ON APPROXIMATION BY GUPTA TYPE GENERAL FAMILY OF OPERATORS

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Abstract. In this paper, we study Gupta type family of positive linear operators, which have a wide range of many well known linear positive operators e.g. Phillips, Baskakov-Durrmeyer, Baskakov-Szász, Szász-Beta, Lupaş-Beta, Lupaş-Szász, genuine Bernstein-Durrmeyer, Link, Păltănea, Miheşan-Durrmeyer, link Bernstein-Durrmeyer etc. We first establish direct results in terms of usual modulus of continuity having order 2 and Ditzian-Totik modulus of smoothness and then study quantitative Voronovkaya theorem for the weighted spaces of functions. Further, we establish Grüss-Voronovskaja type approximation theorem and also derive Grüss-Voronovskaja type asymptotic result in quantitative form.

1. Introduction

In theory of approximation, it is noted that linear positive operators are very important to approximate integrable functions. In last decade many researchers ([1], [6], [13], [14], [20], [21] and [22] etc.) have studied various type of summation-integral type operators to approximate functions of different classes. In this direction, recently Gupta [9] introduced and defined the following general family of set of positive Linear operators for $f : [0, \infty) \to R$ for parameters $\alpha, \beta > 0$ as:

$$\Im^{\sigma,\rho}_{n,\tau}(\zeta, x) = \sum_{\nu=1}^{\infty} \ell^{\tau}_{n,\nu}(x) \int_0^{\infty} \ell^{\sigma+1,\rho}_{n,\nu-1}(v)\zeta(v)dv + m^{\tau}_{n,0}\zeta(0), \tag{1.1}$$

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This paper is dedicated to Professor H. M. Srivastava.

where

$$\ell_{n,\nu}^{\tau}(x)=\frac{(\tau)_k}{k!}\frac{\left(\frac{nx}{\tau}\right)^{\nu}}{\left(1+\frac{nx}{\tau}\right)^{\tau+k}} \tag{1.2}$$

and

$$\ell_{n,\nu-1}^{\sigma+1,\rho}(v)=\frac{n}{\sigma.B(\nu\rho,\sigma\rho+1)}.\frac{\left(\frac{nv}{\sigma}\right)^{\nu\rho-1}}{\left(1+\frac{nv}{\sigma}\right)^{\sigma\rho+\nu\rho+1}}$$

having rising factorial $(\tau)_\nu=\tau(\tau+1)...(\tau+\nu-1)$ and $(\tau)_0=1$. These sequence of positive linear operators reproduce linear functions. For different values and limiting conditions of parameters α and σ, we obtain various linear positive operators e.g. Phillips, Baskakov-Durrmeyer, Baskakov-Szász, Szász-Beta, Lupaş-Beta, Lupaş-Szász, genuine Bernstein-Durrmeyer, Link, Pătănea, Miheşan-Durrmeyer, link Bernstein-Durrmeyer operators etc.

For $x\geq 0$, the operators (1.1) can be written in the following form as

$$\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)=\sum_{\nu=0}^{\infty}\ell_{n,\nu}^{\tau}(x)\varrho_{n,\nu}^{\sigma,\rho}(\zeta),$$

where $\ell_{n,\nu}^{\tau}(x)$ is given in (1.2) and

$$\varrho_{n,\nu}^{\sigma,\rho}(\zeta)=\begin{cases}\int_0^\infty \ell_{n,\nu-1}^{\sigma+1,\rho}(v)\zeta(t)dv, & 1\leq\nu<\infty\\ \zeta(0), & \nu=0.\end{cases}$$

For various values of parameters τ,σ and ρ, we obtain following cases of well known operators by operators (1.1):

(1) If $\tau=\sigma=n,\rho=1$, we find the Baskakov-Durrmeyer operators preserving constant functions studied by Finta [5].
(2) If $\tau=\sigma=-n,\rho=1$, we obtain genuine Bernstein-Durrmeyer polynomials which were introduced by Chen [2] and Goodman-Sharma [8].
(3) If $\tau=\sigma\to\infty,\rho=1$ we get well known Phillips operators which were studied by Finta and Gupta in [6].
(4) If $\tau\neq\sigma$ and $\tau=n,\sigma\to\infty,\rho=1$, we obtain the Baskakov-Szász operators which were studied by Agrawal and Mohammad [1].
(5) If $\tau\neq\sigma$ and $\tau\to\infty,\sigma=n,\rho=1$ we find the Szász-Beta operators considered by Gupta and Noor [11].
(6) If $\tau\neq\sigma$ and $\tau=nx,\sigma=n,\rho=1$, we obtain the Lupaş-Beta operators studied in [12].

(7) If $\tau \neq \sigma$ and $\tau = nx, \sigma \to \infty, \rho = 1$ we obtain the Lupaş-Szász operators which were developed and studied by Govil et al. in [7].
(8) If $\tau = \sigma$, we get the Srivastava-Gupta operators which were studied by Malik [17].
(9) If $\tau = \sigma \to \infty, \rho > 0$, we obtain the Păltănea operators which were considered and studied in [19].
(10) If $\sigma \to \infty, \rho > 0$, and for each value of τ such that $k\rho$ is a positive integer, we get the Maheşan-Durrmeyer operators which were studied and proposed by Kajla in [15].

In first section, we give moment estimates and some basic definitions, auxiliary results to prove our main theorems. In the last ten years, an interesting property of positive linear operators is to find the estimate of their differences using $K-$ functional approach and in terms of appropriate modulus of continuity. The work is been done in this direction by Gupta and Tachev [10] etc. Motivated to this study, we find and study rate of convergence of these operators i.e. estimate of error in terms of the usual modulus of continuity of order 2 and weighted modulus of continuity using Peetre's K-functional approach. Further, we derive quantitative Voronovkaya-type theorem for the class of weighted spaces of functions. Recently, Kajla et al. [16], Malik [17] studied Grüss-Voronovskaja type results for $q-$ Szasz operators and different kind of Gupta type operators for various classes of functions. So, it is very interesting topic to study Grüss-Voronovskaja asymptotic result for general family of operators (1.1) using Grüss type inequality.

2. Estimates of moments

Lemma 1. [9] *The rth moment $\Im^{\sigma,\rho}_{n,\tau}(e_r,x), e_r = v^r, r = 1,2,...$ satisfy the following equation:*
$\Im^{\sigma,\rho}_{n,\tau}(e_r,x)$

$$= \frac{\Gamma(\sigma\rho - r + 1)}{\Gamma(\sigma\rho + 1)}\left(\frac{\sigma}{n}\right)^r\left(1 + \frac{nx}{\tau}\right)^{-\tau}\sum_{k=1}^{\infty}\frac{(\tau)_k}{k!}.\frac{\Gamma(k\rho + r)}{\Gamma(k\rho)}.\frac{(nx)^k}{(\tau + nx)^k}.$$

Let us define central moments of rth order by $\mu^{\sigma,\rho}_{n,\tau,r}(x) = \Im^{\sigma,\rho}_{n,\tau}((v - x)^r, x)$.

Remark 1. *Using above lemma, we obtain $\Im^{\sigma,\rho}_{n,\tau}(e_0,x) = 1, \Im^{\sigma,\rho}_{n,\tau}(e_1,x) = x$ and*

$$\Im^{\sigma,\rho}_{n,\tau}(e_2,x) = \frac{\sigma}{\sigma\rho - 1}\left[\rho x^2\left(1 + \frac{1}{\tau}\right) + \frac{(\rho + 1)x}{n}\right].$$

Hence, central moments are given by

$$\mu^{\sigma,\rho}_{n,\tau,1}(x) = 0$$

and

$$\mu^{\sigma,\rho}_{n,\tau,2}(x) = \frac{\sigma}{\sigma\rho - 1}\left[\frac{(\sigma\rho+\tau)x^2}{\tau\sigma} + \frac{(\rho+1)x}{n}\right].$$

3. Direct results

Let $\Lambda_B[0,\infty)$ denotes the class of bounded and continuous functions on $[0,\infty)$. For $\zeta \in \Lambda_B[0,\infty)$ and $\delta > 0$, the usual modulus of continuity of order j is defined as:

$$\omega_j(\zeta,\delta) = \sup_{0\le h\le\delta}\sup_{x\in[0,\infty)} |\Delta^j_h \zeta(x)|,$$

where Δ^j_h is the jth order forward difference. For $j = 1$, we call it the usual modulus of continuity, denoted by $\omega(\zeta,\delta)$.

Further, let $\varkappa^2 = \{p \in \Lambda_B[0,\infty) : p', p'' \in \Lambda_B[0,\infty)\}$. For $\zeta \in \Lambda_B[0,\infty)$, the $K-$ functional is defined as

$$K_2(\zeta,\delta) = \inf_{p\in\varkappa^2}\{\|\zeta - p\| + \delta\|p''\|\},$$

where $\delta > 0$ and the norm $\|.\|$ is the sup- norm on $[0,\infty)$.

By ([3], p.177, Theorem 2.4) $\exists$ absolute constant $C > 0$ such that

$$K_2(\zeta,\delta) \le C\omega_2(\zeta,\sqrt{\delta}),$$

where $\omega_2(\zeta,\delta)$ is the usual modulus of smoothness of order 2 of ζ.

Now, we discuss the error estimate by linear positive operators $\Im^{\sigma,\rho}_{n,\tau}(\zeta,.)$, in terms of the second order modulus of continuity which has been stated by Gupta [9]. We have the following error estimate:

Theorem 1. *If $\zeta \in \Lambda_B[0,\infty)$ and $x \in [0,\infty)$ $\exists$ an absolute constant $C > 0$ such that*

$$|\Im^{\sigma,\rho}_{n,\tau}(\zeta,x) - \zeta(x)| \;\le\; C\omega_2\left(\zeta, \sqrt{\frac{(\sigma\rho+\tau)x^2}{\tau(\sigma\rho-1)} + \frac{\sigma(\rho+1)x}{n(\sigma\rho-1)}}\right).$$

Proof. Let $p \in \varkappa^2$ and $x, v \in [0,\infty)$. Using Taylor's series expansion, we write

$$p(v) = p(x) + (v-x)p'(x) + \int_x^v (v-u)p''(u)du. \tag{3.1}$$

Applying $\Im^{\sigma,\rho}_{n,\tau}$ on the both sides of (3.1), we get

$$\begin{aligned}
|\Im_{n,\tau}^{\sigma,\rho}(p,x)-p(x)| &= \left(\Im_{n,\tau}^{\sigma,\rho}\left|\int\limits_x^v (v-u)p''(u)du\right|,x\right)\\
&\leq \Im_{n,\tau}^{\sigma,\rho}((v-x)^2,x)\|p'\| \leq \mu_{n,\tau,2}^{\sigma,\rho}(x))\|p'\|\\
&\leq \frac{\sigma}{\sigma\rho-1}\left[\frac{(\sigma\rho+\tau)x^2}{\tau\sigma}+\frac{(\rho+1)x}{n}\right]\|p'\|, \qquad (3.2)
\end{aligned}$$

in view of Remark 1. Now, using definition of operators (1.1) and Lemma 1, we obtain

(3.3)

$$|\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)| \leq \sum_{k=1}^{\infty}\ell_{n,k}^{\tau}(x)\int_0^{\infty}\ell_{n,k-1}^{\sigma+1,\rho}(v)|\zeta(v)|dv+m_{n,0}^{\tau}|\zeta(0)| \leq \|\zeta\|.$$

Using (3.2) and (3.3) we reach to

$$\begin{aligned}
|\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)-\zeta(x)| &\leq |\Im_{n,\tau}^{\sigma,\rho}(\zeta-p,x)-(\zeta-p)(x)|+|\Im_{n,\tau}^{\sigma,\rho}(p,x)-p(x)|\\
&\leq 2\|\zeta-p\|+\frac{\sigma}{\sigma\rho-1}\left[\frac{(\sigma\rho+\tau)x^2}{\tau\sigma}+\frac{(\rho+1)x}{n}\right]\|p'\|.
\end{aligned}$$

Now, taking infimum over all $p \in \varkappa^2$ on both sides of above equation and in view of inequality $K_2(\zeta,\delta) \leq C\omega_2(\zeta,\sqrt{\delta})$, we get the required result.

□

The Ditzian-Totik modulus of smoothness of order 2 is defined as:

$$\omega_{\varphi}^2(\zeta,\delta)=\sup_{0\leq h\leq\delta}\ \sup_{x\pm h\varphi(x)\in[0,\infty)}|\zeta(x+h\varphi(x))-2\zeta(x)+\zeta(x+h\varphi(x))|,$$

where $\varphi(n,\tau,\sigma,\rho,x)=\sqrt{x\left(1+\frac{n(\sigma\rho+\tau)}{\tau\sigma(\rho+1)}x\right)}$.

The corresponding $K-$functional is defined as:

$$K_{2,\varphi}(\zeta,\delta^2)=\inf_{g\in V_{\infty}^2(\varphi)}\{\|\zeta-h\|+\delta^2\|\varphi^2h''\|\},$$

where $\varkappa_{\infty}^2(\varphi)=\{h\in\Lambda_B[0,\infty): h'\in AC_{loc}[0,\infty): \varphi^2h''\in\Lambda_B[0,\infty)\}$.

By ([4], Theorem 2.11) $\exists$ an absolute constant $C>0$ such that $C^{-1}\omega_{\varphi}^2(\zeta,\delta)\leq K_{2,\varphi}(\zeta,\delta^2)\leq C\omega_{\varphi}^2(\zeta,\delta)$.

Now, we estimate an error in terms of weighted Ditzian-Totik modulus of smoothness using $K-$functional approach.

Theorem 2. *If* $\zeta\in\Lambda_B[0,\infty)$, *and* $x\in[0,\infty)$ $\exists$ *a constant* $C>0$ *such that*

$$\|\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)-\zeta(x)\| \leq C\omega_{\varphi}^{2}\left(\zeta,\sqrt{\frac{\sigma(\rho+1)}{n(\sigma\rho-1)}}\right).$$

Proof. Let $p \in \varkappa_{\infty}^{2}$ and $x, v \in [0,\infty)$. Using Taylor's series expansion, we write

$$p(v) = p(x) + (v-x)p'(x) + \int_{x}^{v}(v-u)p''(u)du. \tag{3.4}$$

Applying $\Im_{n,\tau}^{\sigma,\rho}$ on the both sides of (3.4), we get

$$\begin{aligned}
|\Im_{n,\tau}^{\sigma,\rho}(p,x)-p(x)| &= \left(\Im_{n,\tau}^{\sigma,\rho}\left|\int_{x}^{v}(v-u)p''(u)du\right|,x\right)\\
&\leq \frac{\Im_{n,\tau}^{\sigma,\rho}((v-x)^2,x)}{\varphi^2(x)}\|\varphi^2 p'\|\\
&\leq \frac{\sigma(\rho+1)}{n(\sigma\rho-1)}\|\varphi^2 p'\|, \qquad (3.5)
\end{aligned}$$

in view of Remark 1. Now, using definition of operators (1.1) and Lemma 1, we obtain

$$|\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)| \leq \sum_{k=1}^{\infty}\ell_{n,k}^{\tau}(x)\int_{0}^{\infty}\ell_{n,k-1}^{\sigma+1,\rho}(v)|\zeta(v)|dv + m_{n,0}^{\tau}|\zeta(0)| \leq \|\zeta\|. \tag{3.6}$$

Using (3.5) and (3.6) we reach to

$$\begin{aligned}
|\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)-\zeta(x)| &\leq |\Im_{n,\tau}^{\sigma,\rho}(\zeta-p,x)-(\zeta-p)(x)| + |\Im_{n,\tau}^{\sigma,\rho}(p,x)-p(x)|\\
&\leq 2\|\zeta-p\| + \frac{\sigma(\rho+1)}{n(\sigma\rho-1)}\|\varphi^2 p'\|.
\end{aligned}$$

Now, taking infimum over all $p \in \varkappa_{\infty}^{2}$ on both sides of above equation and in view of inequality $K_2(\zeta,\delta) \leq C\omega_2(\zeta,\sqrt{\delta})$, we get the required result.

□

4. Quantitative Voronovskaja asymptotic result

Let $\Lambda[0,\infty)$ be the space of all continuous functions defined on $[0,\infty)$ and $\mho$ be the a subspace of $\Lambda[0,\infty)$, which contains polynomials. Let $\Pi_2[0,\infty)$ be the set of all continuous functions ζ defined on $[0,\infty)$

such that $|\zeta(x)| \leq M_\zeta(1+x^2)$, where $M_\zeta > 0$ is a constant depending on ζ. Also, let $\Lambda_2[0,\infty)$ denotes the space of all continuous functions in $\Pi_2[0,\infty)$. Consider $\Lambda_2^*[0,\infty)$ is the subspace of all functions $\zeta \in \Lambda_2[0,\infty)$ for which $\lim\limits_{x\to\infty} \dfrac{|\zeta(x)|}{1+x^2}$ is finite. The norm on $\Pi_2[0,\infty)$ is defined as $\|\zeta\|_2 = \sup\limits_{x\in[0,\infty)} \dfrac{|\zeta(x)|}{1+x^2}$.

The weighted modulus of continuity is defined as:

$$\Omega(\zeta,\delta) = \sup_{0\leq h<\delta, x\in[0,\infty)} \frac{\zeta(x+h)-\zeta(x)}{(1+h^2)(1+x^2)}$$

for the functions $\zeta \in \Lambda_2[0,\infty)$.

It was shown that for every $\zeta \in \Lambda_2^*[0,\infty)$ there holds:

$$\Omega(\zeta, n\delta) \leq 2(1+n)(1+\delta^2)\Omega(\zeta,\delta), n>0. \tag{4.1}$$

Using the above definition and in view of (4.1), we may write

$$\begin{aligned} &|\zeta(z)-\zeta(x)| \\ &\leq (1+x^2)(1+(z-x)^2)\Omega(\zeta,|z-x|) \\ &\leq 2\left(1+\frac{|z-x|}{\delta}\right)(1+\delta^2)\Omega(\zeta,\delta)(1+x^2)(1+(z-x)^2). \quad (4.2) \end{aligned}$$

Now, we discuss a local error estimate in terms of the weighted modulus of continuity to establish a quantitative Voronovkaya-type theorem.

Theorem 3. *Let E be a subspace of $\Lambda[0,\infty)$ which contains polynomials then for the function $\zeta \in \Lambda^r[0,\infty)$, the space of $r-$ times continuously differentiable functions, we have*

$$\Im_{n,\tau}^{\sigma,\rho}(|R_r(\zeta,u,x)|;x) \leq \frac{16}{r!}(1+x^2)\Omega(\zeta^{(r)},\delta)\left(\mu_{n,\tau,r}^{\sigma,\rho}(x) + \frac{1}{\delta^4}\mu_{n,\tau,r+4}^{\sigma,\rho}(x)\right),$$

where $R_r(\zeta,u,x) = \frac{(u-x)^r}{r!}\left(\zeta^{(r)}(\xi)-\zeta^{(r)}(x)\right), t<\xi<x$ is the remainder term in the Taylor's series expansion.

Proof. On an application of Taylor's series expansion and properties of weighted modulus of continuity, the proof of the theorem follows. □

The quantitative Voronovkaya-type theorem for the class of weighted spaces of functions is as follows:

Theorem 4. *If $\zeta \in E$ and $\zeta'' \in \Lambda_2^*[0,\infty)$ then for $x \in [0,\infty)$ there holds:*

$$\left|\Im^{\sigma,\rho}_{n,\tau}(\zeta,x)-\zeta(x)-\tfrac{1}{2}\zeta''(x)\mu^{\sigma,\rho}_{n,\tau,2}(x)\right|$$
$$\leq 16(1+x^2)\Omega\left(\zeta'',\left(\frac{\mu^{\sigma,\rho}_{n,\tau,6}(x)}{\mu^{\sigma,\rho}_{n,\tau,2}(x)}\right)^{1/4}\right)\mu^{\sigma,\rho}_{n,\tau,2}(x).$$

Proof. Using Taylor's series expansion we may write

$$\zeta(v)=\zeta(x)+\zeta'(x)(v-x)+\frac{\zeta''(x)}{2}(v-x)^2+R_2(\zeta,v,x),$$

where

$$R_2(\zeta,v,x)=\frac{(v-x)^2}{2}(\zeta''(\xi)-\zeta''(x)$$

and ξ is a number lying between v and x.
Operating $\Im^{\sigma,\rho}_{n,\tau}$ on both sides of above equation we get
$$\Im^{\sigma,\rho}_{n,\tau}(\zeta,x)-\zeta(x)-\zeta(x)[\Im^{\sigma,\rho}_{n,\tau}(1,x)-1]-\zeta'(x)\Im^{\sigma,\rho}_{n,\tau}((v-x),x)$$
$$-\frac{\zeta''(x)}{2}\Im^{\sigma,\rho}_{n,\tau}((v-x)^2,x)=\Im^{\sigma,\rho}_{n,\tau}(R_2(\zeta,v,x),x).$$

Using Theorem 3 and Remark 1, we obtain

$$\begin{aligned}
&|\Im^{\sigma,\rho}_{n,\tau}(\zeta,x)-\zeta(x)-\tfrac{\zeta''(x)}{2}\Im^{\sigma,\rho}_{n,\tau}((v-x)^2,x)|\\
&\leq\ 8(1+x^2)\Omega(\zeta'',\delta)\left(\Im^{\sigma,\rho}_{n,\tau}((v-x)^2,x)+\frac{1}{\delta^4}\Im^{\sigma,\rho}_{n,\tau}((v-x)^6,x)\right)\\
&\leq\ 8(1+x^2)\Omega(\zeta'',\delta)\Im^{\sigma,\rho}_{n,\tau}((v-x)^2,x)\left(1+\frac{1}{\delta^4}\frac{\Im^{\sigma,\rho}_{n,\tau}((v-x)^6,x)}{\Im^{\sigma,\rho}_{n,\tau}((v-x)^2,x)}\right).
\end{aligned}$$

Choosing $\delta=\left(\frac{\Im^{\sigma,\rho}_{n,\tau}((v-x)^6,x)}{\Im^{\sigma,\rho}_{n,\tau}((v-x)^2,x)}\right)^{1/4}$, we get
$$|\Im^{\sigma,\rho}_{n,\tau}(\zeta,x)-\zeta(x)-\tfrac{\zeta''(x)}{2}\mu^{\sigma,\rho}_{n,\tau,2}(x)|$$
$$\leq 16(1+x^2)\Omega\left(\zeta'',\left(\frac{\mu^{\sigma,\rho}_{n,\tau,6}(x)}{\mu^{\sigma,\rho}_{n,\tau,2}(x)}\right)^{1/4}\right)\mu^{\sigma,\rho}_{n,\tau,2}(x).$$

Hence the theorem is proved. □

5. Grüss-Voronovskaja type asymptotic result

In this section, first we derive a Grüss-Voronovskaja type approximation theorem and then prove the Grüss-Voronovskaja type asymptotic result.

Theorem 5. *If* $\zeta, p \in E \cap \Lambda_2^*[0,\infty)$ *and* $\zeta'', p'' \in E \cap \Lambda_2^*[0,\infty)$ *then for fixed* $x \geq 0$ *there holds:*

$$|\Im_{n,\tau}^{\sigma,\rho}(\zeta p, x) - \Im_{n,\tau}^{\sigma,\rho}(\zeta, x)\Im_{n,\tau}^{\sigma,\rho}(p, x)| \leq \wp_\zeta(x)\wp_p(x),$$

where

$$\wp_\zeta(x) = \sqrt{32(1+x^2)\Omega\left(\zeta'', \left(\mu_{n,\tau,4}^{\sigma,\rho}(x)\right)^{1/4}\right) + 32(1+C)\|\zeta\|_2(1+x^2)^2\Omega\left(\zeta, \left(\mu_{n,\tau,4}^{\sigma,\rho}(x)\right)^{1/4}\right)},$$

$\wp_p(x)$ *is the analogue of* $\wp_\zeta(x)$ *and* C *is a constant.*

Proof. Let us define $\Im_{n,\tau}^{\sigma,\rho}(\zeta, p, x) = \Im_{n,\tau}^{\sigma,\rho}(\zeta p, x) - \Im_{n,\tau}^{\sigma,\rho}(\zeta, x)\Im_{n,\tau}^{\sigma,\rho}(p, x)$.

By using Cauchy-Schwarz inequality, we have

$$|\Im_{n,\tau}^{\sigma,\rho}(\zeta, p, x)| \leq \sqrt{\Im_{n,\tau}^{\sigma,\rho}(\zeta,\zeta,x)}\sqrt{\Im_{n,\tau}^{\sigma,\rho}(p,p,x)}.$$

In view of (4.2), we reach to

$$\begin{aligned} |\Im_{n,\tau}^{\sigma,\rho}(\zeta, x) - \zeta(x)| &\leq 2(1+x^2)(1+\delta^2)\Omega(\zeta,\delta) \\ &\times \quad \Im_{n,\tau}^{\sigma,\rho}\left(\left(1+\frac{|z-x|}{\delta}\right)(1+(z-x)^2), x\right). \end{aligned} \tag{5.1}$$

Let us define $\emptyset(x,z,\delta) = \left(1+\frac{|z-x|}{\delta}\right)(1+(z-x)^2)$. So,

$$\emptyset(x,z,\delta) = \begin{cases} 2(1+\delta^2), & |z-x| < \delta \\ 2(1+\delta^2)\frac{(z-x)^4}{\delta^4}, & |z-x| \geq \delta \end{cases}$$

Now combining both cases for all $x, z \geq 0$, we get

$$\emptyset(x,z,\delta) \leq 2(1+\delta^2)\left[1+\frac{(z-x)^4}{\delta^4}\right]. \tag{5.2}$$

Combining (5.1)-(5.2), we obtain

$$|\Im_{n,\tau}^{\sigma,\rho}(\zeta, x) - \zeta(x)| \leq 16(1+x^2)\left(1+\frac{1}{\delta^4}\mu_{n,\tau,4}^{\sigma,\rho}(x)\right)\Omega(\zeta,\delta). \tag{5.3}$$

We can write

$$\begin{aligned} \Im_{n,\tau}^{\sigma,\rho}(\zeta,\zeta,x) &= \Im_{n,\tau}^{\sigma,\rho}(\zeta^2, x) - \zeta^2(x) + \zeta^2(x) - \left(\Im_{n,\tau}^{\sigma,\rho}\right)^2(\zeta, x) \\ &= \Im_{n,\tau}^{\sigma,\rho}(\zeta^2, x) - \zeta^2(x) + \left(\zeta(x) - \Im_{n,\tau}^{\sigma,\rho}(\zeta,x)\right)\left(\zeta(x) + \Im_{n,\tau}^{\sigma,\rho}(\zeta,x)\right). \end{aligned}$$

Now,

$$\frac{\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)}{1+x^2} \leq \frac{\|\zeta\|_2\Im_{n,\tau}^{\sigma,\rho}(1+v^2,x)}{1+x^2} \leq \frac{\|\zeta\|_2 C(1+x^2)}{1+x^2} \quad = \quad C\|\zeta\|_2.$$

So we have

$$\left|\Im_{n,\tau}^{\sigma,\rho}(\zeta,\zeta,x)\right|$$
$$\leq \left|\Im_{n,\tau}^{\sigma,\rho}(\zeta^2,x)-\zeta^2(x)\right| \;+\; \left|\zeta(x)-\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)\right|(\|\zeta\|_2+M\|\zeta\|_2)(1+x^2).$$

Further, using (5.3), we reach to

$$\begin{aligned}|\Im_{n,\tau}^{\sigma,\rho}(\zeta,\zeta,x)| &\leq 16(1+x^2)\left(1+\frac{1}{\delta^4}\mu_{n,\tau,4}^{\sigma,\rho}(x)\right)\Omega(\zeta^2,\delta)\\ &+ 16(1+C)\|\zeta\|_2(1+x^2)^2\left(1+\frac{1}{\delta^4}\mu_{n,\tau,4}^{\sigma,\rho}(x)\right)\Omega(\zeta,\delta).\end{aligned}$$

Choosing $\delta=\left(\mu_{n,\tau,4}^{\sigma,\rho}(x)\right)^{\frac{1}{4}}$, we get

$$\begin{aligned}|\Im_{n,\tau}^{\sigma,\rho}(\zeta,\zeta,x)| &\leq 32(1+x^2)\Omega(\zeta^2,\left(\mu_{n,\tau,4}^{\sigma,\rho}(x)\right)^{\frac{1}{4}})\\ &+ 32(1+C)\|\zeta\|_2(1+x^2)^2\Omega(\zeta,\left(\mu_{n,\tau,4}^{\sigma,\rho}(x)\right)^{\frac{1}{4}}).\end{aligned}$$

We find similar estimate for $|\Im_{n,\tau}^{\sigma,\rho}(p,p,x)|$, which completes the proof of the theorem.

□

The Grüss-Voronovskaja type asymptotic result is as follows:

Theorem 6. *Let $\zeta,p\in\mho,\zeta'',p''\in\Lambda_2^*[0,\infty)$ such that $\zeta p\in\mho,(\zeta p)''\in\Lambda_2^*[0,\infty)$, then at any point $x\in[0,\infty)$, we have*
$$n\left|\Im_{n,\tau}^{\sigma,\rho}(\zeta p,x)-\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)\Im_{n,\tau}^{\sigma,\rho}(p,x)-\mu_{n,\tau,2}^{\sigma,\rho}(x)\zeta'(x)p'(x)\right|$$
$$\begin{aligned}&\leq 16(1+x^2)n\mu_{n,\tau,2}^{\sigma,\rho}(x)\left\{\Omega\left((\zeta p)'',\left(\frac{\mu_{n,\tau,6}^{\sigma,\rho}(x)}{\mu_{n,\tau,2}^{\sigma,\rho}(x)}\right)^{1/4}\right)\right\}\\ &+\|\zeta\|_2(1+x^2)\left\{\Omega\left(p'',\left(\frac{\mu_{n,\tau,6}^{\sigma,\rho}(x)}{\mu_{n,\tau,2}^{\sigma,\rho}(x)}\right)^{1/4}\right)\right\}\\ &+\|p\|_2(1+x^2)\left\{\Omega\left((\zeta p)'',\left(\frac{\mu_{n,\tau,6}^{\sigma,\rho}(x)}{\mu_{n,\tau,2}^{\sigma,\rho}(x)}\right)^{1/4}\right)\right\}+n\hbar_n(\zeta)\hbar_n(p),\end{aligned}$$

where

$$\hbar_n(\zeta)=\frac{1}{2}\|\zeta''\|_2(1+x^2)\left(2\mu_{n,\tau,2}^{\sigma,\rho}(x)+\frac{2x}{1+x^2}\mu_{n,\tau,3}^{\sigma,\rho}(x)+\frac{x}{1+x^2}\mu_{n,\tau,4}^{\sigma,\rho}(x)\right)$$

and $\hbar_n(p)$ is the analogue of $\hbar_n(\zeta)$.

Proof. By Taylor's series expansion of ζ, we have

$$
\begin{aligned}
&\Im_{n,\tau}^{\sigma,\rho}(\zeta p,x)-\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)\Im_{n,\tau}^{\sigma,\rho}(p,x)-\Im_{n,\tau}^{\sigma,\rho}(p,x)\zeta'(x)p'(x)\\
&\quad= \Im_{n,\tau}^{\sigma,\rho}(\zeta p,x)-\zeta(x)p(x)-\frac{\Im_{n,\tau}^{\sigma,\rho}((v-x)^2,x)}{2}(\zeta(x)p(x))''\\
&\quad= -\zeta(x)\left[\Im_{n,\tau}^{\sigma,\rho}(p,x)-p(x)-\frac{\Im_{n,\tau}^{\sigma,\rho}((v-x)^2,x)}{2}p''(x)\right]\\
&\quad- p(x)\left[\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)-\zeta(x)-\frac{\Im_{n,\tau}^{\sigma,\rho}((v-x)^2,x)}{2}\zeta''(x)\right]\\
&\quad+ (p(x)-\Im_{n,\tau}^{\sigma,\rho}(p,x)).(\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)-\zeta(x))\\
&\quad= \kappa_1+\kappa_2+\kappa_3+\kappa_4, \text{say.}
\end{aligned}
$$

So we may write

$$
\begin{aligned}
&|\Im_{n,\tau}^{\sigma,\rho}(\zeta p,x)-\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)\Im_{n,\tau}^{\sigma,\rho}(p,x)-\Im_{n,\tau}^{\sigma,\rho}(p,x)\zeta'(x)p'(x)|\\
&\qquad\qquad\leq |\kappa_1|+|\kappa_2|+|\kappa_3|+|\kappa_4|.
\end{aligned}
$$

Using Theorem 4, we get

$$|\kappa_1| \leq 16(1+x^2)\Omega\left((\zeta g)'',\left(\frac{\mu_{n,\tau,6}^{\sigma,\rho}(x)}{\mu_{n,\tau,2}^{\sigma,\rho}(x)}\right)^{1/4}\right)\mu_{n,\tau,2}^{\sigma,\rho}(x).$$

$$|\kappa_2| \leq 16|\zeta(x)|(1+x^2)\Omega\left(p'',\left(\frac{\mu_{n,\tau,6}^{\sigma,\rho}(x)}{\mu_{n,\tau,2}^{\sigma,\rho}(x)}\right)^{1/4}\right)\mu_{n,\tau,2}^{\sigma,\rho}(x).$$

$$|\kappa_3| \leq 16|p(x)|(1+x^2)\Omega\left(\zeta'',\left(\frac{\mu_{n,\tau,6}^{\sigma,\rho}(x)}{\mu_{n,\tau,2}^{\sigma,\rho}(x)}\right)^{1/4}\right)\mu_{n,\tau,2}^{\sigma,\rho}(x).$$

Next, since $\zeta\in\Lambda_2^*[0,\infty)$ we can write

$$\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)-\zeta(x)=\zeta'(x)\mu_{n,\tau,1}^{\sigma,\rho}(x)+\frac{1}{2}\Im_{n,\tau}^{\sigma,\rho}(\zeta''(\xi)(v-x)^2,x)$$

and hence we get

$$
\begin{aligned}
|\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)-\zeta(x)| &\leq \frac{1}{2}\Im_{n,\tau}^{\sigma,\rho}(|\zeta''(\xi)|(v-x)^2,x)\\
&\leq \|\zeta''\|_2\frac{1}{2}\Im_{n,\tau}^{\sigma,\rho}((1+\xi^2)(v-x)^2,x),
\end{aligned}
$$

where $v<\xi<x$.

If ξ lies between v and x, then we get $1+\xi^2\leq 1+x^2$. So we obtain in this case

$$|\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)-\zeta(x)| \leq \frac{\|\zeta''\|_2(1+x^2)}{2}\mu_{n,\tau,2}^{\sigma,\rho}(x).$$

Moreover, if ξ lies between x and v, then we get $1+\xi^2 \leq 1+v^2$. So we get

$$\begin{aligned}|\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)-\zeta(x)| &\leq \frac{\|\zeta''\|_2}{2}\Im_{n,\tau}^{\sigma,\rho}((1+v^2)(v-x)^2,x)\\ &= \frac{\|\zeta''\|_2}{2}((1+x^2)\mu_{n,\tau,2}^{\sigma,\rho}(x)+2x\mu_{n,\tau,3}^{\sigma,\rho}(x)+\mu_{n,\tau,4}^{\sigma,\rho}(x).\end{aligned}$$

Therefore, on combining the both cases of ξ we obtain

$$\begin{aligned}&|\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)-\zeta(x)|\\ &\leq \frac{\|\zeta''\|_2(1+x^2)}{2}\{2\mu_{n,\tau,2}^{\sigma,\rho}(x)+\frac{2x}{1+x^2}\mu_{n,\tau,3}^{\sigma,\rho}(x)+\frac{1}{1+x^2}\mu_{n,\tau,4}^{\sigma,\rho}(x)\}\\ &:= \Gamma_n(\zeta).\end{aligned}$$

Similarly, we estimate $|\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)-\zeta(x)| \leq \Gamma_n(p)$. Hence, we reach to

$$\begin{aligned}&n\left|\Im_{n,\tau}^{\sigma,\rho}(\zeta p,x)-\Im_{n,\tau}^{\sigma,\rho}(\zeta,x)\Im_{n,\tau}^{\sigma}(p,x)-\mu_{n,\tau,2}^{\sigma,\rho}(x)\zeta'(x)p'(x)\right|\\ &\leq 16(1+x^2)n\Omega\left((\zeta p)'',\left(\frac{\mu_{n,\tau,6}^{\sigma,\rho}(x)}{\mu_{n,\tau,2}^{\sigma,\rho}(x)}\right)^{1/4}\right)\mu_{n,\tau,2}^{\sigma,\rho}(x)\\ &+ 16\|\zeta\|_2(1+x^2)^2n\Omega\left(p'',\left(\frac{\mu_{n,\tau,6}^{\sigma,\rho}(x)}{\mu_{n,\tau,2}^{\sigma,\rho}(x)}\right)^{1/4}\right)\mu_{n,\tau,2}^{\sigma,\rho}(x)\\ &+ 16\|p\|_2(1+x^2)^2n\Omega\left(\zeta'',\left(\frac{\mu_{n,\tau,6}^{\sigma,\rho}(x)}{\mu_{n,\tau,2}^{\sigma,\rho}(x)}\right)^{1/4}\right)\mu_{n,\tau,2}^{\sigma,\rho}(x)\\ &+ n\Gamma_n(\zeta)\Gamma_n(p).\end{aligned}$$

Hence, the result of theorem is established.

□

Remark 2. *In previous years many authors have studied Stancu variant of operators by introducing two parameters α,β. They modified $f\left(\frac{nu+\alpha}{n+\beta}\right)$ and $f\left(\frac{j+\alpha}{n+\beta}\right)$, instead of $f(u), f\left(\frac{k}{n}\right)$ for integral and discrete operators respectively. In particular, $\alpha=\beta=0$, we arise at ordinary operators. So, we can also study the analysis to improve constant of approximation for these operators although the order of approximation can not be increased.*

References

[1] Agrawal, P.N., Mohammad, A.J.: Linear combination of a new sequence of linear positive operators. Revista de la UMA42(2), 57–65 (2001).

[2] Chen, W.: On the modified Durrmeyer-Bernstein operator (handwritten, in chinese, 3 pages). In: Report of the Fifth Chinese Conference on Approximation Theory, Zhen Zhou, China (1987).

[3] DeVore, R. A., Lorentz, G.G.: Constructive Approximation, Springer, Berlin (1993).
[4] Ditzian, Z., Totik, V.: Moduli of Smoothness, Springer-Verlag, NewYork, 1987.
[5] Finta, Z.: On converse approximation theorems, J. Math. Anal. Appl. 312, 159–180 (2005).
[6] Finta, Z., Gupta, V.: Direct and inverse estimates for Phillips type operators, J. Math. Anal. Appl., 303(2), 627–642 (2005).
[7] Govil, N. K., Gupta, V., Soybaş, D.: Certain new classes of Durrmeyer type operators, Appl. Math. Comput. 225, 195–203 (2013).
[8] Goodman, T. N. T., Sharma, A.: A modified Bernstein-Schoenberg operator, In: Sendov, Bl(ed.) Constructive Theory of Functions, Varna, 1987, pp. 166–173. Publ. House Bulgar. Acad. Sci, Soia (1988).
[9] Gupta, V.: A note on the general family of operators preserving linear functions, Rev. R. Acad. Cienc. Exactas Fís. Nat. Ser. A Mat. RACSAM, 113, 3717–3725(2019).
[10] Gupta, V., Tachev G.: A note on the differences of two positive linear operators, Construct. Math. Anal. 2(1), 1–7 (2019).
[11] Gupta, V., Noor, M. A.: Convergence of derivatives for certain mixed Szász-Beta operators, J. Math. Anal. Appl. 321(1), 1–9 (2006).
[12] Gupta, V., Rassias, ThM, Yadav, R.: Approximation by Lupaş-Beta integral operators, Appl. Math. Comput. 236(1), 19–26 (2014).
[13] Gupta, V., Srivastava, H. M.: A General Family of the Srivastava-Gupta Operators Preserving Linear Functions, European J. of Pure and Appl. Math. 11(3), 575-579 (2018).
[14] Gupta, V., López-Pellicer, M., Srivastava, H.M.: Convergence Estimates of a Family of Approximation Operators of Exponential Type, Filomat 34 (13), 4329-4341 (2020).
[15] Kajla, A.: Direct estimates of certain Mihesan-Durrmeyer type operators, Adv. Oper. Theory 2(2), 162–178 (2017).
[16] Kajla, A., Mohiuddine S. A., Alotaibi, A.: Blending-type approximation by Lupas-Durrmeyer-type operators involving Pólya distribution, Math. Methods in the Appl. Sci., https://doi.org/10.1002/mma.7368.
[17] Malik, Neha: On approximation properties of Gupta-type operators, The Journal of Analysis, J. Anal. (2019) doi:10.1007/s41478-019-00195-z.
[18] Mohiuddine, S.A.: Approximation by bivariate generalized Bernstein-Schurer operators and associated GBS operators. Adv. Differ. Equ. 2020 676 (2020). https://doi.org/10.1186/s13662-020-03125-7
[19] Păltănea, R.: Modified Szász-Mirakjan operators of integral form, Carpathian J. Math. 24(3), 378–385 (2008).
[20] Srivastava, H.M., Gupta V.: A certain family of summation-integral type operators, Math. Comput. Modelling, 37, 1307–1315 (2003).
[21] Srivastava, H.M., Gupta, V.: Rate of convergence for the Bézier variant of the Bleimann-Butzer-Hahn operators, Applied Mathematics Letters, 18(8), 849-857 (2005).
[22] Srivastava, H. M., Mursaleen, M., Khan, Asif: Generalized equi-statistical convergence of positive linear operators and associated approximation theorems, Math. Comput. Model. 55 (2012), 2040–2051.