Delayed impulsive SDEs driven by multiplicative fBm noise and additive fBm noise

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Abstract

The stability and boundedness for delayed impulsive SDEs driven by fBm are studied in this paper. Two kinds of noises, i.e, additive fBm noise and multiplicative fBm noise are both taken into consideration. By using stochastic Lyapunov technique and impulsive control theory, sufficient criteria for pth moment exponential stability and mean square ultimate boundedness are derived, for two kinds of fBm driven delayed impulsive SDEs, respectively. As application, the obtained results are used to do practical synchronization w.r.t. a class of chaotic systems, in which the response system is perturbed by additive fBm noises. Finally, A Chua chaotic oscillator is given to verify the validity and applicability of the derived results.

Keywords: fractional Brownian motion, delayed impulse, stability, boundedness, practical synchronization.

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1. Introduction

Compared with ordinary differential equations (ODEs), stochastic differential equations (SDEs) are better tools to deal with the ubiquitous environmental disturbances and uncertainties in dynamical systems. In 1968, Mandelbrot [1] originally studied fractional Brownian motion (fBm), which is an effective tool to model the property of long range dependence and the phenomenon of self-similarity, it is widely applied in the fields such as hydrology [2], finance [3], telecommunication [4], etc. Thus there have been many researchers studied S-DEs driven by fBm, for instance [5, 6, 7, 8, 9, 10, 11, 12, 13, 14, 15] and the reference there in. The property of fBm $B^H(t)$ relies heavily on the Hurst parameter H. If $H \in (\frac{1}{2}, 1)$, it exhibits the property of long range dependence, sometimes also called long memory. If $H \in (0, \frac{1}{2})$, it exhibits a short memory. If $H = \frac{1}{2}$, it reduces to a Brownian motion (Bm). Although fBm is the generalization of Bm, the properties of fBm and Bm varies very different. Contrast with Bm, fBm is neither Markov nor semimartingale, thus the classical stochastic analysis theories are not applicable to fBm when $H \neq \frac{1}{2}$. Therefore, it is a challenging and difficult problem to study fBm driven SDEs, new theories and methods are required. Fortunately, there have been some new attempts in this direction, for example, in 2000 year, Duncan et al. [16] initially introduced a new type of stochastic integral w.r.t. fBm using the Wick product, this definition satisfy the following property: $\mathbb{E}\int_0^t f(s)dB^H(s) = 0$. The advantages of this definition are two-fold. Firstly, the above property coincides with a common assumption: the random perturbation in SDEs should not affect the mean rate of change. Secondly, it may facilities some calculations when we do stability analysis. Thus in this paper, we utilize this definition.

On the other hand, impulses, as effective tools to describe the abrupt changes in systems, has been widely studied for a long time. The earlier works on impulsive systems can be founded in [17, 18]. Since impulses in systems have effects (active effects or negative effects) to the performance of systems, stability analysis of impulsive systems is always a research focus. During the past decades, the existence and stability analysis for different types of impulsive SDEs (ISDEs) have been investigated by many researchers[19, 20, 21, 22, 23, 24, 25, 26]. However, most of the above mentioned stability results concerned with ISDEs driven by Bm, as for ISDEs driven by fBm, the stability results are very few. On the other hand, as we know, Lyapunov second method is a useful tool to do stability analysis in ODEs. During the last decades, many researchers aim to generalize the Lyapunov method from ODEs to SDEs. When the SDEs is driven by Bm , the stochastic Lyapunov method was established in the pioneering work by Khasmiskii [27]. Besides, stability analysis for some more general SDEs driven by Bm can be found in monograph by Mao [28]. In 2014, Zeng et al. [29] established a new stochastic Lyapunov technique to evaluate the stochastic stability of SDEs driven by fBm. However, to the best of our knowledge, stochastic Lyapunov stability criteria for fBm driven SDEs with delayed impulses have not been derived yet.

In this paper, we aim to establish some new criteria for stability and boundedness of delayed impulsive SDEs with fBm. The main contributions are list as follows. 1) Delayed impulsive SDEs with fBm are investigated as a first attempt. 2) Two kinds of noises, i.e, multiplicative fBm noise and additive fBm noise, are both taken in to consideration, pth moment exponentially stability criteria and bounded analysis results are presented correspondingly. 3) as application, some new criteria for practical synchronization are derived when the response systems are perturbed by additive fBm noises.

This paper is organized as follows. In Sect. 2, basic definitions and notations needed in this paper are introduced. In Sect. 3, the sufficient conditions of exponential stability and mean square ultimate boundedness for delayed impulsive SDEs with multiplicative fBm noise and additive fBm noise are derived, respectively. In Sect. 4, the criteria of practical synchronization are derived, and the corresponding impulsive controllers are designed. Moreover, a Chua chaotic oscillator is presented to verify the validity of the theoretical results. In Sect. 5, conclusions are drawn.

2. Preliminaries

The common notations and their descriptions in this paper are listed in the following table.

Table 1: Common notations and their descriptions

Notation	Description
$\mathbb{N}$	the set of natural numbers
$\mathbb{Z}_+$	the set of positive integer numbers
$\mathbb{R}$	the set of real numbers
$\mathbb{R}_+$	the set of positive real numbers
$\mathbb{R}^n$	the n-dimensional real space
$\mathbb{R}^{n\times m}$	the $n \times m$-dimensional real space
E	the unit matrix with proper dimensions
$*$	the symmetric block in the symmetric matrix
$\mathbb{E}$	mathematical expectation

Let $(\Omega, \mathcal{F}, \mathbb{P})$ be a complete probability space with a filtration $(\mathcal{F} = \mathcal{F}_t)_{t\geq 0}$ satisfying usual conditions. For $p > 0$, let $\mathbb{S}_p = \{x \in \mathbb{R}^n : |x| < p\}$. Let $B^H(t), (H \in (0,1), t \geq 0)$ be a fBm, which satisfies:

$$\mathbb{E}(B^H(t)) = 0, \quad \forall t \in \mathbb{R}_+,$$

and

$$\mathbb{E}(B^H(t)B^H(s)) = \frac{1}{2}\big(|t|^{2H} + |s|^{2H} - |t-s|^{2H}\big), \quad \forall s, t \in \mathbb{R}_+.$$

There are different types of definitions of stochastic integral w.r.t. fBm. For instance, Wiener type integration was defined in [30], Stratonovich type integration was introduced in [31, 32]. However, the stochastic integral $\int_0^t f(s)\delta B^H(s)$ defined above, in general, dose not satisfy the following property: $\mathbb{E}\int_0^t f(s)\delta B^H(s) = 0$, which is important when dealing with SDEs driven by fBm. In order to overcome this deficiency, Duncan et al. [16] introduced a new type of stochastic integral w.r.t. fBm by using the Wick product, now we recall this definition.

Definition 1. [16] *Let $f(t), t \in [0,T]$ be a stochastic process such that $f \in L(0,T)$. For any time interval $[0,T]$, denote $\pi : 0 = t_0 < t_1 < ... < t_n = T$, denote $|\pi| := \max_i(t_{i+1} - t_i)$, let $f^\pi(t) = f(t_i)$ if $t_i \leq t < t_{i+1}$. The stochastic integral of $f(t)$ w.r.t. fBm is defined by*

$$\int_0^T f(s)dB^H(s) = \lim_{|\pi|\to 0} \sum_{i=0}^{n-1} f^\pi(t_i) \diamond (B^H(t_{i+1}) - B^H(t_i)),$$

where $H \in (\frac{1}{2}, 1)$ and $\diamond$ denotes the Wick product, this integral satisfies the following property:

$$\mathbb{E}\int_0^T f(s)dB^H(s) = 0.$$

Moreover, if $V \in \mathbb{C}^{1,2}([t_0, +\infty) \times \mathbb{S}_p; \mathbb{R}_+)$, then it follows from fractional Itô formula [16] that

$$\begin{aligned} dV(t, x(t)) = &\big[V_t(t, x(t)) + V_x(t, x(t))f(t, x(t)) \\ &+ V_x x(t, x(t))g(t, x(t)) \int_0^t \phi(v, s)g(v, x(v))dv\big]dt \\ &+ V_x(t, x(t))g(t, x(t))dB^H(t) \\ := &L^H V(t, x(t))dt + V_x(t, x(t))g(t, x(t))dB^H(t), \end{aligned}$$

where $\phi(s,t) = H(2H-1)|s-t|^{2H-2}$.

We end this section by recalling some other fundamental definitions and lemmas.

Definition 2. [33] *The equilibrium solution of system (1) is said to be pth moment exponentially stable if there exist positive constants c_0 and λ*

$$\mathbb{E}[\|x(t; t_0, x_0)\|^p] \leq c_0|x_0|^p e^{-\lambda(t-t_0)}, t \geq t_0,$$

for all $x_0 \in \mathbb{R}^n$.

Definition 3. [34] *The equilibrium solution of system is said to be uniformly ultimately bounded (UUB) in mean square if there exists a positive constant b such that*

$$\lim_{t\to+\infty} \mathbb{E}[\|x(t)\|^2] \leq b.$$

Definition 4. *Define $x(t)$ to be a state of the drive system, $y(t)$ to be a state of the response system, $e(t) = y(t) - x(t)$ to be an error function. We say the drive system and the response system are practically synchronized if there exists a positive constant ϵ such that*

$$\lim_{t\to+\infty} \mathbb{E}[\|e(t)\|^2] \leq \epsilon.$$

where ϵ is the synchronization error.

Definition 5. [35] *The number of impulsive times $N(t,T)$ in the interval (t,T) has the following upper bound and lower bound:*

$$\frac{T-t}{\tau^*} - N_0 \leq N(t,T) \leq \frac{T-t}{\tau^*} + N_0,$$

where $N_0 \in \mathbb{R}_+$ is the chatter bound and τ^* is the average dwell time (ADT).

Lemma 1. [36] *Let $X \in \mathbb{R}^n$, $Y \in \mathbb{R}^n$, and a scalar $\epsilon > 0$, then it holds that*

$$X^TY + Y^TX \leq \epsilon X^TX + \epsilon^{-1}Y^TY.$$

3. Main results

In this section, two kinds of fractional noise in SDEs with delayed impulses are considered, i.e., the multiplicative fBm noise and the additive fBm noise. For the multiplicative fBm noise case, the exponential stability in pth moment criteria are established. For the additive fBm case, the UUB in mean square criteria are presented.

3.1. Delayed impulsive SDEs with multiplicative fBm noise.

In this subsection, we study a class of delayed impulsive SDEs with multiplicative fBm noise in the form

$$\begin{cases} dx(t) = f(t, x(t))dt + g(t, x(t))dB^H(t), t \neq t_k, \\ x(t^+) = I(x(t^- - \tau)), t = t_k, \tau = \tau_k, \end{cases} \tag{1}$$

where $x(t) \in \mathbb{R}^n$ is the system state, $B^H(t)$ is the n-dimensional fBm, $x(t_k^-)$ and $x(t_k^+)$ are left and right limit of $x(t)$ at time t_k, respectively. We assume that the state of system (1) is left continuous, i.e., $x(t) = x(t^-)$. We assume that function f, g and I is smooth enough such that system (1) exists a unique solution on $(t_0, +\infty)$. Let $\tau_k = (1-\delta)(t_k - t_{k-1}), (\delta \in (0,1], k \in \mathbb{Z}_+)$ denote delays of impulses, note that the delay τ_k is depend on both impulsive sequence $\{t_k\}$ and the delayed impulse parameter $\delta \in (0,1]$, in particular, if $\delta = 1$, then $\tau_k = 0$. If $t_k = k$, then τ_k changes into a constant delay $\tau = 1 - \delta$. The time sequences $\{t_k, k \in \mathbb{N}\}$ satisfy $0 = t_0 < t_1 < ... < t_k = T$.

Remark 1. We note that in system (1), function $g(t, x(t))$ is depend on the system state $x(t)$, thus system (1) perturbed by a multiplicative noise. Multiplicative noise [37] frequently appeared in microscope images, synthetic aperture radar images and laser images etc.

In the following, the exponential stability criteria are established by Lyapunov technique.

Theorem 1. *Let the Lyapunov fucntion $V(t, x(t))$ of system (1) satisfying:*
(i) $V(t, x(t))$ is continuously once differentiable in t and twice in $x(t)$ on each of the intervals $(t_{k-1}, t_k] \times \mathbb{R}^n, k \in \mathbb{Z}_+$,
(ii) there exist positive constants p, c_1 and $c_2 > 0$ satisfying

$$c_1\|x\|^p \leq V(t, x(t)) \leq c_2\|x\|^p, \tag{2}$$

(iii) there exists a positive constant c_3 satisfying

$$L^H V(t, x) \leq -c_3 V(t, x), \tag{3}$$

(iv) there exists a constant d satisfying

$$V(t_k^+, x(t_k^+)) \leq \exp(d) V(t_k^- - \tau_k, x(t_k^- - \tau_k)), \tag{4}$$

where $V(t_k^-, x(t_k^-))$, $V(t_k^+, x(t_k^+))$ are left limit and right limit of $V(t_k, x(t_k))$, respectively, let $V(t_k^-, x(t_k^-)) = V(t_k, x(t_k))$,
(v) the ADT τ^* of impulses satisfies

$$\tau^* > \frac{d}{c_3 \delta}. \tag{5}$$

Then the equilibrium solution of system (1) is pth moment exponentially stable.

Proof. Let $x(t) = x(t, t_0, x_0)$ be the solution of system (1) through (t_0, x_0). We will use induction to show that $\forall t \in (t_k, t_{k+1}], k \in \mathbb{N}$,

$$\mathbb{E}V(t, x(t)) \leq V_0 \exp(-c_3(t - \sum_{i=0}^{k} \tau_i - t_0) + kd), \tag{6}$$

where $V_0 = V(t_0, x(t_0))$, $\tau_0 = 0$.

For $t \in [t_0, t_1]$, utilizing fractional Itô formula to $\exp(c_3(t - t_0))V(t, x(t))$, it follows that

$$\begin{aligned} &\exp(c_3(t - t_0))V(t, x(t)) \\ &\qquad = V_0 + \int_{t_0}^{t} \exp(c_3(s - t_0))\big(c_3 V(s, x(s)) + L^H(s, x(s))\big) ds \\ &\qquad\quad + \int_{t_0}^{t} \exp(c_3(s - t_0)) V_x(s, x(s)) g(s, x(s)) ds, \end{aligned} \tag{7}$$

taking expectation on both sides of (7), it can be get that

$$\begin{aligned} &\mathbb{E} \exp(c_3(t - t_0))V(t, x(t)) \\ &\qquad = V_0 + \mathbb{E} \int_{t_0}^{t} \exp(c_3(s - t_0))\big(c_3 V(s, x(s)) + L^H(s, x(s))\big) ds, \end{aligned} \tag{8}$$

combing with condition(iii) yields that

$$\mathbb{E}[V(t, x(t))] \leq V_0 \exp(-c_3(t - t_0)), \quad t \in [t_0, t_1],$$

thus inequality (6) holds for $k = 0$. For $k = m$, assuming that ineq. (6) holds, namely,

$$\mathbb{E}V(t, x(t)) \leq V_0 \exp(-c_3(t - \sum_{i=0}^{m} \tau_i - t_0) + md), \tag{9}$$

now we show it holds for $k = m+1$, noting that $t_{m+1} - \tau_{m+1} = \delta(t_{m+1} - t_m) \in (t_m, t_{m+1}]$, thus from (9) we have

$$\mathbb{E}V(t_{m+1} - \tau_{m+1}, x(t_{m+1} - \tau_{m+1})) \leq V_0 \exp(-c_3(t_{m+1} - \sum_{i=0}^{m+1} \tau_i - t_0) + md).$$

Applying condition (iv), it follows that

$$\mathbb{E}V(t_{m+1}^+, x(t_{m+1}^+)) \leq V_0 \exp\Big(- c_3(t_{m+1} - \sum_{i=0}^{m+1} \tau_i - t_0) + (m+1)d \Big).$$

Similarly, $\forall t \in (t_{m+1}, t_{m+2}]$, by utilizing fractional Itô formula to $\exp(c_3(t - t_{m+1}))V(t, x(t))$, taking expectation and combining with condition (iii) yields that

$$\begin{aligned} \mathbb{E}V(t, x(t)) &\leq \mathbb{E}V(t_{m+1}^+, x(t_{m+1}^+)) \exp(-c_3(t - t_{m+1})) \\ &\leq V_0 \exp\Big(- c_3(t - \sum_{i=0}^{m+1} \tau_i - t_0) + (m+1)d \Big). \end{aligned} \tag{10}$$

Hence by induction, inequality (6) holds for any $k \in \mathbb{N}$.

Noting that

$$\sum_{i=0}^{k} \tau_i = \sum_{i=1}^{k} (1-\delta)(t_i - t_{i-1}) = (1-\delta)(t_k - t_0),$$

and

$$\begin{aligned} -c_3(t - \sum_{i=0}^{k} \tau_i - t_0) &= -c_3(t - t_0 - (1-\delta)(t_k - t_0)) \\ &= -c_3\delta(t - t_0) - c_3(1-\delta)(t - t_k) \\ &\leq -c_3\delta(t - t_0). \end{aligned}$$

Thus

$$\mathbb{E}[V(t, x(t))] \leq V_0 \exp(-c_3\delta(t - t_0) + N(t, t_0)d),$$

where $N(t,t_0)$ denotes the number of impulse times in interval $(t_0,t]$, in terms of Definition 5, we have

$$-c_3\delta(t-t_0)+N(t,t_0)d \le N_0 d + (\frac{d}{\tau^*}-c_3\delta)(t-t_0),$$

thus

$$\mathbb{E}[V(t,x(t))] \le V_0 \exp(N_0 d + (\frac{d}{\tau^*}-c_3\delta)(t-t_0)),$$

it then follows from condition (i) that

$$\mathbb{E}[\|x\|^p] \le c_0\|x_0\|^p \exp(-\lambda(t-t_0)),$$

where $c_0 = \frac{c_2}{c_1}e^{N_0 d}, \lambda = -\frac{d}{\tau^*}+c_3\delta$, where $\lambda > 0$ is guaranteed by inequality (15), this completes the proof of Theorem 1.

3.2. Delayed impulsive SDEs with additve fBm noise.

In this subsection, the following delayed impulsive SDEs with additve fBm noise is considered.

$$\begin{cases} dx(t) = f(t,x(t))dt + \sigma(t)dB^H(t), t \neq t_k, \\ x(t^+) = I(x(t^- - \tau)), t = t_k, \tau = \tau_k, \end{cases} \tag{11}$$

where $\sigma(t)$ is independent of system state $x(t)$, thus system (11) perturbed by an additive noise. Additive noises are used to represent external noises such as environment disturbances.

In what follows, the sufficient conditions for UUB in mean square are established by Lyapunov technique.

Theorem 2. *Let the Lyapunov function $V(t,x(t))$ of system (1) satisfying:*
(i) $V(t,x(t))$ is continuously once differentiable in t and twice in $x(t)$ on each of the intervals $(t_{k-1},t_k] \times \mathbb{R}^n, k \in \mathbb{Z}_+$,
(ii′) there exist positive constants c_1, c_2 such that

$$c_1\|x\|^2 \le V(t,x(t)) \le c_2\|x\|^2, \tag{12}$$

(iii′) there exist positive constants c_3, c_4 such that

$$L^H V(t,x) \le -c_3 V(t,x) + c_4, \tag{13}$$

(iv) there exists a positive constant d such that

$$V(t_k^+, x(t_k^+)) \leq \exp(d) V(t_k^- - \tau_k, x(t_k^- - \tau_k)), \tag{14}$$

(v) the ADT τ^* of impulses satisfies

$$\tau^* > \frac{d}{c_3 \delta}. \tag{15}$$

Then the equilibrium solution of system (11) is mean square UUB .

Proof. We will use induction to show that for all $t \in (t_k, t_{k+1}], k = 2, 3, ...,$ the following estimation holds,

$$\begin{aligned}
\mathbb{E}V(t, x(t)) \leq{}& (V_0 - \frac{c_4}{c_3}) \exp\Big(- c_3(t - \sum_{i=1}^{k} \tau_i - t_0) + kd\Big) \\
&+ \frac{c_4}{c_3} \sum_{j=1}^{k-1} \exp\Big(- c_3(t - t_j - \sum_{i=j+1}^{k} \tau_i) + (k + 1 - j)d\Big) \\
&- \frac{c_4}{c_3} \sum_{j=1}^{k-1} \exp\Big(- c_3(t - t_j - \sum_{i=j+1}^{k} \tau_i) + (k - j)d\Big) \\
&+ \frac{c_4}{c_3} \exp(-c_3(t - t_k) + d) - \frac{c_4}{c_3} \exp(-c_3(t - t_k)) + \frac{c_4}{c_3}.
\end{aligned} \tag{16}$$

For $t \in (t_0, t_1]$, using fractional Itô formula to $\exp(c_3(t - t_0)V(t, x(t))$, taking expectation and combining with condition (iii′) yields that

$$\begin{aligned}
&\mathbb{E}\Big[\exp\big((c_3(t - t_0)\big)V(t, x(t))\Big] \\
&= \mathbb{E}V_0 + \mathbb{E}\Big\{ \int_{t_0}^{t} \exp\big(c_3(s - t_0)\big)\big[c_3 V(s, x(s)) + L^H V(s, x(s))\big] ds\Big\} \\
&\leq \mathbb{E}V_0 + c_4 \int_{t_0}^{t} \exp\big(c_3(s - t_0)\big) ds \\
&= \mathbb{E}V_0 + \frac{c_4}{c_3}\big(\exp(c_3(t - t_0)) - 1\big),
\end{aligned} \tag{17}$$

or

$$\mathbb{E}V(t, x(t)) \leq (\mathbb{E}V_0 - \frac{c_4}{c_3}) \exp(-c_3(t - t_0)) + \frac{c_4}{c_3}, \tag{18}$$

where $V_0 = V(t_0, x(t_0))$, noting that $t_1 - \tau_1 \in (t_0, t_1]$, together with condition (iv), we arrive at

$$\mathbb{E}V(t_1^+, x(t_1^+)) \leq (\mathbb{E}V_0 - \frac{c_4}{c_3}) \exp(-c_3(t_1 - \tau_1 - t_0) + d) + \frac{c_4}{c_3} \exp(d). \tag{19}$$

For $t \in (t_1, t_2]$, using fractional Itô formula to $\exp(c_3(t-t_1)V(t, x(t))$, similarly, we get

$$\mathbb{E}V(t, x(t)) \leq (\mathbb{E}V(t_1^+, x(t_1^+)) - \frac{c_4}{c_3}) \exp(-c_3(t - t_1)) + \frac{c_4}{c_3}, \tag{20}$$

combining with inequality (19), we obtain

$$\begin{aligned}\mathbb{E}V(t, x(t)) \leq{}& (V_0 - \frac{c_4}{c_3}) \exp\Big(- c_3(t - \tau_1 - t_0) + d\Big) \\ &+ \frac{c_4}{c_3} \exp\Big(- c_3(t - t_1) + d\Big) - \frac{c_4}{c_3} \exp(-c_3(t - t_1)) + \frac{c_4}{c_3}.\end{aligned} \tag{21}$$

Similarly, for all $t \in (t_2, t_3]$, it can be get that

$$\begin{aligned}\mathbb{E}V(t, x(t)) \leq{}& (V_0 - \frac{c_4}{c_3}) \exp\Big(- c_3(t - \tau_2 - \tau_1 - t_0) + 2d\Big) \\ &+ \frac{c_4}{c_3} \exp\Big(- c_3(t - \tau_2 - t_1) + 2d\Big) \\ &- \frac{c_4}{c_3} \exp(-c_3(t - \tau_2 - t_1) + d) \\ &+ \frac{c_4}{c_3}(-c_3(t - t_2) + d) - \frac{c_4}{c_3}(-c_3(t - t_2)) + \frac{c_4}{c_3},\end{aligned} \tag{22}$$

thus ineq. (16) holds for $k = 2$. For $k = m$, assuming that ineq. (16) holds, now we show it holds for $k = m + 1$.

Noting that $t_{m+1} - \tau_{m+1} \in (t_m, t_{m+1}]$, thus from inequality (16) we have

$$\begin{aligned}&\mathbb{E}V(t_{m+1} - \tau_{m+1}, x(t_{m+1} - \tau_{m+1})) \\ &\quad\leq (V_0 - \frac{c_4}{c_3}) \exp\Big(- c_3(t_{m+1} - \tau_{m+1} - \sum_{i=1}^{m} \tau_i - t_0) + md\Big) \\ &\quad\quad+ \frac{c_4}{c_3} \sum_{j=1}^{m-1} \exp\Big(- c_3(t_{m+1} - \tau_{m+1} - t_j - \sum_{i=j+1}^{m} \tau_i) + (m + 1 - j)d\Big) \\ &\quad\quad- \frac{c_4}{c_3} \sum_{j=1}^{m-1} \exp\Big(- c_3(t_{m+1} - \tau_{m+1} - t_j - \sum_{i=j+1}^{m} \tau_i) + (m - j)d\Big) \\ &\quad\quad+ \frac{c_4}{c_3} \exp(-c_3(t_{m+1} - \tau_{m+1} - t_m) + d) - \frac{c_4}{c_3} \exp(-c_3(t_{m+1} - \tau_{m+1} - t_m)) + \frac{c_4}{c_3}.\end{aligned} \tag{23}$$

taking condition (iv) into consideration, we arrive at

$$\begin{aligned}
\mathbb{E}V(t_{m+1}^+, x(t_{m+1}^+)) &\le \exp(d)\mathbb{E}V(t_{m+1}-\tau_{m+1}, x(t_{m+1}-\tau_{m+1})) \\
&\le (V_0 - \frac{c_4}{c_3})\exp\Big(-c_3(t_{m+1} - \sum_{i=1}^{m+1}\tau_i - t_0) + (m+1)d\Big) \\
&\quad + \frac{c_4}{c_3}\sum_{j=1}^{m}\exp\Big(-c_3(t_{m+1} - t_j - \sum_{i=j+1}^{m+1}\tau_i) + (m+2-j)d\Big) \\
&\quad - \frac{c_4}{c_3}\sum_{j=1}^{m}\exp\Big(-c_3(t_{m+1} - t_j - \sum_{i=j+1}^{m+1}\tau_i) + (m+1-j)d\Big) \\
&\quad + \frac{c_4}{c_3}\exp(d).
\end{aligned} \tag{24}$$

For all $t \in (t_{m+1}, t_{m+2}]$, by utilizing fractional Itô formula to $\exp(c_3(t - t_{m+1}))V(t, x(t))$, taking expectation and combining with condition (iii$'$) yields that

$$\begin{aligned}
\mathbb{E}V(t, x(t)) &\le (\mathbb{E}V(t_{m+1}^+, x(t_{m+1}^+)) - \frac{c_4}{c_3})\exp(-c_3(t - t_{m+1})) + \frac{c_4}{c_3} \\
&\le (V_0 - \frac{c_4}{c_3})\exp\Big(-c_3(t - \sum_{i=1}^{m+1}\tau_i - t_0) + (m+1)d\Big) \\
&\quad + \frac{c_4}{c_3}\sum_{j=1}^{m}\exp\Big(-c_3(t - t_j - \sum_{i=j+1}^{m+1}\tau_i) + (m+2-j)d\Big) \\
&\quad - \frac{c_4}{c_3}\sum_{j=1}^{m}\exp\Big(-c_3(t - t_j - \sum_{i=j+1}^{m+1}\tau_i) + (m+1-j)d\Big) \\
&\quad + \frac{c_4}{c_3}\exp(-c_3(t - t_{m+1}) + d) - \frac{c_4}{c_3}\exp(-c_3(t - t_{m+1})) + \frac{c_4}{c_3},
\end{aligned} \tag{25}$$

thus by induction, inequality (16) holds for all $t \in (t_k, t_{k+1}], k = 2, 3, \ldots$. Noting that

$$\begin{aligned}
-c_3(t - t_j - \sum_{i=j+1}^{k}\tau_i) &= -c_3(t - t_j - \sum_{i=j+1}^{k}(1-\delta)(t_i - t_{i-1})) \\
&= -c_3(t - t_j - (1-\delta)(t_k - t_j)) \\
&= -c_3(\delta(t - t_j) + (1-\delta)(t - t_k)) \\
&\le -c_3\delta(t - t_j).
\end{aligned} \tag{26}$$

Moreover, by Definition 5, we have $N(t, t_0) = k, N(t, t_j) = k - j$, and $N(t, t_0) \le$

$\frac{t-t_0}{\tau^*} + N_0$, $N(t,t_j) \leq \frac{t-t_j}{\tau^*} + N_j$, we have

$$\begin{aligned}
\mathbb{E}V(t,x(t)) \leq (V_0 - \frac{c_4}{c_3}) \exp\Big(&- (c_3\delta - \frac{d}{\tau^*})(t-t_0) + N_0 d\Big) \\
&+ \frac{c_4}{c_3} \sum_{j=1}^{N(t,t_1)} \exp\Big(- (c_3\delta - \frac{d}{\tau^*})(t-t_j) + (N_j+1)d\Big) \\
&- \frac{c_4}{c_3} \sum_{j=1}^{N(t,t_1)} \exp\Big(- (c_3\delta - \frac{d}{\tau^*})(t-t_j) + (N_j)d\Big) \\
&+ \frac{c_4}{c_3} \exp(-c_3(t-t_k)+d) - \frac{c_4}{c_3}\exp(-c_3(t-t_k)) + \frac{c_4}{c_3},
\end{aligned} \tag{27}$$

thus $\lim_{t\to+\infty} \mathbb{E}V(t,x(t)) \leq \frac{c_4}{c_3}$, using condition (ii), we get

$$\lim_{t\to+\infty} \mathbb{E}\|x(t)\|^2 \leq \frac{c_4}{c_3 c_1}, \tag{28}$$

in terms of Definition 3, system (11) is UUB in mean square with ultimate bound $\frac{c_4}{c_3 c_1}$, the proof is completed.

4. Applications

New criteria for practical synchronization are derived in this section, the response systems are perturbed by additive fBm noises. We consider the following chaotic system:

$$dx(t) = Ax(t)dt + Bf(t,x(t))dt, t \neq t_k, \tag{29}$$

where $A, B \in \mathbb{R}^{n\times n}$, the map $f : \mathbb{R}^n \to \mathbb{R}^n$ satisfies Lipschitz condition, i.e., $|f(x(t)) - f(y(t))| \leq L|x(t) - y(t)|, \forall x(t), y(t) \in \mathbb{R}^n$, where $L = \text{diag}\{l_1, l_2, ..., l_n\}$ is a nonnegative constant matrix.

The corresponding response systems are considered in the form

$$\begin{cases} dy(t) = Ay(t)dt + Bf(t,y(t))dt + \sigma(t)dB^H(t), t = t_k, \\ \Delta y(t) = u(t^-), t = t_k, \end{cases} \tag{30}$$

where $\Delta y(t) = y(t^+) - y(t^-)$ and $\sigma(t) \leq \sigma$.

Remark 2. We can find some possible applications in real world in which the response system is perturbed by delayed impulses and additive fBm noises. For example, in the missile tracking system, the disturbance may be caused due to abrupt strong turbulent airflow and propagation delay, this part of disturbance can be modeled by delayed impulses. Meanwhile, the response system can be perturbed by environmental noise, this part of noise can be modeled by fBm.

Let $e(t) = y(t) - x(t)$, consider the impulsive controller with delay in the form

$$u(t^-) = Ke(t^- - \tau) - e(t^-), \quad t = t_k, \tau = \tau_k, \tag{31}$$

where the constant matrix K is the control gain.

Then the error system is given by:

$$\begin{cases} de(t) = Ae(t)dt + B\tilde{f}(t, e(t))dt + \sigma dB^H(t), t \neq t_k, \\ e(t^+) = Ke(t^- - \tau), t = t_k, \tau = \tau_k, \end{cases} \tag{32}$$

where $\tilde{f}(t, e(t)) = f(t, y(t)) - f(t, x(t))$. In what follows, the sufficient conditions to assure the error system UUB in mean square is established.

Theorem 3. Assume that there exist constants $\epsilon > 0, c_4 > 0$, $d > 0$, $\delta > 0$ and $\lambda > c_4$ such that

$$\begin{pmatrix} A + A^T + \epsilon L^T L - c_4 I & B \\ * & -\epsilon I \end{pmatrix} < 0, \tag{33}$$

and

$$\begin{pmatrix} -\exp(d) & K^T \\ * & -I \end{pmatrix} < 0, \tag{34}$$

and

$$\tau^* > \frac{d}{(\lambda - c_4)\delta}. \tag{35}$$

Then system (32) is UUB in mean square.

Proof. Chose the Lyapunov candidate $V(t, e(t)) = \exp(-\lambda t)e^T(t)e(t)$, where $\lambda > c_4$. Denote by $c_5 = \sup \exp(-\lambda t)t^{2H-1}$. It is obvious that $V(t, e(t))$ satisfy condition (i) and condition (ii′) in Theorem 2. Moreover, in terms of Lemma 1 and LMI (33), we have

$$
\begin{aligned}
L^H V(t, e(t)) = & -\lambda \exp(-\lambda t) e^T(t) e(t) + 2\exp(-\lambda t) e^T(t)[Ae(t) \\
& + B\tilde{f}(t, e(t))] + 2\sigma^2 \exp(-\lambda t) \int_0^t \phi(t, s) ds \\
\leq & -\lambda \exp(-\lambda t) e^T(t) e(t) + \exp(-\lambda t)[e^T(t)(A + A^T) e(t) \\
& + \epsilon^{-1} e^T(t) B B^T e(t) + \epsilon F^T(e(t)) F(e(t))] \\
& + 2\sigma^2 \exp(-\lambda t) t^{2H-1} \\
\leq & -\lambda \exp(-\lambda t) e^T(t) e(t) + \exp(-\lambda t)[e^T(t)(A + A^T \\
& + \epsilon^{-1} B B^T + \epsilon L^T L) e(t)] + 2\sigma^2 \exp(-\lambda t) t^{2H-1} \\
\leq & -(\lambda - c_4) \exp(-\lambda t) e^T(t) e(t) + 2\sigma^2 \exp(-\lambda t) t^{2H-1} \\
\leq & -(\lambda - c_4) V(t, e(t)) + 2\sigma^2 c_5.
\end{aligned}
\tag{36}
$$

On the other hand, by virtue of LMI (34), we have $K^T K - \exp(d) < 0$, which implies that

$$
\begin{aligned}
V(t^+, e(t^+)) &= \exp(-\lambda t) e^T(t^- - \tau) K^T K e(t^- - \tau) \\
&\leq \exp(d) \exp(-\lambda t) e^T(t^- - \tau) e(t^- - \tau) \\
&\leq \exp(d) \exp(-\lambda(t - \tau)) e^T(t^- - \tau) e(t^- - \tau) \\
&= \exp(d) V(t^- - \tau, e(t^- - \tau)),
\end{aligned}
\tag{37}
$$

it then follows from inequality (35) Theorem 2 that the error system (32) is UUB in mean square, this completes the proof.

Example 1. Consider the following Chua chaotic oscillator.

$$
\begin{cases}
dx_1(t) = [u(x_2(t) - x_1(t) + f(x_1(t))]dt, \\
dx_2(t) = [x_1(t) - x_2(t) + x_3(t)]dt, \\
dx_3(t) = [-vx_2(t)]dt,
\end{cases}
\tag{38}
$$

where nonlinear function $f(x_1(t)) = m_1 x_1(t) + 0.5(m_0 - m_1)\big(|x_1(t)+1| - |x_1(t) - 1|\big)$ and u, v, m_0, m_1 are given constants.

Rewrite system (38) of the form

$$
dx(t) = [Ax(t) + Bf(x(t))]dt,
\tag{39}
$$

where $x=(x_1,x_2,x_3)^T$,

$$A=\begin{pmatrix} -u+um_1 & u & 0 \\ 1 & -1 & 1 \\ 0 & -v & 0 \end{pmatrix}, \quad B=u(m_0-m_1)E,$$

$$f(x(t))=\begin{pmatrix} f_1(x_1(t)) \\ 0 \\ 0 \end{pmatrix},$$

in which $f_1(x_1(t))=0.5(|x_1(t)+1|-|x_1(t)-1|)$.

Chose the parameters with (cf.[38]) $u=10$, $v=18.432$, $m_0=-1.4554$, $m_1=-0.7853$, the initial condition is given by $x(0)=(0.11,0.2,-0.3)^T$, with the above parameters, system (39) is chaotic with double scroll attractor as shown in Fig. 1.

Consider system (39) as the drive system, consider system (30) (in which $t_k=k,\sigma=1,H=0.85$) as the response system, consider system (32) as the error system. We will show that system (39) and system (30) achieving synchronization with the control gain K. Firstly, choose the Lipschitz matrix $L=\text{diag}\{1,0,0\}$, chose $c_4=0.02$ such that LMI (33) holds, let $d=0.01$, then the control gain can be chosen as $K=\sqrt{d}I=0.1E$, by simple calculation, we can choose suitable $\lambda>c_4$ such that inequality (35) holds for all $\delta\in(0,1]$, then system (30) and system (39) achieve synchronization. Fig.2 shows the evolution of each variable of system (39) and system (30). Fig.3 depicts the trajectory of synchronization error $\|e(t)\|^2=|e_1(t)|^2+|e_2(t)|^2+|e_3(t)|^2$ under the conditions $K=0.1E$ and $\tau=0.1$.

5. Conclusion

In this paper, the stability, boundedness and synchronization for delayed impulsive SDEs with fBm have been investigated. Two kinds of noises, i.e, additive fBm noise and multiplicative fBm noise are both taken in to consideration. Sufficient criteria for pth moment exponential stability and mean square

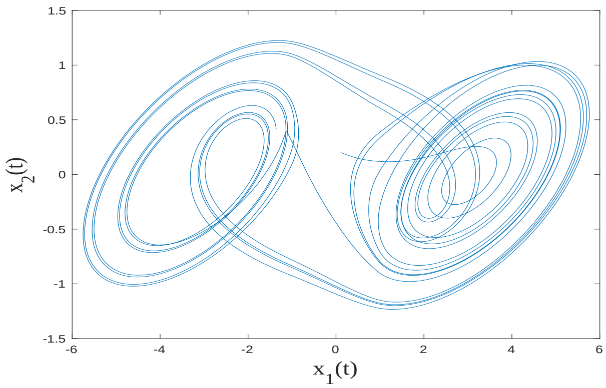

Fig. 1: Chaotic attractor of system(39).

ultimate boundedness have been derived, for two kinds of fBm driven stochastic differential equations with delayed impulses, respectively. As application, the obtained results are used to do practical synchronization with respect to a class of chaotic systems, in which the response systems are perturbed by additive fBm noises. The corresponding impulsive controllers are also designed. Examples have been given to illustrate the validity of the derived results at length.

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Compliance with ethical standards

Conflict of Interest: The authors declare that they have no conflict of interest.

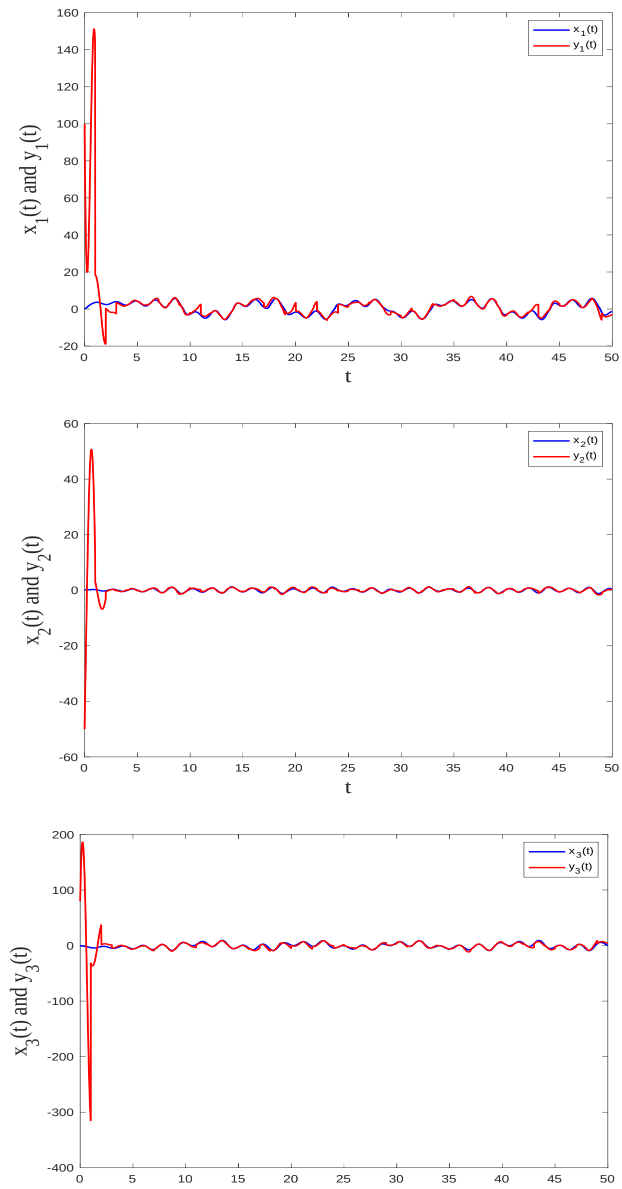

Fig. 2: The evolution of each variable of drive system (39) and response system (30).

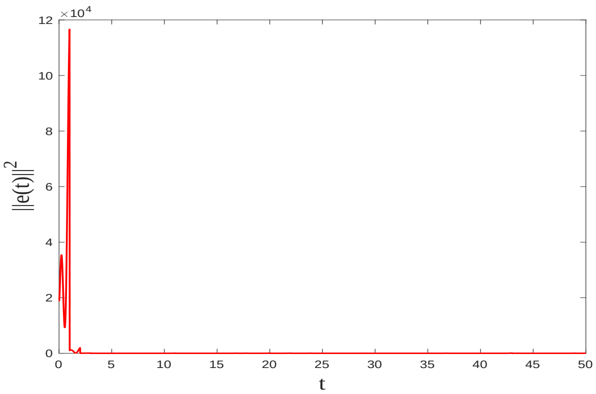

Fig. 3: Trajectory of synchronization error $\|e(t)\|^2$ when $K = 0.1I, \tau = 0.1$

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