New Lump Solutions for Spatio-Temporal Dispersion (1+1)-Dimensional Ito-Equation

Maha S.M .Shehata [1], Ahmet Bekir [2] , Emad H.M. Zahran [3]

[1] Department of Mathematics, Zagazig University, Faculty of Science, Zagazig, Egypt;
[2]Neighbourhood of Akcaglan, Imarli Street, Number: 28/4, 26030, Eskisehir, Turkey
[3]Departments of Mathematical and Physical Engineering, Benha University, Faculty of Engineering, Shubra, Egypt

ABSTRACT

From point of view of two different schemas several new impressive lump solutions to (1+1)-dimensional Ito equation have been established. The first schema is the Paul-Painleve approach method (PPAM) which will be applied perfectly to extract multiple lump solutions of this model, while the second schema is the famous one of the ansatze method and has personal profile named the Ricatti-Bernolli Sub-ODE method. In related subject the numerical solutions corresponding to all lump solutions achieved via each method have been demonstrated individually in the framework of the variational iteration method (VIM).

Keywords: The spatio-temporal dispersion (1+1)-dimensional Ito equation; the Paul-Painleve approach method; the Ricatti-Bernolli Sub-ODE method; the variational iteration method; Lump solutions; numerical solutions.

1. Introduction

In mathematics, Ito, s calculus is used to find a time-dependent function of a stochastic process, it seems as the stochastic calculus counterpart of the chain rule and the best well known application for it is the derivation of the Black-Scholes equation for option values. Moreover, Ito diffusion is a solution to a specific type of stochastic differential equation. That equation is similar to the Langevin equation used in physics to describe the Brownian motion of a particle subjected to a potential in a viscous fluid. Ito quantum stochastic differential equation, Ito for white noise analysis and in quantum stochastic calculus represent general form of Ito calculus and in the last few decades it has same application as the renormalization problem in physics and the representation theory of Lie algebras.

The constructed (1+1)-dimensionally Ito equation is the general form of the bilinear KdV equation which plays vital role in many phenomena arising in various branches of nonlinear science . In the last few decades some studied are demonstrated for this model see for example Zabusky and Kruskal [1] who observed unusual nonlinear interactions among "solitary-wave pulses" propagating in nonlinear dispersive media when he established the numerical solutions of the Korteweg-de Vries equation which describe the one-dimensional long-time asymptotic behavior of small but finite amplitude, Ito [2] who extracted the N -soliton solution and the inverse scattering form for the higher order Sawada-Kotera equation as well as the N -soliton solutions, the Bäcklund transformation and the inverse scattering form for higher order modified KdV equation. In related subject recent studies to this model have been established through significant articles see for example; Bhrawy, et al. [3] who applied the extended F-expansion method based on computerized symbolic computation technique to extracting the hyperbolic and triangular solutions for the (1+1)-dimensional and (2+1)-dimensional Ito equations, Hu, et al. [4] who extracted multiple cosh-solutions of the (1+1)-dimension Ito equation using the systematic method, Liu, et al. [5] who achieved two

classes of mixed type exact solutions to the (2+1)-dimensional Ito-equation using the test function based on the Hirota's bilinear form, Ma, [6] who extracted abounded exact interaction solutions, including lump-soliton, lump kink and lump periodic solutions of the Hirota-Satsuma-Ito-equation via conducting symbolic computations with maple of Hirota bilinear form.
Various significant articles have been established to discuss the soliton dynamics of different non-linear evolution equations [8-28].

The main target of this work, implementing the PPAM [29-32] to achieve distinct types of lump solutions to the (1+1)-dimensionally Ito-equation as well as other various types of lump solutions via RBSOM [33]. In addition the numerical solutions of this model have been established using the VIM [34-36].

This article is organized as follows: in Section 2, we will describe the PPAM and implement its application in Section 3. In Section 4, we will propose the mathematical analysis of RBSOM and implement its application in section 5. In Section 6, the VIM is presented and its applications to construct the numerical solutions are established in Section 7. In Section 8 briefly conclusions has been drawn.

2. The PPAM

To discuss the mathematical analysis of this method, let us firstly invesitigate the general form of any nonlinear evolution equation hence, let us introduce H as a function of $\varphi(x,t)$ and its partial derivatives as,

$$H(\varphi,\varphi_x,\varphi_t,\varphi_{xx},\varphi_{tt},......)=0 \tag{1}$$

that contained the highest order derivatives and nonlinear terms. This equation under the transformation $\varphi(x,t)=\varphi(\xi),\xi=x-wt$ will be converted to the this ODE:

$$Z(\varphi',\varphi'',\varphi''',......)=0 \tag{2}$$

where, Z is a function related to $\varphi(\xi)$ and its total derivatives, while $' = \frac{d}{d\xi}$.
The exact solution for equation (2) in the framework of PPAM [29-32] can be proposed as,

$$\varphi(\xi)=A_0+A_1R(X)e^{-N\xi}, X=S(\xi) \tag{3}$$

or

$$\varphi(\xi)=A_0+A_1R(X)e^{-N\xi}+A_2R^2(X)e^{-2N\xi}, X=S(\xi) \tag{4}$$

where $X=S(\xi)=C_1-\frac{e^{-N\xi}}{N}$, and $R(X)$ in equations (3) and (4) satisfy the Riccati equation in the form $R_X-AR^2=0$ and its solution is,

$$R(X)=\frac{1}{AX+X_0} \tag{5}$$

Consequently,

$$\varphi_\xi=-N\,e^{-N\xi}R+S_\xi e^{-N\xi}R_X \tag{6}$$

$$\varphi_{\xi\xi} = N^2 e^{-N\xi} R - 2NS_{\xi} e^{-N\xi} R_X + S_{\xi\xi} e^{-N\xi} R_X + S_{\xi}^2 e^{-N\xi} R_{XX} \tag{7}$$

$$\begin{aligned}\varphi_{\xi\xi\xi} =& - N^3 Re^{-N\xi} + 3N^2 R_X S_{\xi} e^{-N\xi} - 3NR_{XX} S_{\xi} e^{-N\xi} - \\ & 3NR_X^2 S_{\xi\xi} e^{-N\xi} + 3R_X R_{XX} S_{\xi\xi} + R_{XXX} S_{\xi} e^{-N\xi} + R_X^3 S_{\xi\xi\xi} e^{-N\xi}\end{aligned} \tag{8}$$

3. Application

We will implement an efficient solver to retrieve lump solutions to the (1+1)-dimensionally Ito-equation in the framework of the PPAM.

The (1+1)-dimensionally Ito-equation which is the general form of the bilinear KdV equation [1-3] can be written as,

$$v_{tt} + v_{xxxt} + 3(v_x v_t + v v_{xt}) + 3v_{xx} \int_{\infty}^{x} v_t dx^j = 0 \tag{9}$$

Under the effect of this transformation $v = \varphi_x$ the last equation became,

$$\varphi_{xtt} + \varphi_{xxxxt} + 3(\varphi_{xx}\varphi_{xt} + \varphi_x \varphi_{xxt}) + 3\varphi_{xxx}\varphi_t = 0 \tag{10}$$

When one seek to choose $\xi = x - wt$ that represents the traveling wave solutions with moving coordinate and substitute into equation (10) it will be reduced to nonlinear ordinary differential equation of the form,

$$-w\varphi''' + \varphi^{(v)} - 3w(\varphi''\varphi'' + \varphi'\varphi''') - 3w\varphi'''\varphi' = 0 \tag{11}$$

Integrating Equation (11) once we obtain,

$$\varphi''' + 3\varphi'^2 - w\varphi' = 0 \, . \tag{12}$$

Via inserting $\psi = \varphi'$ we get,

$$\psi'' + 3\psi^2 - w\psi = 0 \, . \tag{13}$$

The homogenous balance applied between ψ'', ψ^2 implies $M + 2 = 2M \Rightarrow M = 2$ hence, the solution according to the proposed method is;

$$\psi(\xi) = A_0 + A_1 \varphi e^{-N\xi} + A_2 \varphi^2 e^{-2N\xi} \, . \tag{14}$$

Consequently,

$$\psi' = - NA_1 e^{-N\xi} \varphi - (AA_1 + 2A_2 N) e^{-2N\xi} \varphi^2 - 2AA_2 e^{-3N\xi} \varphi^3 \tag{15}$$

$$\begin{aligned}\psi'' =& 6A^2 A_2 e^{-4N\xi} \varphi^4 + (2A^2 A_1 + 10AA_2 N) e^{-3N\xi} \varphi^3 + \\ & (3AA_1 N + 2A_2 N^2) e^{-2N} \varphi^2 + N^2 A_1 e^{-N\xi} \varphi\end{aligned} \tag{16}$$

Substitute about ψ', ψ'' into Eq. (13) and equating the coefficients of various powers of $e^{-N\xi}\varphi(x)$ to zero; we obtain this system of equations,

$$
\begin{aligned}
&2A^2A_1 + A_2^2 = 0,\\
&2A^2A_1 + 10AA_2N + 6A_1A_2 = 0,\\
&3AA_1N + 4A_2N^2 + 3A_1^2 + 6A_0A_2 - wA_2 = 0,\\
&N^2 + 6A_0 - w = 0,\\
&3A_0^3 - wA_0 = 0.
\end{aligned} \tag{17}
$$

From the last equation of the system (17) it is clear that $w = 3A_0^2$, by substituting about it in the third and fourth equations of this system and solve it we get many complicated results from which only four results are valid and the remaining will be refused because in these results either $A = 0$ or $A_1 = A_2 = 0$ and for simplicity we take $A_0 = 1$.

The valid results are,

$$
\begin{aligned}
&(1)\quad \mathrm{A} = \frac{3i}{11}\sqrt{1933+1116\sqrt{3}},\, N = \frac{i}{396}\left(3877\sqrt{1933+1116\sqrt{3}} - (1933+1116\sqrt{3})^{\frac{3}{2}}\right),\\
&A_1 = 2\left(7+4\sqrt{3}\right),\, A_2 = \frac{6}{11}\left(-116-67\sqrt{3}\right)
\end{aligned} \tag{18}
$$

$$
\begin{aligned}
&(2)\quad \mathrm{A} = \frac{-3i}{11}\sqrt{1933+1116\sqrt{3}},\, N = \frac{-i}{396}\left(3877\sqrt{1933+1116\sqrt{3}} - (1933+1116\sqrt{3})^{\frac{3}{2}}\right),\\
&A_1 = 2\left(7+4\sqrt{3}\right),\, A_2 = \frac{6}{11}\left(-116-67\sqrt{3}\right)
\end{aligned} \tag{19}
$$

$$
\begin{aligned}
&(3)\quad A_1 = \frac{2}{279}\left(20+\frac{121}{1933+1116\sqrt{3}}\right),\, N = \frac{-i}{36}\left(\frac{121}{(1933+1116\sqrt{3})^{\frac{3}{2}}} - \frac{3877}{\sqrt{1933+1116\sqrt{3}}}\right),\\
&A_2 = \frac{2}{279}\left(\frac{3200+\frac{1361613}{(1933+1116\sqrt{3})^3} - \frac{43285693}{(1933+1116\sqrt{3})^2} - \frac{7172500}{1933+1116\sqrt{3}}}{51894}\right),\, \mathrm{A} = \frac{3i}{\sqrt{1933+1116\sqrt{3}}}
\end{aligned} \tag{20}
$$

$$
\begin{aligned}
&(4)\quad A_1 = \frac{2}{279}\left(20+\frac{121}{1933+1116\sqrt{3}}\right),\, N = \frac{i}{36}\left(\frac{121}{(1933+1116\sqrt{3})^{\frac{3}{2}}} - \frac{3877}{\sqrt{1933+1116\sqrt{3}}}\right),\\
&A_2 = \frac{2}{279}\left(\frac{3200+\frac{1361613}{(1933+1116\sqrt{3})^3} - \frac{43285693}{(1933+1116\sqrt{3})^2} - \frac{7172500}{1933+1116\sqrt{3}}}{51894}\right),\, \mathrm{A} = \frac{-3i}{\sqrt{1933+1116\sqrt{3}}}
\end{aligned} \tag{21}
$$

For simplicity and similarity we will take only one result of these results and extracting the corresponding solutions and plot it say the firs and the first result.

For the first results which is,

$$\text{A} = \frac{3i}{11}\sqrt{1933+1116\sqrt{3}}, N = \frac{i}{396}\left(3877\sqrt{1933+1116\sqrt{3}} - (1933+1116\sqrt{3})^{\frac{3}{2}}\right),$$

$$A_1 = 2\left(7+4\sqrt{3}\right), A_2 = \frac{6}{11}\left(-116-67\sqrt{3}\right)$$

This result can be simplified to be,

$$\text{A} = 17\text{i}, N = 2\text{i}, A_1 = 28, A_2 = -127 \tag{22}$$

Thus the solution is,

$$\psi(\xi) = A_0 + A_1 R(X)e^{-N\xi} + A_2 R^2(X)e^{-2N\xi},$$

$$\varphi(\xi) = A_0 + A_1\left(\frac{1}{AX+X_0}\right)e^{-N\xi} + A_2\left(\frac{1}{AX+X_0}\right)^2 e^{-2N\xi} \tag{23}$$

Where $X = C_1 - \frac{e^{-N\xi}}{N}$, and put $C_1 = 1, X_0 = 1$ then equation (33) become,

$$\psi(\xi) = 1 + A_1\left(\frac{e^{-N\xi}}{A\left(1-\frac{e^{-N\xi}}{N}\right)+1}\right) + A_2\left(\frac{e^{-N\xi}}{A\left(1-\frac{e^{-N\xi}}{N}\right)+1}\right)^2 \tag{24}$$

$$\psi(\xi) = 1 + 28\left(\frac{e^{-2i\xi}}{17i\left(1-\frac{e^{-2i\xi}}{2i}\right)+1}\right) - 127\left(\frac{e^{-2i\xi}}{17i\left(1-\frac{e^{-2i\xi}}{2i}\right)+1}\right)^2 \tag{25}$$

$$\begin{aligned}\psi(\xi) = 1 + 28&\left(\frac{[4\text{Cos}\,2\xi - 68\,\text{Sin}\,2\xi - 34] - i[68\,\text{Cos}\,2\xi + 4\,\text{Sin}\,2\xi]}{1449 + 34\,\text{Cos}\,2\xi + 1156\,\text{Sin}\,2\xi}\right) \\ -127&\left(\frac{[4\text{Cos}\,2\xi - 68\,\text{Sin}\,2\xi - 34] - i[68\,\text{Cos}\,2\xi + 4\,\text{Sin}\,2\xi]}{1449 + 34\,\text{Cos}\,2\xi + 1156\,\text{Sin}\,2\xi}\right)^2\end{aligned} \tag{26}$$

$$\begin{aligned}\text{Re}\,\psi(\xi) = 1 + 28&\left(\frac{4\,\text{Cos}\,2\xi - 68\,\text{Sin}\,2\xi - 34}{1449 + 34\,\text{Cos}\,2\xi + 1156\,\text{Sin}\,2\xi}\right) \\ -127&\left(\frac{4624\,\text{Sin}\,2\xi - 272\,\text{Cos}\,2\xi + 5796}{\left(1449 + 34\,\text{Cos}\,2\xi + 1156\,\text{Sin}\,2\xi\right)^2}\right)\end{aligned} \tag{27}$$

$$\operatorname{Im}\psi(\xi) = -28\left(\frac{68\operatorname{Cos}2\xi + 4\operatorname{Sin}2\xi}{1449 + 34\operatorname{Cos}2\xi + 1156\operatorname{Sin}2\xi}\right) - 127\left(\frac{1140\operatorname{Sin}4\xi - 272\operatorname{Cos}4\xi + 2312\operatorname{Cos}2\xi + 272\operatorname{Sin}2\xi}{\left(1449 + 34\operatorname{Cos}2\xi + 1156\operatorname{Sin}2\xi\right)^2}\right) \tag{28}$$

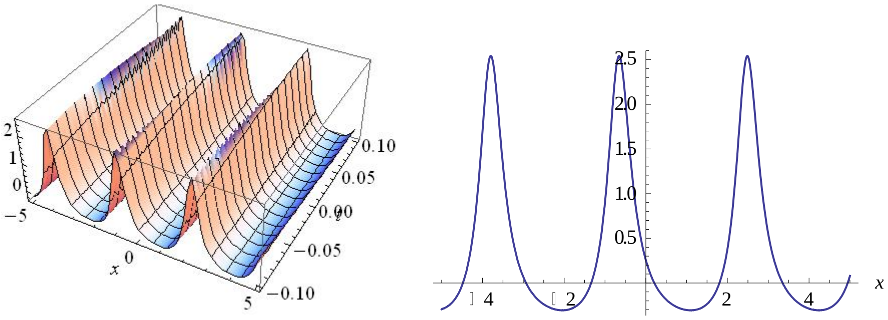

Fig.1 : The plot of the Lump solution Re. part Eq.(27) in two and three dimensions when:

$$\mathrm{A} = 17\mathrm{i}, N = 2\mathrm{i}, A_1 = 28, A_2 = -127, X_0 = w = C_1 = 1$$

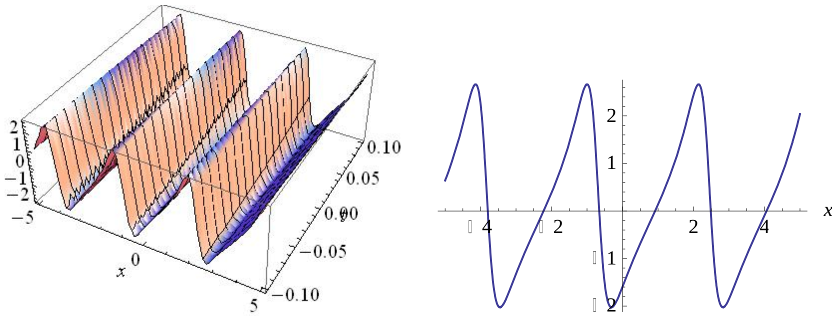

Fig.2 : The plot of the Lump solution Im. part Eq.(28) in two and three dimensions when:

$$\mathrm{A} = 17\mathrm{i}, N = 2\mathrm{i}, A_1 = 28, A_2 = -127, X_0 = w = C_1 = 1$$

By the same manner we can extracting the corresponding solutions for the other three results and plot them.

4. Mathematical analysis of the RBSOM

Step (1) The solution of Eq. (2) according to the RBSOM is,

$$\psi' = a\psi^{2-n} + b\psi + c\psi^n \tag{29}$$

Where a, b, c, and n are constants to be determined later.

Step (2) From Eq. (29) and by directly calculating, we get

$$\psi'' = ab(3-n)\psi^{2-n} + a^2(2-n)\psi^{3-2n} + nc^2\psi^{2n-1} + bc(n+1)\psi^n + (2ac+b^2)\psi, \tag{30}$$

Remark: When $ac \neq 0$ and $n = 0$, Eq. (29) is a Riccati equation. When $a \neq 0$, $c = 0$ and $n \neq 1$, Eq. (29) is a Bernoulli equation. Obviously, the Riccati equation and Bernoulli equation are special cases of Eq. (29). Because Eq. (29) is firstly proposed, we call Eq. (29) the Riccati-

Bernoulli Sub-ODE equation in order to avoid introducing new terminology. Eq. (3) has the following solutions:

Case (1) When n= 1, the solution of Eq. (29) is

$$\psi(\xi) = C\, e^{(a+b+c)\xi} \tag{31}$$

Case (2) When n ≠ 1, b = 0 and c = 0, the solution of Eq. (29) is

$$\psi(\xi) = \left(a(n-1)(\xi + C)\right)^{\frac{1}{(1-n)}}, \tag{32}$$

Case (3) When n ≠ 1, b ≠ 0 and c = 0, the solution of Eq. (29) is

$$\psi(\xi) = \left(-\frac{a}{b} + Ce^{b(n-1)\xi}\right)^{\frac{1}{(n-1)}}, \tag{33}$$

Case (4) When n ≠ 1, a ≠ 0 and $b^2 - 4ac < 0$, the solution of Eq. (29) is

$$\psi(\xi) = \left(\frac{-b}{2a} + \frac{\sqrt{4ac-b^2}}{2a}\tan\frac{(1-n)\sqrt{4ac-b^2}}{2}(\xi + C)\right)^{\frac{1}{(1-n)}}, \tag{34}$$

and

$$\psi(\xi) = \left(\frac{-b}{2a} + \frac{\sqrt{4ac-b^2}}{2a}\cot\frac{(1-n)\sqrt{4ac-b^2}}{2}(\xi + C)\right)^{\frac{1}{(1-n)}}, \tag{35}$$

Case (5) When n ≠ 1, a ≠ 0 and $b^2 - 4ac > 0$, the solution of Eq. (29) is

$$\psi(\xi) = \left(\frac{-b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}\coth\frac{(1-n)\sqrt{b^2-4ac}}{2}(\xi + C)\right)^{\frac{1}{(1-n)}}, \tag{36}$$

and

$$\psi(\xi) = \left(\frac{-b}{2a} + \frac{\sqrt{b^2-4ac}}{2a}\tanh\frac{(1-n)\sqrt{b^2-4ac}}{2}(\xi + C)\right)^{\frac{1}{(1-n)}}, \tag{37}$$

Case (6) When n ≠ 1, a ≠ 0 and $b^2 - 4ac = 0$ the solution of Eq. (29) is

$$\psi(\xi) = \left(\frac{1}{a(n-1)(\xi + C)} - \frac{b}{2a}\right)^{1/(1-n)}. \tag{38}$$

where C is an arbitrary constant.

Step (3) Substituting the derivatives of u into Eq. (29) yields an algebraic equation of u. Noticing the symmetry of the right-hand item of Eq. (29) and setting the highest power exponents of ψ to be equivalence in Eq. (29), m can be determined. Comparing the coefficients of ψ^i yields a set of algebraic equations for a, b, c, and C. Solving the set of algebraic equations and substituting m, a, b, c, C, $\xi = (x - wt)$ to Eq.(31)-(38), we can get the traveling wave solutions of Eq.(2).

Moreover, we will give a Bäcklund transformation of the RBSOM which is important extension that improves and gives power to this method.

5. Bäcklund transformation of the Riccati-Bernoulli equation

In this sub-paragraph we will give brief description for the Bäcklund transformation as follow,

Let us consider $\psi_{n-1}(\xi)$ and $\psi_n(\xi)(\psi_n(\xi)=\psi_n(\psi_{n-1}(\xi)))$ are the solution of equation (29) then,

$$\frac{d\psi_n(\xi)}{d\xi}=\frac{d\psi_n(\xi)}{d\psi_{n-1}(\xi)}\frac{d\psi_{n-1}(\xi)}{d\xi}=\frac{d\psi_n(\xi)}{d\psi_{n-1}(\xi)}(a\psi_{n-1}^{2-m}+b\psi_{n-1}+c\psi_{n-1}^{m}).$$

This tends to,

$$\frac{d\psi_n(\xi)}{(a\psi_n^{2-m}+b\psi_n+c\psi_n^{m})}=\frac{d\psi_{n-1}(\xi)}{(a\psi_{n-1}^{2-m}+b\psi_{n-1}+c\psi_{n-1}^{m})}.$$

By integrating both sides of this equation with respect to ξ and simplifying it we obtain the Backlund transformation of equation (29) which is,

$$\psi_n(\xi)=\left(\frac{-cD_1+aD_2(\psi_{n-1}(\xi))^{1-m}}{bD_1+aD_2+aD_1(\psi_{n-1})^{1-m}}\right)^{\frac{1}{1-m}}. \tag{39}$$

Where D_1, D_2 are arbitrary constants.

According to this transformation when we obtain the solution of any equation by using the RBSOM we can generate new infinite sequence of solutions to this equation, consequently an infinite sequence of solution for equation (2) could be realized.

6. Application:

From the point power of view of the RBSOM, via inserting ψ'' Eq. (30) into Eq. (13) mentioned above and equating the coefficients of different powers of ψ^i after suitable choice of n we get this system of equations,

$$\begin{aligned}&2a^3+a^2=0,\\&2a^2b+ab=0,\\&6ab+6a^2c+2a^2c+ab^2+3b^2+6ac-wa=0,\\&6abc+2abc+b^3+6bc-wb=0,\\&3c^2+2ac^2+b^2c-wc=0.\end{aligned} \tag{40}$$

From which we can easily obtain these results,

$$a=0\ ,\ a=-1/2\ ,b=0,\ c=w/2 \tag{41}$$

These achieved results implies case (4) and case (5) which are,

Case (4) when n ≠ 1, a ≠ 0 and $b^2-4ac<0$, $w\prec 0$, the solution of Eq. (29) is

$$\psi(\xi)=\left(\frac{-b}{2a}+\frac{\sqrt{4ac-b^2}}{2a}\tan\frac{(1-n)\sqrt{4ac-b^2}}{2}(\xi+C)\right)^{\frac{1}{(1-n)}} \tag{42}$$

and

$$\psi(\xi)=\left(\frac{-b}{2a}+\frac{\sqrt{4ac-b^2}}{2a}\cot\frac{(1-n)\sqrt{4ac-b^2}}{2}(\xi+C)\right)^{\frac{1}{(1-n)}} \tag{43}$$

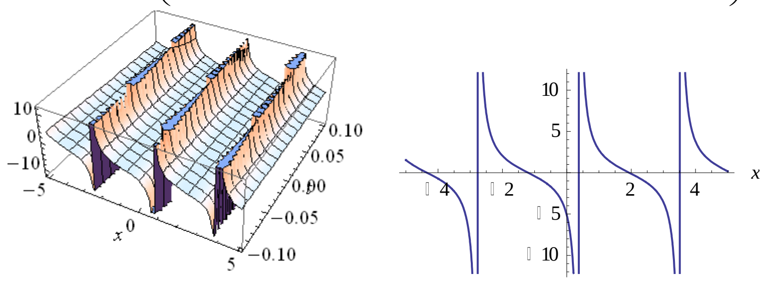

Fig. 3 : The plot of the Lump solution Eq.(42) in two and three dimensions when: $a=-1/2\,, b=0,\; c=-1, w=-2, C=1$

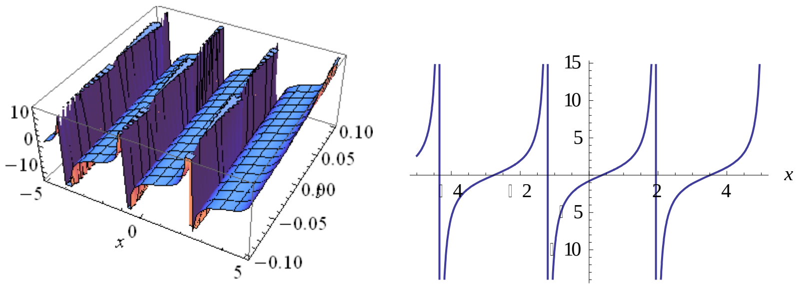

Fig. 4 : The plot of the Lump solution Eq.(43) in two and three dimensions when: $a=-1/2\,, b=0,\; c=-1, w=-2, C=1$

Case (5) when n ≠ 1, a ≠ 0 and $b^2-4ac>0,\; w\succ 0$, the solution of Eq. (29) is

$$\psi(\xi)=\left(\frac{-b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\coth\frac{(1-n)\sqrt{b^2-4ac}}{2}(\xi+C)\right)^{\frac{1}{(1-n)}}, \tag{44}$$

$$\psi(\xi)=\left(\frac{-b}{2a}+\frac{\sqrt{b^2-4ac}}{2a}\tanh\frac{(1-n)\sqrt{b^2-4ac}}{2}(\xi+C)\right)^{\frac{1}{(1-n)}}, \tag{45}$$

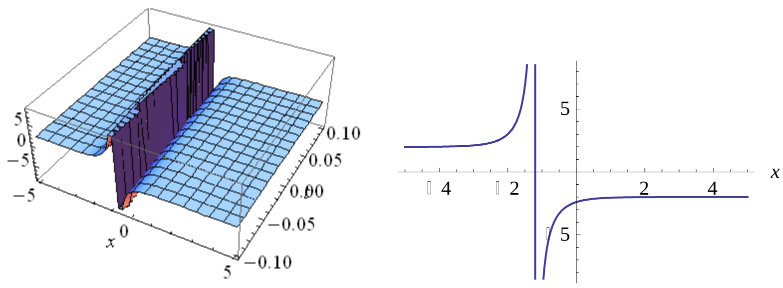

Fig. 5 : The plot of the Lump solution Eq.(44) in two and three dimensions when: $a=-1/2\,, b=0,\; w=2, c=C=1$

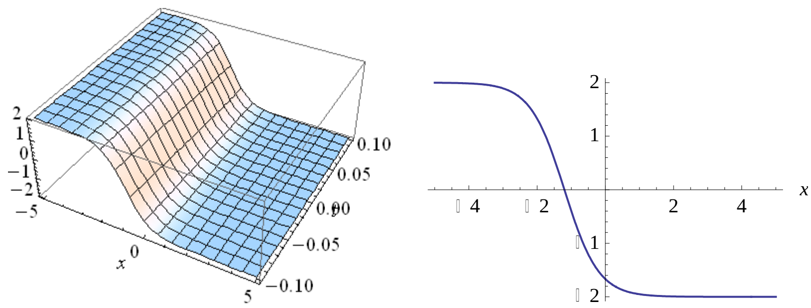

Fig. 6 : The plot of the Lump solution Eq.(45) in two and three dimensions when:
$a = -1/2\ , b = 0,\ w = 2, c = C = 1$

Moreover in the framework of Bäcklund transformation we can generate infinite sequence of solution for each achieved solution.

7. The VIM schemas

To investigate VIM algorithm, let us assume this differential equation,

$$LR + NR = g(\xi). \tag{46}$$

Where $g(\xi)$ is nonhomogeneous function and the operators L, N related to the linear and the nonlinear respectively.

The correction functional for equation (46) according to the VIM is;

$$\psi_{m+1}(\xi) = \psi_m(\xi) + \int_0^{\xi} \lambda(t)(L\psi_m(t) + N\tilde{\psi}_m(t) - f(t))dt. \tag{47}$$

Where λ is a general Lagrange's multiplier, which can be determined using variational theory, from Eq. (47) the following relations will be extracted:

For the 1-st order ODE in the form,

$$\psi' + q(\xi)\psi = p(\xi),\ \psi(0) = \rho, \tag{48}$$

For which $\lambda = -1$, the correction function implies this iteration rule;

$$\psi_{m+1}(\xi) = \psi_m(\xi) - \int_0^{\xi} (\psi_m'(t) + q(t)\psi_m(t) - p(t))dt. \tag{49}$$

The 2-nd order ODE in the form,

$$\psi''(\xi) + c\psi'(\xi) + dh(\xi) = f(\xi),\quad \psi(0) = \rho, \psi'(0) = \eta. \tag{50}$$

For which $\lambda = t - x$, the correction function implies this iteration rule;

$$\psi_{m+1}(\xi) = \psi_m(\xi) + \int_0^{\xi} (t - x)(\psi_m''(t) + c\psi_m'(t) + d\psi_m - f(t))dt. \tag{51}$$

The 3-rd order ODE in the form,

$$\psi'''(\xi) + c\psi''(\xi) + d\psi'(\xi) + eH(\xi) = f(\xi),\ \ H(0) = \rho, \psi'(0) = \eta, \psi''(0) = \sigma, \tag{52}$$

For which $\lambda = -\frac{1}{2!}(t-x)^2$, the correction function implies this iteration rule;

$$\psi_{m+1}(\xi) = \psi_m(\xi) - \frac{1}{2!}\int_0^{\xi}(t-x)^2(\psi_m'''(t) + c\psi_m''(t) + d\psi_m'(t) + e\psi_m - f(t))dt, \tag{53}$$

Hence,for the general form of ODE,

$$\psi^{(m)} + g(\psi', \psi'', \psi''', ..., \psi^{(m-1)}) = f(\xi), \psi(0) = \rho_0, \psi'(0) = \rho_1, \psi''(0) = \rho_2 ... \psi^{m-1}(0) = \rho_{m-1}, \tag{54}$$

The lagrange multiplier λ takes the general form $\lambda = \frac{(-1)^m}{(m-1)!}(t-x)^{m-1}$, which implies this general iteration rule,

$$\psi_{m+1}(\xi) = \psi_m(\xi) + \frac{(-1)^m}{(m-1)!}\int_0^{\xi}(t-x)^{m-1}\left(\psi^{(m)} + g(\psi', \psi'', \psi''', ..., \psi^{(m-1)}) - f(t)\right)dt, \tag{55}$$

Furthermore, the zeros approximation $\psi_0(\xi)$ can be perfectly selected to be,

$$\psi_0(\xi) = \psi_0(0) + \psi'(0)\xi + \frac{1}{2!}\psi''(0)\xi^2 + \frac{1}{3!}\psi'''(0)\xi^3 .... + \frac{1}{(m-1)!}\psi^{m-1}(0)\xi^{m-1} \tag{56}$$

where m is the rank of the ODE.

8. Application:

This section involves the implementing of the VIM to construct the numerical solutions corresponding to the exact solutions using the above schemas individually. The numerical solution corresponding to the first solution achieved using PPAM which is,

$$\psi(\xi) = 1 + 28\left(\frac{e^{-2i\xi}}{17i\left(1 - \frac{e^{-2i\xi}}{2i}\right) + 1}\right) - 127\left(\frac{e^{-2i\xi}}{17i\left(1 - \frac{e^{-2i\xi}}{2i}\right) + 1}\right)^2$$

That possesses these values of the constants, $A = 17i, N = 2i, A_1 = 28, A_2 = -127, X_0 = w = C_1 = 1$

Thus, the first iteration in the framework of the VIM is,

$$\psi_0(\zeta) = \psi(0) + \xi\psi'(0),\ \psi_0(\xi) = 0.8 - 6\xi - i(1.8 + 6.8\xi), \tag{57}$$

$$\begin{aligned} &\psi_1(\xi) = \psi_0(\xi) - \int_0^{\xi}\left(\psi_0'' + 3\psi_0^2 - w\psi_0\right)dt, \\ &\psi_1 = 0.8 - 6\xi - i(1.8 + 6.8\xi) - \int_0^{\xi} 3[(0.8 - 6t) - i(1.8 + 6.8t)]^2 - [(0.8 - 6t) - i(1.8 + 6.8t)]dt. \end{aligned} \tag{58}$$

$$\text{Re}\,\psi_1 = 0.8 - 16.84\xi - 25.3\xi^2 - 82.24\xi^3. \tag{59}$$

$$\text{Im}\ \psi_1 = -3.8 - 19.5\xi^2 - 81.3\xi^3. \tag{60}$$

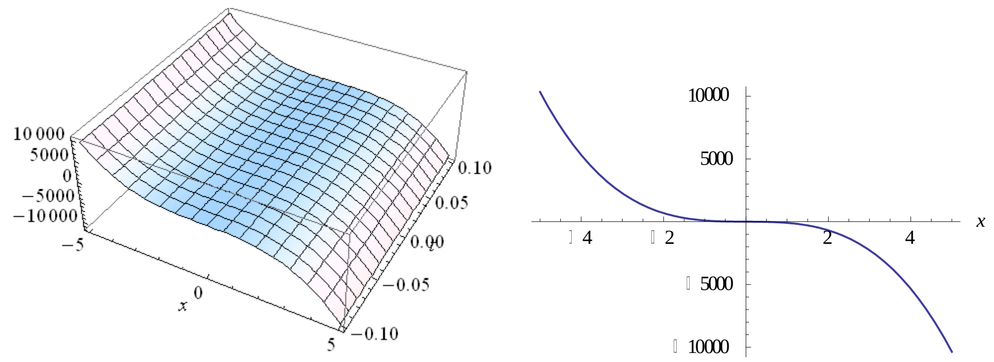

Fig. 7 : The plot of the numerical solution Re. Part Eq.(59) in two and three dimensions when:

$$\mathrm{A} = 17\mathrm{i}, N = 2\mathrm{i}, A_1 = 28, A_2 = -127, X_0 = w = C_1 = 1$$

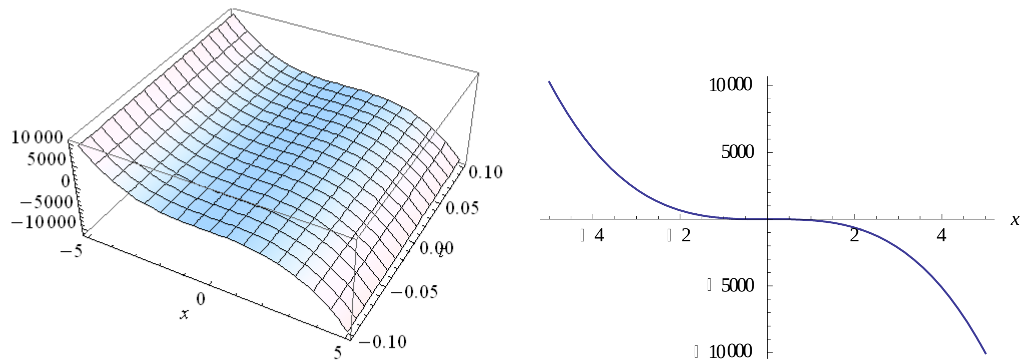

Fig. 8 : The plot of the numerical solution Im. Part Eq.(60) in two and three dimensions when:

$$\mathrm{A} = 17\mathrm{i}, N = 2\mathrm{i}, A_1 = 28, A_2 = -127, X_0 = w = C_1 = 1$$

By the same manner we can construct the numerical solutions corresponding to the other three cases.

The numerical solutions corresponding to the first and the fourth solution that achieved using RBSODM.
Firstly, for the first solution that achieved in terms of these constants $a=-1/2, b=0, w=2, c=C=1$ which is;

$$\psi(\zeta) = -2\tan(\xi+1)$$

Thus, the first iteration in the framework of the VIM is,

$$\psi_0(\zeta) = \psi(0) + \xi\psi'(0), \ \psi_0(\xi) = -0.03 - 2\xi, \tag{61}$$

$$\psi_1(\xi) = \psi_0(\xi) - \int_0^{\xi} \left(\psi_0'' + 3\psi_0^2 - w\psi_0 \right) dt,$$

$$\psi_1 = -0.03 - 2\xi - \int_0^{\xi} 3[-0.03 - 2t]^2 - [-0.03 - 2t] dt. \tag{62}$$

$$\psi_1 = -0.03 - 2.03\xi - 1.2\xi^2 - 4\xi^3.$$

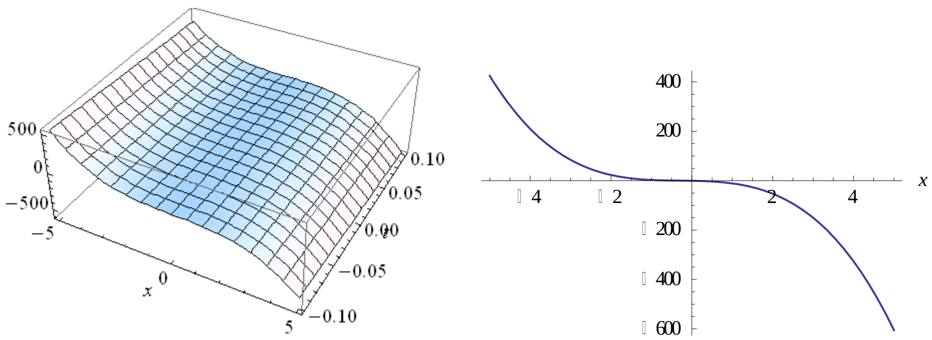

Fig. 9 : The plot of the numerical solution Eq.(62) in two and three dimensions when:

$$a=-1/2\ ,b=0,\ w=-2,c=C=1$$

Secondly, for the fourth solution that achieved in terms of these constants $a=-1/2\ ,b=0,\ w=2,c=C=1$ which is;

$$\psi(\zeta)=-2\tanh(\xi+1)$$

Thus, the first iteration in the framework of the VIM is,

$$\psi_0(\zeta)=\psi(0)+\xi\psi'(0),\ \psi_0(\xi)=-1.5-\xi, \tag{63}$$

$$\psi_1(\xi)=\psi_0(\xi)-\int_0^{\xi}\left(\psi_0''+3\psi_0^2-w\psi_0\right)dt,$$

$$\psi_1=-1.5-\xi-\int_0^{\xi}3[-1.5-t]^2-[-1.5-t]dt. \tag{64}$$

$$\psi_1=-1.5-9.3\xi-3.5\xi^2-\xi^3.$$

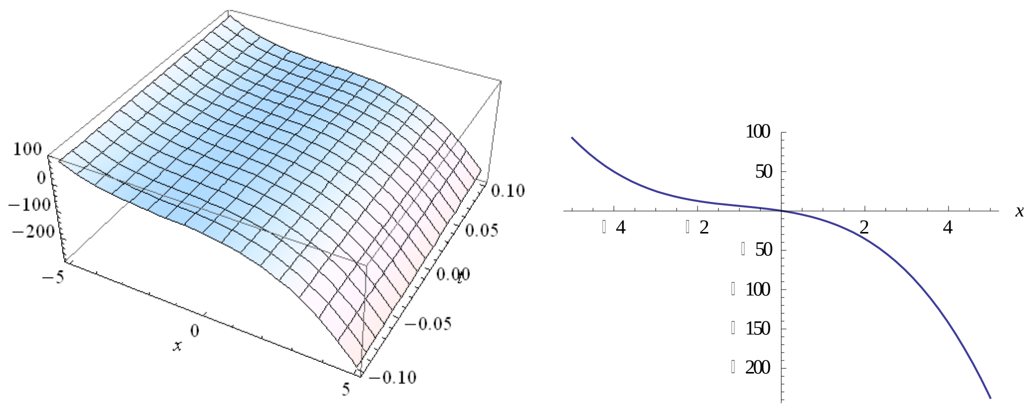

Fig. 10 : The plot of the numerical solution Eq.(64) in two and three dimensions when:

$$a=-1/2\ ,b=0,\ w=2,c=C=1$$

For all the last three cases, the successive iterations to the VIM could been easily obtained as;

$$\psi_2(\xi)=\psi_1(\xi)-\int_0^{\xi}\left(\psi_1''+3\psi_1^2-\psi_1\right)dt,$$

$$\psi_3(\xi)=\psi_2(\xi)-\int_0^{\xi}\left(\psi_2''+3\psi_2^2-\psi_2\right)dt, \tag{63}$$

....

$$\psi_{N+1}(\xi)=\psi_N(\xi)-\int_0^{\xi}\left(\psi_N''+3\psi_N^2-\psi_N\right)dt.$$

Using the fact that the exact solution is obtained by using $\psi(\xi)=\lim\limits_{\xi\to\infty}\psi_N(\xi)$

Conclusion

In the framework of two distinct and impressive different methods we established the lump solutions of the spatio temporal dispersion (1+1)-dimensional Ito equation. These two various manners are the PPAM which has been applied perfectly to extract new lump solutions for this model figures (1-2), while the other one is the RBSODM which also success to demonstrate other new lump solutions of this model in Figures (3-6). The two schemas are implemented at the same time and parallel. Furthermore, the numerical solutions corresponding for the lump solutions emergence from these two methods have been extracted in the framework of the VIM in Figures (7-10). The achieved new lump solutions which weren't realized are new and express the novelty of these results. These new achieved solutions will be representing new perceptions of the lump solutions which more significant compared with that achieved lastly by [3-6].

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