A fractional-order love dynamical model with time delay for synergic couple : Stability analysis and Hopf bifurcation

S. Panigrahi[1,*], S. Chand[2], S. Balamuralitharan[3]

1 Department of Mathematics, Siksha 'O' Anusandhan(Deemed to be University), Khandagiri Square, Bhubaneswar−751030, Odisha, India

2 Center for Applied Mathematics and Computing, Siksha 'O' Anusandhan(Deemed to be University), Khandagiri Square, Bhubaneswar-751030, Odisha, India

3 Bharath Institute of Higher Education and Research (BIHER) Selaiyur-600073, TamilNadu, INDIA

Abstract

We investigate the fractional order love dynamic model with time delay for synergic couples in this manuscript. The quantitative analysis of the model has been done where the asymptotic stability of the equilibrium points of the model have been analyzed. Under the impact of time delay, the Hopf bifurcation analysis of the model has been done. The stability analysis of the model has been studied with the reproduction number less than or greater than 1. By using Laplace transformation, the analysis of the model has been done. The analysis shows that the fractional order model with a time delay can sufficiently improve the components and invigorate the outcomes for either stable or unstable criteria. In this model, all unstable cases are converted to stable cases under neighbourhood points. For all parameters, the reproduction ranges have been described. Finally, to illustrate our derived results numerical simulations have been carried out by using MATLAB. Under the theoretical outcomes from parameter estimation, the love dynamical system is verified.

Keywords: Love dynamics; Stability, Hopf bifurcation; Time-delay; Fractional differential equation ;Caputo fractional derivative.

Mathematics Subject Classification (2010) : $93A30; 37N35; 34D20; 34E05; 34K28; 34K37$

1. Introduction

In 1988 and 1944 Stogatz [18],[19] discussed a love affair model in his paper and book respectively. After that several researchers such as Rinaldi[10], Rinaldi and Gragnani[11], Gragnani et.al[4], Rinaldi[12] have proposed more realistic mathematical models for love dynamics. Sprott[17] discussed the dynamics of a love triangle which produce a chaotic behaviour. Wauer et. al[21] studied the love dynamics for time varying fuctuations. Rinaldi

Email addresses: santoshi.panigrahi1994@gmail.com (S. Panigrahi[1,*]), mami_chand@yahoo.co.in (S. Chand[2]), balamuralitharan.maths@bharathuniv.ac.in (S. Balamuralitharan[3])

et. al[13] constructed full catalog of possible love stories among two individuals.

Delay differential equations and fractional differential equations have gained considerable importance due to their applications in various fields such as biology, chemistry, neural systems, etc. The stability and bifurcation analysis of Ebola Virus and Corona virus models with time delay have done by Liu and Zhang[6], Radha and Balamuralitharan[9] respectively. By using predictor-corrector method Bhalekar and Daftardar-Gejji[1] have solved nonlinear delay differential equations of fractional order. The constructions of next generation matrices for compartmental epidemic models have discussed in Diekmann et al.[2]. The development on fractional order and partial differential equations can be found in the monograph by Podlubny[8] and the references therein. Gomez-Aguilar[3] have discussed the analytical and numerical solutions of nonlinear alcoholism model in his paper. Sene[15] have studied the solutions of fractional diffusion equations and Cattaneo-Hristov diffusion model in his paper. Liao and Ran[5] have studied the love dynamical models with nonlinear couples and time delays. Son and Park[16] have investigated the time delay effect on the love dynamical model for both synergic couple and non-synergic couple.

Many researchers are also working in the field of fractional delay differential equations. The different fractional order models with time delay have studied in Preethi Latha et. al.[7] , Wang et. al[20], Santoshi et. al [14].

For the analysis of complex mathematical models, stability analysis is an established tool.The solutions and their stability change as the parameters in the system vary is understood by Hopf bifurcation analysis of a dynamical system. This motivates us to do the stability analysis and Hopf bifurcation analysis of the Love dynamical model for synergic couples. The summary of main contribution of the new results in this paper are described below:

- In this manuscript, we formulate a fractional order love dynamical model with time delay for synergic couple.
- We discuss the asymptotic stability of the equilibrium points of the model by using Laplace transformation,.
- We analyze the Hopf bifurcation of the model.
- By next generation matrix method, we find the reproduction number R_0.
- Reproduction number show that the system is stable and controllable without Hopf bifurcation.
- Finally, we do the numerical simulations to verify the theoretical results by using predictor-corrector scheme.

As far as we could possibly know, there is no such writing present in which the stability analysis and Hopf bifurcation of the fractional order love dynamical model with time delay has been done.

The rest of this paper is coordinated as follows. In section 2, the fractional order love dynamical model with time delay is formulated. The model is stabilized by Laplace transformation strategy and the Hopf bifurcation analysis for the model has been done in section 3. Moreover, the reproduction number for the system has been done and the stability and controllability criteria of the system have been studied. Section 4 consists the numerical simulations which illustrate our theoretical results. Concluding comments with overall discussions are given in section 5.

2. Model Formulation

Son and Park[16] have defined the following love dynamical model for synergic couple,

$$x_1'(t) = -\alpha_1 x_1(t) + R_1(x_2(t-\tau)) + \{1 + S_1(x_1(t))\}\gamma_1 A_2 \tag{1}$$

$$x_2'(t) = -\alpha_2 x_2(t) + R_2(x_1(t-\tau)) + \{1 + S_2(x_2(t))\}\gamma_2 A_1 \tag{2}$$

Where $x_i (i = 1, 2)$ is the measure of the love of an individuals for his/her partner j $(j = 2, 1)$, $\alpha_i > 0$ is a forgetting coefficient (i.e. x_i decays exponentially when an individual loses his partner) for $i = 1, 2$, $R_i(i = 1, 2)$ is the return function (Reaction of i to the partner's love x_j), $\gamma_i > 0$ is reactiveness to the appeal for $i = 1, 2$, $A_j(j = 1, 2)$ is the partner's appeal and $S_i(x_i)$ is the synergic function.

Rinaldi and Gragnani[11] have specified the secure return i.e.

$$R_i^s(x_j) = \begin{cases} \beta_i x_j/(1 + x_j) & for \;\; x_j \geq 0, \\ \beta_i x_j/(1 - x_j) & for \;\; x_j < 0. \end{cases} \tag{3}$$

Gragnani et.al[4] have specified the non-secure return i.e.

$$R_i^n(x_j) = \begin{cases} \beta_i x_j(1 - x_j^8)/\{(1 + x_j)(1 + x_j^8)\} & for \;\; x_j \geq 0, \\ \beta_i x_j/(1 - x_j) \;\; for \;\; x_j < 0. \end{cases} \tag{4}$$

where $\beta_i > 0$ is a reactiveness to the love. Gragnani[4] has described the synergic function as

$$S_i(x_i) = \begin{cases} \sigma_i x_i^8/(1 + x_i^8) & for \;\; x_i \geq 0, \\ 0 \;\; for \;\; x_i < 0. \end{cases} \tag{5}$$

where σ_i is the synergism coefficient.

Here, we have discussed the fractional order love dynamical model with time delay and checked the long term behaviour of the synergic couple. The model is,

$$D^{\alpha}x_1(t) = -\alpha_1 x_1(t) + R_1(x_2(t-\tau)) + \{1 + S_1(x_1(t))\}\gamma_1 A_2 \tag{6}$$
$$D^{\alpha}x_2(t) = -\alpha_2 x_2(t) + R_2(x_1(t-\tau)) + \{1 + S_2(x_2(t))\}\gamma_2 A_1 \tag{7}$$

with initial functions $x_1(t) = \phi_1(t)$, $x_2(t) = \phi_2(t)$, $t \in [-\tau, 0]$.

Where $x_i (i = 1, 2)$ is the measure of the love of an individuals for his/her partner j $(j = 2, 1)$. $\alpha_i > 0$ is a forgetting coefficient (i.e. x_i decays exponentially when an individual loses his partner) where $i = 1, 2$. $R_i(i = 1, 2)$ is the return function (Reaction of i to the partner's love x_j). $\gamma_i > 0$ is reactiveness to the appeal where $i = 1, 2$. $A_j(j = 1, 2)$ is the partner's appeal.

[8] D^{α} denotes the Caputo fractional order derivative and defined as follows,

$$D^{\alpha}f(t) = \frac{1}{\Gamma(\alpha - n)} \int_0^t \frac{f^{(n)}(\tau)}{(t-\tau)^{\alpha+1-n}} d\tau$$

where $\alpha \in R$, $n - 1 < \alpha < n$, $n \in N$ and f is a continuous function.

3. Quantitave analysis

3.1. Stability Analysis

Let (x_1^*, x_2^*) be an equilibrium point (1-2) .
So,

$$-\alpha_1 x_1^* + R_1(x_2^*) + (1 + S_1(x_1^*))\gamma_1 A_2 = 0 \tag{8}$$

$$-\alpha_2 x_2^* + R_2(x_1^*) + (1 + S_2(x_2^*))\gamma_2 A_1 = 0 \tag{9}$$

Let $\xi_1 = x_1 - x_1^*$, $\xi_2 = x_2 - x_2^*$, $x_{1,\tau} = x_1(t-\tau)$, $x_{2,\tau} = x_2(t-\tau)$, $\xi_{1,\tau} = \xi_1(t-\tau)$, $\xi_{2,\tau} = \xi_2(t-\tau)$.

By using above transformations, we will find the following linearized equation for (1-2).

$$D^{\alpha}(\xi_1) = R_1'(x_2^*)\xi_{2,\tau} - \alpha_1\xi_1 + \gamma_1 A_2 S_1'(x_1^*)\xi_1 \tag{10}$$

$$D^{\alpha}(\xi_2) = R_2'(x_1^*)\xi_{1,\tau} - \alpha_2\xi_2 + \gamma_2 A_1 S_2'(x_2^*)\xi_2 \tag{11}$$

Let's take the Laplace transform on both sides of (10-11),

$$s^{\alpha}X_1(s) = s^{\alpha-1}\phi_1(0) - \alpha_1 X_1(s) + R_1'(x_2^* e^{-s\tau})X_2(s) + R_1'(x_2^*)e^{-s\tau}\int_{-\tau}^{0} e^{-st}\xi_2(t)dt + \gamma_1 A_2 S_1'(x_1^*)X_1(s) \tag{12}$$

$$s^{\alpha}X_2(s) = s^{\alpha-1}\phi_2(0) - \alpha_2 X_2(s) + R_2'(x_1^* e^{-s\tau})X_1(s) + R_2'(x_1^*)e^{-s\tau}\int_{-\tau}^{0} e^{-st}\xi_1(t)dt + \gamma_2 A_1 S_2'(x_2^*)X_2(s) \tag{13}$$

where $L[\xi_1(t)] = X_1(s)$, $L[\xi_2(t)] = X_2(s)$, $\bar{\xi_1}(0) = \phi_1(0)$, $\bar{\xi_2}(0) = \phi_2(0)$.

Then (12-13) can be written as

$$\begin{bmatrix} s^\alpha + \alpha_1 - a_1 & -ae^{-s\tau} \\ -be^{-s\tau} & s^\alpha + \alpha_2 - a_2 \end{bmatrix} \begin{bmatrix} X_1(s) \\ X_2(s) \end{bmatrix} = \begin{bmatrix} s^{\alpha-1}\phi_1(0) + R_1'(x_2^*)e^{-s\tau}\int_{-\tau}^0 e^{-st}\xi_2(t)dt \\ s^{\alpha-1}\phi_2(0) + R_2'(x_1^*)e^{-s\tau}\int_{-\tau}^0 e^{-st}\xi_1(t)dt \end{bmatrix} \quad (14)$$

Let $\triangle\ (s) = \begin{bmatrix} s^\alpha + \alpha_1 & -ae^{-s\tau} \\ -be^{-s\tau} & s^\alpha + \alpha_2 \end{bmatrix}$

where $a_1 = \gamma_1 A_2 S_1'(x_1^*)$, $a_2 = \gamma_2 A_1 S_2'(x_2^*)$ $a = R_1'(x_2^*)$, $b = R_2'(x_1^*)$

$\triangle\ (s)$ is considered as characteristic matrix of system ((1)-(2)). So, the characteristic equation is ,

$$F(s) = s^{2\alpha} + (\alpha_1 + \alpha_2 - a_1 - a_2)s^\alpha + (\alpha_1\alpha_2 - \alpha_1 a_2 - a_1\alpha_2 + a_1 a_2) - abe^{-2s\tau} = 0. \quad (15)$$

Let $s = iv$. So the (15) becomes

$$(iv)^{2\alpha} + (\alpha_1 + \alpha_2 - a_1 - a_2)(iv)^\alpha + (\alpha_1\alpha_2 - a_1\alpha_2 - \alpha_1 a_2 + a_1 a_2) - abe^{-2iv\tau} = 0 \quad (16)$$

By putting $s = v(cos(\frac{\pi}{2}) + isin(\frac{\pi}{2})), v > 0$ and separating real and imaginary parts of the above equation , we get

$$v^{2\alpha}cos\alpha\pi + (\alpha_1 + \alpha_2 - a_1 - a_2)v^\alpha cos\frac{\alpha\pi}{2} = abcos2v\tau - (\alpha_1\alpha_2 - a_1\alpha_2 - \alpha_1 a_2 + a_1 a_2) \quad (17)$$

$$v^{2\alpha}sin\alpha\pi + (\alpha_1 + \alpha_2 - a_1 - a_2)v^\alpha sin\frac{\alpha\pi}{2} = -absin2v\tau \quad (18)$$

Squaring and adding ((17)-(18)), we get

$$\begin{aligned} & v^{4\alpha} + (\alpha_1 + \alpha_2 - a_1 - a_2)^2 v^{2\alpha} - (a^2b^2) - (\alpha_1\alpha_2 - a_1\alpha_2 - \alpha_1 a_2 + a_1 a_2)^2 \\ + \quad & 2v^{3\alpha}(\alpha_1 + \alpha_2 - a_1 - a_2)cos(\frac{\alpha\pi}{2}) = -2ab\alpha_1\alpha_2 cos2v\tau \end{aligned}$$

$$\tau = \frac{1}{2v}\left[2n\pi \pm cos^{-1}\left\{\frac{\left(v^{4\alpha}+(\alpha_1+\alpha_2-a_1-a_2)^2v^{2\alpha}-(a^2b^2)-(\alpha_1\alpha_2-a_1\alpha_2-\alpha_1a_2+a_1a_2)^2+2v^{3\alpha}(\alpha_1+\alpha_2-a_1-a_2)cos(\frac{\alpha\pi}{2})\right)}{-2ab(\alpha_1\alpha_2-a_1\alpha_2-\alpha_1a_2+a_1a_2)}\right\}\right],\ n = 0,1,2,...$$

Thus

$$\tau_1 = \frac{1}{2v}\left[2n\pi + cos^{-1}\left\{\frac{\left(v^{4\alpha}+(\alpha_1+\alpha_2-a_1-a_2)^2v^{2\alpha}-(a^2b^2)-(\alpha_1\alpha_2-a_1\alpha_2-\alpha_1a_2+a_1a_2)^2+2v^{3\alpha}(\alpha_1+\alpha_2-a_1-a_2)cos(\frac{\alpha\pi}{2})\right)}{-2ab(\alpha_1\alpha_2-a_1\alpha_2-\alpha_1a_2+a_1a_2)}\right\}\right]\quad n = 0,1,2,..$$

$$\tau_2 = \frac{1}{2v}\left[2n\pi - cos^{-1}\left\{\frac{\left(v^{4\alpha}+(\alpha_1+\alpha_2-a_1-a_2)^2v^{2\alpha}-(a^2b^2)-(\alpha_1\alpha_2-a_1\alpha_2-\alpha_1a_2+a_1a_2)^2+2v^{3\alpha}(\alpha_1+\alpha_2-a_1-a_2)cos(\frac{\alpha\pi}{2})\right)}{-2ab(\alpha_1\alpha_2-a_1\alpha_2-\alpha_1a_2+a_1a_2)}\right\}\right],\quad n = 0,1,2,...$$

Let $\tau_0 = \min\{\tau_1, \tau_2\}$.

Theorem 3.1.1: The equilibrium point (x_1^*, x_2^*) of the love dynamical model (6)-(7) is asymptotical stable when $\tau < \tau_0$.

3.2. Hopf Bifurcation Analysis

By differentiating (15) with respect to τ, we get

$$\frac{ds}{d\tau} = \frac{-2sabe^{-2s\tau}}{2\alpha s^{2\alpha-1} + (\alpha_1 + \alpha_2 - a_1 - a_2)\alpha s^{\alpha-1} + 2ab\tau e^{-2s\tau}} \tag{19}$$

Now consider the numerator term,

$$\begin{aligned} -2sabe^{-2s\tau} &= -2abvsin2v\tau - i2abvcos2v\tau \\ &= M_1 + iM_2. \end{aligned}$$

Where $M_1 = -2abvsin2v\tau$, $M_2 = -2abvcos2v\tau$.

Now we consider the denominator term

$$\begin{aligned} & 2\alpha s^{2\alpha-1} + (\alpha_1 + \alpha_2 - a_1 - a_2)\alpha s^{\alpha-1} + 2ab\tau e^{-2s\tau} \\ = & \left(2\alpha v^{2\alpha-1} cos\frac{(2\alpha-1)\pi}{2} + (\alpha_1 + \alpha_2 - a_1 - a_2)\alpha v^{\alpha-1} cos\frac{(\alpha-1)\pi}{2} + 2ab\tau cos2v\tau\right) \\ + & i\left(2\alpha v^{2\alpha-1} sin\frac{(2\alpha-1)\pi}{2} + (\alpha_1 + \alpha_2 - a_1 - a_2)\alpha v^{\alpha-1} sin\frac{(\alpha-1)\pi}{2} - 2ab\tau sin2v\tau\right) \\ = & N_1 + iN_2 \end{aligned}$$

where,

$$N_1 = 2\alpha v^{2\alpha-1} cos\frac{(2\alpha-1)\pi}{2} + (\alpha_1 + \alpha_2 - a_1 - a_2)\alpha v^{\alpha-1} cos\frac{(\alpha-1)\pi}{2} + 2ab\tau cos2v\tau$$

$$N_2 = 2\alpha v^{2\alpha-1} sin\frac{(2\alpha-1)\pi}{2} + (\alpha_1 + \alpha_2 - a_1 - a_2)\alpha v^{\alpha-1} sin\frac{(\alpha-1)\pi}{2} - 2ab\tau sin2v\tau$$

So, (19) becomes,

$$\left(\frac{ds}{d\tau}\right)|_{\tau=\tau_0, v=v_0} = \frac{M_1 + iM_2}{N_1 + iN_2}$$

$$Re\left(\frac{ds}{d\tau}\right)|_{\tau=\tau_0, v=v_0} = \frac{M_1N_1 + M_2N_2}{N_1^2 + N_2^2}.$$

From the above discussion we get the following Theorem.

Theorem 3.2.1: The love dynamical model from (6)-(7) has Hopf bifurcation when $\tau = \tau_0$.

By using next generation matrix method[2], the basic reproduction number R_0 can be calculated for the synergic love dynamical model (6)-(7). After linearizing the system we get R_0 for the system (6)-(7) is $\frac{\alpha_1\gamma_2 A_1}{\alpha_2}$.

Note:

- When $R_0 < 1$, the love dynamical model (6)-(7) is stable.
- When $R_0 > 1$, the love dynamical model (6)-(7) is unstable.

4. Numerical Simulations

By using predictor-corrector scheme[1], we have done the numerical simulation which illustrates our result in this section. Graphs are done by using MATLAB.Here $\alpha_1 = 0.3, \alpha_2 = 0.5, \gamma_1 = 0.01, \gamma_2 = 0.02, A_1 = 25, A_2 = 50, \beta_1 = 1, \beta_2 = 1.1, \sigma_1 = 0.4, \sigma_2 = 0.3$. To calculate the critical value and the bifurcation point, let's choose $v_0 = 0.1$. So, $\tau_0 = 1.5$. The phase diagram of fractional order system (6)-(7) shows in figure 1, which is asymptotically stable, when $\tau = 0.9(< \tau_0)$ and $\alpha = 0.5$ with initial values (0.05,0.03). We have changed all the parameter values for the stability criteria. The parameter estimation was carried out from the previous studies. The model (6)-(7) undergoes Hopf bifurcation which is depicted in figure 2 for $\tau = 1.5(= \tau_0)$ and $\alpha = 0.5$ with initial values (0.05,0.03). The numerical simulations of the system (6)-(7) have showed in figure 3 and also it calculates the ranges of reproduction number R_0. R_0 portrays how decreasing the transmission rate can change the system dynamics from the limit cycle to stable focus. All ranges of reproduction number value have been zoned in figure 3 and shows that the system is stable and controllable without Hopf bifurcation. Figure 4 shows the diagrams of the system (6)-(7) for parameter estimation of α_1, α_2. Figure 5 shows the diagrams of the system (6)-(7) for parameter estimation of γ_1, γ_2.

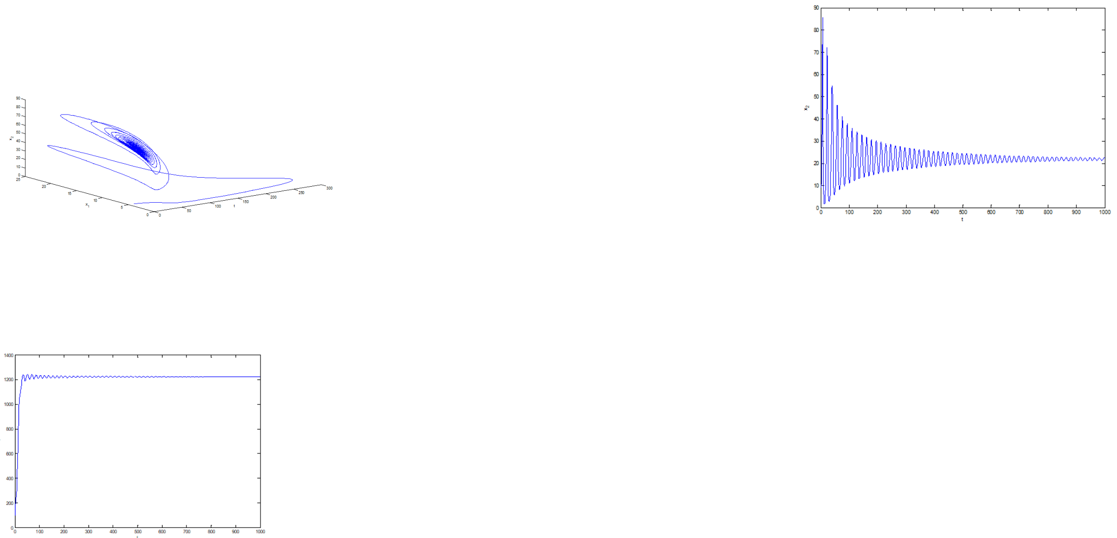

Figure 1: When $\tau = 0.9$, the phase diagrams of the system (6)-(7) is asymptotically stable

Figure 2: When $\tau = 1.5$, the phase diagrams of the system (6)-(7) undergoes Hopf bifurcation

Figure 3: The ranges of Reproduction number R_0 for the system (6)-(7)

Figure 4: Diagrams for the system (6)-(7) for parameter estimation of $\alpha_1 = 0.3$ and $\alpha_2 = 0.5$

Figure 5: Diagrams for the system (6)-(7) for parameter estimation of $\gamma_1 = 0.01$ and $\gamma_2 = 0.02$

5. Conclusion

In this manuscript, we have analyzed a fractional order time delay love dynamical model for synergic couple. By using the Laplace transform technique, the stability of the model has been analyzed. We have found a formula which gives the critical value τ_0. The equilibria points will be asymptotically stable for all positive time delay values under some conditions have been demonstrated. As per our study, When the time delay greater than the critical value $\tau > \tau_0$, the model (6)-(7) undergoes Hopf bifurcation. The system (6)-(7) is stable under the parameter values when $R_0 = \frac{\alpha_1 \gamma_2 A_1}{\alpha_2} < 1$. The system (6)-(7) is unstable under the parameter values, when $R_0 = \frac{\alpha_1 \gamma_2 A_1}{\alpha_2} > 1$. The accuracy and efficiency of the model have been proved by numerical simulations. The theoretical results have been illustrated by the numerical simulations. When $\xi < \xi_0$, the system (6)-(7) is asymptotically stable(see figure 1). When $\xi = \xi_0$, the system damps the oscillatory behavior of solutions and undergoes Hopf bifurcation(see figure 2). Without Hopf bifurcation the system is stable and controllable(see figure 3). Figure 4 and 5 describes the parameter estimations of the system.

References

[1] Bhalekar, S., Daftardar-Gejji, V., *A predictor-corrector scheme for solving delay differential equations of fractional order*, Journal of Fractional Calculus and Applications , 1(5)(2011)1-9.

[2] Diekmann, O., Heesterbeek, J. A. P., Roberts, M. G., *The construction of next-generation matrices for compartmental epidemic models*, Journal of the royal society interface, 7(2010) 873–885.

[3] Gomez-Aquilar, J. F., *Analytical and numerical solutions of a nonlinear alcoholism model via variable order fractional differential equations*, Physics A, 494(2018) 52-75.

[4] Gragnani, A., Rinaldi, S. and Feichtinger, G., *Cyclic dynamics in romantic relationships*, International Journal of Bifurcation and Chaos, 7(11)(1997), 2611-2619.

[5] Liao , X., Ran, J., *Hopf bifurcation in love dynamical models with nonlinear couples and time delays*, Chaos, Soliton and Fractals, 31(2007) 853-865.

[6] liu, H., Zhang, Z., *Dynamics of two time delays differential equation model to HIV latent infection*, Physics A 514(2019) 384-395.

[7] PreethiLatha, V., Rihan, F. A., Rakkiyappan, R., Velmurugan, G., *A fractional order delay differential model for Ebola infection and CD8+ T cells response: Stability analysis and Hopf bifurcation*, International Journal of Biomathematics, 10(8)(2017) 22 pages.

[8] Podlubny, I., *Fractional differential equations*, Academic Press, California, USA, 1999.

[9] Radha, M., Balamuralitharan, S., *A study on COVID-19 transmission dynamics: Stability analysis of SEIR model with Hopf bifurcation for effect of time delay*, Advances in difference equations, (2020) 2020:523 .

[10] Rinaldi, S., *Love dynamics: The case of linear couple*, Applied Mathematics and Computation, 95(1998) 181-192.

[11] Rinaldi, S., Gragnani, A., *Love Dynamics between secure individuals : A modeling Approach*, Nonlinear Dynamics, psychology and Life sciences, 2 (4) (1998).

[12] Rinaldi, S. , *Laura and Petrarch: An intriguing case of cyclic love dynamics*, SIAM Journal on Applied Mathematics , 58(4)(1998) 1205-1221.

[13] Rinaldi, S. , Rossa, F.D. and Derecole, F., *Love and appeal in standard couples*, International Institute for Applied Systems Analysis, Interim report, IR-10-047 (2011).

[14] Panigrahi, S., Chand, S. and Balamuralitharan, S., *Stability and Hopf bifurcation analysis of fractional-order nonlinear financial system with time delay.* Math Meth Appl Sci 2021;1-11. https://doi.org/10.1002/mma.7705 .

[15] Sene, N., *Solutions of fractionals diffusion equations and Cattaneo-Hristov diffusion model*, International journal of Analysis and Applications, 17 (2)(2019) 191-207 .

[16] Son, W., Park, Y. , *Time delay effect on the love dynamical model*, Journal ofbthe Korean Physical Society, 59(3)(2011), 2197-2204.

[17] Sprott, J. C., *Dynamical models of Love*, Nonlinear Dynamics, psychology and Life Sciences, 8, (3)(2004).

[18] Strogatz, S. H., *Love affairs and differential equations*, Mathematics Magazine, 61 (35)(1988).

[19] Strogatz, S.H, *Nonlinear Dynamics and Chaos*, Persus Books, Reading, 1994.

[20] Wang, H., Yu, Y. , Wen, G. , Zhang, S., Yu, J. *Global stability analysis of fractional order Hopfield neural networks with time delay*, Neuro computing, 154(2015) 15-23.

[21] Wauer, J., Schwarzer, D., Cai, G. Q., Lin, Y.K. *Dynamical models of love with time varying fluctuations*, Applied Mathematics and Computation, 188 (2007) 1535-1548.