# Riemann-Liouville Fractional Operators of Bicomplex Order and Its Properties

Mahesh Puri Goswami[1], Raj Kumar[2]
Department of Mathematics and Statistics,
Mohanlal Sukhadia University, Udaipur-313001, INDIA
E-mail: [1]*maheshgoswami1989@gmail.com*, [2]*rajsantoshmaths@gmail.com*

**Abstract**

In this paper, we construct the Riemann-Liouville fractional integral and differential operator of bicomplex order and illustrate some examples to calculate the fractional integration and differentiation of bicomplex order of some elementary functions. Also, we discuss some properties of these operators by proving analogues of the Leibniz and chain rules for these operators.

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# 1 Introduction

In the last few years, there has been a remarkable development in subject of bicomplex numbers and fractional calculus. The bicomplex number system has reached different results in complex number systems and also, the fractional calculus has given many multidimensional and radical changes in the field of calculus.

Bicomplex number system is a special kind of generalization of complex number system as they are complex numbers appearing with another imaginary unit whose coefficients are also complex numbers. Bicomplex numbers were firstly defined by *Segre* [42] in 1892. *Segre* described an infinite set whose elements were later called bicomplex number, tri-complex number, $\cdots$, n-complex numbers. The bicomplex numbers extend the complex numbers into a four-dimensional space and can be recognized as four-dimensional vectors, in the similar way that complex numbers are recognized as two-dimensional vectors. Also, bicomplex numbers can be conceptualized as ordered pairs of two complex numbers.

The initial development of bicomplex numbers basically rests on the pairs of complex numbers. The theory of bicomplex numbers has many applications in various branches of science [1, 2, 3, 21]. *Goyal* et al. [23], extended gamma and beta functions to bicomplex variables. *Luna* et al. [16] presented algebra, geometry, and analysis of bicomplex holomorphic functions. Efforts have been made to define generalized bicomplex Riemann zeta function [31], bicomplex Dirichlet series [33] and its integral representation [32], some properties of bicomplex numbers [34]. Topological modules over the ring of bicomplex numbers were developed in [35]. *Agarwal* et al. [4] generalized Mellin transform in bicomplex space with application and discussed solution of Maxwell's equation in bicomplex space [5]. Fibonacci sequence with coefficients from bicomplex numbers [25] and bicomplex Fibonacci numbers and their generalization were presented in [24]. Common fixed point theorems in the frame work of bicomplex valued metric space $(X, d)$ and max function for the partial order in bicomplex valued metric $d$ were obtained [29].

Recently, the bicomplex version of Enström-Kakeya theorem and some of its consequences was introduced [11]. The concept of triple Laplace transform in bicomplex space was given in [22].

Some fractal-type sets in the four dimensional space were constructed [14]. The bicomplex finite element method that can be applied for wave propagation problems in various environments was presented [39]. A necessary and sufficient condition for the general principle of convergence of infinite product of bicomplex number, some of its consequences and validity of the theorems were given in [13]. Work has been done on some new fixed point theorems for contractive maps that satisfied Mizoguchi-Takahashi's condition in the setting of bicomplex-valued metric spaces with existence and uniqueness of the solution of a non-linear integral equation [26].

With this, if we talk about fractional calculus, then we can say that fractional calculus is not a new topic. Its history is almost as old as that of classical calculus. Fractional operators were viewed in their early life by various mathematicians with an inferior view and we can even say that pure mathematicians have contributed more than applied mathematicians in its early stages. Applied scientists, mathematicians, and engineers over the last few decades have realized that differential equations with fractional operators provide a simple framework for discussing the problems associated with various real life problems, under which the fractional operators exists such as viscoelastic systems, signal processing, diffusion processes, control processing, fractional stochastic systems, allometry in biology and ecology and many more [30]. With non-local and non-singular kernel [7] and without non-singular kernel [43], new fractional derivatives were defined. Series representations for models of fractional calculus [19] and mean value theorem and Taylor's theorem for fractional derivatives [18] and properties of fractional derivatives [17] involving Mittag-Leffler kernel were given. Many concepts have been proposed via generalized fractional operators [10, 28].

Some of useful applications of fractional calculus were given in [12]. Incomplete Riemann-Liouville fractional integral operators and hypergeometric functions were described in [38]. In [20], A fundamental connection with classical fractional calculus was given. Some applications of differential equations equipped with fractional operators [6], special functions and solution of differential equations in frame of fractional calculus [45] were established . Many researcher have done work in the direction of generalization of a solution for a class of integro-differential equations with non-separated integral boundary conditions [44]. A new fractional derivative, which is based on Caputo-type derivative with a smooth kernel was proposed in [40].

Many interesting properties of the incomplete I-functions associated with the Marichev-Saigo-Maeda fractional operators were investigated [27]. Several other ways have been proposed to define fractional integral and differential operators. There are many operators among them which are conform to the definition of Riemann-Liouville fractional operators, while many such operators have also been defined which do not conform to the definition of Riemann-Liouville fractional operators. For example, the Grünwald-Letnikov fractional differintegral is defined by the limit of a convergent series which is equivalent to the definition of Riemann-Liouville fractional operators [37]; meanwhile, the Caputo fractional derivative is defined by switching the order of the operations on the right-hand side of (2.19), and it is not equivalent to Riemann-Liouville definition [8, 9].

Classification of the paper as follows:
In Section 2, we summarize basic facts on bicomplex numbers and fractional calculus according to our purpose. In Section 3, we define the Riemann-Liouville fractional operators of bicomplex order and illustrate some examples. In Section 4, we prove an analogue of properties of these operators including Leibniz and chain rules. In last Section, we conclude the paper.

# 2 Preliminaries

## 2.1 Bicomplex Numbers

The bicomplex number (see, e.g. [16]) defined as

$$w = z_1 + jz_2; \quad z_1, z_2 \in \mathbb{C}. \tag{2.1}$$

Let $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$, then (2.1) becomes

$$w = x_1 + iy_1 + jx_2 + ijy_2; \quad x_1, y_1, x_2, y_2 \in \mathbb{R} \tag{2.2}$$

equipped with the multiplication defined by $j^2 = -1, \quad ij = ji = k, \quad i^2j^2 = 1$.

Using (2.1) and (2.2), the set of bicomplex numbers $\mathbb{C}_2$ can be given as

$$\mathbb{C}_2 = \{w = z_1 + jz_2 : z_1, z_2 \in \mathbb{C}\}, \tag{2.3}$$
$$\mathbb{C}_2 = \{w = x_1 + iy_1 + jx_2 + ijy_2 : x_1, y_1, x_2, y_2 \in \mathbb{R}\}. \tag{2.4}$$

By manipulating $i$ and $j$ we may have other representations also. Let us consider following sets

$$\mathbb{C}(i) = \{z = x + iy \ : \ x, y \in \mathbb{R}\} \tag{2.5}$$
$$\mathbb{C}(j) = \{z = x + jy \ : \ x, y \in \mathbb{R}\} \tag{2.6}$$
$$\mathbb{D} = \{w = x_1 + ijy_2 \ : \ x_1, y_2 \in \mathbb{R}\}. \tag{2.7}$$

Both $\mathbb{C}(i)$ and $\mathbb{C}(j)$ are isomorphic fields of complex numbers and $\mathbb{D}$ is set of hyperbolic numbers. The algebraic operations for bicomplex numbers are exactly the same as the operations for complex numbers.

### 2.1.1 Zero Divisors

$w = z_1 + jz_2 \neq 0$ be a zero divisor if both $z_1$ and $z_2$ are non-zero but the sum $z_1^2 + z_2^2 = 0$. In fact, all zero divisors $w = z_1 + jz_2$ in $\mathbb{C}_2$ are characterized by the equations $z_1^2 = -z_2^2$, i.e. $z_1 = \pm iz_2$. Thus all zero divisors are of the form

$$w = \lambda(1 \pm ij) \tag{2.8}$$

for any $\lambda \in \mathbb{C}(i) \setminus \{0\}$. The set of all zero divisors in $\mathbb{C}_2$ is said to be *null-cone* $\mathcal{O}_2$ and defined as

$$\mathcal{O}_2 = \{w = z_1 + jz_2 \in \mathbb{C}_2 : z_1 = \pm iz_2 \ ; \ z_1^2 + z_2^2 = 0\}. \tag{2.9}$$

By applying simple calculations we can prove that $e_1$ and $e_2$ are two linearly independent zero divisors, rest are some complex multiple of either $e_1$ or $e_2$ [15], where

$$e_1 = \frac{1 + ij}{2}, \ e_2 = \frac{1 - ij}{2}. \tag{2.10}$$

### 2.1.2 Idempotent Representation

An element $w$ is said to be an idempotent element if $w^2 = w$. As we all know that in real and complex number field there are only two idempotent elements 0 and 1. In bicomplex number system, rather than these two, $e_1$ and $e_2$ (defined in (2.10)) are also idempotent elements i.e.

$$e_1^2 = e_1, \quad e_2^2 = e_2, \tag{2.11}$$

using these two idempotent elements we have a unique representation for every bicomplex number, known as *idempotent representation*, and given as

$$\begin{aligned} w = z_1 + jz_2 &= (z_1 - iz_2)e_1 + (z_1 + iz_2)e_2 \\ &= w_1e_1 + w_2e_2. \end{aligned} \tag{2.12}$$

Infact, for every $w \in \mathbb{C}_2$ (2.12) can be written as

$$w = \mathcal{P}_1(w)e_1 + \mathcal{P}_2(w)e_2, \tag{2.13}$$

where the projections $\mathcal{P}_1, \mathcal{P}_2 : \mathbb{C}_2 \to \mathbb{C}(i)$ are defined as $\mathcal{P}_1(z_1 + jz_2) = z_1 - iz_2$ and $\mathcal{P}_2(z_1 + jz_2) = z_1 + iz_2$.

### 2.1.3 Bicomplex Differentiation

The definition of differentiability of a bicomplex differentiable function can be seen as similar as in complex field. But the approaches may different. Let $f$ be a function defined on a set $S \subseteq \mathbb{C}_2$, then the derivative of the function $f$ at a point $w_0 \in S$ is the limit, if it exists,

$$\lim_{\substack{w \to w_0 \\ w - w_0 \notin \mathcal{O}_2}} \frac{f(w) - f(w_0)}{w - w_0} \quad \text{or} \quad \lim_{\substack{\triangle w \to 0 \\ \triangle w \notin \mathcal{O}_2}} \frac{f(w + \triangle w) - f(w)}{\triangle w}$$

and we write

$$f'(w_0) = \lim_{\substack{w \to w_0 \\ w - w_0 \notin \mathcal{O}_2}} \frac{f(w) - f(w_0)}{w - w_0} = \lim_{\substack{\triangle w \to 0 \\ \triangle w \notin \mathcal{O}_2}} \frac{f(w + \triangle w) - f(w)}{\triangle w} \tag{2.14}$$

for $w$ in the domain of $f$ such that $w - w_0$ is an invertible bicomplex number.

### 2.1.4 Bicomplex Integration

Let $w = z_1 + jz_2 \approx (z_1, z_2)$ be a bicomplex number, where $z_1$ and $z_2$ are complex numbers. Consider a bicomplex function

$$f(w) = f(z_1, z_2) = f_1(z_1, z_2) + jf_2(z_1, z_2) = (f_1(z_1, z_2), f_2(z_1, z_2))$$

and let $\Upsilon$ be a four dimensional piecewise continuously differentiable curve in a set $S \subseteq \mathbb{C}_2$. Then the bicomplex integration of bicomplex function $f$ is defined as a line integral, that is evaluated with respect to some four-dimensional curve $\Upsilon$ in $\mathbb{C}_2$. More specifically, the bicomplex integration is defined as

$$\int_\Upsilon f(w) \cdot dw; \quad dw = (dz_1, dz_2). \tag{2.15}$$

If we represents $\Upsilon$ in parametric form $w(t) = (z_1(t), z_2(t))$, where $r \le t \le s$. Then (2.15) can be rewritten as

$$\int_\Upsilon f(w) \cdot dw = \int_r^s f(w(t)) \cdot w'(t)dt. \tag{2.16}$$

Here $w'(t)$ may discontinuous at some points. $\Upsilon$ can be taken as a curve made up of two component curves $\gamma_1$ and $\gamma_2$ in $\mathbb{C}$ i.e.

$$\Upsilon = (\gamma_1, \gamma_2). \tag{2.17}$$

## 2.2 Fractional Calculus

Origin of fractional calculus was from a question under which it was asked that can $n$ be any number: fractional, irrational, or complex in $\frac{d^n y}{dx^n}$. The question was answered affirmatively latter. *Leibniz, Euler, Laplace, Lacroix,* and *Fourier* made mention of derivatives of arbitrary order, but the first use of fractional operations was made by *Abel* in 1823 [36]. By the time of the $21^{st}$ century many different fractional operators have been defined. For example, Riemann-Liouville operators, Caputo operators, Atangana-Baleanu operators, Prabhakar operators etc. The origin of the Riemann-Liouville operator is primarily Abel equation based. Defining the operator on the basis of Abel equation is mentioned in [41] [p. 28-37].

Riemann-Liouville operators are the most common way of defining fractional calculus. Now, we discuss Riemann-liouville operators firstly for complex order and then define for bicomplex order.

### 2.2.1 Riemann-Liouville Fractional Integral and Derivative of Complex Order

**Definition 2.1** ([36])**.** *"Let $\alpha = a+ib \in \mathbb{C}(i)$ with $Re(\alpha) > 0$ and let $f(x)$ be piecewise continuous on $J' = (0,\infty)$ and integrable on any finite subinterval of $J = [0,\infty)$. Then for $t > 0$ we call*

$$ {}_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(x)(t-x)^{\alpha-1} dx \tag{2.18}$$

*the Riemann-Liouville fractional integral of $f$ of order $\alpha$, where $\Gamma$ denotes complex gamma function.*

Let us denote $\mathbf{C}$ as the class of functions defined in above definition.

**Definition 2.2.** *Let $f$ be a function of class $\mathbf{C}$ and $\beta = c+id \in \mathbb{C}$. Let $m = \lfloor Re(\beta) \rfloor + 1$, where $\lfloor x \rfloor$ denote the greatest integer less or equal to $x$. Then the fractional derivative of $f$ of order $\beta$ is given as*

$$D^\beta f(t) = D^m\{D^{-v} f(t)\}, \quad Re(\beta) > 0, \quad t > 0 \tag{2.19}$$

$$D^\beta f(t) = \frac{1}{\Gamma(v)} \frac{d^m}{dt^m} \int_0^t f(x)(t-x)^{v-1} dx, \tag{2.20}$$

*(if exists) where $v = m - \beta$ s.t. $0 < Re(v) \leq 1$.*

In particular, if $\mathrm{Re}(\beta) = c = 0$ then (2.19) converted into fractional differentiation of purely imaginary order. In the case of purely imaginary order the fractional derivatives defined similarly to (2.20) by the formula

$$D^{id} f(t) = \frac{1}{\Gamma(1-id)} \frac{d}{dt} \int_0^t f(x)(t-x)^{-id} dx. \tag{2.21}$$

It is worth noting here that fractional integration cannot be defined as in (2.18) in the way that we have introduced fractional derivative in (2.21) because of the divergence of the integral for $\alpha = id$. So the fractional integration is defined as follows

$$D^{-id} f = \frac{d}{dt} D^{-1-id} f.$$

Therefore,

$$D^{-id} f(t) = \frac{1}{\Gamma(1+id)} \frac{d}{dt} \int_0^t f(x)(t-x)^{id} dx. \tag{2.22}$$

Now in order to complete the definition for all complex order we have identity operator as follows

$$D^0 f = D^{-0} f = f.\text{''} \tag{2.23}$$

# 3 Riemann-Liouville Fractional Integration and Differentiation of Bicomplex Order

In this section, we introduce a new concept of integration and differentiation in which we discuss Riemann-Liouville fractional operators of bicomplex order.

Let $w = z_1 + jz_2 \in \mathbb{C}_2$ ; $z_1, z_2 \in \mathbb{C}(i)$, where $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$. Then using idempotent representation of $w$, we have

$$\begin{aligned} w &= (z_1 - iz_2)e_1 + (z_1 + iz_2)e_2 \\ \text{or } w &= w_1 e_1 + w_2 e_2, \end{aligned} \tag{3.1}$$

where

$$w_1 = z_1 - iz_2 \text{ and } w_2 = z_1 + iz_2 \tag{3.2}$$

$$\Rightarrow w_1 = (x_1 + iy_1) - i(x_2 + iy_2) \text{ and } w_2 = (x_1 + iy_1) + i(x_2 + iy_2)$$

$$\Rightarrow w_1 = (x_1 + y_2) + i(y_1 - x_2) \text{ and } w_2 = (x_1 - y_2) + i(y_1 + x_2),$$

where $e_1$ and $e_2$ have the same meaning as we introduced in (2.10).

We introduce Riemann-Liouville fractional integration of bicomplex order as

$${}_0D_t^{-w} f(t) = \frac{1}{\varGamma(w)} \int_0^t f(x)(t-x)^{w-1} dx, \tag{3.3}$$

where $w = z_1 + jz_2 \in \mathbb{C}_2$ with $\text{Re}(z_1) > |\text{Im}(z_2)|$ and $\varGamma$ is bicomplex gamma function [23]. The definition of Riemann-Liouville fractional integration of bicomplex order is well justified by the following theorem:

**Theorem 3.1.** *Let $w = z_1 + jz_2 \in \mathbb{C}_2$ with $Re(z_1) > |Im(z_2)|$ and let $f$ be a function of class* **C**. *Then for $t > 0$,*

$${}_0D_t^{-w} f(t) = \frac{1}{\varGamma(w)} \int_0^t f(x)(t-x)^{w-1} dx. \tag{3.4}$$

*Proof.* Consider the Riemann-Liouville fractional integral of $f$ of order $\alpha \in \mathbb{C}(i)$ as we defined in (2.18)

$${}_0D_t^{-\alpha} f(t) = \frac{1}{\Gamma(\alpha)} \int_0^t f(x)(t-x)^{\alpha-1} dx. \tag{3.5}$$

Now replacing $\alpha$ by $w \in \mathbb{C}_2$ in the LHS of (3.5) and using representation (3.1), we have

$$\begin{aligned} {}_0D_t^{-w} f(t) &= {}_0D_t^{(-w_1e_1 - w_2e_2)} f(t) \\ &= \{{}_0D_t^{-w_1} f(t)\}e_1 + \{{}_0D_t^{-w_2} f(t)\}e_2 && (3.6) \\ &= \left\{ \frac{1}{\Gamma(w_1)} \int_0^t f(x)(t-x)^{w_1-1} dx \right\} e_1 + \left\{ \frac{1}{\Gamma(w_2)} \int_0^t f(x)(t-x)^{w_2-1} dx \right\} e_2. && (3.7) \end{aligned}$$

Both integrals exist and well define by the hypothesis, because

$$\begin{aligned} &\text{Re}(z_1) > |\text{Im}(z_2)| \\ \Rightarrow\ & x_1 > |y_2| \\ \Rightarrow\ & x_1 > y_2 \text{ and } x_1 > -y_2 \\ \Rightarrow\ & x_1 - y_2 > 0 \text{ and } x_1 + y_2 > 0 \\ \Rightarrow\ & \text{Re}(w_1) > 0 \text{ and } \text{Re}(w_2) > 0. \end{aligned}$$

Hence, integrals defined in (3.7) are meaningful. Now, using the concept of bicomplex gamma function [23] in (3.7), we have

$$\begin{aligned} {}_0D_t^{-w} f(t) &= \frac{1}{\Gamma(w_1e_1 + w_2e_2)} \int_0^t f(x)(t-x)^{w_1e_1+w_2e_2-1} dx \\ &= \frac{1}{\Gamma(w)} \int_0^t f(x)(t-x)^{w-1} dx. \end{aligned}$$

□

*Remark* 3.2. Since the condition $\mathrm{Re}(z_1) > |\mathrm{Im}(z_2)|$ can be seen as $x_1 > |y_2|$. Hence, for such $w \in \mathbb{C}_2$ defined in above theorem let us introduce the set $S = \{w : H_\rho(w)$ represents a right half plane $x_1 > |y_2|\}$, where $H_\rho(w) :=$ Hyperbolic projection of $w$. Geometric interpretation is shown in figure 1.

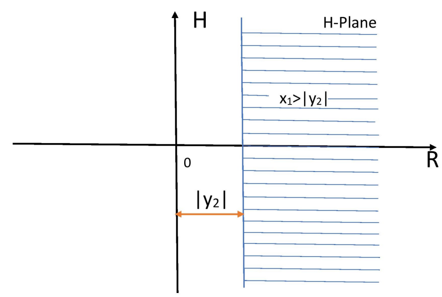


Figure : 1

For simplicity let ${}_0D_t^{-w} = D^{-w}$ in our further discussion.
Let us consider different cases :

1. $w = z_1 + j0 = z_1$,
   in this case (3.4) coincides with the case of complex order (2.18).

2. $w = 0 + jz_2 = jz_2$ with $\mathrm{Im}(z_2) \neq 0$,
   in this case inegration cannot be defined because of the divergence of either integral described in (3.7) (particularly for $y_2 \in \mathbb{Z}$.)

3. $w = 0 + jz_2 = jz_2$ with $\mathrm{Im}(z_2) = 0$,
   this case will be define further by help of differential operator.

4. $w = x_1 + ijy_2 \in \mathbb{D}$,
   for such $w$ integration (3.4) exists if $w \in D_w$, where the new set $D_w$ can be define as : $D_w = \{x_1 + ijy_2 : x_1 > 0, x_1^2 - y_2^2 > 0\}$ shown in figure 2.

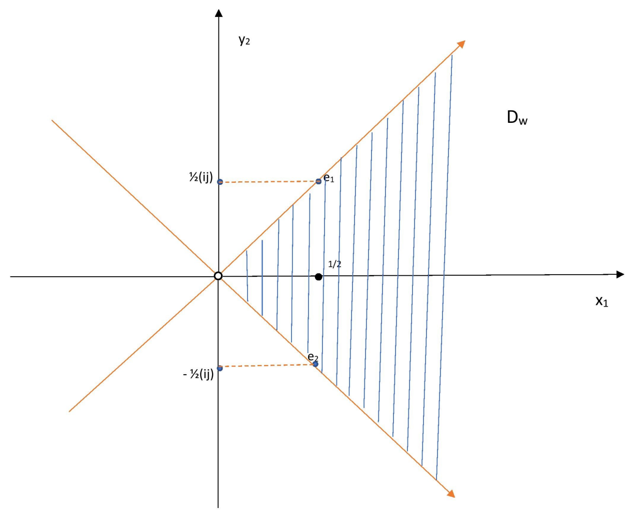


Figure: 2

A bicomplex number defined as $w = z_1 + jz_2$ admits several other forms of representations and based on these representations, we can also demonstrate the condition proposed above in different ways shown in table 1.

| s.n. | **Bicomplex Number** | **Transformed Condition** |
|---|---|---|
| 1 | $w = (x_1 + iy_1) + j(x_2 + iy_2) = z_1 + jz_2$ | $\mathrm{Re}_i(z_1) > \lvert\mathrm{Im}_i(z_2)\rvert$ |
| 2 | $w = (x_1 + jx_2) + i(y_1 + jy_2) = \zeta_1 + i\zeta_2$ | $\mathrm{Re}_j(\zeta_1) > \lvert\mathrm{Im}_j(\zeta_2)\rvert$ |
| 3 | $w = (x_1 + ky_2) + i(y_1 - kx_2) = \rho_1 + i\rho_2$ | $\mathrm{Re}_k(\rho_1) > \lvert\mathrm{Im}_k(\rho_1)\rvert$ |
| 4 | $w = (x_1 + ky_2) + j(x_2 - ky_1) = \sigma_1 + j\sigma_2$ | $\mathrm{Re}_k(\sigma_1) > \lvert\mathrm{Im}_k(\sigma_1)\rvert$ |
| 5 | $w = (x_1 + iy_1) + k(y_2 - ix_2) = \omega_1 + k\omega_2$ | $\mathrm{Re}_i(\omega_1) > \lvert\mathrm{Re}_i(\omega_2)\rvert$ |
| 6 | $w = x_1 + iy_1 + jx_2 + ijy_2$ | $\mathrm{Re}_i(w) > \lvert\mathrm{Im}_{ij}(w)\rvert$ |

Table 1: Conditions for Riemann-Liouville fractional integration according to different way of writing a bicomplex number.

In table 1, $\mathrm{Re}_l(g)$ represents real part of $g$ w.r.t. $l$ and $\mathrm{Im}_l(g)$ represents imaginary part of $g$ w.r.t. $l$, where $l = i, j, k$ and $g = z_1, z_2, \zeta_1, \zeta_2 \cdots$ etc. Now we introduce Riemann-Liouville fractional differentiation of bicomplex order as

$$D^w f(t) = \frac{1}{\Gamma(m-w)} \frac{d^m}{dt^m} \int_0^t f(x)(t-x)^{m-w-1} dx,$$

where $f$ is a function of class $\mathbf{C}$, $w = z_1 + jz_2 = w_1e_1 + w_2e_2 \in \mathbb{C}_2$ with $\mathrm{Re}(z_1) > 0$, $m_1 = \lfloor \mathrm{Re}(w_1) \rfloor + 1$, $m_2 = \lfloor \mathrm{Re}(w_2) \rfloor + 1$, and $m = m_1e_1 + m_2e_2$. The definition of Riemann-Liouville fractional differentiation of bicomplex order is well justified by the following theorem:

**Theorem 3.3.** *Let $f$ be a function of class* **C** *and let $w = z_1 + jz_2 = w_1e_1 + w_2e_2 \in \mathbb{C}_2$ with $Re(z_1) > 0$. Let $m_1 = \lfloor Re(w_1)\rfloor + 1$, $m_2 = \lfloor Re(w_2)\rfloor + 1$ and $m = m_1e_1 + m_2e_2$. Then the fractional derivative of $f$ of order $w$ define as*

$$D^w f(t) = \frac{1}{\Gamma(m-w)}\frac{d^m}{dt^m}\int_0^t f(x)(t-x)^{m-w-1}dx. \tag{3.8}$$

*Proof.* Replacing $\beta$ in LHS of (2.19) by $w \in \mathbb{C}_2$ and using (3.1), we have

$$\begin{aligned} D^w f(t) &= \{D^{w_1e_1+w_2e_2} f(t)\} \\ &= \{D^{w_1} f(t)\}e_1 + \{D^{w_2} f(t)\}e_2. \end{aligned} \tag{3.9}$$

We suppose (i) $m_1 = \lfloor \mathrm{Re}(w_1)\rfloor + 1$ and $m_2 = \lfloor \mathrm{Re}(w_2)\rfloor + 1$
. (ii) $v_1 = m_1 - w_1$ and $v_2 = m_2 - w_2$ to obtain

$$\begin{aligned} D^w f(t) &= \{D^{m_1}\{D^{-v_1} f(t)\}\}e_1 + \{D^{m_2}\{D^{-v_2} f(t)\}\}e_2 \\ &= \left\{\frac{1}{\Gamma(v_1)}\frac{d^{m_1}}{dt^{m_1}}\int_0^t f(x)(t-x)^{v_1-1}dx\right\}e_1 + \left\{\frac{1}{\Gamma(v_2)}\frac{d^{m_2}}{dt^{m_2}}\int_0^t f(x)(t-x)^{v_2-1}dx\right\}e_2, \end{aligned} \tag{3.10}$$

which is meaningful since $\mathrm{Re}(v_1) > 0$ and $\mathrm{Re}(v_2) > 0$ by the definition of $v_1$ and $v_2$. Using properties of idempotent representation, we can write (3.10) as follows.

$$D^w f(t) = \frac{1}{\Gamma(v_1e_1 + v_2e_2)}\frac{d^{(m_1e_1+m_2e_2)}}{dt^{(m_1e_1+m_2e_2)}}\int_0^t f(x)(t-x)^{(v_1e_1+v_2e_2)-1}dx.$$

Putting values of $v_1$ and $v_2$ and let $m = m_1e_1 + m_2e_2$, we have

$$\begin{aligned} D^w f(t) &= \frac{1}{\Gamma((m_1-w_1)e_1 + (m_2-w_2)e_2)}\frac{d^{(m_1e_1+m_2e_2)}}{dt^{(m_1e_1+m_2e_2)}}\int_0^t f(x)(t-x)^{((m_1-w_1)e_1+(m_2-w_2)e_2)-1}dx \\ &= \frac{1}{\Gamma((m_1e_1+m_2e_2) - (w_1e_1+w_2e_2))}\frac{d^{(m_1e_1+m_2e_2)}}{dt^{(m_1e_1+m_2e_2)}}\int_0^t f(x)(t-x)^{((m_1e_1+m_2e_2)-(w_1e_1+w_2e_2))-1}dx \\ &= \frac{1}{\Gamma(m-w)}\frac{d^m}{dt^m}\int_0^t f(x)(t-x)^{m-w-1}dx. \end{aligned}$$

□

Let us consider different cases :

1. $w = z_1 + j0 = z_1$,
   in this case (3.8) coincides with the case of complex order (2.20).

2. $w = 0 + jz_2 = jz_2$ with $\mathrm{Im}(z_2) \neq 0$,
   in this case one component represents integration while other component represents differentiation of complex order in (3.9) and can be solve easily using (2.18) and (2.20).

3. $w = 0 + jz_2 = jz_2$ with $\mathrm{Im}(z_2) = 0$,
   in this case from (3.8), we obtain

$$D^{jx_2} f(t) = \frac{1}{\Gamma(1-jx_2)}\frac{d}{dt}\int_0^t f(x)(t-x)^{-jx_2}dx, \tag{3.11}$$

which is conform to (2.21). In this situation we can define integration as follows

$$D^{-jx_2}f = \frac{d}{dt}D^{-1-jx_2}f.$$

Therefore,

$$D^{-jx_2}f(t) = \frac{1}{\Gamma(1+jx_2)}\frac{d}{dt}\int_0^t f(x)(t-x)^{jx_2}dx. \tag{3.12}$$

4. $w = x_1 + ijy_2 \in \mathbb{D}$,
   in this case result can be derived in the similar way as we did for (3.8).

Now, we will state some examples to calculate the fractional integration and differetiation of bicomplex order.

**Example 3.4.** *Find the Riemann-Liouville fractional integration and differentiation of bicomplex order of $f(t) = t^u$, where $u > -1$.*

*Solution.* **Integration:** Let $w = z_1 + jz_2 = w_1e_1 + w_2e_2 \in \mathbb{C}_2$ with $\mathrm{Re}(z_1) > |\mathrm{Im}(z_2)|$. On putting $f(t) = t^u$ in (3.6), we have

$$D^{-w}t^u = \{D^{-w_1}t^u\}e_1 + \{D^{-w_2}t^u\}e_2. \tag{3.13}$$

Firstly, we deal with $D^{-w_1}t^u$,

$$D^{-w_1}t^u = \frac{1}{\Gamma(w_1)}\int_0^t x^u(t-x)^{w_1-1}dx, \quad w_1 \in \mathbb{C}(i) \tag{3.14}$$

$$= \frac{B(u+1,w_1)}{\Gamma w_1}t^{u+w_1} \qquad [36, \text{p.36}]\ .$$

$$= \frac{\Gamma(u+1)}{\Gamma(u+w_1+1)}t^{u+w_1}; \quad (u+1) > 0, \tag{3.15}$$

where $B$ is complex beta function. Similarly, we can find $D^{-w_2}t^u$ as

$$D^{-w_2}t^u = \frac{\Gamma(u+1)}{\Gamma(u+w_2+1)}t^{u+w_2}; \quad (u+1) > 0. \tag{3.16}$$

On substituting values from (3.15) and (3.16) into (3.13) we get

$$D^{-w}t^u = \left\{\frac{\Gamma(u+1)}{\Gamma(u+w_1+1)}t^{u+w_1}\right\}e_1 + \left\{\frac{\Gamma(u+1)}{\Gamma(u+w_2+1)}t^{u+w_2}\right\}e_2$$

$$= \frac{\Gamma(u+1)}{\Gamma(u+w+1)}t^{u+w}. \tag{3.17}$$

Where we used concept of bicomplex gamma function given in [23]. In particular, for $u = 0$ we have from (3.17)

$$D^{-w}(1) = \frac{\Gamma(1)}{\Gamma(w+1)}t^w$$

$$= \frac{1}{\Gamma(w+1)}t^w. \tag{3.18}$$

Similar to the results obtained for real and complex [36, p.47], here we are also getting the same kind of surprising results under which the bicomplex fractional integral of a constant is not a multiple of $t$.

**Differentiation :** Let $f(t) = t^u$, where $u > -1$ and $w = z_1 + jz_2 = w_1e_1 + w_2e_2 \in \mathbb{C}_2$ with $\text{Re}(z_1) < |\text{Im}(z_2)|$. Then using (3.9), we have

$$\begin{aligned} D^w t^u &= \{D^{w_1} t^u\}e_1 + \{D^{w_2} t^u\}e_2 \\ &= \{D^{m_1}\{D^{-v_1} t^u\}\}e_1 + \{D^{m_2}\{D^{-v_2} t^u\}\}e_2, \end{aligned} \tag{3.19}$$

where $m_1 = \lfloor \text{Re}(w_1) \rfloor + 1,\ m_2 = \lfloor \text{Re}(w_2) \rfloor + 1,\ v_1 = m_1 - w_1,$ and $v_2 = m_2 - w_2$. Now,

$$\begin{aligned} (i)\ D^{w_1} t^u &= D^{m_1}\{D^{-v_1} t^u\} \\ &= D^{m_1}\left\{\frac{\Gamma(u+1)}{\Gamma(u+v_1+1)} t^{u+v_1}\right\} \qquad \text{(By (3.15))} \\ &= \frac{\Gamma(u+1)}{\Gamma(u+v_1+1)} \frac{d^{m_1}}{dx^{m_1}} t^{u+v_1} \\ &= \frac{\Gamma(u+1)}{\Gamma(u+v_1+1)} \cdot \frac{\Gamma(u+v_1+1)}{\Gamma(u+v_1-m_1+1)} t^{(u+v_1-m_1)} \\ &= \frac{\Gamma(u+1)}{\Gamma(u-w_1+1)} t^{u-w_1} \\ \Rightarrow D^{w_1} t^u &= \frac{\Gamma(u+1)}{\Gamma(u-w_1+1)} t^{u-w_1}. \end{aligned} \tag{3.20}$$

Similarly,

$$(ii)\ D^{w_2} t^u = \frac{\Gamma(u+1)}{\Gamma(u-w_2+1)} t^{u-w_2}. \tag{3.21}$$

Putting values from (3.20) and (3.21) into (3.19), we have

$$\begin{aligned} D^w t^u &= \left\{\frac{\Gamma(u+1)}{\Gamma(u-w_1+1)} t^{u-w_1}\right\} e_1 + \left\{\frac{\Gamma(u+1)}{\Gamma(u-w_2+1)} t^{u-w_2}\right\} e_2 \\ &= \frac{\mathit{\Gamma}(u+1)}{\mathit{\Gamma}(u-w+1)} t^{u-w}. \end{aligned} \tag{3.22}$$

In particular, for $u = 0$ we have

$$\begin{aligned} D^w(1) &= \frac{\mathit{\Gamma}(1)}{\mathit{\Gamma}(-w+1)} t^{-w} \\ &= \frac{1}{\mathit{\Gamma}(1-w)} t^{-w} \end{aligned} \tag{3.23}$$

which is of course not zero. Again, we get very different results in which bicomplex fractional differentiation of a constant is not zero.

**Example 3.5.** *Find the Riemann-Liouville fractional integration and differentiation of $f(t) = \log t$ of bicomplex order.*

*Solution.* Let us start with integration,

**Integration:** Let $w = z_1 + jz_2 = w_1e_1 + w_2e_2 \in \mathbb{C}_2$ with $\text{Re}(z_1) > |\text{Im}(z_2)|$.

$$D^{-w} \log t = \{D^{-w_1} \log t\}e_1 + \{D^{-w_2} \log t\}e_2 \tag{3.24}$$

Consider $D^{-w_1}\log t$,

$$D^{-w_1}\log t = \frac{1}{\Gamma(w_1)}\int_0^t \log y(t-y)^{w_1-1}dy.$$

If we make the change of variable $y = tx$, then

$$D^{-w_1}\log t = \frac{t^{w_1}}{\Gamma(w_1+1)}\log t + \frac{t^{w_1}}{\Gamma(w_1)}\int_0^1 (1-x)^{w_1-1}\log x dx. \tag{3.25}$$

Since

$$\int_0^1 p^{\alpha-1}(1-p)^{\beta-1}\log p dp = B(\alpha,\beta)\left[\psi(\alpha)-\psi(\alpha+\beta)\right]; \quad \mathrm{Re}(\alpha), \mathrm{Re}(\beta) > 0, \tag{3.26}$$

where $\psi$ is the complex digamma function. Thus if we let $\alpha = 1$ in (3.26),

$$\begin{aligned}\int_0^1 (1-p)^{\beta-1}\log p dp &= B(1,\beta)\left[\psi(1)-\psi(1+\beta)\right]\\ &= B(1,\beta)\left[-\gamma-\psi(1+\beta)\right],\end{aligned} \tag{3.27}$$

where $\gamma$ is Euler's constant. Using (3.27) into (3.25), we have

$$D^{-w_1}\log t = \frac{t^{w_1}}{\Gamma(w_1+1)}\left[\log t - \gamma - \psi(w_1+1)\right]. \tag{3.28}$$

Similarly, we can find $D^{-w_2}\log t$ as

$$D^{-w_2}\log t = \frac{t^{w_2}}{\Gamma(w_2+1)}\left[\log t - \gamma - \psi(w_2+1)\right]. \tag{3.29}$$

Substituting (3.28) and (3.29) into (3.24), we have

$$\begin{aligned}D^{-w}\log t &= \left\{\frac{t^{w_1}}{\Gamma(w_1+1)}\left[\log t-\gamma-\psi(w_1+1)\right]\right\}e_1 + \left\{\frac{t^{w_2}}{\Gamma(w_2+1)}\left[\log t-\gamma-\psi(w_2+1)\right]\right\}e_2\\ &= \frac{t^w}{\mathit{\Gamma}(w+1)}\left[\log t-\gamma-\Psi(w+1)\right],\end{aligned} \tag{3.30}$$

where $\Psi$ is bicomplex digamma function.
**Differentiation :**
In the similar manner for $w = z_1 + jz_2 = w_1e_1 + w_2e_2 \in \mathbb{C}_2$ with $\mathrm{Re}(z_1) < 1 - |\mathrm{Im}(z_2)|$, we can obtain

$$D^w \log t = \frac{t^{-w}}{\mathit{\Gamma}(1-w)}[\log t - \gamma - \Psi(1-w)]. \tag{3.31}$$

Riemann-Liouville integration and differentiation of appropriate bicomplex order of some elementary functions are given in table 2.

*Remark* 3.6. For $E_t(w,a)$, $C_t(w,a)$, and $S_t(w,a)$ see e.g. [36, p.314-320].

*Remark* 3.7. For $E_t(-w,a)$, $C_t(-w,a)$, and $S_t(-w,a)$ see e.g. [36, p.48-49].

| S.N. | **Function** | $D^{-w}$ **Integration** | $D^{w}$ **Differentiation** |
|---|---|---|---|
| 1 | $t^u \log t$ | $\frac{\Gamma(u+1)t^{w+u}}{\Gamma(u+w+1)}[\log t$ $+\Psi(u+1)-\Psi(u+w+1)]$ | $\frac{\Gamma(u+1)}{\Gamma(u-w+1)}t^{u-w}[\log t$ $+\Psi(u+1)-\Psi(u-w+1)]$ |
| 2 | $e^{at}$, $a \in \mathbb{R}$ | $D^{-w}e^{at} = t^w e^{at}\gamma^*(w, at)$ $= E_t(w, a)$ | $D^w e^{at} = E_t(-w, a)$ |
| 3 | $\cos at$, $a \in \mathbb{R}$ | $D^{-w}\cos at = C_t(w, a)$ $= \frac{1}{\Gamma(w)}\int_0^t x^{(w-1)}\cos a(t-x)dx$ | $D^w \cos at = C_t(-w, a)$ |
| 4 | $\sin at$, $a \in \mathbb{R}$ | $D^{-w}\sin at = S_t(w, a)$ $= \frac{1}{\Gamma(w)}\int_0^t x^{(w-1)}\sin a(t-x)dx$ | $D^w \sin at = S_t(-w, a)$. |

Table 2: Riemann-Liouville integration and differentiation of bicomplex order of some elementary functions.

# 4 Some Properties of Riemann-Liouville Fractional Operators of Bicomplex Order:

In this section, we discuss some useful properties viz. the law of exponent and Leibniz rule for Riemann-Liouville integral and differential operators.

**Theorem 4.1.** *Let $f(t)$ and $g(t)$ be piecewise continuous functions on $J' = (0, \infty)$ and integrable on any finite subinterval of $J = [0, \infty)$ and $w = z_1 + jz_2 = w_1e_1 + w_2e_2 \in \mathbb{C}_2$ with $Re(z_1) > 0$. Then for $t > 0$,*

$$D^w(f(t) + g(t)) = D^w f(t) + D^w g(t).$$

*Proof.*

$$\begin{aligned}
D^w(f(t)+g(t)) &= D^{(w_1e_1+w_2e_2)}(f(t)+g(t)) \\
&= \{D^{w_1}(f(t)+g(t))\}e_1 + \{D^{w_2}(f(t)+g(t))\}e_2 \\
&= \{D^{m_1}\{D^{-v_1}(f(t)+g(t))\}\}e_1 + \{D^{m_2}\{D^{-v_2}(f(t)+g(t))\}\}e_2 \\
&= \{\{D^{m_1}\{D^{-v_1}f(t)\}\} + \{D^{m_1}\{D^{-v_1}g(t)\}\}\}e_1 \\
&\quad + \{\{D^{m_2}\{D^{-v_2}f(t)\}\} + \{D^{m_2}\{D^{-v_2}g(t)\}\}\}e_2 \\
&= \{\{\{D^{m_1}\{D^{-v_1}f(t)\}\}e_1 + \{\{D^{m_2}\{D^{-v_2}f(t)\}\}e_2\} \\
&\quad + \{\{\{D^{m_1}\{D^{-v_1}g(t)\}\}e_1 + \{\{D^{m_2}\{D^{-v_2}g(t)\}\}e_2\} \\
&= \{\{D^{w_1}f(t)\}e_1 + \{D^{w_2}f(t)\}e_2\} + \{\{D^{w_1}g(t)\}e_1 + \{D^{w_2}g(t)\}e_2\} \\
&= D^{(w_1e_1+w_2e_2)}f(t) + D^{(w_1e_1+w_2e_2)}g(t) \\
&= D^w f(t) + D^w g(t).
\end{aligned}$$

where $m_1,\ m_2,\ v_1$, and $v_2$ are same as taken before. □

**Theorem 4.2.** *Let $f(t)$ and $g(t)$ be piecewise continuous functions on $J' = (0, \infty)$ and integrable on any finite subinterval of $J = [0, \infty)$ and $w = z_1 + jz_2 \in \mathbb{C}_2$ with $Re(z_1) > |Im(z_2)|$. Then for $t > 0$*

$$D^{-w}(f(t) + g(t)) = D^{-w} f(t) + D^{-w} g(t).$$

**Theorem 4.3** (The law of exponents for integration)**.** *Let $f$ be a piecewise continuous function on $J$. Let $w = z_1 + jz_2 = w_1e_1 + w_2e_2 \in \mathbb{C}_2$ and $\xi = z_3 + jz_4 = \xi_1e_1 + \xi_2e_2 \in \mathbb{C}_2$ with $Re(w_1), Re(w_2), Re(\xi_1), Re(\xi_2) > 0$ and $Re(w_1 + \xi_1), Re(w_2 + \xi_2) > 0$. Then*

$$D^{-w}[D^{-\xi}f(t)] = D^{-(\xi+w)}f(t) = D^{-\xi}[D^{-w}f(t)]. \tag{4.1}$$

*Proof.* By applying definition of integration :

$$\begin{aligned}
\textbf{(i)} \quad D^{-w}[D^{-\xi}f(t)] &= \frac{1}{\Gamma(w)}\int_0^t (t-x)^{w-1}\left[\frac{1}{\Gamma(\xi)}\int_0^x (x-y)^{\xi-1}f(y)dy\right]dx \\
&= \frac{1}{\Gamma w\Gamma\xi}\int_0^t (t-x)^{w-1}dx\int_0^x (x-y)^{\xi-1}f(y)dy \\
&= \left\{\frac{1}{\Gamma w_1\Gamma\xi_1}\int_0^t (t-x)^{w_1-1}dx\int_0^x (x-y)^{\xi_1-1}f(y)dy\right\}e_1 \\
&\quad + \left\{\frac{1}{\Gamma w_2\Gamma\xi_2}\int_0^t (t-x)^{w_2-1}dx\int_0^x (x-y)^{\xi_2-1}f(y)dy\right\}e_2.
\end{aligned} \tag{4.2}$$

Now,

$$\begin{aligned}
\frac{1}{\Gamma w_1\Gamma\xi_1}\int_0^t (t-x)^{w_1-1}dx\int_0^x (x-y)^{\xi_1-1}f(y)dy &= \frac{1}{\Gamma w_1\Gamma\xi_1}\int_0^t\int_0^x (t-x)^{w_1-1}(x-y)^{\xi_1-1}f(y)dxdy \\
&= \frac{1}{\Gamma w_1\Gamma\xi_1}\int_0^t\int_y^t (t-x)^{w_1-1}(x-y)^{\xi_1-1}f(y)dydx \\
&= \frac{1}{\Gamma w_1\Gamma\xi_1}\int_0^t f(y)dy\int_y^t (t-x)^{w_1-1}(x-y)^{\xi_1-1}dx \\
&= \frac{1}{\Gamma w_1\Gamma\xi_1}B(w_1,\xi_1)\int_0^t (t-y)^{w_1+\xi_1-1}f(y)dy \\
&= \frac{1}{\Gamma(w_1+\xi_1)}\int_0^t (t-y)^{w_1+\xi_1-1}f(y)dy,
\end{aligned} \tag{4.3}$$

where $B$ is complex beta function. Similarly,

$$\frac{1}{\Gamma w_2\Gamma\xi_2}\int_0^t (t-x)^{w_2-1}dx\int_0^x (x-y)^{\xi_2-1}f(y)dy = \frac{1}{\Gamma(w_2+\xi_2)}\int_0^t (t-y)^{w_2+\xi_2-1}f(y)dy. \tag{4.4}$$

Using (4.3) and (4.4) into (4.2), we have

$$\begin{aligned}
D^{-w}[D^{-\xi}f(t)] &= \left\{\frac{1}{\Gamma(w_1+\xi_1)}\int_0^t (t-y)^{w_1+\xi_1-1}f(y)dy\right\}e_1 + \left\{\frac{1}{\Gamma(w_2+\xi_2)}\int_0^t (t-y)^{w_2+\xi_2-1}f(y)dy\right\}e_2 \\
&= \frac{1}{\Gamma(w+\xi)}\int_0^t (t-y)^{w+\xi-1}f(y)dy \\
&= D^{-\xi}[D^{-w}f(t)].
\end{aligned}$$

Hence,

$$D^{-w}[D^{-\xi}f(t)] = \frac{1}{\Gamma(\xi+w)}\int_0^t (t-y)^{\xi+w-1}f(y)dy = D^{-\xi}[D^{-w}f(t)] \tag{4.5}$$

$$\textbf{(ii)} \quad D^{-(\xi+w)}f(t) = \frac{1}{\Gamma(\xi+w)}\int_0^t (t-y)^{\xi+w-1}f(y)dy. \tag{4.6}$$

(4.5) and (4.6) now implies the truth of (4.1) . ∎

The similar property can be proved for differentiation by imposing some additional restrictions on $f$. Before this let us define a new class of functions $\acute{\mathbf{C}}$ which contains all the functions $f(t)$ of the form

$$t^{\lambda}\eta(t) \text{ or } t^{\lambda}\log(t)\eta(t) \tag{4.7}$$

where $\lambda > -1$ and $\eta(t) = \sum_{n=0}^{\infty} a_n t^n$ has a radius of convergence $R > 0$.

**Theorem 4.4** (The law of exponents for differentiation)**.** *Consider* $w = w_1 e_1 + w_2 e_2 \in \mathbb{C}_2$ *and* $\xi = \xi_1 e_1 + \xi_2 e_2 \in \mathbb{C}_2$ *with* $Re(w_1), Re(w_2), Re(\xi_1), Re(\xi_2) < \lambda + 1$ *and* $Re(w_1 + \xi_1) < \lambda + 1, Re(w_2 + \xi_2) < \lambda + 1$. *Let* $f(t)$ *be of class* $\acute{\mathbf{C}}$ *and* $X$ *be a positive number less than* $R$. *Then*

$$D^{w}\left[D^{\xi} f(t)\right] = D^{(w+\xi)} f(t) = D^{\xi}\left[D^{w} f(t)\right] \tag{4.8}$$

*for all* $t \in (0, X]$.

*Proof.* If $f(t) = t^{\lambda}\eta(t)$, then from (3.22)

$$D^{\xi} f(t) = t^{\lambda-\xi} \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-\xi)} t^n, \tag{4.9}$$

and if $f(t) = t^{\lambda}\log(t)\eta(t)$, then from table 2,

$$\begin{aligned} D^{\xi} f(t) &= t^{\lambda-\xi}(\log t) \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-\xi)} t^n \\ &\quad + t^{\lambda-\xi} \sum_{n=0}^{\infty} a_n \left[\Psi(n+\lambda+1) - \Psi(n+\lambda+1-\xi)\right] \times \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-\xi)} t^n. \end{aligned} \tag{4.10}$$

Since, by hypothesis $\mathrm{Re}(\xi_1), \mathrm{Re}(\xi_2) < \lambda + 1$, $D^{\xi} f(t) \in \acute{\mathbf{C}}$ in both cases.
Thus, from (4.9)

$$\begin{aligned} D^{w}\left[D^{\xi} f(t)\right] &= \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-\xi)} \times \left[\frac{\Gamma(n+\lambda+1-\xi)}{\Gamma(n+\lambda+1-\xi-w)}\right] t^{n+\lambda-\xi-w} \\ &= \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-(\xi+w))} t^{n+\lambda-(\xi+w)} \end{aligned} \tag{4.11}$$

which is precisely $D^{w+\xi}f(t)$. From (4.10) and then using table 2, we have

$$\begin{aligned} D^{w}\left[D^{\xi}f(t)\right] &= \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-\xi)} \frac{\Gamma(n+\lambda+1-\xi)}{\Gamma(n+\lambda+1-\xi-w)} \\ &\quad \times \left[\log t + \Psi(n+\lambda+1-\xi) - \Psi(n+\lambda+1-(\xi+w))\right] t^{n+\lambda-(\xi+w)} \\ &\quad + \sum_{n=0}^{\infty} a_n \left[\Psi(n+\lambda+1) - \Psi(n+\lambda+1-\xi)\right] \times \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-\xi)} \\ &\quad \times \left[\frac{\Gamma(n+\lambda+1-\xi)}{\Gamma(n+\lambda+1-(\xi+w))} t^{n+\lambda-(\xi+w)}\right] \\ &= \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-(\xi+w))} \\ &\quad \times \left[\log t + \left\{\Psi(n+\lambda+1-\xi) - \Psi(n+\lambda+1-(\xi+w))\right\}\right. \\ &\quad \left. + \left\{\Psi(n+\lambda+1) - \Psi(n+\lambda+1-\xi)\right\}\right] \times t^{n+\lambda-(\xi+w)} \qquad (4.12) \\ &= t^{\lambda-(\xi+w)}(\log t) \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-(\xi+w))} t^n + t^{\lambda-(\xi+w)} \\ &\quad \times \sum_{n=0}^{\infty} a_n \frac{\Gamma(n+\lambda+1)}{\Gamma(n+\lambda+1-(\xi+w))} \times \left[\Psi(n+\lambda+1) - \Psi(n+\lambda+1-(\xi+w))\right] t^n, \end{aligned} \tag{4.13}$$

which is again precisely $D^{\xi+w}f(t)$. In the similar manner we can prove another equality $D^{(w+\xi)}f(t) = D^{\xi}\left[D^{w}f(t)\right]$. Thus the proof is complete. □

**Theorem 4.5** (Leibniz rule for integration)**.** *Let $f$ be continuous on [0, X] and let $g$ be analytic at $a$ for all $a \in [0, X]$. Then for $w = z_1 + jz_2 = w_1e_1 + w_2e_2 \in \mathbb{C}_2$ with $Re(z_1) > 1 + |Im(z_2)|$ and $0 < t \leq X$,*

$$D^{-w}\left[f(t)g(t)\right] = \sum_{k=0}^{\infty} \binom{-w}{k} \left[D^{k}g(t)\right] \left[D^{-w-k}f(t)\right]. \tag{4.14}$$

*Proof.* Firstly, we see that if $f$ is continuous on $[0, X]$ and $g$ is analytic at $a$ for all $a \in [0, X]$, then $fg$ is certainly of class $\mathbf{C}$ and for $\mathrm{Re}(z_1) > 1 + |\mathrm{Im}(z_2)|$, the fractional integral

$$D^{-w}\left[f(t)g(t)\right] = \frac{1}{\Gamma w} \int_0^t (t-x)^{w-1} f(x)g(x)dx \qquad \text{exists.} \tag{4.15}$$

Since $g$ is analytic function, we may write its Taylor series as

$$\begin{aligned} g(x) &= \sum_{k=0}^{\infty} (-1)^k \frac{D^k g(t)}{k!} (t-x)^k \\ &= g(t) + \sum_{k=1}^{\infty} (-1)^k \frac{D^k g(t)}{k!} (t-x)^k. \end{aligned} \tag{4.16}$$

The series (4.16) converges for all $x$ in an interval that properly contains $[0, t]$ and hence uniformly on $[0,t]$. Substituting (4.16) into (4.15) we have

$$\begin{aligned} D^{-w}\left[f(t)g(t)\right] &= \frac{1}{\Gamma w}\int_0^t (t-x)^{w-1}f(x)\left\{g(t)+\sum_{k=1}^{\infty}(-1)^k\frac{D^kg(t)}{k!}(t-x)^k\right\}dx \\ &= g(t)\left\{\frac{1}{\Gamma w}\int_0^t (t-x)^{w-1}f(x)dx\right\}+\frac{1}{\Gamma w}\int_0^t (t-x)^{w-1}f(x)\times\sum_{k=1}^{\infty}(-1)^k\frac{D^kg(t)}{k!}(t-x)^kdx. \\ &= g(t)D^{-w}f(t)+\frac{1}{\Gamma w}\int_0^t (t-x)^{w-1}f(x)\times\sum_{k=1}^{\infty}(-1)^k\frac{D^kg(t)}{k!}(t-x)^kdx. \qquad (4.17) \end{aligned}$$

Since $f$ is continuous on $[0,X]$ and $\mathrm{Re}(z_1)>1+|\mathrm{Im}(z_2)|$, therefore, $(t-x)^{w-1}f(x)$ is bounded on $[0,t]$. Hence we may interchange the order of integration and summation in (4.17) to obtain

$$\begin{aligned} D^{-w}\left[f(t)g(t)\right] &= \sum_{k=0}^{\infty}(-1)^k\frac{D^kg(t)}{k!}\int_0^t (t-x)^{w+k-1}f(x)dx \\ &= \sum_{k=0}^{\infty}(-1)^k\frac{\Gamma(w+k)}{k!\Gamma(w)}\left[D^kg(t)\right]\left[D^{-w-k}f(t)\right] \\ &= \sum_{k=0}^{\infty}\binom{-w}{k}\left[D^kg(t)\right]\left[D^{-w-k}f(t)\right] \qquad [36, p.298] \qquad (4.18) \end{aligned}$$

□

*Remark* 4.6. The only reason we assumed $g$ analytic for all points $a$ in $[0,X]$ was to guarantee the uniform convergence of series (4.16) for $x\in[0,X]$.

Now we shall attempt to prove analogous results for fractional derivatives.

**Theorem 4.7** (Leibniz rule for differentiation)**.** *Let $f$ and $g$ be two functions of the class $\acute{\mathbf{C}}$. Then for $w=z_1+jz_2=w_1e_1+w_2e_2\in\mathbb{C}_2$ with $Re(z_1)>|Im(z_2)|$*

$$D^{w}\left[f(t)g(t)\right]=\sum_{k=0}^{\infty}\binom{w}{k}\left[D^kg(t)\right]\left[D^{w-k}f(t)\right]. \qquad (4.19)$$

*Proof.* If $f,g\in\acute{\mathbf{C}}$ then by replacing $-w$ with $w$ in (4.18) , we obtain the Leibniz rule for differentiation. □

Apart from these, there are some other properties which we can get from mere observation. Some of these are as follows :

- Both fractional integral and differential operators are non-local operator (as it is defined on an interval).
- Calculating time-fractional derivative of a function $f(t)$ at some $t=t_1$ requires all the past history, i.e. all $f(t)$ from $t=0$ to $t=t_1$.
- Calculating space-fractional derivative of a function $f(x)$ at $x=x_0$ requires all non-local values of $f(x)$.

# 5 Conclusion

In this paper, we have performed some detailed analysis of the Riemann-Liouville fractional integral and differential operators of bicomplex order. We demonstrated these operators by calculating some examples. We established some important properties of these operators including Leibniz and chain rules.

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