nodes. After that, an operational matrix, that makes our computation significantly fast, has been derived. In some theorems, the convergence of the presented method has been shown and the rate of convergence has been derived. The most important benefit of this method is that its complexity is very low and does not depend on the number of monitoring dates. The numerical results confirm the accuracy and efficiency of the presented numerical algorithm.

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1. Introduction

Barrier options are popular instruments that play an important role in financial securities markets where the option pricing is the most important problem, i.e. to evaluate a fair price for the option. The Nobel Prize-winning Black-Scholes option valuation theory motivates using classical methods for solving partial differential equations (PDE's) [1]. In computational finance, numerous nonstandard numerical methods are proposed and successfully applied for pricing options [2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12]. Numerical methods are often preferred to closed-form solutions as they could be more easily extended or adapted to satisfy all the financial requirements of the option contracts and continuously changing conditions imposed by financial institutions and over-the-counter market for controlling trading of derivatives.

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Kunitomo and Ikeda [13] obtained general pricing formulas for European double barrier options with curved barriers but like for a variety of path-dependent options and corporate securities most formulas are obtained for restricted cases as continuous monitoring or single barrier [5]. The discrete monitoring is essential as the trading year is considered to consist of 250 working days and a week of 5 days. Thus, taking for one year $T = 1$, the application of barriers occurs with a time increment of 0.004 daily and 0.02 weekly.

Several other different contracts with discrete time monitoring are characterized by updating the initial conditions, such as Parisian options and occupation time derivatives [14]. We remark that although most real contracts specify fixed times for monitoring the asset price, academic researchers have focused mainly on continuous time monitoring models as the analysis of fixed barriers could be treated mathematically using some techniques such as the reflection principle [15]. For example, Li utilize reflection principle of Brownian motion to obtain the conditional density of stock price process and then obtain option price by evaluation of expectation of discounted payoff in [16]. Broadie et. al. have found an explicit correction formula for discretely monitored option with one barrier [17]. However, these three well-known methods [6, 10, 11] have not been still applied in the presence of two barriers, i.e. a discrete double barrier option.

Initially classical quantitative methods in finance have been explored for pricing barrier options. This includes standard lattice techniques, i.e. the binomial and trinomial trees of Kamrad and Ritchken [18], Boyle and Lau [19], Kwok [15], Heyen and Kat [20], Tian [21], Dai and Lyuu [22]. In lattice methods, at first, the time interval divided in too small time step and then by generating price tree the price of the option is obtained. If at each time step there are two possible movements (down and up) for the price the method is called binomial. The trinomial method is like binomial but at each time step the stock price has three possible states: up, down and middle. The negative point of these methods is that they are time-consuming. In fact, for getting more accuracy of these methods, we must increase the number of time steps that increase CPU time. To solve this problem, Ahn et al. [23] used adaptive mesh model approach that had been introduced before by Figlewski and Gao [24]. This method improves the efficiency of the trinomial method.

Another approach that is widely applied for option pricing is simulation Monte Carlo methods. The base of this algorithm is to simulate great numbers of stock price sample path and then by computing the payoff at the maturity time estimate expectation value of discounted payoff. For more information we refer interested readers to [25, 26, 27, 28, 29, 30].

Although it could not be claimed that it is impossible to be found an exact or closed-form solution of the Black-Scholes equation [31] for the valuation of discrete double barrier knock-out call option, it is sure that there is a substantial difference in the option prices between continuous and discrete monitoring even for 1 000 000 monitoring dates. This could be trivially tested for a single barrier knock-in and knock-out option using formulas [3], [13], or the correction formula [17], for double barrier knock-out options with the numerical algorithm [5] or with a high-order accurate finite difference scheme [11]. It is well-known in the literature the relation when comparing the price of continuous and discretely monitored barrier options with the corresponding vanilla option with the same parameters and absence of rebates. The discrete monitoring considerably complicates the analysis of barrier options [17] and their pricing often requires nonstandard method as those presented in [2, 5, 7, 11]. Difficulties of pricing double barrier options emerge even in the case of continuous monitoring where some drawbacks of closed-form formulas could be clearly observed. The analytical solutions of such options are usually expressed as an infinite series of reflections and presented with Fourier series. For fixed barriers contracts the Fourier series solution gives the same answer when all the terms have been added up but the main drawback is that the rate of convergence of the sum to the solution can be quite different, depending on the time to expiry.

Recently, different types of analytical and semi-analytical solution for pricing barrier options have been obtained by using integral transforms. Geman and Yor found the Laplace transform of the price of barrier option and then compute invert Laplace transform by numerical integration [32]. Plesser in [33] proposed an analytical formula by using contour integration for calculating invert Laplace transform. Fusai et al. convert single barrier option pricing to a Wiener-Hopf integral equation by the help of Z-transform and then found an analytical solution for it [3]. In [34], Broadie and Yamamoto proposed a numerical algorithm based on the numerical computation of convolution integral by using double exponential formula and fast Gauss transform. In [35] a method based on Fourier-cosine series expansions presented.

Farnoosh et al. considered double barrier options that their parameters are not constant. They used some change of variables to transform the time-dependent coefficient partial differential equations in each monitoring interval to constant ones and then solve it by recursive formula and Romberg integration technique [36, 37], while in [38] projection methods have been explored. Numerical integration and quadrature methods have been implemented for option pricing in [39, 5].

The main objective of this paper is to present a new efficient computational method for valuation of discrete barrier options based on Lagrange interpolation on Jacobi nodes that have not only a simpler computer implementation but also differ with minimum memory requirements and extreme short computational times.

This article is organized as follows. In Section 2 we formulate the mathematical model for the valuation of barrier options under the classical Black-Scholes framework. In Section 3 we briefly list definitions for Jacobi polynomials. In section 4 we propose a new efficient numerical method where an orthogonal Lagrange interpolation is utilized and a suitable operational matrix form has been obtained for pricing discrete double barrier options. One of the main advantages of this algorithm is that it does not depend on the number of monitoring dates. In section 5, the convergence of the presented method has shown. In the next Section 6, we observe numerical errors of order 10^{-4} and 10^{-6} in the maximum norm for different computational experiments according to the number of node points. The obtained results are in good agreement with other benchmark values in literature and this confirms the efficiency and accuracy of the presented numerical algorithm.

2. The Pricing Model

Let r be the risk-free interest rate, σ volatility, and S_0 initial stock price. We suppose that underlying stock price S_t is an Ito process satisfies in Geometric Brownian motion as follows:

$$S_t = S_0 e^{(r-\frac{1}{2}\sigma^2)t+\sigma B_t}$$

where B_t is standard Brownian motion under the neutral measure P. A European option is named call (put) option if it gives the holder the right of buy (sell) of the underlying stock at an exercise price E on the maturity time T. A double barrier option is an option that has two level barrier price: upper barrier U and lower barrier L. Depends on the type of barrier option if the stock price hits each of barriers, the option comes in or out of the money. A knock-out (knock-in) barrier option is a barrier option that comes out (in) of money if the underlying stock price cross each of barriers before the maturity time. If the barrier levels only monitored at specific predetermined dates $t_1, t_2, ..., t_M$, the barrier option is said discrete barrier option. In this article, we work on knock-out discrete double barrier call option. In addition, the monitoring dates are assumed equally spaced, i.e. $t_m = m\tau, i = 1, 2, ..., m$ where $\tau = T/M$. If the barriers are not hit by stock price in monitoring dates, the payoff of the option at the maturity time will be $\max(S_T - E, 0)$.

The price of an option can be considered as the expectation value of discounted payoff at the expiration time. The price of option $\mathcal{P}(S, t, m-1)$ at time time $t \in (t_{m-1}, t_m)$, under the the Black-Scholes model, fulfills in the following partial differential equations

$$-\frac{\partial \mathcal{P}}{\partial t} + rS\frac{\partial \mathcal{P}}{\partial S} + \frac{1}{2}\sigma^2 S^2 \frac{\partial^2 \mathcal{P}}{\partial S^2} - r\mathcal{P} = 0, \tag{1}$$

subject to the initial conditions:

$$\mathcal{P}(S, t_0, 0) = (S - E)\,\mathbf{1}_{(\max(E,L) \le S \le U)}$$

$$\mathcal{P}(S, t_m, 0) = \mathcal{P}(S, t_m, m-1)\,\mathbf{1}_{(L \le S \le U)};\ m = 1, 2, ..., M-1\ ,$$

where $\mathcal{P}(S, t_m, m-1) := \lim_{t \to t_m} \mathcal{P}(S, t, m-1)$.

By denoting $\mu = r - \frac{\sigma^2}{2}$, $c^2 = \frac{\sigma^2}{2}$, $a = -\frac{\mu}{\sigma^2}$, $b = a\mu + a^2\frac{\sigma^2}{2} - r$, $E^* = \ln\left(\frac{E}{L}\right)$, $\theta = \ln(U/L)$, and change of variables $z = \ln(S/L)$, $\mathcal{P}(S, t, n) = e^{az+bt}u(z, t, n)$ equation (1) is reduced to the heat equation:

$$\frac{\partial u}{\partial t} - c^2\frac{\partial^2 u}{\partial x^2} = 0 \tag{2}$$

with the initial conditions as below:

$$u(z, t_0, 0) = Le^{-az}\left(e^z - e^{E^*}\right)\mathbf{1}_{(\delta \le z \le \theta)},$$

$$u(z, t_m, 0) = u(z, t_m, m-1)\,\mathbf{1}_{(0 \le z \le \theta)};\ m = 1, 2, ..., M-1\ .$$

Equation (2) has analytical solution:

$$u(z, t, m) = \begin{cases} L\int_\delta^\theta k(z-\xi, \tau)e^{-az}\left(e^z - e^{E^*}\right)(\xi)\,d\xi; m = 0 \\ \int_0^\theta k(z-\xi, \tau)u(z, t_m, m-1)(\xi)\,d\xi;\ m = 1, 2, 3, ..., M \end{cases} \tag{3}$$

where

$$k(z, t) = \frac{1}{\sqrt{4\pi c^2 t}}e^{-\frac{z^2}{4c^2 t}}. \tag{4}$$

We define $u_m(z)$ as following recursive formula:

$$u_1(z) = \int_0^{\theta} k(z-\xi,\tau)u_0(\xi)\,d\xi \tag{5}$$

$$u_m(z) = \int_0^{\theta} k(z-\xi,\tau)u_{m-1}(\xi)\,d\xi;\ m = 2,3,...,M \tag{6}$$

where

$$u_0(z) = Le^{-\alpha z}\left(e^z - e^{E^*}\right)\mathbf{1}_{(\delta\le z\le\theta)}, \tag{7}$$

Therefore, the price of the knock-out discrete double barrier option can be obtained as follows:

$$\mathcal{P}(S_0, t_M, M-1) \simeq e^{az_0+bt}u_M(z_0) \tag{8}$$

where $z_0 = \ln\left(\frac{S_0}{L}\right)$.

3. Jacobi Polynomials

Let $w^{(\alpha,\beta)}(x) = (1-x)^{\alpha}(1+x)^{\beta},\ \alpha,\beta > -1$, and $L^2_{w^{(\alpha,\beta)}}(-1,1)$ be Hilbert space with the following inner product and norm:

$$< f,g >_{w^{(\alpha,\beta)}} = \int_{-1}^{1} f(x)g(x)w(\alpha,\beta)(x)dx, \tag{9}$$

$$\|f\|_{w^{(\alpha,\beta)}} = \sqrt{< f,f >_{w^{(\alpha,\beta)}}}. \tag{10}$$

The Jacobi polynomials, $J_i^{(\alpha,\beta)}(x)$ are orthogonal polynomials in $L^2_{w^{(\alpha,\beta)}}(-1,1)$, i.e;

$$\int_{-1}^{1} J_i^{(\alpha,\beta)}(x)J_j^{(\alpha,\beta)}(x)w(\alpha,\beta)(x)dx = \lambda_i\delta_{ij}, \tag{11}$$

where $\lambda_i = \|J_i^{(\alpha,\beta)}\|^2_{w^{(\alpha,\beta)}}$. These polynomials that set an orthogonal basis in $L^2_{w^{(\alpha,\beta)}}(-1,1)$, are eigenfunctions of singular Sturm-Liouville operator $\mathcal{SL}$:

$$\mathcal{SL}(\varphi) := (x^2-1)\frac{d^2\varphi}{dx^2} + Q_{\alpha,\beta}(x)\frac{d\varphi}{dx}$$

where $Q_{\alpha,\beta}(x) = \alpha-\beta+(\alpha+\beta+2)x$. Jacobi polynomials could be obtained by Rodrigues' formula:

$$J_n^{(\alpha,\beta)}(x) = \frac{(-1)^n}{w^{(\alpha,\beta)}2^n n!}\frac{d^n}{dx^n}\left[(\ w^{(\alpha,\beta)}\ )^n\right]$$

They also satisfy in following three-term recurrence relation:

$$J_0^{(\alpha,\beta)}(x) = 1,\ J_1^{(\alpha,\beta)}(x) = \frac{1}{2}(\alpha+\beta+2)x + \frac{1}{2}(\alpha-\beta) \tag{12}$$

$$J_{i+1}^{(\alpha,\beta)}(x) = \left(a_i^{(\alpha,\beta)}x - b_i^{(\alpha,\beta)}J_i^{(\alpha,\beta)}(x)\right) - c_i^{(\alpha,\beta)}J_{i-1}^{(\alpha,\beta)}(x) \tag{13}$$

where:

$$a_i^{(\alpha,\beta)} = \frac{(2i+\alpha+\beta+1)(2i+\alpha+\beta+2)}{2(i+1)(n+\alpha+\beta+1)} \tag{14}$$

$$b_i^{(\alpha,\beta)} = \frac{(\beta^2-\alpha^2)(2n+\alpha+\beta+1)}{2(i+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)} \tag{15}$$

$$c_i^{(\alpha,\beta)} = \frac{(n+\alpha)(n+\beta)(2n+\alpha+\beta+2)}{(i+1)(n+\alpha+\beta+1)(2n+\alpha+\beta)}. \tag{16}$$

The roots of Jacobi polynomials are real and distinct. In addition, all of them lie in the interval $[-1,1]$. On the other hand, they have density properties, i.e. for any subinterval of $[-1,1]$ there exists $N \in \mathbb{N}$ such that for all $n \geq N$ Jacobi polynomials J_n has at least one root in this subinterval. Because of this appropriate features, they play important role in spectral methods and approximation theory.

4. Pricing by orthogonal Lagrange interpolation

In this section we consider Π_n as vector space of all polynomials with real coefficients of degree less or equal to n, set points $\{x_i^{\alpha,\beta}\}_{i=0}^n$ as roots of $(n+1)$-th Jacobi polynomial $J_{n+1}^{(\alpha,\beta)}$ that are shifted to $[0,\theta]$ and $\mathcal{I}_n^{\alpha,\beta} : C[0,\theta] \to \Pi_n$ as orthogonal polynomial interpolation projection operator, that is defined as follows:

$$\mathcal{I}_n^{\alpha,\beta}(f) = \sum_{i=0}^{n} f(x_i^{\alpha,\beta})\mathcal{L}_i(x) \tag{17}$$

where $\mathcal{L}_i(x)$ is the i-th Lagrange polynomial basis function defined on $\{x_i^{\alpha,\beta}\}_{i=0}^n$:

$$\mathcal{L}_i(x) = \prod_{j=0, j\neq i}^{n} \frac{(x-x_j^{\alpha,\beta})}{(x_i^{\alpha,\beta}-x_j^{\alpha,\beta})}. \tag{18}$$

For any $f \in C^{n+1}[0,\theta]$ we have:

$$||\mathcal{I}_n^{\alpha,\beta}(f) - f||_{max} \leq ||Q_n||_{max} \frac{||f^{(n+1)}||_{max}}{(n+1)!} \tag{19}$$

where $Q_n = \prod_{j=0}^n (x - x_j^{\alpha,\beta})$.

Let integral operator $\mathcal{K} : C[0,\theta] \to C[0,\theta]$ is defined as below:

$$\mathcal{K}(u)(z) := \int_0^\theta \kappa(z-\xi,\tau)u(\xi)d\xi. \tag{20}$$

where kernel function κ is defined in (4). As κ is a continuous kernel function, hence linear integral operator $\mathcal{K}$ is a bounded and compact operator on $C[0,\theta]$, i.e. a linear operator that maps every bounded subset of $C[0,\theta]$ to a precompact subset of $C[0,\theta]$. We rewrite equations (5) and (6) with the aid of operator $\mathcal{K}$ as follows:

$$u_1 = \mathcal{K}u_0 \tag{21}$$

$$u_m = \mathcal{K} u_{m-1} \quad m = 2, 3, ..., M. \tag{22}$$

Now, we define

$$\tilde{u}_{1,n} = \mathcal{I}_n^{\alpha,\beta} \mathcal{K} \left(u_0 \right) \tag{23}$$

$$\tilde{u}_{m,n} = \mathcal{I}_n^{\alpha,\beta} \mathcal{K} \left(\tilde{u}_{m-1} \right) = \left(\mathcal{I}_n^{\alpha,\beta} \mathcal{K} \right)^m \left(u_0 \right) \ , \ m \geq 2 \tag{24}$$

where $\mathcal{I}_n^{\alpha,\beta}\mathcal{K}$ is as follows:

$$(\mathcal{I}_n^{\alpha,\beta}\mathcal{K})(u) = \mathcal{I}_n^{\alpha,\beta}\left(\mathcal{K}(u)\right).$$

Since, $\tilde{u}_{m,n} \in \Pi_n$ for $m \geq 1$, we can write

$$\tilde{u}_{m,n} = \sum_{i=0}^{n} a_{mi} \mathcal{L}_i(z) = \Phi_n'(x) G_m,$$

where $G_m = [a_{m0}, a_{m1}, \cdots, a_{mn}]'$ and $\Phi_n = [\mathcal{L}_m, \mathcal{L}_m, \cdots, \mathcal{L}_n]'$. From equation (24) we obtain

$$\tilde{u}_{m,n} = (\mathcal{I}_n^{\alpha,\beta}\mathcal{K})^{m-1} \left(\tilde{u}_{1,n} \right). \tag{25}$$

Since dimension of Π_n is finite, then the linear operator $\mathcal{I}_n^{\alpha,\beta}\mathcal{K}$ on Π_n can be represented By a matrix that is here denoted K. Consequently matrix form of equation (25) is represented as follows:

$$\tilde{u}_{m,n} = \Phi_n' K^{m-1} G_1. \tag{26}$$

In above relation matrix operator K and the vector G_1 are easily obtained as:

$$G_1 = [a_{10}, a_{11}, \cdots, a_{1n}]'$$

$$K = \left(k_{ij}\right)_{n \times n}$$

where

$$a_{1i} = \int_{\delta}^{\theta} \kappa(x_i^{\alpha,\beta} - \xi, \tau) u_0(\xi) d\xi \ , \ 0 \leq i \leq n.$$

$$k_{ij} = \int_{0}^{\theta} \kappa(x_i^{\alpha,\beta} - \xi, \tau) \mathcal{L}_{j-1}(\xi) d\xi \ .$$

Thus, we could approximate the price of a knock-out discrete double barrier option by following formula:

$$\mathcal{P}\left(S_0, t_M, M-1\right) \simeq e^{\alpha z_0 + \beta t} \tilde{u}_{M,n}\left(z_0\right) \tag{27}$$

where $z_0 = \ln\left(\frac{S_0}{L}\right)$ and $\tilde{u}_{M,n}$ is obtained from (26).

Remark 1. In (26) the parameter m, the number of monitoring dates, arise as the power of the matrix K. Therefore, it could be said that the consuming time of the algorithm is not impacted by changing m. In fact, the complexity of this algorithm is $\mathcal{O}(n^2)$ that only depends on the number of basis functions. On the other hand computing (26) needs only to compute $n^2 + n$ integral against double integral in [38, 8] that makes presented method far faster than them.

5. Convergence of numerical scheme

In this section we discuss the convergence of our presented method in following three theorems.

Theorem 1. *The operator $\mathcal{I}_n^{\alpha,\beta}\mathcal{K}$ is pointwise convergent to operator $\mathcal{K}$ on $C[0,\theta]$.*

Proof:. Let $\varphi \in C[0,\theta]$ then

$$
\begin{aligned}
||(\mathcal{I}_n^{\alpha,\beta}\mathcal{K}-\mathcal{K})\varphi||_{max} &= \sum_{i=0}^{n}\left(\int_0^{\theta}\kappa(x_i^{\alpha,\beta}-\xi,\tau)\varphi(\xi)d\xi\right)\mathcal{L}_i(z)-\int_0^{\theta}\kappa(z-\xi,\tau)\varphi(\xi)d\xi \\
&\leq \frac{||Q_n||_{max}}{(n+1)!}\left\|\int_0^{\theta}\frac{\partial^{n+1}\kappa(z-\xi,\tau)}{\partial z^{n+1}}\varphi(\xi)d\xi\right\|_{max} \\
&\leq \frac{\theta||Q_n||_{max}}{(n+1)!}\left\|\frac{\partial^{n+1}\kappa(z-\xi,\tau)}{\partial z^{n+1}}\right\|_{max}||\varphi||_{max}. \qquad (28)
\end{aligned}
$$

In addition, we have

$$
\frac{\partial^{n+1}\kappa(z-\xi,\tau)}{\partial z^{n+1}} = \frac{1}{\sqrt{4\pi c^2\tau}}\frac{\partial^{n+1}e^{-\frac{(z-\xi)^2}{4c^2\tau}}}{\partial z^{n+1}}. \qquad (29)
$$

Now by change of variable $y=\frac{(z-\xi)}{\sqrt{4c^2\tau}}$ and considering $C_0=\frac{1}{\sqrt{4c^2\tau}}$, $C_1=\frac{1}{\sqrt{4\pi c^2\tau}}$ we obtain:

$$
\frac{\partial^{n+1}\kappa(z-\xi,\tau)}{\partial z^{n+1}} = C_1\ C_0^{n+1}\frac{d^{n+1}e^{-y^2}}{dy^{n+1}} = C_1\ C_0^{n+1}H_{n+1}(y)e^{-y^2}, \qquad (30)
$$

where $H_n(y)$ is Hermite Polynomial of degree n. By using following inequality about hermite polynomials (see [40],page 324)

$$
|H_{n+1}(y)| < k_0\sqrt{(n+1)!}2^{(n+1)/2}e^{y^2/2}, \qquad (31)
$$

where $k_0 = 1.086435$, we obtain the following inequality:

$$
\frac{\partial^{n+1}\kappa(z-\xi,\tau)}{\partial z^{n+1}} \leq k_0C_1C_0^{n+1}\sqrt{(n+1)!}2^{(n+1)/2}e^{-y^2/2} \qquad (32)
$$

and consequently:

$$
\left\|\frac{\partial^{n+1}\kappa(z-\xi,\tau)}{\partial z^{n+1}}\right\|_{max} \leq k_0C_1C_0^{n+1}\sqrt{(n+1)!}2^{(n+1)/2}. \qquad (33)
$$

On the other hand

$$
\|Q_n\|_{max} \leq \theta^{(n+1)} \qquad (34)
$$

by substituting (33) and (34) in (28), it follows that:

$$
||(\mathcal{I}_n^{\alpha,\beta}\mathcal{K}-\mathcal{K})\varphi||_{max} \leq \theta k_0C_1||\varphi||_{max}\left(\frac{(2\theta^2C_0^2)^{n+1}}{(n+1)!}\right)^{1/2}. \qquad (35)
$$

Since

$$\lim_{n\to\infty} \frac{(2\theta^2 C_0^2)^{n+1}}{(n+1)!} = 0 \tag{36}$$

we conclude

$$\lim_{n\to\infty} ||(\mathcal{I}_n^{\alpha,\beta}\mathcal{K} - \mathcal{K})\varphi||_{max} = 0, \tag{37}$$

which finishes the proof. □

Theorem 2. *The operator $\mathcal{I}_n^{\alpha,\beta}\mathcal{K}$ is norm convergent to operator $\mathcal{K}$ on $C[0,\theta]$, i.e.*

$$\lim_{n\to\infty} ||\mathcal{I}_n^{\alpha,\beta}\mathcal{K} - \mathcal{K}|| = 0. \tag{38}$$

Proof:. Let $\mathcal{B}$ denotes

$$\mathcal{B} = \{\varphi \in C[0,\theta] |\, \|\varphi\|_{max} \leq 1\},$$

then we have

$$||(\mathcal{I}_n^{\alpha,\beta}\mathcal{K} - \mathcal{K})|| = \sup_{\varphi\in\mathcal{B}} ||(\mathcal{I}_n^{\alpha,\beta}\mathcal{K} - \mathcal{K})\varphi||_{max}, \tag{39}$$

that by relation (35) tends to zero when $n \to \infty$. □

Following theorem guarantees the convergence of our presented method:

Theorem 3. *The function $\tilde{u}_{M,n}$ is convergent to u_M uniformly in $[0,\theta]$.*

Proof:. At the first we note that

$$||(\mathcal{I}_n^{\alpha,\beta}\mathcal{K})^M - \mathcal{K}^M|| \leq \left\|\mathcal{I}_n^{\alpha,\beta}\mathcal{K}\right\| ||(\mathcal{I}_n^{\alpha,\beta}\mathcal{K})^{M-1} - \mathcal{K}^{M-1}|| - ||\mathcal{I}_n^{\alpha,\beta}\mathcal{K} - \mathcal{K}||||\mathcal{K}||^{M-1} \tag{40}$$

By regarding the above-mentioned relation, (38), and mathematical induction we reach to

$$\lim_{n\to\infty} ||(\mathcal{I}_n^{\alpha,\beta}\mathcal{K})^M - \mathcal{K}^M|| = 0. \tag{41}$$

as a result, ones obtains

$$\|\tilde{u}_{M,n} - u_M\| = \left\|(\mathcal{I}_n^{\alpha,\beta}\mathcal{K})^M u_0 - \mathcal{K}^M u_0\right\| \leq \left\|(\mathcal{I}_n^{\alpha,\beta}\mathcal{K})^M - \mathcal{K}^M\right\| ||u_0||. \tag{42}$$

From (41) the right hand side of (42) tends to zero as $n \to \infty$. This completes the proof of theorem. □

Remark 2. Since the convergence rate of our algorithm depends on decay rate of

$$\left(\frac{(2\theta^2 C_0^2)^{n+1}}{(n+1)!}\right)^{\frac{1}{2}} \sim \exp\left(-\frac{n}{2}log(n)\right),$$

the convergence rate of presented numerical algorithm is supergeometric (see [41], page 25).

$\alpha \backslash \beta$	-0.8	-0.5	0	0.5	0.8
-0.8	$8.6074e-06$	$8.1718e-06$	$2.2103e-05$	$1.8770e-05$	$9.5120e-06$
-0.5	$8.7606e-06$	$7.8929e-06$	$1.2852e-05$	$3.8600e-05$	$4.6071e-05$
0	$2.7040e-05$	$2.5788e-05$	$2.2103e-05$	$5.3726e-05$	$9.3608e-05$
0.5	$9.1438e-05$	$9.5461e-05$	$9.2619e-05$	$8.2564e-05$	$1.0667e-04$
0.8	$1.4675e-04$	$1.6600e-04$	$1.7498e-04$	$1.6536e-04$	$1.5456e-04$

Table 1: The maximum norm error for $n = 25$ of example(1) with $L = 95$ and $M = 125$.

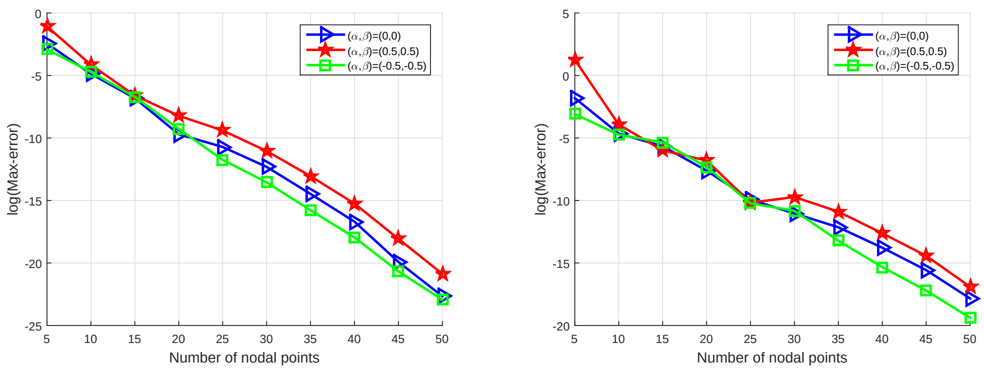
6. Computational Results

In order to show the efficiency and accuracy of the presented numerical method, in some examples, we solve barrier option pricing problem and compare it with some other ones. The method has been programmed in MATLAB 2015 and numerical results have been obtained on a personal computer with a 3.2 GHz Intel Core i5 processor and 8-gigabit DDR3 memory.

Test problem 1. *At the first test problem, we consider the price of a discrete double barrier option. The parameters is supposed to be* $r = 0.05$, $\sigma = 0.25$, $T = 0.5$, $S_0 = 100$, *and* $E = 100$. *The upper barrier is set* 120 *and lower barriers are set various values:* 80,90, 95, 99,*and* 99.5. *Numerical results for various number of monitoring dates are obtained by presented method with* $\alpha = -0.5, \beta = -0.5, n = 25$ *and compared with reported results of other numerical schemes such as Milev method based on numerical integration [5], Crank-Nicholson [42], trinomial based on tree methods, adaptive mesh model (AMM) and QUAD-K200 as benchmark based on quadrature methods [43]. The compared results are provided in table 2 that demonstrate our results coincide with the benchmark. moreover, CPU time of presented method is far less than others and does not depend on the number of monitoring dates against other methods. In table 1 the maximum norm error of presented method for various values of* α *and* β *has been provided. It is easy to see that least error is obtained for* $(\alpha = -0.5, \beta = -0.5)$ *that is not by chance. Actually, it is due to the fact that* $||Q_n(x)||_{max}$ *gets its minimum when nodal points are roots of Chebyshev polynomials. In figure 1 we have plotted maximum norm of error for* $M = 25, 125$ *and in figure 2 the estimated price and error of it have been plotted For* $L = 80$ *and* $M = 125$.

M	l L	PM ($\alpha=-0.5, \beta=-0.5$) (n=25)	Milev (200)	Milev (400)	Trinomial	AMM-8	Benchmark
	80	2.4499	-	-	2.4439	2.4499	2.4499
	90	2.2028	-	-	2.2717	2.2027	2.2028
5	95	1.6831	1.6831	1.6831	1.6926	1.6830	1.6831
	99	1.0811	1.0811	1.0811	0.3153	1.0811	1.0811
	99.9	0.9432	0.9432	0.9432	-	0.9433	0.9432
CPU		0.035 s	1 s	5 s			
	80	1.9420	-	-	1.9490	1.9419	1.9420
	90	1.5354	-	-	1.5630	1.5353	1.5354
25	95	0.8668	0.8668	0.8668	0.8823	0.8668	0.8668
	99	0.2931	0.2931	0.2931	0.3153	0.2932	0.2931
	99.9	0.2023	0.2023	0.2023	-	0.2024	0.2023
CPU		0.035 s	8 s	30 s			
	80	1.6808	-	-	1.7477	1.6807	1.6808
	90	1.2029	-	-	1.2370	1.2028	1.2029
125	95	0.5532	0.5528	0.5531	0.5699	0.5531	0.5532
	99	0.1042	0.1042	0.1042	0.1201	0.1043	0.1042
	99.9	0.0513	0.0513	0.0513	-	0.0513	0.0513
CPU		0.035 s	35 s	150 s			

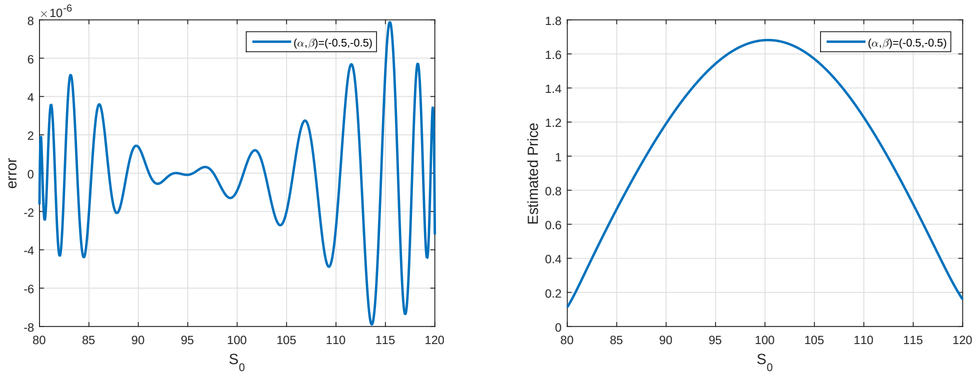
Table 2: The price of double barrier call option of test problem 1: $T = 0.5$, $r = 0.05$, $\sigma = 0.25$, $S_0 = 100$, $E = 100$.



(a) $M = 125$ (b) $M = 250$

Figure 1: $Max - error$ for example (1) with $L = 80$.

Test problem 2. *In the second test problem, the price of a knock-out discrete double barrier option with* $r = 0.05$*,* $\sigma = 0.25$*,* $T = 0.5$*,* $E = 100$*, and* $m = 5$ *for various values of initial price is evaluated. The lower and upper barriers are placed at* 95 *and* 110 *respectively. The numerical results are compared with Milev method, Crank-Nicholson, and Monte Carlo simulation method in [44].*



(a) Error (b) Estimated Price

Figure 2: The error and estimated Price in example(1) with $L = 80$ and $M = 125$.

S_0	PM ($\alpha = -0.5, \beta = -0.5,$) ($n = 25$)	Crank-Nicolson (1000)	Milev (400)	Milev (1000)	MC (st.error) with 10^7 paths
95	0.174498	0.1656	0.174503	0.174498	-
95.0001	0.174499	$\simeq$ 0.1656	0.174501	0.174499	0.17486 (0.00064)
95.5	0.182428	0.1732	0.182429	0.182428	0.18291 (0.00066)
99.5	0.229349	0.2181	0.229356	0.229349	0.22923 (0.00073)
100	0.232508	0.2212	0.232514	0.232508	0.23263 (0.00036)
100.5	0.234972	0.2236	0.234978	0.234972	0.23410 (0.00073)
109.5	0.174462	0.1658	0.174463	0.174462	0.17426 (0.00063)
109.9999	0.167394	$\simeq$ 0.1591	0.167399	0.167394	0.16732 (0.00062)
110	0.167393	0.1591	0.167398	0.167393	-
CPU	0.035 s	Minutes	1 s	39 s	

Table 3: Price of discrete double barrier option of test problem 2 in 5 monitoring dates: $T = 0.5$, $M = 5$, $r = 0.05$, $\sigma = 0.25$, $E = 100$, $U = 110$ and $L = 95$.

Test problem 3. *Because that the chance of touching the upper barrier level by stock price before the maturity time T when $U \geq 2E$ is insignificant, the price of a single down-and-out call option can be approximated by double ones by considering the upper barrier above $2E$. In this test problem, we want to evaluate price of a single down-and-out barrier call option . We suppose $r = 0.1$, $\sigma = 0.2$, $T = 0.5$, $S_0 = 100$, and $E = 100$. Then we set the lower barriers at 95, 99.5, and 99.9. After that, we approximate the price by double ones with $U = 250$. In table (4) the numerical results are shown and compared with schemes referred in [3, 2, 28] that demonstrate the effectiveness of proposed approach in this case.*

		PM ($\alpha = -0.5, \beta = -0.5$)				
L	M	n=25	n=50	(IR17)	MCH	MC (st.error)
95	25	6.63104	6.63156	6.63156	6.6307	6.63204 (0.0009)
99.5	25	3.35644	3.35558	3.35558	3.3552	3.35584 (0.00068)
99.9	25	3.00897	3.00887	3.00887	3.0095	3.00918 (0.00064)
95	125	6.16940	6.16863	6.16864	6.1678	6.16879 (0.00088)
99.5	125	1.95811	1.96130	1.96130	1.9617	1.96142 (0.00053)
99.9	125	1.50991	1.51020	1.51068	1.5138	1.5105 (0.00046)
CPU		0.038 s	0.051 s			

Table 4: prices of Single barrier option of test problem 3: $T = 0.5$, $r = 0.1$, $\sigma = 0.2$, $S_0 = 100$, $E = 100$, $U = 250$.

Test problem 4. *Here we estimate the price of continuous monitoring call barrier down and out option, $\mathcal{P}c$, with discrete ones, $\mathcal{P}d_m$, using the following formula [17]:*

$$\mathcal{P}c(L) = \mathcal{P}d_m \left(L \ e^{\lambda\sigma\sqrt{\Delta t}} \right), \tag{43}$$

where $\lambda = \zeta(1/2)/\sqrt{2\pi} \simeq 0.5826$ with ζ the Riemann zeta function. The parameters of this problem is considered as $r = 0.1$, $\sigma = 0.3$, $T = 0.2$, $E = 100$, $S = 100$. In table (5) *the option price for different lower barriers is evaluated and compared with continuous monitoring price that is obtained in [17]. As we can see, this estimations is accurate except when the barrier is close to the spot price.*

		PM($\alpha = -0.5, \beta = -0.5, M = 50$)		PM($\alpha = -0.5, \beta = -0.5, M = 125$)	
L	Countinous Barrier	n=25	n=50	n=25	n=50
85	6.308	6.307	6.308	6.306	6.308
88	6.185	6.185	6.185	6.182	6.185
91	5.808	5.808	5.808	5.809	5.808
93	5.277	5.277	5.277	5.277	5.277
95	4.398	4.396	4.397	4.398	4.397
97	3.060	3.067	3.067	3.059	3.059
99	1.171	1.479	1.477	1.265	1.267
CPU		0.038 s	0.051 s	0.038 s	0.051 s

Table 5: Single barrier option pricing with continuous monitoring of Example (4): $T = 0.2$, $r = 0.1$, $\sigma = 0.3$, $S_0 = 100$, $E = 100$, $U = 250$.

7. Conclusion and remarks

In this work, a numerical algorithm with the aid of the Lagrange interpolation on roots of Jacobi polynomials for pricing both single and double barrier call options have been proposed. Thanks to relation (26) CPU time of the presented method is almost changeless when the number of monitoring dates grows. This feature enabled us to apply this method even in the continuous monitoring cases with the help of relation (43). The numerical results show the effectiveness and reliability of the presented method in comparison with other methods. On the other hand, we proved the convergence of this method and obtained convergence rate of the algorithm, which provides the guarantee of reliability of the method for pricing discrete double barrier options.

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