Equivalence between invariant measures and statistical solutions for the 2D non-autonomous magneto-micropolar fluid equations[†]

Yanjiao Li*, Xiaojun Li

Department of Mathematics, School of Science, Hohai University,

Nanjing, Jiangsu 210098, China

Abstract: In this article, we aim to investigate the regularity of statistical solution for the 2D non-autonomous magneto-micropolar fluid equations as well as the relationship between invariant measures and statistical solutions. Firstly, to get the regularity of the statistical solution, we prove the existence and regularity of the pullback attractor for the equations. Then we obtain the statistical solution possesses some regularity properties by using regularity of the pullback attractor. Finally, we prove the statistical solution is actual an invariant measure for the equations.

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1 Introduction

In a turbulent flow, almost all physical quantities vary so rapidly in space and time that the actual instantaneous values of these quantities cannot be determined. Instead, people usually measures the moments, or some averaged values of physical quantities. In other words, a statistical description of the turbulent flow is available. Statistical solution is a rigorous mathematical notion, which has been introduced to formalize the object of ensemble average. Foias and Prodi in [1] first introduced the concept of time-dependent statistical solutions (or simply statistical solutions), which is a family of time-parameterized probability measures defined on the phase space of Navier-Stokes equations and describes the probability distribution of the velocity field of the flow at each time. Moreover, statistical solutions represent the time evolution of the probability distribution functions associated with the fluid flows and are closely relevant to the

*Corresponding author.

E-mail: yanjiaoli2013@163.com(Y. Li), lixjun05@hhu.edu.cn (X. Li).

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invariant measures defined on the phase space of the corresponding system. At present, statistical solutions and invariant measures have been extensively used to describe the turbulence in different fluids.

The invariant measures have been investigated in many references (see e.g. [2–8]). For example, Foias et al. in [3] used the notion of Generalized Banach limit, which can link ensemble and time average without "ergodic hypothesis", to construct invariant measures for the Navier-Stokes equations. Wang in [6] studied the existence and upper semi-continuity of invariant measures for a class of uniformly dissipative systems. Based on works [3, 6], Łukaszewicz, Real and Robinson in [4] constructed invariant measures for the continuous dynamical systems. Chekroun and Glatt-Holtz in [2] generalized and simplified the proof of [4] to construct invariant measures for general autonomous dissipative dynamical systems. Łukaszewicz and Robinson in [5] extended the result of [2] to construct invariant measures for a class of non-autonomous dissipative dynamical systems.

There are also a series of references investigating the relationship between invariant measures and statistical solutions or stationary statistical solutions (see e.g. [3, 9–14]). For instance, Foias et al. systematically investigated the relationship between invariant measures, stationary statistical solutions and statistical solutions for the Navier-Stokes equations; Łukaszewicz in [11] investigated the relationship between pullback attractors, invariant measures, and statistical solutions for the 2D non-autonomous Navier-Stokes equations; Kloeden, Marín-Rubio and Real in [10] proved the equivalence between stationary statistical solutions and invariant measures for the 3D autonomous globally modified Navier-Stokes equations. Recently, Zhao, the first author, Łukaszewicz proved the existence of invariant measurers for the 2D non-autonomous magneto-micropolar fluid system in [13] and revealed that the invariant measure satisfies a Liouville-type equation and is actual a statistical solution for the system. Moreover, they also got the partial degenerate regularity of the statistical solution when the Grashof number corresponding to the system was small enough.

In this article, we continue to investigate statistical solution for the following 2D non-autonomous magneto-micropolar fluid equations:

$$\begin{aligned}
&\frac{\partial u}{\partial t} - (\kappa+\chi)\Delta u + u\cdot\nabla u + \nabla(p+\frac{1}{2}|h|^2) = 2\chi\nabla\times\omega + \gamma h\cdot\nabla h + f,\\
&k\frac{\partial \omega}{\partial t} - \mu\Delta\omega + 4\chi\omega + ku\cdot\nabla\omega = 2\chi\nabla\times u + g,\\
&\frac{\partial h}{\partial t} - \alpha\Delta h + u\cdot\nabla h - h\cdot\nabla u = 0,\\
&\mathrm{div} u = 0, \mathrm{div} h = 0,
\end{aligned} \tag{1.1}$$

where the unknown functions $u = (u_1(x,t), u_2(x,t))$, $\omega = \omega_3(x,t)$, $h = (h_1(x,t), h_2(x,t))$, $p = p(x,t)$ denote the velocity, micro-rotational velocity, magnetic field and the pressure, respectively, and

$$\nabla\times u = \frac{\partial u_2}{\partial x_1} - \frac{\partial u_1}{\partial x_2},\ \mathrm{div} u = \frac{\partial u_1}{\partial x_1} + \frac{\partial u_2}{\partial x_2},\ \nabla\times\omega = (\frac{\partial \omega}{\partial x_2}, -\frac{\partial \omega}{\partial x_1}).$$

Moreover, the given functions $f = (f_1(x,t), f_2(x,t))$ and $g = g_3(x,t)$, respectively, denote external force and moments, and κ, χ, γ, k, μ, α are positive constants related to properties of the material. For simplicity, we take $\gamma = k = 1$. As in [13], we consider (1.1) in a bounded domain $\Omega \subset \mathbb{R}^2$ with suitable smooth boundary $\partial\Omega$ and assume it satisfies the following initial and boundary conditions

$$w(x,\tau) = (u(x,\tau), \omega(x,\tau), h(x,\tau)) = (u_\tau(x), \omega_\tau(x), h_\tau(x)),\ x \in \Omega, \tau \in \mathbb{R}, \tag{1.2}$$

$$u(x,t) = \omega(x,t) = h(x,t) = 0,\ (x,t) \in \partial\Omega \times [\tau, +\infty). \tag{1.3}$$

The mathematical studies for system (1.1)-(1.3) have been widely done due to its important physical background (see e.g. [15–23]).

At the present paper, we investigate the equivalence between invariant measure and statistical solution for Eq. (1.1)-(1.3). The first goal is to prove the statistical solution for Eq. (1.1)-(1.3) possesses following regularity properties:

(i) the support of the statistical solution is included in some regular space;

(ii) the statistical solution satisfies a stronger form of Liouville-type equation.

We recall that the support of statistical solution $\{m_t\}_{t\in\mathbb{R}}$ given in [13] is included in the pullback attractor $\hat{\mathcal{A}}_{\mathcal{D}_\sigma^{\hat{H}}}$ of Eq. (1.1)-(1.3). Naturally, to get the regularity of the statistical solution, we first prove the regularity of pullback attractor of the equations and then investigate the regularity of the statistical solution. The second goal is to prove that the statistical solution of Eq. (1.1)-(1.3) is actual an invariant measure for the equations, which is the inverse result of [13, Theorem 4.2], and therefore we can get the equivalence between invariant measure and statistical solution for the equations. We remark that the regularity properties (i) and (ii) play an essential role in this proof (for details see the proof of Theorem 4.3).

The article is organized as follows. In next section, we present some preliminaries and results related to the solution of Eq. (1.1)-(1.3). In section 3, we devote to prove the existence and regularity of pullback attractor for the associated solution operators process $\{S(t,\tau)\}_{t\geqslant\tau}$. In section 4, we first recall the result on the existence of statistical solution of Eq. (1.1)-(1.3) and then we prove the statistical solution possesses some regularity properties. Furthermore, we verify that the statistical solution with these properties is an invariant measure for the equations.

2 Preliminaries

In this section, we first introduce some notations and operators. Then, we give the results of existence and uniqueness of solutions for problem (1.1)-(1.3).

At the present paper, we denote by $\mathcal{V}$ the set of all divergence free vector functions in $(\mathcal{C}_0^\infty(\Omega))^2$ i.e. $\mathcal{V} = \{u = (u_1, u_2) \in (\mathcal{C}_0^\infty(\Omega))^2 : \mathrm{div} u = 0\}$. We will often use the following notations and function spaces:

$H =$ the closure of $\mathcal{V}$ in $(L^2(\Omega))^2$;

$V =$ the closure of $\mathcal{V}$ in $(H^1(\Omega))^2$;

$\hat{H}= H \times L^2(\Omega) \times H$ with norm $\|\cdot\|_{\hat{H}} = \|\cdot\|$ and inner product $(\cdot,\cdot)$ defined by

$$\|w\| = (\|u\|^2 + \|\omega\|^2 + \|h\|^2)^{1/2},$$
$$(w_1, w_2) = (u_1, u_2) + (\omega_1, \omega_2) + (h_1, h_2),\ w_i = (u_i, \omega_i, h_i) \in \hat{H},\ i = 1, 2;$$

$\hat{V}= V \times H_0^1(\Omega) \times V$ with norm $\|\cdot\|_{\hat{V}}$ and inner product $((\cdot,\cdot))$ defined by

$$\|w\|_{\hat{V}} = (\|u\|_V^2 + \|\omega\|_{1,2}^2 + \|h\|_V^2)^{1/2},$$
$$((w_1, w_2)) = (\nabla u_1, \nabla u_2) + (\nabla \omega_1, \nabla \omega_2) + (\nabla h_1, \nabla h_2),\ w_i = (u_i, \omega_i, h_i) \in \hat{V},\ i = 1, 2;$$

$\hat{V}^*$-the dual space of $\hat{V}$;
$\langle\cdot,\cdot\rangle$-the dual paring between $\hat{V}$ and $\hat{V}^*$.

For simplicity, the inner product in $L^2(\Omega)$ or H is also denoted by the same notation $(\cdot,\cdot)$ if no confusion arises. In addition, we use $\mathrm{dist}_M(X, Y)$ to denote the Hausdorff semidistance between $X \subseteq M$ and $Y \subseteq M$ defined by

$$\mathrm{dist}_M(X, Y) = \sup_{x\in X} \inf_{y\in Y} \|x - y\|_M.$$

For any $w = (u, \omega, h) \in \hat{V}$, the operators A and $R : \hat{V} \mapsto \hat{V}^*$ are defined by

$$\langle Aw, \Phi\rangle = (\kappa + \chi)(\nabla u, \nabla\xi) + \mu(\nabla\omega, \nabla\eta) + \alpha(\nabla h, \nabla\zeta),\ \forall\Phi = (\xi, \eta, \zeta) \in \hat{V},$$

$$\langle R(w), \Phi\rangle = -2\chi(\nabla \times \omega, \xi) - 2\chi(\nabla \times u, \eta) + 4\chi(\omega, \eta),\ \forall\Phi = (\xi, \eta, \zeta) \in \hat{V}.$$

Obviously, the operator A is a positive self-adjoint linear elliptic operator with compact inverse in $\hat{V}$. It follows by the classical spectral theory of elliptic operators that there exists a sequence $\{\lambda_n\}_{n=1}^{\infty}$ satisfying

$$0 < \lambda_1 \leqslant \lambda_2 \leqslant \cdots \lambda_n \leqslant \cdots,\ \lambda_n \to +\infty \text{ as } n \to \infty,$$

and a family of elements $\{\xi_n\}_{n=1}^{\infty} \subseteq D(A)$, which forms a basis of $\hat{V}$ and is orthonormal in $\hat{H}$, such that

$$A\xi_n = \lambda_n\xi_n,\ n = 1, 2, \cdots.$$

From this, we have the Poincaré inequality

$$\lambda_1\|w\|^2 \leqslant \|w\|_{\hat{V}}^2,\ \forall w \in \hat{V}. \tag{2.1}$$

In what follows, we denote by P_m the orthogonal projector of $\hat{H}$ onto the space spanned by $\xi_1, \xi_2, \cdots, \xi_m$. Then we get the following inequalities:

$$\|w - P_m w\|^2 \le \lambda_m\|w - P_m w\|_{\hat{V}}^2,\ \forall w \in \hat{V},\ \forall m \in \mathbb{N}^+. \tag{2.2}$$

Moreover, we also need to define two trilinear forms $b_1(\cdot,\cdot,\cdot)$ and $b_2(\cdot,\cdot,\cdot)$ by

$$b_1(u, \phi, \xi) = \sum_{i,j=1}^{2} \int_\Omega u_i \frac{\partial\phi_j}{\partial x_i}\xi_j \mathrm{d}x,\ \forall u,\ \phi,\ \xi \in V,$$

$$b_2(u,\varphi,\eta)=\sum_{i=1}^{2}\int_\Omega u_i\frac{\partial\varphi}{\partial x_i}\eta\mathrm{d}x,\ \forall u\in V,\ \varphi,\ \eta\in H_0^1(\Omega).$$

It is not difficult to testify that $b_1(u,\phi,\xi)$ and $b_2(u,\varphi,\eta)$ satisfy

$$b_1(u,\phi,\xi)=-b_1(u,\xi,\phi),\ b_1(u,\phi,\phi)=0,\ \forall u,\ \phi,\ \xi\in V, \tag{2.3}$$

$$b_2(u,\varphi,\eta)=-b_2(u,\eta,\varphi),\ b_2(u,\varphi,\varphi)=0,\ \forall u\in V,\ \forall\varphi,\ \eta\in H_0^1. \tag{2.4}$$

Using above trilinear forms, we define a bilinear continuous operator $B:\hat{V}\times\hat{V}\mapsto\hat{V}^*$ by

$$\langle B(w,v),\Phi\rangle=b_1(u,\phi,\xi)+b_2(u,\varphi,\eta)+b_1(u,\psi,\zeta)-b_1(h,\phi,\zeta)-b_1(h,\psi,\xi),$$

for any $w=(u,\omega,h)$, $v=(\phi,\varphi,\psi)\in\hat{V}$, $\Phi=(\xi,\eta,\zeta)$. It follows from (2.3) and (2.4) that

$$\langle B(w,v),v\rangle=0,\ \forall w,\ \Psi\in\hat{V}. \tag{2.5}$$

Now, we can write the weak form of Eq. (1.1)-(1.3) as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}w(t)+Aw(t)+B(w(t),w(t))+R(w(t))=F(t)\text{ in }\mathcal{D}'((\tau,+\infty),\hat{V}^*), \tag{2.6}$$

$$w(\tau)=(u(\tau),\omega(\tau),h(\tau))=w_\tau,\ \tau\in\mathbb{R}, \tag{2.7}$$

where $F(t)=(f(t),g(t),0)$.

Definition 2.1 A function w is called a weak solution to problem (2.6)-(2.7), if for any $T>\tau$,

(1) $w\in L^\infty(\tau,T;\hat{H})\cap L^2(\tau,T;\hat{V})$;

(2) w satisfies the equation

$$(\frac{\mathrm{d}}{\mathrm{d}t}w(t),\Psi)+\langle Aw(t),\Psi\rangle+\langle B(w(t),w(t)),\Psi\rangle+\langle R(w(t)),\Psi\rangle=\langle F(t),\Psi\rangle,\ \forall\Psi\in\hat{V},$$

in the distribution sense of $\mathcal{D}'(\tau,+\infty)$.

Moreover, if $w\in L^\infty(\tau,T;\hat{V})\cap L^2(\tau,T;D(A))$ for all $T>\tau$, then w is said to be strong solution of problem (2.6)-(2.7).

The existence and uniqueness of the solution to problem (2.6)-(2.7) have been proved in [20, 21] via Galerkin method, which we present as the following results.

Lemma 2.1

(1) *If $F(t)\in L^2(\tau,T;\hat{V}^*)$ and $w_\tau\in\hat{H}$, then problem* (2.6)-(2.7) *has a unique weak solution $w(t;\tau,w_\tau)$ satisfying*

$$w(t;\tau,w_\tau)\in\mathcal{C}([\tau,T];\hat{H})\cap L^2(\tau,T;\hat{V})\cap L^\infty(\tau,T;\hat{H}),\ \textit{for all } T>\tau. \tag{2.8}$$

(2) *If $F(t)\in L^2(\tau,T;\hat{H})$ and $w_\tau\in\hat{V}$, then problem* (2.6)-(2.7) *has a unique strong solution $w(t;\tau,w_\tau)$ satisfying*

$$w(t;\tau,w_\tau)\in\mathcal{C}([\tau,T];\hat{V})\cap L^2(\tau,T;D(A))\cap L^\infty(\tau,T;\hat{V}),\ \textit{for all } T>\tau. \tag{2.9}$$

In fact, we can obtain that the solution of problem (2.6)-(2.7) depends continuously on its initial value in $\hat{H}$ or $\hat{V}$. Then by Lemma 2.1, we get the family of solution operators

$$S(t,\tau): w_\tau \in X \mapsto S(t,\tau)w_\tau = w(t;\tau,w_\tau) \in X, \ \forall t \geqslant \tau,$$

generates a continuous process $\{S(t,\tau)\}_{t\geqslant\tau}$ on X, $X = \hat{H}$ or $\hat{V}$.

We complete this section with the following lemma which plays an important role in the proof of existence and regularity of pullback attractors and statistical solution for Eq. (2.6)-(2.7).

Lemma 2.2

(1) *There is positive constant c_1 such that*

$$|\langle B(w,v),\Phi\rangle| \leqslant c_1\|Aw\|\|v\|_{\hat{V}}\|\Phi\|, \ \forall w \in D(A), v \in \hat{V}, \Phi \in \hat{H}. \tag{2.10}$$

$$|\langle B(w,v),\Phi\rangle| \leqslant c_1\|u\|^{\frac{1}{2}}\|w\|_{\hat{V}}^{\frac{1}{2}}\|v\|^{\frac{1}{2}}\|v\|_{\hat{V}}^{\frac{1}{2}}\|\Phi\|_{\hat{V}}, \ \forall w,\ v,\ \Phi \in \hat{V}. \tag{2.11}$$

(2) *There are two constants δ_1 and δ_2 such that*

$$\delta_1\|w\|_{\hat{V}}^2 \leqslant \langle Aw,w\rangle + \langle R(w),w\rangle, \ \forall w \in \hat{V}, \tag{2.12}$$

$$\|R(w)\| \leqslant \delta_2\|w\|_{\hat{V}}^2, \forall w \in \hat{V}, \tag{2.13}$$

where $\delta_1 = \min\{\kappa,\mu,\alpha\}$ and δ_2 depends only on χ and Ω.

Item (1) and item(2) of Lemma 2.2 have been prove in [3] and [13] respectively, and we omit the proof here.

3 Existence and regularity of pullback attractors

The main purpose of this section is to establish the existence and regularity of pullback attractors for problem (2.6)-(2.7). We begin by recalling some definitions about the pullback attractors.

Let $\mathcal{P}(X)$ be the family of all subsets of X, and consider a family of nonempty sets $\widehat{D}_0^X = \{D_0^X(t)|t \in \mathbb{R}\} \subseteq \mathcal{P}(X)$. Denote by $\mathcal{D}^X$ a class of families parameterized in time $\widehat{D}^X = \{D^X(t)|t \in \mathbb{R}\} \subseteq \mathcal{P}(X)$.

Definition 3.1 A family of sets $\widehat{D}_0^X = \{D_0^X(t)|t \in \mathbb{R}\} \subseteq \mathcal{P}(X)$ is called pullback $\mathcal{D}^X$-absorbing for the process $\{S(t,\tau)\}_{t\geqslant\tau}$ on X if for any $t \in \mathbb{R}$ and any $\widehat{D}^X = \{D^X(t)|t \in \mathbb{R}\} \subseteq \mathcal{D}^X$, there exists a $\tau_0(t,\widehat{D}^X) \leqslant t$ such that $S(t,\tau)D^X(\tau) \subseteq D_0^X(t)$ for all $\tau \leqslant \tau_0(t,\widehat{D}^X)$.

Definition 3.2 The process $\{S(t,\tau)\}_{t\geqslant\tau}$ on X is called pullback $\widehat{D}_0^X$-asymptotically compact if for any $t \in \mathbb{R}$ and any sequences $\{\tau_n\} \subseteq (-\infty,t]$ and $\{x_n\} \subseteq X$ satisfying $\tau_n \mapsto -\infty$ and $x_n \in D_0^X(\tau_n)$ for all n, the sequence $\{S(t,\tau_n)x_n\}$ is precompact in X. If the process $\{S(t,\tau)\}_{t\geqslant\tau}$ on X is pullback $\widehat{D}^X$-asymptotically compact for any $\widehat{D}^X \in \mathcal{D}^X$, it is said to be pullback $\mathcal{D}^X$-asymptotically compact.

Definition 3.3 A family of sets $\hat{\mathcal{A}}_{\mathcal{D}^X} = \{\mathcal{A}_{\mathcal{D}^X}(t)|t \in \mathbb{R}\} \subseteq \mathcal{P}(X)$ is called the minimal pullback $\mathcal{D}^X$-attractor for process $\{S(t,\tau)\}_{t\geqslant\tau}$ in X if it satisfies the following properties:

(1) for any $t \in \mathbb{R}$, $\mathcal{A}_{\mathcal{D}^X}(t)$ is a nonempty compact subset of X;

(2) $\hat{\mathcal{A}}_{\mathcal{D}^X}$ is invariant in the following sense

$$S(t,\tau)\mathcal{A}_{\mathcal{D}^X}(\tau) = \mathcal{A}_{\mathcal{D}^X}(t), \; \forall\, t \geqslant \tau;$$

(3) $\hat{\mathcal{A}}_{\mathcal{D}^X}$ is pullback $\mathcal{D}^X$-attracting i.e.

$$\lim_{\tau \to -\infty} \mathrm{dist}_X(S(t,\tau)D^X(\tau), \mathcal{A}_{\mathcal{D}^X}(t)) = 0, \; \forall\, \widehat{D}^X = \{D^X(t)|t \in \mathbb{R}\} \in \mathcal{D}^X, t \geqslant \tau;$$

(4) the family of sets $\mathcal{A}_{\mathcal{D}^X}$ is minimal in the sense that if

$$\widehat{O} = \{O(t)|t \in \mathbb{R}\} \subseteq \mathcal{P}(X)$$

is another family of closed sets such that for any $\widehat{D} = \{D(t)|t \in \mathbb{R}\} \in \mathcal{D}$,

$$\lim_{\tau \to -\infty} \mathrm{dist}_X(S(t,\tau)D(\tau), O(t)) = 0,$$

then $\mathcal{A}_{\mathcal{D}^X} \subseteq O(t)$ for $t \in \mathbb{R}$.

3.1 pullback attractors in $\hat{H}$

The pullback asymptotic behavior of the process $\{S(t,\tau)\}_{t \geqslant \tau}$ corresponding to problem (2.6)-(2.7) in $\hat{H}$ have been investigated in our previous article [13], and we only present the main results here. In fact, these results in $\hat{H}$ can also be illustrated in a similar but simpler way than that will be used in $\hat{V}$.

Lemma 3.1 *Assume $F(t) \in L^2(\tau, T; \hat{V}^*)$ with any $T > \tau$. Let $w(t;\tau,w_\tau)$ be a solution to Eq. (2.6)-(2.7) with initial datum $w_\tau \in \hat{H}$. Then there exists some $\sigma \in (0, 2\delta_1\lambda_1)$ such that*

$$\|w(t;\tau,w_\tau)\|^2 \leqslant e^{\sigma(\tau - t)}\|w_\tau\|^2 + \frac{e^{-\sigma t}}{2\delta_1 - \lambda_1^{-1}\sigma}\int_{-\infty}^{t} e^{\sigma s}\|F(s)\|_{\hat{V}^*}^2 \mathrm{d}s, \forall t \geqslant \tau. \tag{3.1}$$

Denote by $\mathcal{D}_\sigma^{\hat{H}}$ the collection of all families of nonempty subsets $\widehat{D}^{\hat{H}}(t) = \{D^{\hat{H}}(t) : t \in \mathbb{R}\} \subseteq \mathcal{P}(\hat{H})$ such that

$$\lim_{\tau \to -\infty} (e^{\sigma\tau} \sup_{w \in D(\tau)} \|w\|^2) = 0. \tag{3.2}$$

Let function $F(\cdot)$ satisfy the following assumption:

($\mathbf{H_1}$) Assume $F(t) \in L^2(\tau, T; \hat{V}^*)$ with any $T > \tau$, and satisfies

$$\int_{-\infty}^{t} e^{\sigma\theta}\|F(\theta)\|_{\hat{V}^*}^2 \mathrm{d}\theta < +\infty, \text{ for some } \sigma \in (0, 2\delta_1\lambda_1).$$

Corollary 3.1 *Let assumption ($\mathbf{H_1}$) hold. Then the family*

$$\widehat{D}_0^{\hat{H}} = \{\bar{\mathcal{B}}(0, r_\sigma^{1/2}(t)) : t \in \mathbb{R}\}$$

is pullback $\mathcal{D}_\sigma^{\hat{H}}$-absorbing for the process $\{S(t,\tau)\}_{t \geqslant \tau}$ in $\hat{H}$, where

$$r_\sigma^{1/2}(t) = 1 + \frac{e^{-\sigma t}}{2\delta_1 - \lambda_1^{-1}\sigma}\int_{-\infty}^{t} e^{\sigma\theta}\|F(\theta)\|_{\hat{V}^*}^2 \mathrm{d}\theta.$$

Theorem 3.1 *Suppose assumption ($\mathbf{H_1}$) holds. Then the process $\{S(t,\tau)\}_{t \geqslant \tau}$ is pullback $\mathcal{D}_\sigma^{\hat{H}}$-asymptotically compact in space $\hat{H}$ and possesses the minimal pullback $\mathcal{D}_\sigma^{\hat{H}}$-attractor*

$$\hat{\mathcal{A}}_{\mathcal{D}_\sigma^{\hat{H}}} = \{\mathcal{A}_{\mathcal{D}_\sigma^{\hat{H}}}(t)|t \in \mathbb{R}\}.$$

3.2 pullback attractors in $\hat{V}$

The purpose of this subsection is to prove the existence of pullback attractors for the process $\{S(t,\tau)\}_{t\geqslant\tau}$ in $\hat{V}$ and the regularity of the pullback attractors. Here, we need the function $F(t)$ to satisfy the following assumption.

($\mathbf{H_2}$) Assume $F(t)\in L^2(\tau,T;\hat{H})$ with any $T>\tau$, and there exists some $\sigma\in(0,2\delta_1\lambda_1)$ such that

$$\int_{-\infty}^{t} e^{\sigma\theta}\|F(\theta)\|^2\mathrm{d}\theta<+\infty.$$

Lemma 3.2 *Let assumption* ($\mathbf{H_2}$) *hold. Then for any fixed* $t\in\mathbb{R}$ *and* $\widehat{D}^{\hat{H}}(t)\in\mathcal{D}_{\sigma}^{\hat{H}}$, *there exists* $\tau_1(\widehat{D}^{\hat{H}},t)<t-3$ *such that for any* $\tau\leqslant\tau_1(\widehat{D}^{\hat{H}},t)$ *and any* $w_\tau\in D^{\hat{H}}(\tau)$, *the solution* $w(\cdot;\tau,w_\tau)$ *satisfies,*

$$\|w(s;\tau,w_\tau)\|^2\leqslant\rho_1(t),\ \forall s\in[t-3,t], \tag{3.3}$$

$$\|w(s;\tau,w_\tau)\|_{\hat{V}}^2\leqslant\rho_2(t),\ \forall s\in[t-2,t], \tag{3.4}$$

$$\int_{s-1}^{s}\|Aw(\theta;\tau,w_\tau)\|^2\mathrm{d}\theta\leqslant\rho_3(t),\ \forall s\in[t-1,t], \tag{3.5}$$

$$\int_{s-1}^{s}\|w'(\theta;\tau,w_\tau)\|^2\mathrm{d}\theta\leqslant\rho_4(t),\ \forall s\in[t-1,t], \tag{3.6}$$

where

$$\rho_1(t):=1+\frac{e^{-\sigma t}}{2\delta_1\lambda_1-\sigma}\int_{-\infty}^{t}e^{\sigma\theta}\|F(\theta)\|^2\mathrm{d}\theta, \tag{3.7}$$

$$\begin{aligned}\rho_2(t):=\max_{s\in[t-2,t]}&\Big\{c(\delta_2,\delta_1,\lambda_1)\Big(\rho_1(s)+\int_{s-1}^{s}\|F(\theta)\|^2\mathrm{d}\theta\Big)\\&\times\exp\Big[c(c_3,\delta_1,\lambda_1)\Big(\rho_1^2(s)+\rho_1(s)\int_{s-1}^{s}\|F(\theta)\|^2\mathrm{d}\theta\Big)\Big]\Big\},\end{aligned} \tag{3.8}$$

$$\rho_3(t):=c(\delta_2,c_3)\Big(\rho_2(t)+\int_{t-2}^{t}\|F(\theta)\|^2\mathrm{d}\theta+\rho_2^2(t)\rho_1(t)\Big), \tag{3.9}$$

$$\rho_4(t):=c(c_1,\delta_2)\Big(\rho_2(t)+\int_{t-2}^{t}\|F(\theta)\|^2\mathrm{d}\theta+\rho_2(t)\rho_3(t)\Big). \tag{3.10}$$

Proof. Let $w_\tau\in\hat{H}$. For each integer $n\geqslant1$, denote by $w_n(t)=w_n(t;\tau,w_\tau)=\sum\limits_{k=1}^{n}d_{n,k}(t)\xi_k$ Galerkin approximate solution of problem (2.6)-(2.7), which satisfies

$$\begin{aligned}\frac{\mathrm{d}}{\mathrm{d}t}(w_n(t),\xi_k)+\langle Aw_n(t),\xi_k\rangle+\langle B(w_n(t),w_n(t)),\xi_k\rangle+\langle R(w_n(t)),\xi_k\rangle&\\=(F(t),\xi_k),\ t>\tau,&\end{aligned} \tag{3.11}$$

$$(w_n(\tau),\xi_k)=(w_\tau,\xi_k). \tag{3.12}$$

Multiplying (3.11) with $d_{n,k}(t)$ and summing from $k=1$ to n, then using (2.5) and (2.12), we have

$$\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\|w_n(\theta)\|^2+\delta_1\|w_n(\theta)\|_{\hat{V}}^2\leqslant\langle F(\theta),w_n(\theta)\rangle. \tag{3.13}$$

Multiplying (3.13) with $e^{\sigma\theta}$ and applying Poincaré inequality (2.1), we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(e^{\sigma\theta}\|w_n(\theta)\|^2)+2\delta_1\lambda_1 e^{\sigma\theta}\|w_n(\theta)\|^2\leqslant 2e^{\sigma\theta}\langle F(\theta),w_n(\theta)\rangle+\sigma e^{\sigma\theta}\|w_n(\theta)\|^2. \tag{3.14}$$

By Cauchy inequality, we get

$$2|\langle F(\theta),w_n(\theta)\rangle|\leqslant(2\lambda_1\delta_1-\sigma)\|w_n(\theta)\|^2+\frac{1}{2\delta_1\lambda_1-\sigma}\|F(\theta)\|^2. \tag{3.15}$$

It follows from (3.14) and (3.15) that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}(e^{\sigma\theta}\|w_n(\theta)\|^2)\leqslant\frac{e^{\sigma\theta}}{2\delta_1\lambda_1-\sigma}\|F(\theta)\|^2,\ \text{a.e. } \theta>t,$$

which implies

$$\|w_n(t)\|^2\leqslant e^{\sigma(\tau-t)}\|w_\tau\|^2+\frac{e^{-\sigma t}}{2\delta_1\lambda_1-\sigma}\int_{-\infty}^{t}e^{\sigma\theta}\|F(\theta)\|^2\mathrm{d}\theta. \tag{3.16}$$

By (3.2), we get there exists $\tau_1(\widehat{D}^{\hat{H}},t)<t-3$ such that, for any $\widehat{D}^{\hat{H}}(t)\in\mathcal{D}_\sigma^{\hat{H}}$ and $\tau\leqslant\tau_1(\widehat{D}^{\hat{H}},t)$,

$$\|w_n(t)\|^2\leqslant 1+\frac{e^{-\sigma t}}{2\delta_1\lambda_1-\sigma}\int_{-\infty}^{t}e^{\sigma\theta}\|F(\theta)\|^2\mathrm{d}\theta,\ w_\tau\in D^{\hat{H}}(\tau). \tag{3.17}$$

Then from Lemma 2.1, the fact that $w_n(t)\rightharpoonup w(t)$ weakly in $L^2(\tau,t;\hat{V})$ and $w(t)\in\mathcal{C}([\tau,T];\hat{H})$, we can take the limit in (3.17) as $n\to\infty$ to get (3.3).

Multiplying Eq. (3.11) with $\lambda_k d_{n,k}(t)$ and then summing from $k=1$ to n, we obtain

$$\begin{aligned}\frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\|w_n(\theta)\|_{\hat{V}}^2+\|Aw_n(\theta)\|^2+\langle B(w_n(\theta),w_n(\theta)),Aw_n(\theta)\rangle+\langle R(w_n(\theta)),Aw_n(\theta)\rangle&\\ =(F(\theta),Aw_n(\theta))&\\ \leqslant\|F(\theta)\|^2+\frac{1}{4}\|Aw_n(\theta)\|^2,\ a.e.\theta>\tau.&\end{aligned} \tag{3.18}$$

According to the definition of operator $B(\cdot,\cdot)$ and using Hölder inequality, Ladyzhenskaya inequality and Young inequality, we conclude that there exists a constant c_3 such that

$$\begin{aligned}|\langle B(w_n(\theta),w_n(\theta)),Aw_n(\theta)|\rangle&\leqslant c_3\|w_n(\theta)\|_{(L^4)^2\times L^4\times((L^4))^2}\|\nabla w_n(\theta)\|_{(L^4)^2\times L^4\times((L^4))^2}\|Aw_n(\theta)\|\\&\leqslant c_3\|w_n(\theta)\|^{\frac{1}{2}}\|\nabla w_n(\theta)\|^{\frac{1}{2}}\|\nabla w_n(\theta)\|^{\frac{1}{2}}\|Aw_n(\theta)\|^{\frac{1}{2}}\|Aw_n(\theta)\|\\&\leqslant c_3\|w_n(\theta)\|^{\frac{1}{2}}\|\nabla w_n(\theta)\|\|Aw_n(\theta)\|^{\frac{3}{2}}\\&\leqslant\frac{1}{4}\|Aw(\theta)\|^2+\frac{27}{4}c_3^4\|w_n(\theta)\|^2\|w_n(\theta)\|_{\hat{V}}^4.\end{aligned} \tag{3.19}$$

Applying (2.13) and Cauchy inequality, we compute

$$\begin{aligned}|\langle R(w_n(\theta)),Aw_n(\theta)\rangle|&\leqslant\delta_2\|w_n(\theta)\|_{\hat{V}}\|Aw_n(\theta)\|\\&\leqslant\frac{1}{4}\|Aw_n(\theta)\|^2+\delta_2^2\|w_n(\theta)\|_{\hat{V}}^2.\end{aligned} \tag{3.20}$$

It then follows from (3.18)-(3.20) that

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\|w_n(\theta)\|_{\hat{V}}^2+\frac{1}{2}\|Aw_n(\theta)\|^2\leqslant 2\|F(\theta)\|^2+\frac{27}{2}c_3^4\|w_n(\theta)\|^2\|w_n(\theta)\|_{\hat{V}}^4+2\delta_2^2\|w_n(\theta)\|_{\hat{V}}^2. \tag{3.21}$$

By setting

$$\begin{aligned}H_n(\theta)&=\|w_n(\theta)\|_{\hat{V}}^2,\\ I_n(\theta)&=2\|F(\theta)\|^2+2\delta_2^2\|w_n(\theta)\|_{\hat{V}}^2,\\ K_n(\theta)&=\frac{27}{2}c_3^4\|w_n(\theta)\|^2\|w_n(\theta)\|_{\hat{V}}^2,\end{aligned}$$

inequality (3.21) implies

$$\frac{\mathrm{d}}{\mathrm{d}\theta}H_n(\theta)\leqslant I_n(\theta)+K_n(\theta)H_n(\theta). \tag{3.22}$$

Apply Gronwall inequality to (3.22) with $\tau\leqslant s-1\leqslant r\leqslant s$, and then we have

$$\begin{aligned}H_n(s)&\leqslant\Big(H_n(r)+\int_{s-1}^{s}I_n(\theta)\mathrm{d}\theta\Big)\exp\Big(\int_{s-1}^{s}K_n(\theta)\mathrm{d}\theta\Big)\\ &=\Big(\|w_n(r)\|_{\hat{V}}^2+\int_{s-1}^{s}I_n(\theta)\mathrm{d}\theta\Big)\exp\Big(\int_{s-1}^{s}K_n(\theta)\mathrm{d}\theta\Big).\end{aligned} \tag{3.23}$$

Integrating (3.23) with respect to r between $s-1$ and s, we obtain

$$H_n(s)\leqslant\Big(\int_{s-1}^{s}\|w_n(r)\|_{\hat{V}}^2\mathrm{d}r+\int_{s-1}^{s}I_n(\theta)\mathrm{d}\theta\Big)\exp\Big(\int_{s-1}^{s}K_n(\theta)\mathrm{d}\theta\Big). \tag{3.24}$$

Now, by (3.13), (2.1) and Cauchy inequality we have

$$\frac{\mathrm{d}}{\mathrm{d}\theta}\|w_n(\theta)\|^2+\delta_1\|w_n(\theta)\|_{\hat{V}}^2\leqslant\frac{\|F(\theta)\|^2}{\delta_1\lambda_1},\ a.e.\ s>\tau. \tag{3.25}$$

Integrating (3.25) over $[s-1,s]$, we can get

$$\|w_n(s)\|^2+\delta_1\int_{s-1}^{s}\|w_n(\theta)\|_{\hat{V}}^2\mathrm{d}\theta\leqslant\|w_n(s-1)\|^2+\frac{1}{\delta_1\lambda_1}\int_{s-1}^{s}\|F(\theta)\|^2\mathrm{d}\theta. \tag{3.26}$$

It follows from (3.26) that

$$\int_{s-1}^{s}\|w_n(r)\|_{\hat{V}}^2\mathrm{d}r+\int_{s-1}^{s}I_n(\theta)\mathrm{d}\theta\leqslant c(\delta_2,\delta_1,\lambda_1)\Big(\|w_n(s-1)\|^2+\int_{s-1}^{s}\|F(\theta)\|^2\mathrm{d}\theta\Big). \tag{3.27}$$

By (3.3) and (3.26), we can also obtain that

$$\begin{aligned}\int_{s-1}^{s}K_n(\theta)\mathrm{d}\theta&=\frac{27}{2}c_3^4\int_{s-1}^{s}\|w_n(\theta)\|^2\|w_n(\theta)\|_{\hat{V}}^2\mathrm{d}\theta\\ &\leqslant c(c_3,\delta_1,\lambda_1)\sup_{\theta\in[s-1,s]}\|w_n(\theta)\|^2\Big(\|w_n(s-1)\|^2+\int_{s-1}^{s}\|F(\theta)\|^2\mathrm{d}\theta\Big).\end{aligned} \tag{3.28}$$

Combining (3.3), (3.24), (3.27)and (3.28), we get that, for any $\tau \leqslant \tau_1(\widehat{D}^{\hat{H}}, t)$ and any $w_\tau \in D^{\hat{H}}(\tau)$,

$$\|w_n(s;\tau,w_\tau)\|_{\hat{V}}^2 \leqslant \rho_2(t),\ \forall s \in [t-2,t], \tag{3.29}$$

where $\rho_2(t)$ is given by (3.8). Integrating (3.21) on $[s-1,s]$, we get that

$$\begin{aligned}\int_{s-1}^{s} \|Aw_n(\theta)\|^2 \mathrm{d}\theta \leqslant & 2\|w_n(s-1)\|_{\hat{V}}^2 + 4\int_{s-1}^{s} \|F(\theta)\|^2 \mathrm{d}\theta + 4\delta_2^2 \int_{s-1}^{s} \|w_n(\theta)\|_{\hat{V}}^2 \mathrm{d}\theta \\ & + 27c_3^4 \int_{s-1}^{s} \|w_n(\theta)\|^2 \|w_n(\theta)\|_{\hat{V}}^4 \mathrm{d}\theta, \end{aligned} \tag{3.30}$$

which implies that

$$\int_{s-1}^{s} \|Aw_n(\theta;\tau,w_\tau)\|^2 \mathrm{d}\theta \leqslant \rho_3(t),\ \forall s \in [t-1,t],\ \tau \leqslant \tau_1(\widehat{D}^{\hat{H}}, t), w_\tau \in D^{\hat{H}}(\tau), \tag{3.31}$$

where $\rho_3(t)$ is given by (3.9).

Now, multiplying (3.11) with $(d_{k,m})'(t)$, and then summing from $k = 1$ to n, we get

$$\begin{aligned}\|w_n'(\theta)\|^2 + \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}\theta}\|w_n(\theta)\|_{\hat{V}}^2 + \langle B(w_n(\theta), w_n(\theta)), w_n'(\theta)\rangle + \langle R(w_n(\theta)), w_n'(\theta)\rangle & \\ = (F(\theta), w_n'(\theta)) & \\ \leqslant \frac{1}{4}\|w_n'(\theta)\|^2 + \|F(\theta)\|^2. & \end{aligned} \tag{3.32}$$

Using (2.10), (2.13) and Cauchy inequality, we have

$$\begin{aligned}\langle B(w_n(\theta), w_n(\theta)), w_n'(\theta)\rangle & \leqslant |\langle B(w_n(\theta), w_n(\theta)), w_n'(\theta)\rangle| \\ & \leqslant c_1\|\nabla w_n(\theta)\|\|Aw_n(\theta)\|\|w_n'(\theta)\| \\ & \leqslant \frac{1}{4}\|w_n'(\theta)\|^2 + c_1^2\|Aw_n(\theta)\|^2\|w_n(\theta)\|_{\hat{V}}^2, \end{aligned} \tag{3.33}$$

and

$$\begin{aligned}\langle R(w(\theta)), w_n'(\theta)\rangle & \leqslant |\langle R(w_n(\theta)), w_n'(\theta)\rangle| \\ & \leqslant \delta_2\|w_n(\theta)\|_{\hat{V}}\|w_n'(\theta)\| \\ & \leqslant \frac{1}{4}\|w_n'(\theta)\|^2 + \delta_2^2\|w_n(\theta)\|_{\hat{V}}^2. \end{aligned} \tag{3.34}$$

Combining (3.32)-(3.34), we get that

$$\|w_n'(\theta)\|^2 + 2\frac{\mathrm{d}}{\mathrm{d}\theta}\|w_n(\theta)\|_{\hat{V}}^2 \leqslant 4\|F(\theta)\|^2 + 4\delta_2^2\|w_n(\theta)\|_{\hat{V}}^2 + 4c_1^2\|w_n(\theta)\|_{\hat{V}}^2\|Aw_n(\theta)\|^2,\ \text{a.e } \theta > \tau. \tag{3.35}$$

Integrating (3.35) between $s-1$ and s, we have

$$\int_{s-1}^{s} \|w_n'(\theta)\|^2 \mathrm{d}\theta \leqslant 2\|w_n(s-1)\|_{\hat{V}}^2 + 4\int_{s-1}^{s} \|F(\theta)\|^2 \mathrm{d}\theta + 4\delta_2^2 \int_{s-1}^{s} \|w_n(\theta)\|_{\hat{V}}^2 \mathrm{d}\theta$$

$$+ 4c_1^2 \sup_{\theta\in[s-1,s]} \|w_n(\theta)\|_{\hat{V}}^2 \int_{s-1}^{s} \|Aw_n(\theta)\|^2 \mathrm{d}\theta. \tag{3.36}$$

From (3.29), (3.31) and (3.36), we can easily deduce that

$$\int_{s-1}^{s} \|w_n'(\theta;\tau,w_\tau)\|^2 \mathrm{d}\theta \leqslant \rho_4(t),\ \tau \leqslant \tau_1(\widehat{D}^{\hat{H}},t),\ w_\tau \in D^{\hat{H}}(\tau), \tag{3.37}$$

where $\rho_4(t)$ is given by (3.10). By Lemma 2.1 and the fact that $w_n(t) \rightharpoonup w(t)$ weakly in $L^2(t-2,t;D(A))$, $w_n'(t) \rightharpoonup w'(t)$ weakly in $L^2(t-2,t;\hat{H})$, and $w(t) \in \mathcal{C}([t-2,t];\hat{V})$, and then taking the limits as $n \to \infty$ in (3.29), (3.31) and (3.37), we can get the desired results. $\square$

Hereinafter, we denote by $\mathcal{D}_\sigma^{\hat{H},\hat{V}}$ the class of all families $\widehat{D}^{\hat{V}}$ of elements of $\mathcal{P}(V)$, where $\widehat{D}^{\hat{V}}$ is defined by $\widehat{D}^{\hat{V}} = \{D^{\hat{H}}(t) \cap \hat{V} | t \in \mathbb{R}\}$, $\widehat{D}^{\hat{H}} = \{D^{\hat{H}}(t) | t \in \mathbb{R}\} \in \mathcal{D}_\sigma^{\hat{H}}$. In addition, if the assumption $(\mathbf{H_2})$ holds, it is obvious that

$$\lim_{\tau\to-\infty} e^{\sigma\tau}\rho_1(t) = 0,$$

that is, the family $\{\bar{\mathcal{B}}(0,\rho_1^{1/2}(t)) | t \in \mathbb{R}\}$ belongs to $\mathcal{D}_\sigma^{\hat{H}}$.

Corollary 3.2 *Let the assumption* $(\mathbf{H_2})$ *hold. Then the family*

$$\widehat{D}_0^{\hat{V}} = \{\bar{\mathcal{B}}(0,\rho_1^{1/2}(t)) \cap \hat{V} | t \in \mathbb{R}\} \subseteq \mathcal{D}_\sigma^{\hat{H},\hat{V}} \subseteq \mathcal{D}_\sigma^{\hat{H}}$$

is pullback $\mathcal{D}_\sigma^{\hat{H},\hat{V}}$*-absorbing for the process* $\{S(t,\tau)\}_{t\geqslant\tau}$ *in space* $\hat{V}$*. Moreover, for any* $t \in \mathbb{R}$ *and any* $\widehat{D}^{\hat{H}} \in \mathcal{D}_\sigma^{\hat{H}}$*, there also exists some* $\tau(\widehat{D}^{\hat{H}},t) < t$ *such that*

$$S(t,\tau)D^{\hat{H}}(\tau) \subseteq D_0^{\hat{V}}(t)\ for\ all\ \tau \leqslant \tau(\widehat{D}^{\hat{H},t}).$$

Next, we prove the pullback asymptotic compactness of the process $\{S(t,\tau)\}_{t\geqslant\tau}$ in $\hat{V}$ by energy method.

Lemma 3.3 *Suppose assumption* $(\mathbf{H_2})$ *holds. Then the process* $\{S(t,\tau)\}_{t\geqslant\tau}$ *is pullback* $\mathcal{D}_\sigma^{\hat{H},\hat{V}}$*-asymptotically compact in* $\hat{V}$*.*

Proof. Given $t \in \mathbb{R}$, a family $\widehat{D}^{\hat{V}} \in \mathcal{D}_\sigma^{\hat{H},\hat{V}}$, a sequence $\{\tau_n\} \subseteq (-\infty,t]$ with $\tau_n \to -\infty$, and a sequence $\{w_{\tau_n}\} \subseteq \hat{V}$ with $w_{\tau_n} \in D^{\hat{V}}(\tau_n)$ for all n, we need to demonstrate that the sequence $\{w(t;\tau_n,w_{\tau_n})\}$ is precompact in $\hat{V}$. For simplicity, we write $w^{(n)} = w^{(n)}(s) = w(s;\tau_n,w_{\tau_n})$.

Lemma 3.2 indicates that there exists some $\tau_1(\widehat{D}^{\hat{H}},t) < t-3$, such that sequences $\{w^{(n)} | \tau_n \leqslant \tau_1(\widehat{D}^{\hat{H}},t)\}$ and $\{(w^{(n)})' | \tau_n \leqslant \tau_1(\widehat{D}^{\hat{H}},t)\}$ are, respectively, uniformly bounded in $L^\infty(t-2,t;\hat{V}) \cap L^2(t-2,t;D(A))$ and in $L^2(t-2,t;\hat{H})$. Then from Aubin-Lions compactness lemma, we get that there exists a subsequence of $\{w^{(n)}\}$, which we do not relabel, and an element $w \in L^\infty(t-2,t;\hat{V}) \cap L^2(t-2,t;D(A))$ with $w' \in L^2(t-2,t;\hat{H})$, such that

$$w^{(n)} \overset{*}{\rightharpoonup} w \text{ weakly-star in } L^\infty(t-2,t;\hat{V}), \tag{3.38}$$

$$w^{(n)} \rightharpoonup w \text{ weakly in } L^2(t-2,t;D(A)), \tag{3.39}$$

$$(w^{(n)})' \rightharpoonup w' \text{ weakly in } L^2(t-2,t;\hat{H}), \tag{3.40}$$

$$w^{(n)} \to w \text{ strongly in } L^2(t-2,t;\hat{V}), \tag{3.41}$$

$$w^{(n)}(s) \to w(s) \text{ strongly in } \hat{V},\ a.e.\ s \in (t-2,t). \tag{3.42}$$

From (3.38)-(3.42) and the fact $w \in \mathcal{C}([t-2,t];\hat{V})$, it obvious that w satisfies the following equation on $[t-2,t]$,

$$\frac{\mathrm{d}}{\mathrm{d}t}(w(t),\Psi) + \langle Aw(t),\Psi\rangle + \langle B(w(t),w(t)),\Psi\rangle + \langle R(w(t)),\Psi\rangle = (F(t),\Psi),\ \forall \Psi \in \hat{V}. \tag{3.43}$$

Since $\{(w^{(n)})'\}$ is uniformly bounded on $L^2(t-2,t;\hat{H})$, then we can get that $\{w^{(n)}\}$ is equicontinuous in $\hat{H}$ as follows:

$$\begin{aligned}
&\|w^{(n)}(t_1) - w^{(n)}(t_2)\| \\
\leqslant& \Big\| \int_{t_1}^{t_2} (w^{(n)})'(s)\mathrm{d}s \Big\| \leqslant \int_{t_1}^{t_2} \|(w^{(n)})'(s)\|\mathrm{d}s \\
\leqslant& (t_2-t_1)^{\frac{1}{2}} \Big(\int_{t_1}^{t_2} \|(w^{(n)})'(s)\|^2 \mathrm{d}s \Big)^{\frac{1}{2}},\ t_1,\ t_2 \in [t-2,t], \text{with } t_2 \geqslant t_1.
\end{aligned}$$

Consider that the sequence $\{w^{(n)}\}$ is uniformly bounded in $\mathcal{C}([t-2,t];\hat{V})$ and the embedding $\hat{V} \hookrightarrow \hat{H}$ is compact. Then, by Ascoli-Arzelá Theorem, we obtain that

$$w^{(n)} \to w \text{ strongly in } \mathcal{C}([t-2,t];\hat{H}). \tag{3.44}$$

Using again the uniform boundedness of $\{w^{(n)}\}$ in $\mathcal{C}([t-2,t];\hat{V})$, we can get

$$w^{(n)}(t_n) \rightharpoonup w(t_*) \text{ weakly in } \hat{V},\ \forall \{t_n\} \subseteq [t-2,t] \text{ with } t_n \to t_*, \tag{3.45}$$

and the weak limit is identified by (3.44). Actually, we can get that

$$w^{(n)} \to w \text{ strongly in } \mathcal{C}([t-2,t];\hat{V}), \tag{3.46}$$

which implies the desired relative compactness. If (3.46) is not true, then there exists a sequence $\{s_n\} \subseteq [t-2,t]$ (without loss of generality we can assume that it converges to some s_*) such that for some $\epsilon_0 > 0$ and each $n \geqslant 1$,

$$\|w^{(n)}(s_n) - w(s_*)\|_{\hat{V}} \geqslant \epsilon_0. \tag{3.47}$$

From (3.45) and using the lower semi-continuity of the norm, we have

$$\|w(s_*)\|_{\hat{V}} \leqslant \liminf_{n\to\infty} \|w^{(n)}(s_n)\|_{\hat{V}}. \tag{3.48}$$

Notice that $\hat{V}$ is a Hilbert space, if we can prove that

$$\|w(s_*)\|_{\hat{V}} \geqslant \limsup_{n\to\infty} \|w^{(n)}(s_n)\|_{\hat{V}}, \tag{3.49}$$

then inequality (3.48)-(3.49) will contradict to (3.47). Thus, we just need to prove inequality (3.49).

Now, using the enstrophy inequality (3.21) for w and $w^{(n)}$, for all $t-2 \leqslant s_1 \leqslant s_2 \leqslant t$ we have that

$$\begin{aligned}
&\|w^{(n)}(s_2)\|_{\hat{V}}^2 + \frac{1}{2}\int_{s_1}^{s_2} \|Aw^{(n)}(\theta)\|^2 \mathrm{d}\theta \\
\leqslant &\|w^{(n)}(s_1)\|_{\hat{V}}^2 + 2\int_{s_1}^{s_2} \|F(\theta)\|^2 \mathrm{d}\theta + 2\delta_2^2 \int_{s_1}^{s_2} \|w^{(n)}(\theta)\|_{\hat{V}}^2 \mathrm{d}\theta \\
&+ \frac{27}{2} c_3^4 \int_{s_1}^{s_2} \|w^{(n)}(\theta)\|^2 \|w^{(n)}(\theta)\|_{\hat{V}}^4 \mathrm{d}\theta,
\end{aligned} \tag{3.50}$$

and

$$\begin{aligned}
&\|w(s_2)\|_{\hat{V}}^2 + \frac{1}{2}\int_{s_1}^{s_2} \|Aw(\theta)\|^2 \mathrm{d}\theta \\
\leqslant &\|w(s_1)\|_{\hat{V}}^2 + 2\int_{s_1}^{s_2} \|F(\theta)\|^2 \mathrm{d}\theta + 2\delta_2^2 \int_{s_1}^{s_2} \|w(\theta)\|_{\hat{V}}^2 \mathrm{d}\theta \\
&+ \frac{27}{2} c_3^4 \int_{s_1}^{s_2} \|w(\theta)\|^2 \|w(\theta)\|_{\hat{V}}^4 \mathrm{d}\theta.
\end{aligned} \tag{3.51}$$

Define

$$\begin{aligned}
Q_n(s) :=&\|w^{(n)}(s)\|_{\hat{V}}^2 - 2\int_{t-2}^{s} \|F(\theta)\|^2 \mathrm{d}\theta - 2\delta_2^2 \int_{t-2}^{s} \|w^{(n)}(\theta)\|_{\hat{V}}^2 \mathrm{d}\theta \\
&- \frac{27}{2} c_3^4 \int_{t-2}^{s} \|w^{(n)}(\theta)\|^2 \|w^{(n)}(\theta)\|_{\hat{V}}^4 \mathrm{d}\theta,
\end{aligned} \tag{3.52}$$

and

$$\begin{aligned}
Q(s) :=&\|w(s)\|_{\hat{V}}^2 - 2\int_{t-2}^{s} \|F(\theta)\|^2 \mathrm{d}\theta - 2\delta_2^2 \int_{t-2}^{s} \|w(\theta)\|_{\hat{V}}^2 \mathrm{d}\theta \\
&- \frac{27}{2} c_3^4 \int_{t-2}^{s} \|w(\theta)\|^2 \|w(\theta)\|_{\hat{V}}^4 \mathrm{d}\theta.
\end{aligned} \tag{3.53}$$

Then we have,

$$\begin{aligned}
Q_n(s_2) - Q_n(s_1) =&\|w^{(n)}(s_2)\|_{\hat{V}}^2 - \|w^{(n)}(s_1)\|_{\hat{V}}^2 - 2\int_{s_1}^{s_2} \|F(\theta)\|^2 \mathrm{d}\theta - 2\delta_2^2 \int_{s_1}^{s_2} \|w^{(n)}(\theta)\|_{\hat{V}}^2 \mathrm{d}\theta \\
&- \frac{27}{2} c_3^4 \int_{s_1}^{s_2} \|w^{(n)}(\theta)\|^2 \|w^{(n)}(\theta)\|_{\hat{V}}^4 \mathrm{d}\theta \\
\leqslant &- \frac{1}{2}\int_{s_1}^{s_2} \|Aw^{(n)}(\theta)\|^2 \leqslant 0, \text{ for all } t-2 \leqslant s_1 \leqslant s_2 \leqslant t.
\end{aligned} \tag{3.54}$$

Thus, for every $n \geqslant 1$, $Q_n(\cdot)$ is a non-increasing function on $[t-2, t]$. Similarly, $Q(\cdot)$ is also a non-increasing function on interval $[t-2, t]$. Now, by relations (3.42) and (3.44), we have $\|w^{(n)}(\theta)\|_{\hat{V}} \to \|w(\theta)\|_{\hat{V}}$ and $\|w^{(n)}(\theta)\|^2 \|w^{(n)}(\theta)\|_{\hat{V}}^4 \to \|w(\theta)\|^2 \|w(\theta)\|_{\hat{V}}^4$ a.e. $\theta \in (t-2, t)$. Furthermore, the boundedness of sequence $\{\|w^{(n)}(\theta)\| \|w^{(n)}(\theta)\|_{\hat{V}}^4\}$ in $L^\infty(t-2, t)$ can be obtained by the boundedness of sequence $\{w^{(n)}\}$ in $L^\infty(t-2, t; \hat{V}) \subseteq L^\infty(t-2, t; \hat{H})$. Therefore, using the Lebesgue Dominated Convergence Theorem, one can show that

$$\int_{t-2}^{s} \|w^{(n)}(\theta)\|^2 \|w^{(n)}(\theta)\|_{\hat{V}}^4 \mathrm{d}\theta \to \int_{t-2}^{s} \|w(\theta)\|^2 \|w(\theta)\|_{\hat{V}}^4 \mathrm{d}\theta, \text{ for any } s \in [t-2, t].$$

Thus we have,

$$Q_n(s) \to Q(s),\ a.e.\, s \in (t-2, t). \tag{3.55}$$

Hence, there exists a sequence $\{\tilde{s}_k\} \subseteq [t-2, s_*]$ such that $\tilde{s}_k \to s_*$ as $k \to \infty$, and

$$\lim_{n\to\infty} Q_n(\tilde{s}_k) = Q(\tilde{s}_k),\ \text{ for all } k.$$

Considering that Q is continuous, for an arbitrary $\varepsilon > 0$, there exists some k_ε such that

$$|Q(\tilde{s}_k) - Q(s_*)| < \frac{\varepsilon}{2},\ \forall k \geqslant k_\varepsilon. \tag{3.56}$$

By (3.55), we can choose $n(k_\varepsilon)$ such that $\forall n \geqslant n(k_\varepsilon)$, it holds

$$s_n \geqslant \tilde{s}_{k_\varepsilon} \text{ and } |Q_n(\tilde{s}_{k_\varepsilon}) - Q(\tilde{s}_{k_\varepsilon})| < \frac{\varepsilon}{2}. \tag{3.57}$$

Since, for any $n \geqslant 1$, Q_n are non-increasing function, we deduce from estimates (3.56)-(3.57) that for all $n \geqslant n(k_\varepsilon)$,

$$\begin{aligned} Q_n(s_n) - Q(s_*) &\leqslant Q_n(\tilde{s}_{k_\varepsilon}) - Q(s_*) \\ &\leqslant |Q_n(\tilde{s}_{k_\varepsilon}) - Q(s_*)| \\ &\leqslant |Q_n(\tilde{s}_{k_\varepsilon}) - Q(\tilde{s}_{k_\varepsilon})| + |Q(\tilde{s}_{k_\varepsilon}) - Q(s_*)| \\ &< \varepsilon, \end{aligned} \tag{3.58}$$

which implies that

$$\limsup_{n\to\infty} Q_n(s_n) \leqslant Q(s_*). \tag{3.59}$$

By (3.38)-(3.42) and (3.59), we get (3.49). The proof is completed. □

Theorem 3.2 *Let assumption* $(\mathbf{H_2})$ *hold. Then there exists the minimal pullback* $\mathcal{D}_\sigma^{\hat{H},\hat{V}}$*-attractor*

$$\hat{\mathcal{A}}_{\mathcal{D}_\sigma^{\hat{H},\hat{V}}} = \{\mathcal{A}_{\mathcal{D}_\sigma^{\hat{H},\hat{V}}}(t) | t \in \mathbb{R}\}$$

for the process $\{S(t,\tau)\}_{t\geqslant\tau}$ *in* $\hat{V}$*. Moreover, we have the following relationship*

$$\mathcal{A}_{\mathcal{D}_\sigma^{\hat{H}}}(t) = \mathcal{A}_{\mathcal{D}_\sigma^{\hat{H},\hat{V}}}(t). \tag{3.60}$$

Proof. Combining Lemma 3.3, Corollary 3.2 and [24, Theorem 3.11], we immediately get the existence of the minimal pullback $\mathcal{D}_\sigma^{\hat{H},\hat{V}}$-attractor $\hat{\mathcal{A}}_{\mathcal{D}_\sigma^{\hat{H},\hat{V}}}$.

We next prove the equality (3.60). For any $t \in \mathbb{R}$, it is obvious holds $\mathcal{A}_{\mathcal{D}_\sigma^{\hat{H},\hat{V}}}(t) \subseteq \mathcal{A}_{\mathcal{D}_\sigma^{\hat{H}}}(t)$. Consequently, we just need to prove $\mathcal{A}_{\mathcal{D}_\sigma^{\hat{H},\hat{V}}}(t) \supseteq \mathcal{A}_{\mathcal{D}_\sigma^{\hat{H}}}(t)$. Firstly, by the pullback attracting property of attractor $\hat{\mathcal{A}}_{\mathcal{D}_\sigma^{\hat{H},\hat{V}}}$ we know

$$\lim_{\tau\to-\infty} \operatorname{dist}_{\hat{V}}\big(S(t,\tau)D^{\hat{V}}(\tau), \mathcal{A}_{\mathcal{D}_\sigma^{\hat{H},\hat{V}}}(t)\big) = 0.$$

Then from Corollary 3.2 and the embedding $\hat{V} \hookrightarrow \hat{H}$, we obtain

$$\lim_{\tau\to-\infty} \operatorname{dist}_{\hat{H}}\big(S(t,\tau)D(\tau), \mathcal{A}_{\mathcal{D}_\sigma^{\hat{H},\hat{V}}}(t)\big) = 0.$$

Finally, it follows from the minimality property of $\hat{\mathcal{A}}_{\mathcal{D}_\sigma^{\hat{H}}}$ that $\mathcal{A}_{\mathcal{D}_\sigma^{\hat{H}}}(t) \subseteq \mathcal{A}_{\mathcal{D}_\sigma^{\hat{H},\hat{V}}}(t)$. The proof is therefore completed. □

4 Invariant measures and regularity of statistical solutions

In this section, we first present result on the existence of invariant measure and statistical solution for Eq. (2.6)-(2.7) proved in [13], and then we prove the regularity of the statistical solution and reveal the equivalence between invariant measure and statistical solution for the equations. Hereinafter, $Pr(X)$ denotes the set of all Borel probability measures on X of $B(X)$ that is the collection of Borel measurable sets of X, and for any $\mu_t \in Pr(X)$, $\mathrm{supp}(\mu_t)$ denotes the smallest closed set E such that $\mu_t(E) = 1$.

To get the regularity of the statistical solution for Eq. (2.6)-(2.7), we need to recall some definitions of statistical solutions. We begin with the definition of a class of test functions.

Definition 4.1 ([13, 14]) We denote by $\mathcal{T}$ the class of real-valued functionals $\Upsilon = \Upsilon(w)$ on $\hat{H}$ that are bounded on bounded subset of $\hat{H}$ and satisfy

(a) for any $w \in \hat{V}$, the Fréchet derivative $\Upsilon'(w)$ exists, that is, for each $w \in \hat{V}$ there exists an element $\Upsilon'(w)$ such that

$$\frac{|\Upsilon(w+v) - \Upsilon(w) - \langle \Upsilon'(w), v\rangle|}{\|v\|_{\hat{V}}} \longrightarrow 0 \ \ as \ \ \|v\|_{\hat{V}} \to 0, \ \ v \in \hat{V};$$

(b) $\Upsilon'(w) \in \hat{V}$ for all $w = (u, \omega, h) \in \hat{V}$, and the mapping $w \longmapsto \Upsilon'(w)$ is continuous and bounded as a function from $\hat{V}$ to $\hat{V}$.

We rewrite the Eq. (2.6) as follows:

$$\frac{\mathrm{d}w(t)}{\mathrm{d}t} = \mathcal{F}(t, w(t)) = F(t) - Aw(t) - B(w(t), w(t)) - R(w(t)). \tag{4.1}$$

Notice that for any $\Upsilon \in \mathcal{T}$, if $w(t)$ is the solution of Eq. (4.1), then it holds

$$\frac{\mathrm{d}}{\mathrm{d}t}\Upsilon(w(t)) = \langle \Upsilon'(w(t)), \mathcal{F}(t, w(t))\rangle. \tag{4.2}$$

Definition 4.2 ([13, 14]) A family of measures $\{m_t\}_{t\in\mathbb{R}} \subseteq Pr(\hat{H})$ is said to be a statistical solution of Eq. (2.6) if it satisfies the following conditions:

(a) the mapping $t \mapsto \int_{\hat{H}} \Gamma(w)\mathrm{d}m_t(w)$ is continuous for all $\Gamma \in \mathcal{C}_{\mathrm{b}}(\hat{H})$ (the collection of all continuous and bounded functions on $\hat{H}$);

(b) for almost $t \in \mathbb{R}$, $\mathrm{supp}(m_t)$ is included in $\hat{H}$ and the function $w \mapsto \langle \phi, \mathcal{F}(w,t)\rangle$ is m_t-integrable for each $\phi \in \hat{V}$. In addition, the mapping

$$t \mapsto \int_{\hat{H}} \langle \phi, \mathcal{F}(w,t)\rangle \mathrm{d}m_t(w)$$

belongs to $L^1_{\mathrm{loc}}(\mathbb{R})$ for all $\phi \in \hat{V}$;

(c) for each $\Upsilon \in \mathcal{T}$, the following Liouville-type equation holds

$$\int_{\hat{H}} \Upsilon(w)\mathrm{d}\rho_t(w) - \int_{\hat{H}} \Upsilon(w)\mathrm{d}m_\tau(w) = \int_\tau^t \int_{\hat{H}} \langle \Upsilon'(w), \mathcal{F}(w,s)\rangle \mathrm{d}m_s(w)\mathrm{d}s,$$

for all $t, \tau \in \mathbb{R}$ with $t \geqslant \tau$.

Before we present the result on the existence of invariant measure and statistical solution of Eq. (2.6)-(2.7), we recall the definition of generalized Banach limit.

Definition 4.3 ([3]) Denote by $\mathcal{S}_+$ the collection of all bounded real-valued functions on $[0, +\infty)$. A linear functional defined on $\mathcal{S}_+$ is called a generalized Banach limit denoted by $\mathrm{LIM}_{t\to+\infty}$, if it satisfies

(1) $\mathrm{LIM}_{t\to+\infty}\mathcal{Y}(t) \geqslant 0$ for all nonnegative functions $\mathcal{Y}(\cdot)$ on $[0, +\infty)$;

(2) $\mathrm{LIM}_{t\to+\infty}\mathcal{Y}(t) = \lim\limits_{t\to+\infty} \mathcal{Y}(t)$ if the usual limit $\lim\limits_{t\to+\infty} \mathcal{Y}(t)$ exists.

Any generalized Banach limit $\mathrm{LIM}_{t\to+\infty}$ possesses the following useful property (see in [2]):

$$|\mathrm{LIM}_{t\to+\infty}\mathcal{Y}(t)| \leqslant \limsup_{t\to+\infty} |\mathcal{Y}(t)|, \quad \forall\, \mathcal{Y}(\cdot) \in B_+. \tag{4.3}$$

In this article, we consider the "pullback" asymptotic behavior of (2.6). Hence, we need generalized limits when τ goes to negative infinity. Toward this end, for any real-valued function $\mathcal{Y}$ defined on $(-\infty, 0]$ and any given Banach limit $\mathrm{LIM}_{t\to+\infty}$, we define by

$$\mathrm{LIM}_{t\to-\infty}\mathcal{Y}(t) = \mathrm{LIM}_{t\to+\infty}\mathcal{Y}(-t).$$

Theorem 4.1 ([13]) *Suppose that assumption* $(\mathbf{H_1})$ *holds. Let* $\mathrm{LIM}_{\tau\to-\infty}$ *be a given generalized Banach limit and* $\mathcal{A}_{\mathcal{D}_\sigma^{\hat{H}}} = \{\mathcal{A}_{\mathcal{D}_\sigma^{\hat{H}}}(t) | t \in \mathbb{R}\}$ *be the pullback* $\mathcal{D}_\sigma^{\hat{H}}$*-attractor given in Theorem 3.1. Then for any continuous mapping* $w : \mathbb{R} \mapsto \hat{H}$ *satisfying* $w(\cdot) \in \mathcal{D}_\sigma^{\hat{H}}$*, there exists a unique family of measures* $\{\mu_t\}_{t\in\mathbb{R}} \subseteq Pr(\hat{H})$ *and* $\mathrm{supp}(\mu_t) \subseteq \mathcal{A}_{\mathcal{D}_\sigma^{\hat{H}}}(t)$ *such that it holds*

$$\begin{aligned}\mathrm{LIM}_{\tau\to-\infty}\frac{1}{t-\tau}\int_\tau^t \psi(S(t,s)w(s))\mathrm{d}s &= \int_{\hat{H}} \psi(w)\mathrm{d}\mu_t(w) \\ &= \int_{\mathcal{A}_{\mathcal{D}_\sigma^{\hat{H}}}(t)} \psi(w)\mathrm{d}\mu_t(w),\end{aligned} \tag{4.4}$$

for each real-valued continuous functional ψ *on* $\hat{H}$ *and satisfies the following generalized invariance*

$$\int_{\hat{H}} \psi(w)\mathrm{d}\mu_t(w) = \int_{\hat{H}} \psi(S(t,\tau)w)\mathrm{d}\mu_\tau(w). \tag{4.5}$$

Moreover, the family of invariant measures $\{\mu_t\}_{t\in\mathbb{R}}$ *is a statistical solution of Eq.* (2.6).

We next investigate the regularity of the statistical solution for Eq. (2.6). As described in [3], here we need a class of time-dependent test functions $\Psi = \Psi(t, w)$, which are continuous real-valued functions defined on $\mathbb{R} \times \hat{V}$ and Fréchet differentiable in the sense that:

(i) there exists some element $\Psi'(s,w) = (\Psi'_s(t,w), \Psi'_w(t,w))$ in $\mathbb{R}\times\hat{V}$ that satisfies

$$\frac{|\Psi(s+q,w+v)-\Psi(s,w)-q\Psi'_s(s,w)-(v,\Psi'_w(w))|}{\|v\|+|q|} \longrightarrow 0 \ \ as \ \ \|v\|+|q|\to 0,$$

(ii) $\Psi'(s,w)$ is continuous from $\mathbb{R}\times\hat{V}$ into $\mathbb{R}\times\hat{V}$, where $\Psi'_w(s,w)$ is uniformly bounded in $\hat{V}$ and $\Psi'_s(s,w)$ has at most a linear growth in $\|w\|$.

Theorem 4.2 *Suppose that assumption* ($\mathbf{H_2}$) *holds. Then the statistical solution* $\{\mu_t\}_{t\in\mathbb{R}}$ *of Eq.* (2.6) *given by Theorem 4.1 possesses the following regularity properties:*

(a) *for any given* $t\geqslant\tau$, $\tau\in\mathbb{R}$, *the support of* μ_t *is included and bounded in* $\hat{V}$, *that is*

$$\mathrm{supp}(\mu_t)\subseteq\{w(t;\tau,w_\tau)\in\hat{V}; \|w(t;\tau,w_\tau)\|_{\hat{V}}\leqslant c(t)\}; \tag{4.6}$$

(b) *for any time-dependent test function* Ψ *satisfying above conditions* (i) *and* (ii), *the following stronger form of Liouville-type equation holds,*

$$\int_{\hat{H}}\Psi(t,w)\mathrm{d}\mu_t(w)-\int_{\hat{H}}\Psi(\tau,w)\mathrm{d}\mu_\tau(w)=\int_\tau^t\int_{\hat{H}}\Psi'_s(s,w)+\langle\Phi'_w(s,w),\mathcal{F}(s,w)\rangle\mathrm{d}\mu_s(w)\mathrm{d}s, \tag{4.7}$$

for every $t\in[\tau,T]$.

Proof. From Lemma 3.2, Theorem 3.2, and Theorem 4.1, we can easily get item (a). Now, we prove the family of measures $\{\mu_t\}_{t\in\mathbb{R}}$ satisfies item (b). For any given time-dependent test function $\Psi(t,w)$ satisfying conditions (i) and (ii), we have

$$\frac{\mathrm{d}}{\mathrm{d}s}\Psi(s,w(s))=\Psi'_s(s,w(s))+\langle\Psi'_w(s,w(s)),\mathcal{F}(s,w(s))\rangle, \tag{4.8}$$

where $w(t)$ is a solution of Eq. (4.1). Integrating (4.8) on $[\tau,t]$ with $t\geqslant\tau$, we get

$$\Psi(t,w(t))-\Psi(\tau,w(\tau))=\int_\tau^t\Psi'_s(s,w(s))+\big\langle\Psi'_w(s,w(s)),\mathcal{F}(s,w(s))\big\rangle\mathrm{d}s. \tag{4.9}$$

For any $\theta<\tau$, we let $w_\theta\in\hat{V}$. It then follows from (4.9) and $w(s)=S(s,\theta)w_\theta$, $s\geqslant\theta$ that

$$\begin{aligned}&\Psi(t,S(t,\theta)w_\theta)-\Psi(\tau,S(\tau,\theta)w_\theta)\\ =&\int_\tau^t\Psi'_s(s,S(s,\theta)w_\theta)+\langle\Psi'_w(s,S(s,\theta)w_\theta),\mathcal{F}(s,S(s,\theta)w_\theta)\rangle\mathrm{d}s.\end{aligned} \tag{4.10}$$

From (4.6), we find that (4.4) and (4.5) also hold for any real-valued continuous functional ψ defined on $\hat{V}$. Then by (4.4), (4.10) and Fubini Theorem, we can obtain

$$\begin{aligned}&\int_{\hat{H}}\Psi(t,w(t))\mathrm{d}\mu_t(w)-\int_{\hat{H}}\Psi(\tau,w(\tau))\mathrm{d}\mu_\tau(w)\\ =&\mathrm{LIM}_{M\to-\infty}\frac{1}{t-M}\int_M^t\Psi(t,S(t,\theta)w_\theta)\mathrm{d}\theta-\mathrm{LIM}_{M\to-\infty}\frac{1}{\tau-M}\int_M^\tau\Psi(\tau,S(\tau,\theta)w_\theta)\mathrm{d}\theta\end{aligned}$$

$$
\begin{aligned}
=&\mathrm{LIM}_{M\to-\infty}\frac{1}{\tau-M}\int_M^\tau \big(\Psi(t,S(t,\theta)w_\theta)-\Psi(\tau,S(\tau,\theta)w_\theta)\big)\mathrm{d}\theta\\
=&\mathrm{LIM}_{M\to-\infty}\frac{1}{\tau-M}\int_M^\tau\int_\tau^t \Psi'_s(s,S(s,\theta)w_\theta)+\big\langle\Psi'_w(s,S(s,\theta)w_\theta),\mathcal{F}(s,S(s,\theta)w_\theta)\big\rangle\mathrm{d}s\mathrm{d}\theta\\
=&\int_\tau^t \mathrm{LIM}_{M\to-\infty}\frac{1}{\tau-M}\int_M^\tau \Psi'_s(s,S(s,\theta)w_\theta)+\big\langle\Psi'_w(s,S(s,\theta)w_\theta),\mathcal{F}(s,S(s,\theta)w_\theta)\big\rangle\mathrm{d}\theta\mathrm{d}s.
\end{aligned}
\tag{4.11}
$$

It follows from (4.4), (4.5) and $S(s,\theta)=S(s,\tau)S(\tau,\theta)w_\theta$ that

$$
\begin{aligned}
&\mathrm{LIM}_{M\to-\infty}\frac{1}{\tau-M}\int_M^\tau \Psi'_s(s,S(s,\theta)w_\theta)+\big\langle\Psi'_w(s,S(s,\theta)w_\theta),\mathcal{F}(s,S(s,\theta)w_\theta)\big\rangle\mathrm{d}\theta\\
=&\mathrm{LIM}_{M\to-\infty}\frac{1}{\tau-M}\int_M^\tau \Psi'_s(s,S(s,\tau)S(\tau,\theta)w_\theta)\\
&+\big\langle\Psi'_w(s,S(s,\tau)S(\tau,\theta)w_\theta),\mathcal{F}(s,S(s,\tau)S(\tau,\theta)w_\theta)\big\rangle\mathrm{d}\theta\\
=&\int_{\hat H}\Psi'_s(s,S(s,\tau)w)+\big\langle\Psi'_w(s,S(s,\tau)w),\mathcal{F}(s,S(s,\tau)w)\big\rangle\mathrm{d}\mu_\tau(w),\\
=&\int_{\hat H}\Psi'_s(s,w)+\big\langle\Psi'_w(s,w),\mathcal{F}(s,w)\big\rangle\mathrm{d}\mu_s(w),
\end{aligned}
\tag{4.12}
$$

which along with (4.11) gets the desired result. □

The following result implies that a regular statistical solution for Eq. (2.6), i.e. a statistical solution satisfies regularity properties in Theorem 4.2, is an invariant measure for the equation.

Theorem 4.3 *Let the family of probability measures $\{\mu_t\}_{t\in\mathbb{R}}\subseteq Pr(\hat H)$ be a regular statistical solution for Eq.* (2.6)*, then μ_t satisfies the generalized invariance* (4.5)*.*

Proof. Since $\mathcal{T}$ is dense in $\mathcal{C}(\hat H)$, for the generalized invariance (4.5), we only need to prove

$$
\int_{\hat H}\Upsilon(w)\mathrm{d}\mu_t(w)=\int_{\hat H}\Upsilon(S(t,\tau)w)\mathrm{d}\mu_\tau(w),\ \forall\,\Upsilon\in\mathcal{T}.
$$

Denote by P_m the Galerkin projection onto the finite-dimensional space $P_m\hat H$ spanned by the first m eigenfunctions of the operator A, and also denote by $\{S_m(t,\tau)\}_{t\geqslant\tau}$ the solution operator corresponding to the Galerkin approximate, that is, $S_m(t,\tau)P_m u_\tau=u_m(t)$ for all $t\geqslant\tau$, where $w_m(t)$ is the solution of the following finite-dimensional equation on $P_m\hat H$:

$$
\frac{\mathrm{d}w_m}{\mathrm{d}t}=P_m\mathcal{F}(t,w_m),\ w_m(\tau)=P_m w_\tau,
$$

where $\mathcal{F}(t,w_m)=F(t)-Aw_m-R(w_m)-B(w_m,w_m)$. Set

$$
\Psi_m(s,w)=\Upsilon(S_m(t,s)P_m w),\ \tau\leqslant s\leqslant t,\ w\in\hat H,
\tag{4.13}
$$

where Υ is any test function in the sense of Definition 4.1. It obvious that $\Psi_m=\Psi_m(s,w)$ is a time-dependent test function allowed in (4.7). Then by (4.7) and (4.13), we get

$$
\int_{\hat H}\Upsilon(P_m w)\mathrm{d}\mu_t(w)-\int_{\hat H}\Upsilon(S_m(t,\tau)P_m w)\mathrm{d}\mu_\tau(w)
$$

$$= \int_\tau^t \int_{\hat{H}} (\Psi_m)'_s(s,w) + \langle \mathcal{F}(s,w), (\Psi_m)'_w(s,w)\rangle \mathrm{d}\mu_s(w)\mathrm{d}s. \tag{4.14}$$

By Lebesgue dominated convergence theorem, we obtain

$$\lim_{m\to+\infty} \int_{\hat{H}} \Upsilon(P_m w)\mathrm{d}\mu_t(w) = \int_{\hat{H}} \Upsilon(w)\mathrm{d}\mu_t(w),$$

and

$$\lim_{m\to+\infty} \int_{\hat{H}} \Upsilon(S_m(t,\tau)P_m w)\mathrm{d}\mu_\tau(w) = \int_{\hat{H}} \Upsilon(S(t,\tau)w)\mathrm{d}\mu_\tau(w).$$

Hence, we can complete the proof if we prove that the right-hand side of (4.14) vanishes as m goes to infinity. By the property $S(t,\tau) = S(t,s)S(s,\tau)w_\tau$ for any $t \geqslant s \geqslant \tau$, we have

$$\Psi_m(s, S_m(s,\tau)P_m w) = \Upsilon(S_m(t,s)S_m(s,\tau)P_m w) = \Upsilon(S_m(t,\tau)P_m w),$$

which is independent of s. Therefore, taking derivative with respect to s of $\Psi_m(s, S_m(s,\tau)P_m w)$, we find for all $s \geqslant \tau$ that

$$(\Psi_m)'_s(s, S_m(s,\tau)P_m w) + \langle P_m\mathcal{F}(s, S_m(s,\tau)P_m w), (\Psi_m)'_w(s, S_m(s,\tau)P_m w)\rangle = 0.$$

Taking $\tau = s$, we get

$$(\Psi_m)'_s(s, P_m w) + \langle P_m\mathcal{F}(s, P_m w), (\Psi_m)'_w(s, P_m w)\rangle = 0.$$

Note that $\Psi_m(s, P_m w) = \Psi_m(s,w)$ for all $w \in \hat{H}$ and $s \in \mathbb{R}$. Then we have

$$(\Psi_m)'_s(s,w) + \langle P_m\mathcal{F}(s, P_m w), (\Psi_m)'_w(s,w)\rangle = 0. \tag{4.15}$$

Therefore, by (4.15), we rewrite the integrand of the right-hand side of (4.14) as

$$\langle \mathcal{F}(s,w) - P_m\mathcal{F}(s,P_m w), (\Psi_m)'_w(s,w)\rangle. \tag{4.16}$$

Note that $\mathcal{F}(s,w) = F(s) - Aw - R(w) - B(w,w)$ and $(\Psi_m)'_w(s,w)$ belongs to $P_m\hat{H}$. Firstly, it is obvious that for any $w \in \mathrm{supp}(\mu_s)$ and all $s \in [\tau,t]$, we have

$$\langle F(s) - P_m F(s), (\Psi_m)'_w(s,w)\rangle \to 0,\ m \to +\infty, \tag{4.17}$$

and

$$\langle P_m A(P_m w) - Aw, (\Psi_m)'_w(s,w)\rangle \to 0,\ m \to +\infty. \tag{4.18}$$

We next prove

$$\int_\tau^t \int_{\hat{H}} \langle R(P_m w) - R(w), (\Psi_m)'_w(s,w)\rangle \mathrm{d}\mu_s \mathrm{d}s,$$

and

$$\int_\tau^t \int_{\hat{H}} \langle B(P_m w, P_m w) - B(w,w), (\Psi_m)'_w(s,w)\rangle \mathrm{d}\mu_s \mathrm{d}s$$

go to zero as m goes to infinity. Using integration by parts and the definition of $R(w)$, we have for all $\Phi = (\xi,\eta,\zeta) \in \hat{V}$

$$\langle R(w), \Phi\rangle = -2\chi(\nabla\times\omega, \xi) - 2\chi(\nabla\times u, \eta) + 4\chi(\omega,\eta),$$

$$= -2\chi(\omega, \nabla\times\xi) - 2\chi(u, \nabla\times\eta) + 4\chi(\omega,\eta).$$

For any $\varphi \in H_0^1(\Omega)$, we have $\|\nabla\times\varphi\|^2 \leqslant \|\nabla\varphi\|^2$. Indeed, by some direct calculations we can get that

$$\nabla\times(\nabla\times\varphi) = -\Delta\varphi + \nabla(\mathrm{div}\varphi),\ \forall\varphi\in H_0^1(\Omega). \tag{4.19}$$

Taking inner product of (4.19) with φ gets

$$\|\nabla\times\varphi\|^2 \leqslant \|\nabla\varphi\|^2,\ \forall\varphi\in H_0^1(\Omega). \tag{4.20}$$

Then by (4.20) and (2.2), we there exists some c_4 such that

$$\begin{aligned}
&\langle R(P_m w) - R(w), (\Psi_m)'_w(s,w)\rangle\\
\leqslant& 2\chi\|w - P_m w\|\|(\Psi_m)'_w(s,w)\|_{\hat V} + 2\chi\|w - P_m w\|\|(\Psi_m)'_w(s,w)\|_{\hat V}\\
&+ 4\chi\|P_m w - w\|\|(\Psi_m)'_w(s,w)\|\\
\leqslant& c_4\lambda_m^{-\frac12}\|P_m w - w\|_{\hat V}\|(\Psi_m)'_w(s,w)\|_{\hat V}\\
\leqslant& c_4\lambda_m^{-\frac12}\|w\|_{\hat V}\|(\Psi_m)'_w(s,w)\|_{\hat V}.
\end{aligned} \tag{4.21}$$

By the definition of the operator $B(\cdot,\cdot)$, (2.1), (2.2), and (2.11), we get

$$\begin{aligned}
&\langle B(P_m w, P_m w) - B(w,w), (\Psi_m)'_w(s,w)\rangle\\
=&\langle B(P_m w - w, P_m w), (\Psi_m)'_w(s,w)\rangle + \langle B(w, P_m w - w), (\Psi_m)'_w(s,w)\rangle\\
\leqslant& c_1\|P_m w - w\|^{\frac12}\|(P_m w - w)\|_{\hat V}^{\frac12}\|P_m w\|^{\frac12}\|P_m w\|_{\hat V}^{\frac12}\|(\Psi_m)'_w(s,w)\|_{\hat V}\\
&+ c_1\|w\|^{\frac12}\|w\|_{\hat V}^{\frac12}\|P_m w - w\|^{\frac12}\|P_m w - w\|_{\hat V}^{\frac12}\|(\Psi_m)'_w(s,w)\|_{\hat V}\\
\leqslant& 2c_1\|w\|^{\frac12}\|w\|_{\hat V}^{\frac12}\|P_m w - w\|^{\frac12}\|P_m w - w\|_{\hat V}^{\frac12}\|(\Psi_m)'_w(s,w)\|_{\hat V}\\
\leqslant& 2c_1\lambda_1^{-\frac14}\lambda_m^{-\frac14}\|w\|_{\hat V}\|P_m w - w\|_{\hat V}\|(\Psi_m)'_w(s,w)\|_{\hat V}\\
\leqslant& 2c_1\lambda_1^{-\frac14}\lambda_m^{-\frac14}\|w\|_{\hat V}^2\|(\Psi_m)'_w(s,w)\|_{\hat V}.
\end{aligned} \tag{4.22}$$

For any $w \in \mathrm{supp}(\mu_s)$ and all $s\in[\tau,t]$, we get from (4.6) that the right-hand sides of (4.21) and (4.22) go to zero as $m\to+\infty$. This implies the result. □

Conflict of interest statement

This work does not have any conflicts of interest.

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